Surface Charges in Conductor Plates Carrying Constant Currents

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In this work we analyze the case of resistive conductor plates carrying constant currents, utilizing surface charge distributions. We obtain the electric potential in the plates and in the space surrounding them. We obtain a non-vanishing electric field outside the conductors. We compare the theoretical results with experimental data present in the literature.

1. INTRODUCTION

The existence of an electric field outside stationary resistive conductors carrying constant currents is an important subject neglected by most authors. In this work we calculate this field theoretically in a simple geometry, comparing our results with experimental data.

Consider a metallic circuit and a test charge placed near it. The stationary test charge will induce an electrostatic surface charge distribution on the circuit located next to it. This will give rise to an induced electrostatic electric field yielding an attraction between the circuit and the test charge. This electrostatic field is a zeroth-order effect, since it does not depend on the velocity (nor acceleration) between the free charge and the circuit. If the circuit is a large metallic plate and the test charge q is close to its center at a distance d from the plate then by the method of images the

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electrostatic force between them is given by $F = q^2/(16\pi\varepsilon_o d^2)$, where $\varepsilon_o = 8.85 \times 10^{-12} \, \mathrm{C^2 N^{-1} m^{-2}}$ is called the permittivity of free space.

What happens when we now pass a constant current through the stationary resistive plate connected to a battery? The electric field that maintains the current against Ohmic resistance is generated by a surface charge distribution on the plate. This surface charge distribution is maintained by the battery, and generates as well an electric field outside the conductor. This electric field is called of first-order (it is proportional to the current—or to the drifting velocity of the conduction charges). This last effect is the aim of this article. That is, our goal is to calculate the potential and electric field in the plate and in the space surrounding it when a constant current flows through the resistive plate.

There is already a number of cases presented in the literature discussing these surface charges: [1, pp. 125–133; 2; 3, pp. 336–337; 4–11]. Here we present other cases not considered in all their details previously.

We wish to emphasize here that this electric field outside a resistive current carrying wire exists according to standard (Maxwell's) theory. Here we obtain it utilizing the usual formulas for the potential and electric field found in most classical textbooks.

2. SINGLE PLATE

We consider the case of conducting plates from the point of view of surface charge distributions generating the electric fields.

The geometry we are considering is that of a rectangular plate of length $2L_x$ in the x direction and $2L_z$ in the z direction. The geometric center of the plate is located at $(x, y, z) = (0, y_0, 0)$, where y_0 is a constant. The plate is parallel to the xz plane, passing through $y = y_0$. We assume that the current flows uniformly from $-L_x$ to $+L_x$. We also assume that the surface charge density is linear along x,[12]

$$\sigma(x) = \alpha x + \beta \tag{1}$$

Note that in general the surface charge is a general function of the x and z coordinates, $\sigma = \sigma(x, z)$. We neglect the dependence on z as an approximation for $L_z \gg |\vec{r}|$, where $\vec{r}$ is the observation point.

The electric potential is readily given from the surface charge σ above by:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma \, da'}{|\vec{r} - \vec{r}'|}$$

where the integral is evaluated over the whole charge distribution. We are interested in the potential at the symmetric plane $z = 0$:

$$\phi(x, y, 0) = \frac{1}{4\pi\epsilon_0} \int_{-L_z}^{L_z} \int_{-L_x}^{L_x} \frac{\alpha x' + \beta}{\sqrt{(x-x')^2 + (y-y_0)^2 + z'^2}} \, dx' \, dz' \qquad (2)$$

We solve these integrals utilizing the approximation $L_z \gg L_x \gg \sqrt{x^2 + y^2}$. The final result is as follows:

$$\phi(x, y, 0) = \frac{1}{\epsilon_0} \left[(\alpha x + \beta) \left(\frac{L_x}{\pi} - \frac{|y - y_0|}{2} \right) + \frac{\beta L_x}{\pi} \ln \frac{2L_z}{L_x} \right.$$
$$\left. - \frac{x^2}{2\pi L_x} \left(\frac{2\alpha x}{3} + \beta \right) + \frac{(y - y_0)^2}{2\pi L_x} (2\alpha x + \beta) \right] \qquad (3)$$

We define the constants λ_1 and λ_2: $\lambda_1 = L_x/\pi$ and $\lambda_2 = (L_x/\pi) \ln(2L_z/L_x)$. With these constants we can write the electric potential for this single plate situated at the $y = y_0$ plane as given by:

$$\phi(x, y, 0) = \frac{1}{\epsilon_0} \left[(\alpha x + \beta) \left(\lambda_1 - \frac{|y - y_0|}{2} \right) + \beta \lambda_2 \right] \qquad (4)$$

In order to test the coherence of our procedure we revert the argument. From the electric potencial (4) we obtain the electric field by $\vec{E} = -\nabla \phi$. Applying Gauss's law to a small cylinder centered on the plate we obtain the usual boundary condition relating the normal component of the electric field, E_y, to the surface charge density, σ, namely: $\varepsilon_0 E_y(\lim y \to y_o^+) - \varepsilon_o E_y(\lim y \to y_o^-) = \sigma$. And this yields exactly the same charge distribution on the plates as that given by our starting point, Eq. (1). We checked our calculations by a similar procedure in the other cases.

