The Relationship between Mach's Principle and the Principle of Physical Proportions

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Mach's principle is compared with the principle of physical proportions. Laws that are compatible and others not compatible with the latter principle are discussed. Avenues for the implementation of this principle are also outlined.

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1 Newtonian Mechanics and Mach's Principle

In his book *Mathematical Principles of Natural Philosophy* (1687) Newton laid the foundations of classical mechanics [1]. In the *Scholium* after the Definitions in the beginning of this book Newton defined absolute time, absolute space and absolute motion, the concepts to be employed in his laws. According to Newton, absolute time flows equably without relation to anything external, while relative time is some sensible and external measure of duration by means of the motion of bodies; absolute space remains always similar and immovable without relation to anything external, while relative space is some movable dimension or measure of the absolute spaces which our senses determine by its position to bodies; and absolute (relative) motion is the translation of a body from one absolute (relative) place to another. We can thus say that relative time is a measure of duration by means of motion of material bodies (like the angle of rotation of the earth relative to the fixed stars), relative space is a measure of dimension by means material bodies (as the distance between two bodies measured by a material rule; or the relative order of three bodies).

In order to distinguish absolute from relative motion, Newton performed the famous bucket experiment, also presented in this *Scholium*: when the bucket and the water are at rest relative to the earth, the surface of the water remains flat and horizontal; when the bucket and the water rotate together relative to the earth with a constant angular velocity, the water rises up the sides of the vessel, forming a concave figure. Newton attributed this real and observed curvature to the absolute rotation of the water relative to absolute space, not to the rotation of the water relative to ambient bodies (earth and distant stars).

Leibniz, Berkeley and Mach rejected these concepts, proposing that only relative time, relative space and relative motion could be perceived by the senses and produce observed effects. Accordingly, only these relative concepts

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should appear in the laws of physics. For references and discussion see the author's monograph, *Relational Mechanics* [2, Chapters 5 and 6].

Mach expressed these ideas clearly in 1883 in his book *The Science of Mechanics* [3]. In place of Newton's absolute space, Mach proposed the frame of distant stars, that is, the frame in which the distant stars are seen to be at rest [3, pp. 285-6 and 336-7]. In place of Newton's absolute, time Mach proposed the angle of rotation of the earth relative to the fixed stars [3, pp. 273, 287 and 295]. According to Mach the curvature of the water in Newton's bucket experiment was due only to its rotation relative to the distant stars, not to its rotation relative to absolute and empty space [3, pp. 279 and 283-4]. Two key statements by Mach in this connection are as follows [3, pp. 279 and 284]: "Try to fix Newton's bucket and rotate the heaven of fixed stars and then prove the absence of centrifugal forces;" and "The principles of mechanics can, indeed, be so conceived, that even for relative rotations centrifugal forces arise."

The ideas expressed by Mach became generally known by the name "Mach's principle." Formulations of this principle by different authors are presented in *Relational Mechanics* [2, Section 6.8]. The main idea is that only motions of bodies relative to one another should enter in the laws of physics. No effects should arise due to specific motions of bodies relative to empty space.

2 The Principle of Physical Proportions

We concur with Leibniz, Berkeley and Mach on this problem, and as a generalization of their ideas [4] we propose the principle of physical proportions (PPP). Mach advocated doing away with all absolute quantities of motion (reducing local, absolute quantities to global, relational quantities). Here we advocate the abolition of all absolute quantities, whatsoever. In classical physics, space and time are absolute, as well as mass, electrical charge, *etc*. We propose that none of these absolute quantities should appear in the laws of physics, but only ratios of these quantities.

We formulate the principle as follows: (1) All laws of physics must depend only on the ratio of known quantities of the same type. This principle can also be understood in four further ways in order to clarify its meaning: (2) In the laws of physics, no absolute concepts should appear, only ratios of known magnitudes of the same type should be present; (3) Dimensional constants should not appear in the laws of physics; (4) The universal constants (such as G, c, h, k_B, ...) must depend on cosmological or microscopic properties of the universe; (5) All laws of physics and all measurable effects must be invariant under scale transformations of any kind (length, time, mass, charge, *etc*.).

This principle shows similarities with the principle of homogeneity, which was introduced by the Greeks. The idea of dimension had its origins in ancient Greek geometry. It was considered then that lines had one dimension, surfaces had two dimensions and solids had three dimensions [5, Vol. 1, pp. 158-9 and Vol. 3, pp. 262-3] and [6]. These dimensions were related to the rule or principle of homogeneity, according to which only magnitudes of the same kind could be added or equated, and only such magnitudes could have a numeric ratio (it is not possible to divide a volume by a length, for instance) [6]. Heath has called this principle the principle of similitude, and has also spoken of the theory of proportions [5, Vol. 1, pp. 137 and 351; Vol. 2, pp. 112-113 and 187]. The geometrical notion of dimension was extended by Fourier to include physical dimensions [7, §§160-161].