3. TWO PARALLEL PLATES

With this result we are now able to analyze Jefimenko's experiments which proved the existence of an electric field outside stationary resistive conductors carrying constant currents.[13, 14] In his experiments he could show the existence of electric fields outside conductors by utilizing grass seeds as test-particles. These seeds, neutral in natural state, polarize themselves in the presence of an electric field and align along the electric field lines (in analogy with iron fillings which map magnetic fields). This was an

ingenious idea. With this procedure he could overcome the large electro-static force mentioned above, which would arise if he had placed a charged body near the conductor (the zeroth-order electrostatic force is usually much larger than the first-order one[8]). With the grass seeds the zeroth-order force does not appear and he was able to show conclusively the existence of the first-order electric field outside the conductor. We first consider Fig. 1 of [13] and Plate 6 of [14]. In these cases we have a constant current flowing uniformly along the x axis of a conductor of resistivity g in the form of a parallelepiped of lengths $2L_x$, $2a$ and $2L_z$. Accordingly there will be free charges only along its outer surfaces located at $y = \pm a$ (considering the thick conductor centered at $(x, y, z) = (0, 0, 0)$). At both sides the free charges will be given by Eq. (1). The superposition of the two charged planes situated in $y = a$ and $y = -a$, utilizing Eq. (3) and replacing y_0 by a or by $-a$, yields the potential in the plane $z = 0$ as given by:

$$\phi(x, y, 0) = \frac{1}{\epsilon_0}\left[(\alpha x + \beta)\left(2\lambda_1 - \frac{|y-a|+|y+a|}{2}\right) + 2\beta\lambda_2 \right] \qquad (5)$$

The electric field is readily given by $\vec{E} = -\nabla\phi$. The lines of electric field $\psi(x, y)$ such that $\nabla\psi \cdot \nabla\phi = 0$ can be obtained by the method described in Sommerfeld's book, [1, p. 128]. They are given by the following equation:

$$\psi(x, y, 0) = \begin{cases} x^2 + 2\beta x/\alpha - y^2 + 4\lambda_1 y, & y > a \\ -ay & -a < y < a \\ x^2 + 2\beta x/\alpha - y^2 - 4\lambda_1 y & y < -a \end{cases} \qquad (6)$$

In Fig. 1 we plot this function with the approximation $L_z/L_x = L_x/a = 6$, in order to have similar dimensions as in Jefimenko's experiment. This theoretical figure is very similar to Jefimenko's experimental one, giving support to our calculation.

Given Eq. (5) and the following boundary conditions:

$$\phi(x = -L_x, y = \pm a) = V/2 \qquad \text{and} \qquad \phi(x = L_x, y = \pm a) = -V/2$$

we can relate α and β with the given potential difference V, if necessary.

In Fig. 2 we plotted the equipotential lines for the two plates as given by Eq. (5) in the approximation $L_z/L_x = L_x/a = 6$. This can be compared with another experiment in which equipotential lines outside resistive conductors carrying constant currents were mapped utilizing an electronic
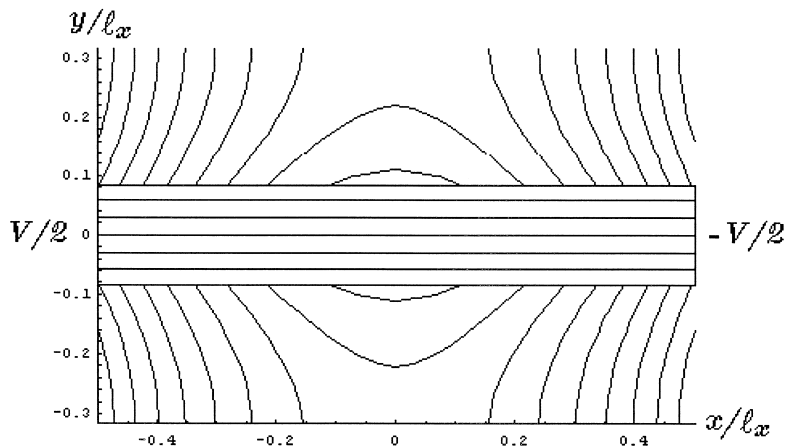
Fig. 1. Electric field lines for two thin plates separated by a distance $2a$, or for a single plate with thickness $2a$. The current goes from left (at potential $\phi = V/2$) to right ($\phi = -V/2$).

electrometer [15 and 14, p. 301]. A radioactive alpha-source was utilized to ionize the air at the point where the field was to be measured. The alpha-source acquired the same potential as the field at that point. The potential was measured with an electronic electrometer connected to the alpha-source. There is a striking analogy between our theoretical Fig. 2 and Fig. 3a of [15] (or Fig. 9.11a of [14]). This lends support to our calculation.
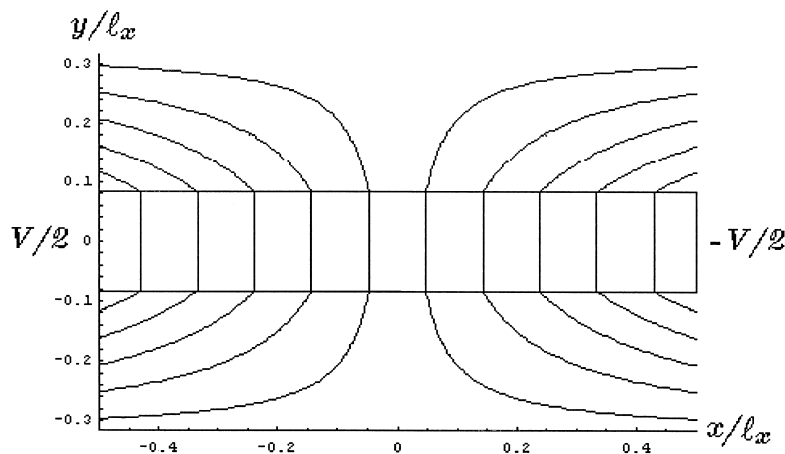

Fig. 2. Equipotentials for two thin plates separated by a distance $2a$, or for a single plate with thickness $2a$. The current goes from left (at potential $\phi = V/2$) to right ($\phi = -V/2$).

4. FOUR PARALLEL PLATES

We now wish to model Figs. 5 and 6 of [13], or the second result of Plate 6 of [14]. We have essentially a transmission line in which the current flows uniformly along the x axis of a parallelepiped of conductivity g_1 and thickness $b-a$, returning uniformly along another parallel parallelepiped of the same thickness but conductivity g_2. The centers of the two conductors are separated by a distance $b+a$. In this case there will be free charges in the four planes situated at $y = \pm a$ and $y = \pm b$.

4.1. Opposite Potentials

In this case both conductors have the same finite conductivity $g_1 = g_2 = g$. We assume that the potentials are exactly opposite in the two thick plates, for any x. We define two new constants, namely: $\kappa_1 = 1/(\pi L_x/4a - 1)$ and $\kappa_2 = 1/(\pi L_x/2a - 1)$.

With $\sigma(y = \pm a) = \pm \alpha x \pm \beta$, $\sigma(y = \pm b) = \pm 2a\alpha x \lambda_2/(1 - 2b\lambda_1)$ (so that the potential doesn't depend on y in the regions $a < y < b$ and $-b < y < -a$) and utilizing Eq. (3) with appropriate y_0's for each of the four plates, the potential becomes:

$$\phi(x, y, 0) = \begin{cases} [(\alpha x + \beta) + (b - y)(\alpha x \kappa_1 + \beta \kappa_2)]/\epsilon_0, & y > b \\ a(\alpha x + \beta)/\epsilon_0, & a < y < b \\ y(\alpha x + \beta)/\epsilon_0 & -a < y < a \\ -a(\alpha x + \beta)/\epsilon_0 & -b < y < -a \\ -[(\alpha x + \beta) + (b + y)(\alpha x \kappa_1 + \beta \kappa_2)]/\epsilon_0 & y < -b \end{cases} \quad (7)$$

The electric lines of force are listed below, for each region.