The principle of physical proportions presented here is thus related to the principle of homogeneity introduced by the ancient Greeks in geometry. It should therefore be extended to physics in a new way, a way not implemented by Newton, Fourier, *etc.*

Perhaps the PPP will not be feasible in all laws of physics; but it can at least be utilized as a guiding principle in order to explore more deeply the known laws and see their possible limitations. It seems plausible that whenever a law can be put in this form, with known terms and ratios, a better understanding of the physical principles involved will be achieved.

3 Laws that Satisfy the PPP

There are laws of physics that satisfy the PPP. The law of the lever is a prime example. It can be written as follows: two weights P_1 and P_2 at distances d_1 and d_2 from a fulcrum remain in horizontal static equilibrium (relative to the surface of the earth) when $P_1/P_2 = d_2/d_1$. Only ratios of local weights and local distances are relevant here. No fundamental constants appear in this law. Doubling all lengths or all weights (or gravitational masses) in the universe does not affect the equilibrium of the lever.

The law of the inclined plane also satisfies this principle. Consider a frictionless triangle ABC in a vertical plane with its side AC parallel to the horizon and the two bodies above hanging on sides AB and BC, respectively, connected by a string. They will be in equilibrium relative to the surface of the earth when $P_1/P_2 = AB/BC$. Once more, only ratios of weights and of known lengths are involved here.

Another example is the law of floating bodies discovered by Archimedes. Consider a homogeneous solid body of density ρ_S lower than the density of the fluid ρ_F in which its floats. The condition of equilibrium (no motion relative to the fluid) is obtained when

$$\frac{V_B}{V_T} = \frac{\rho_S}{\rho_F}. \tag{1}$$

Here V_B is the submersed volume of the body (below the surface of the fluid) and V_T its total volume. Only ratios of known volumes and known densities appear in the law. No fundamental constants are involved in this law. Doubling all densities in the universe will not affect the ratio V_B/V_T.

Another example involves communicating vessels filled with liquids. If the cross-sectional area of vessel 1 (2) is A_1 (A_2) and if the forces P_1 (P_2), respectively, are applied on the vessels' free surfaces, equilibrium (no motion relative to the surface of the earth) will result if $P_1 / P_2 = A_1 / A_2$.

There are also dynamical laws which satisfy this principle. One example is Kepler's second law of planetary motion: Areas swept out by the radius vector from the sun to the planet in equal times are equal [8, p. 135]. In other words, the area is proportional to the time. In algebraic terms if one planet describes an area A_1 in time t_1 and area A_2 in time t_2 then $A_1 / A_2 = t_1 / t_2$.

Another example is Newton's second law of motion coupled with his third law. Consider two bodies of inertial masses m_{i1} and m_{i2} interacting with one another along a straight line. If they are subjected to accelerations a_1 and a_2 relative to an inertial system of reference, from Newton's laws we obtain (assuming constant inertial masses): $m_{i1} / m_{i2} = -a_2 / a_1$.

4 Laws that do Not Satisfy the PPP

The majority of physical laws do not comply with the PPP. A number of examples were discussed in earlier work [4]. Here we briefly present some of them.

The free fall acceleration a near the surface of the earth according to classical mechanics is given by $a = GM_{ie} / R_e^2$, where $G = 6.67 \times 10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}$ is the constant of gravitation, $M_{ie} = 5.98 \times 10^{24}\,\text{kg}$ is the earth's inertial mass and $R_e = 6.37 \times 10^6\,\text{m}$ its average radius. This acceleration is known to be independent of the mass of the falling body. Hence there is no ratio of masses in this law, and the acceleration of free fall would then be a measure of the absolute value of the earth's mass: doubling this mass would double the acceleration of free fall, independent of what happens to the mass of the test body, to the mass of stars and galaxies, *etc*. This shows that not only space and time are absolute in classical mechanics, but inertial mass is as well.

The flattening of the earth due to its diurnal rotation is also an example of this absolute aspect of mass in classical mechanics [4].

The law of elastic force is a further example of a law that does not comply with the PPP. Consider a spring of relaxed length ℓ_o and elastic constant *k*. A body of weight *P* can be suspended in static equilibrium when this spring is fixed vertically, provided that its final length ℓ satisfies the relation $P = k(\ell - \ell_o)$. There are no ratios of weights here. This law is correct in the sense that it describes the behaviour of springs. (It is valid as long as the lengthening of the spring is not so great as to become irreversible.) But because it does not satisfy the PPP, it must be regarded as incomplete.