$$\psi(x, y, 0) = \begin{cases} x^2 + 2\beta x \kappa_2/\alpha \kappa_1 - y^2 + 2y(b + a/\kappa_1) & y > b \\ -ay & a < y < b \\ x^2 + 2\beta x/\alpha - y^2 & -a < y < a \\ ay & -b < y < -a \\ x^2 + 2\beta x \kappa_2/\alpha \kappa_1 - y^2 - 2y(b + a/\kappa_1) & y < -b \end{cases} \quad (8)$$
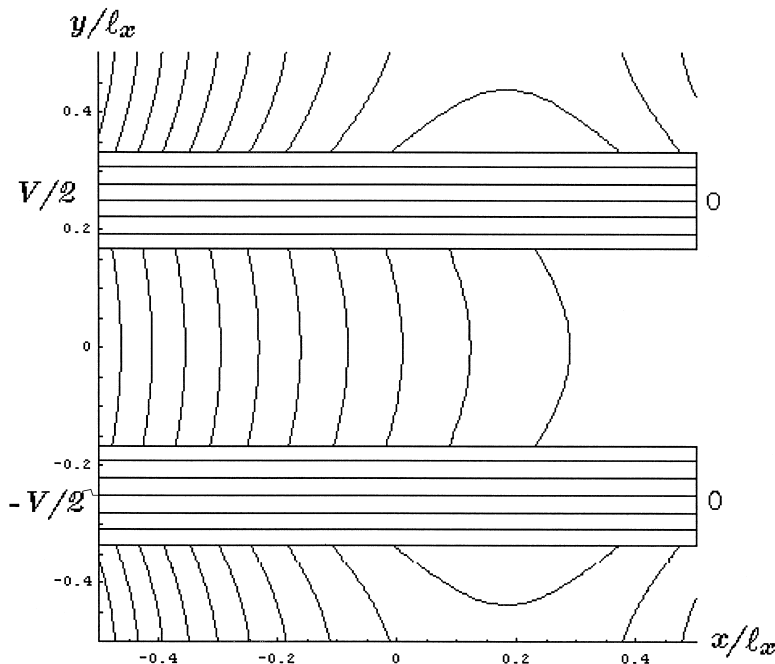
Fig. 3. Electric field lines for four thin plates, or for two plates with thickness $b-a$. The current goes from left ($\phi = V/2$) to right ($\phi = 0$) in the upper thick plate, and returns in the bottom one (from $\phi = 0$ to $\phi = -V/2$, right to left).

In Fig. 3 we plot this function in the approximation $L_z/L_x = L_x/a = 2L_x/b = 6$, in order to have similar dimensions as in Jefimenko's experiment. The upper plate has the potential at its boundaries given by $\phi(-L_x, a < y < b) = V/2$ and $\phi(L_x, a < y < b) = 0$, while the lower plate has the potential at its boundaries given by $\phi(-L_x, -b < y < -a) = -V/2$ and $\phi(L_x, -b < y < -a) = 0$. There is a striking analogy with Jefimenko's experimental result.

4.2. Perfect Conductor Plate

Now, suppose that the lower plate is a perfect conductor. That is, suppose it is submitted to the same constant potential $\phi(x, -b < y < -a) = \Phi$ in it's whole extension along the x axis. This experimental result is shown in Fig. 6 of [13] (in his case, $g_1 \ll g_2$). To model this case we consider four planes, located at $y = b$, $y = a$, $y = -a$ and $y = -b$, such that they have the following surface charges, respectively, $\sigma_b = \alpha_b x + \beta_b$, $\sigma_a = \alpha_a x + \beta_a$, $\sigma_{-a} = \alpha_{-a} x + \beta_{-a}$ and $\sigma_{-b} = \alpha_{-b} x + \beta_{-b}$.

The potential must not depend on y in the region $a < y < b$, and must be a constant in the region $-b < y < -a$. From this we find:

$$\alpha_a = \alpha_b \frac{2\lambda_1 - b}{a} \qquad \beta_a = \frac{\beta_b(2\lambda_1 + 2\lambda_2 - b) - \Phi\epsilon_0}{a}$$

$$\alpha_{-a} = -\alpha_a \qquad\qquad \beta_{-a} = -\beta_a \qquad\qquad\qquad (9)$$

$$\alpha_{-b} = \alpha_b \qquad\qquad\quad \beta_{-b} = \beta_b$$

With Eq. (3) and the appropriate replacements of y_0's we get:

$$\phi(x, y, 0) = \begin{cases} [(\alpha_b x + \beta_b)(4\lambda_1 - b - y) + 4\lambda_2\beta_b]/\epsilon_0 - \Phi & y > b \\ 2a(\alpha_a x + \beta_a)/\epsilon_0 + \Phi & a < y < b \\ (\alpha_a x + \beta_a)(a + y)/\epsilon_0 + \Phi & -a < y < a \\ \Phi & -b < y < -a \\ (\alpha_b x + \beta_b)(b + y)/\epsilon_0 + \Phi & y < -b \end{cases}$$

$$(10)$$

The lines of electric field are given by:

$$\psi(x, y, 0) = \begin{cases} x^3 + 2\beta_b x/\alpha_b - y^2 + 2(4\lambda_1 - b)\,y & y > b \\ -ay & a < y < b \\ x^2 + 2\beta_a x/\alpha_a - y^2 - 2ay & -a < y < a \\ -a^2 & -b < y < -a \\ x^2 + 2\beta_b x/\alpha_b - y^2 - 2by & y < -b \end{cases} \quad (11)$$