The great majority of laws of physics do not comply with the principle of physical proportions. Whenever we encounter physical laws expressed in terms of equalities in which there appear local constants (such as the spring constant *k*, the dielectric constant ε of the material, *etc*.) or universal constants (such as *G*, ε_o, k_B, *h*, *etc*.), they must be incomplete, although correct. Examples include: the law of ideal gases, $PV = k_B NT = RnT$ (*P* being the pressure, *V* the volume, $k_B = 1.38\times10^{-23}\,\text{JK}^{-1}$ Boltzmann's constant, N the number of atoms or molecules, T the temperature, $R = 8.3\,\text{JK}^{-1}\text{mol}^{-1}$ the universal gas constant, and n the number of moles); the velocity of sound, $v_s = \sqrt{B/\rho}$ (B being the bulk modulus of the fluid with density ρ); Ohm's law, $V = RI$ (where V is the voltage or potential difference between two points A and B of a conductor of resistance R in which the constant current I flows), *etc.*

5 Implementation of the PPP

We now discuss a method for implementing this principle in order to make natural laws complete. We first consider hydrostatics and Archimedes's principle. Although Eq. (1) satisfies the principle, we will discuss an incomplete form of this law.

It is easy to imagine how people unaware of Archimedes's results might arrive at a correct but incomplete law when experimenting with floating bodies. They might set a piece of ice, cork, wood, *etc.* afloat only in water, and observe that the ratio of the submersed to the total volume was proportional to the density of the material, namely

$$\frac{V_B}{V_T} = A\rho_S\,, \tag{2}$$

where A would be a constant of proportionality with dimensions of the inverse of density. This constant would be the same for all solid bodies specified above. This equation is correct dimensionally and is invariant under unit transformation (the numerical value of A will depend on the system of units employed, for instance $A = 10^3\,\text{kg/m}^3$ or $A = 2.2\times10^3$ lb/m^3, but the form of the equation will be the same in all systems of units).

Although this law correctly describes the behaviour of bodies floating in water, it is incomplete. In order to transform this law into one that is compatible with the PPP, it would be necessary to discover if A was of cosmological, local or microscopic origin. Specifically, it would be necessary to discover if $1/A$ was proportional to the mean density of mass in the universe, to the density of the local fluid in which the solid was floating, or to the density of the molecules composing the fluid, for instance. By floating the same solids in different fluids like liquid mercury, gasoline and alcohol it would be possible to arrive at $A = 1/\rho_F$. The situation would then be described by Eq. (1) and the law could be considered complete.

Relational mechanics completely satisfies Mach's principle and the more general PPP [2, 4]. It is based on Weber's law for gravitation and electromagnetism [9]. Weber's force depends only on the relative distance between the interacting bodies, on their relative radial velocity and on their relative radial acceleration, so that it is completely relational. Relational mechanics is also based on the principle of dynamical equilibrium [10, 2 Section 8.1]: The sum of all forces of any nature (gravitational, electric, magnetic, elastic, nuclear, *etc.*) acting on any body is always zero in all frames of reference. As the sum of all forces is zero, only ratios of forces will be detectable or measurable. The system of units (MKSA, cgs, *etc.*) to be employed is not relevant. Moreover, the unit or dimension of the forces can be arbitrarily chosen.

According to the theory of relational mechanics [2, Sections 8.4, 8.5 and 9.2], the acceleration of free fall a_{me} of a test body of gravitational mass m_g toward the earth is given by

$$\frac{a_{me}}{a_o} = \left(\frac{2}{2^n \xi}\right)\left(\frac{R_o}{r}\right)^2\left(\frac{M_{ge}}{M_{go}}\right).$$

Here M_{ge} is the gravitational mass of the earth, r is the distance of the test body from the center of the earth, $R_o = c/H_o$ is the radius of the known universe (c is the value of light velocity in vacuum and H_o is Hubble's constant), M_{go} is the gravitational mass of the known universe (mass inside a sphere of radius R_o) and $a_o = R_o H_o^2$ is a fundamental acceleration characteristic of the universe. In this expression there are only ratios of accelerations, distances and gravitational masses. Doubling all distances or all masses in the universe, for example, will not affect the ratio a_{me}/a_o. According to this expression, the acceleration of free fall is independent of the mass of the test body, as known since Galileo. On the other hand, it shows that this acceleration is not only directly proportional to the mass of the earth, as known by Newton, but also inversely proportional to the mass of the distant galaxies. We can double a_{me} either by doubling the mass of the earth (compared to any standard, without simultaneously affecting the masses of the distant galaxies according to this standard), or by halving the masses of the distant galaxies (compared to any standard, without simultaneously affecting the mass of the earth according to this standard). If the distance of the test body from the earth is doubled, the acceleration of free fall decreases by a factor of 4. According to the expression above, the same should happen even if the distance between the earth and the test body is not changed, but the size of the known universe is shrunk by a factor 2. The meaning of a_o is not yet clear, but it must be the acceleration of some material object. Perhaps it is the average acceleration of all bodies in the universe, or some other as yet unknown acceleration. In the future, it may prove interesting to investigate the relationship between this acceleration and the acceleration introduced by Jaakkola in his research on cosmology and gravitation [11]. In any event, the important aspect of the result above is that only ratios of gravitational masses, of distances and of accelerations are relevant here.