They are shown in Fig. 4 with the approximation above and the same dimensions as in Fig. 3. The constant potential in the lower plate is $\Phi = -V/2$. Once more there is a striking analogy with Jefimenko's experimental result.

5. DISCUSSION AND CONCLUSION

The theoretical results presented in this paper have never been obtained before. The only calculation which had been given in the literature for these geometries is that of Eq. (7) and even so only for the region between the plates [14, pp. 303–304]. He also obtained the free charges but only at the internal surfaces $y = \pm a$. He did not analyze the free charges at
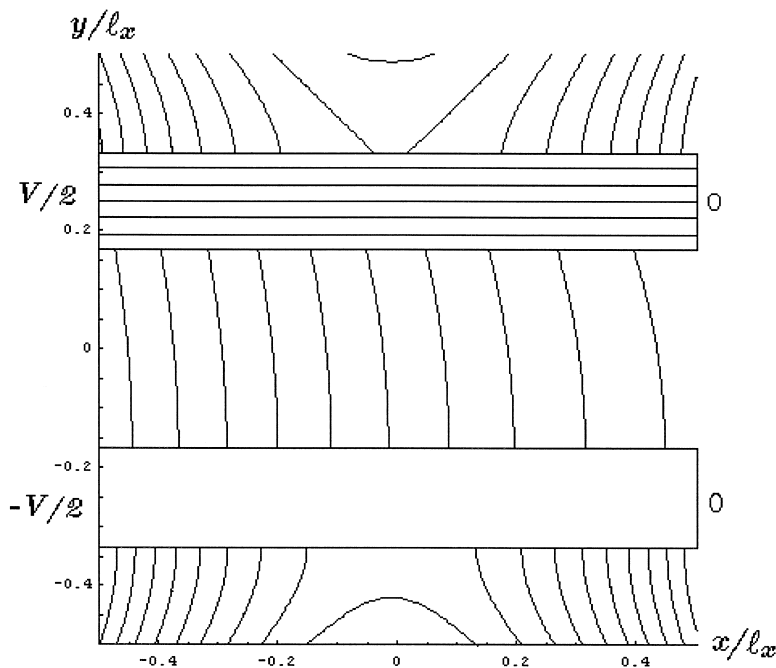
Fig. 4. Electric field lines for two thick plates, the bottom one being a perfect conductor. The current goes from left ($\phi = V/2$) to right ($\phi = -V/2$) in the upper thick plate, and returns in the bottom one (at a constant potential $\phi = -V/2$).

the external surfaces $y = \pm b$ nor the potential and electric field outside the plates ($y > |b+a|$).

The case treated by Heald[2] can also be compared to Jefimenko's results, Fig. 4 of [13]. Heald's geometry consisted of an infinite cilinder with poloidal current (the battery was a thin line parallel to the axis of the cylinder). Then he calculated the solution to Laplace's equation for the electric potencial and obtained also the electric field and the surface charge distribution as function of the polar angle.

The theoretical plots presented here are in excellent agreement with Jefimenko's experiments. The calculations presented in this paper may be considered as a complement to his brilliant work. We mapped theoretically the electric field in different geometries and compared our results with his grass seeds experiments. We also mapped theoretically the equipotentials and compared our result with his measurements utilizing an electronic electrometer.

The only published experiment known to us which tried to measure directly a force between a stationary resistive wire carrying a constant

current and a charged metal foil placed nearby is due to Sansbury.[16] He utilized a torsion balance with a foil charged to approximately $0.5 \times 10^{-9} C$ and when a current of $900\,A$ was passed in the conductor he could detect a force of approximately $10^{-7} N$, although he was not able to make precise measurements. We suspect that what he observed was due to the first-order electric field being discussed here. Further discussions of his experiment with different approaches can be found in [17–20; 21, Sec. 6.10; 22–25; and 8].

The most important fact to emphasize here is the existence of an electric field outside stationary resistive conductors carrying constant currents.

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