The implementation of the PPP as regards the flattening of the earth has already been discussed in a recent essay [4], where we also discussed applications of the PPP to electromagnetism and the equation of ideal gases. There it was shown that these laws in their present form do not comply with this principle, indicating that they may be incomplete. Possible ideas and avenues for completing them were also outlined.

6 Discussion

In closing, it may be appropriate to quote a very pertinent passage from the last chapter of Amitabha Ghosh's book *Origin of Inertia* [11]:

> When I find school students nowadays solving mechanics problems involving pulleys, inclined planes, rockets, cars, I cannot but help think of the early summer of 1956. I had just completed my high school in a remote village of Bengal and was waiting for my admission to the district college for the Intermediate Science programme. My father thought that the time might be better utilized if I were to get some prior exposure to science. In those days, up to Class-10 there were hardly any science topics in the programme, and we had absolutely no introduction to mechanics. Even the terms like velocity, acceleration, momentum, *etc.* were totally unfamiliar to us in the high school. One of my cousins had finished his Intermediate Science and was a trainee in a steel plant. He came to spend a few weeks at our home, and first introduced me to the names of Newton and Galileo. He gave me my first ever lesson in elementary kinematics, the parallelogram laws of the addition of forces and motion parameters. Soon afterwards, I was introduced to the laws of motion by another young postgraduate in Mathematics from the village. By then he had left Mathematics and was studying Law, but had returned to spend his summer vacation at home.
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> I remember the tremendous mental block I had in conceiving of the basic concepts. By that time, I was familiar with multiplying physical quantities by numbers. Somehow, ideas of velocity and acceleration, which involved length and time, I could grasp. What was very difficult for me at that time was to conceive of the idea of one physical quantity being multiplied by another physical quantity. For me the stumbling block was the concept of momentum—the product of mass and velocity. I can still remember the utter exasperation of the young law student who had already completed a Master of Science in Mathematics from Calcutta University. He was completely baffled by my difficulty. It took a long time for me to accept the concept of momentum.

As we saw above, Prof. Ghosh's difficulty in conceiving the idea of linear momentum reflects a deeper problem in the laws of physics themselves. According to the PPP we should only have ratios of quantities of the same type. When we examine the problem more closely, we see that it makes no sense to multiply a mass by a velocity. These are two completely different physical concepts, with different units and operational definitions for their measurements. The most we can say is that, by definition, the linear momentum μ of a body 1 is to the linear momentum of a body 2 as the ratio of their masses m multiplied by the ratio of their velocities v, namely:

$$\frac{\mu_1}{\mu_2}=\frac{m_1}{m_2}\frac{v_1}{v_2}. \tag{3}$$

According to the principle of homogeneity of the ancient Greeks, only magnitudes of the same dimension should be added or equated. The same must be valid for physical magnitudes, as postulated here by the PPP. How should the concept of velocity be handled? Instead of defining it as the ratio of a length by a time interval, as is usually done, the same procedure as above should be utilized, as indicated by Mach [3, p. 273]: "A motion is termed uniform in which equal increments of space described correspond to equal increments of space described by some motion with which we form a comparison, as the rotation of the earth. A motion may, with respect to another motion, be uniform. But the question whether a motion is *in itself* uniform, is senseless." Accordingly, the ratio of velocities should be defined operationally as $v_1 / v_2 = (s_1 / s_2)(t_2 / t_1)$, where v means velocity and s space described in time t. When the ratio v_1 / v_2 is a constant in time, we can say by definition that the motion of body 1 is uniform in comparison with the motion of body 2. The same should be applied to other magnitudes. For instance, instead of defining density as the ratio of mass to volume, only ratios of densities should be defined. That is, the ratio of density of two bodies 1 and 2 should be defined as the ratio of their masses multiplied by the inverse ratio of their volumes, namely: $\rho_1 / \rho_2 = (m_1 / m_2)(V_2 / V_1)$.

Because not all laws of physics are written in terms of ratios of known quantities of the same type, they must be incomplete. The ideas presented in this work may help to indicate possible ways to complete these laws.

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