Electric potential for a resistive toroidal conductor carrying a steady azimuthal current

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In this work we treat a resistive toroidal conductor carrying a steady azimuthal current. We calculate the electric potential everywhere in space. We also present the electric field inside and outside the toroid and the surface charge distribution along the conductor. We compare our theoretical result with Jefimenko's experiment.

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I. INTRODUCTION

In the past few years there has been a renewed interest in the electric field inside and outside Ohmic conductors carrying steady currents. The analytical cases known in the literature include coaxial cables (Ref. [1], pp. 125–130), (Ref. [2], pp. 318 and 509–511), [3,4], (Ref. [5], pp. 336–337), and [6]; a long solenoid with azimuthal current (Ref. [2], p. 318) and [7]; transmission lines (Ref. [8], p. 262) and with more detail in [9]; a long straight wire [10]; and conductor plates carrying steady currents [11]. Sometimes this problem is referred to as Merzbacher's puzzle [12] and (Ref. [5], pp. 336–337).

Here we consider a case not yet solved in the literature, namely, to find the electric field inside and outside a resistive toroidal conductor carrying a steady azimuthal current. We have three goals in mind. The authors who analyzed long straight conductors found that the potential and surface charge density are linear functions of the longitudinal variable z [13]. As pointed out by Griffiths (Ref. [5], pp. 336–337), this is a peculiar result since the answer depends on the geometry of the circuit, that is, on the return conductor and on the location of the battery. So our first goal in order to avoid the ambiguity of the return conductor is to utilize a toroidal conductor where the battery is clearly localized. Our second goal is to find a solution for the potential due to a current distributed in a finite volume of space, clearly creating an electric field outside the Ohmic conductor. The authors above who obtained an electric field outside the conductor considered normally infinite straight conductors. The exception are Jefimenko and Heald, who dealt with a curved conductor (Ref. [2], p. 318) and [7]. But their solution is also idealized because the cylindrical resistive sheet with azimuthal current had an infinite length. The only author who completely solved a problem with the current bounded in a finite volume is Jackson [3], who considered a coaxial cable of finite length. But as he considered a return conductor of zero resistivity, he obtained an electric field only inside the cable, with no electric field outside it. Our third goal is to obtain a solution that can be compared with a known experimental result [14]. This will be discussed in the final section of the paper.

II. DESCRIPTION OF THE PROBLEM

Consider a stationary toroidal Ohmic conductor (greater radius R and smaller radius r_0) with a steady current I, constant over the length of the conductor. We assume that the conductor has uniform resistivity, and the current is in the azimuthal direction, flowing along the circular loop. The toroid is centered on the plane $z=0$, z being its axis of symmetry. There is a battery located at $\varphi=\pi$ rad maintaining constant potentials at its extremities; see Fig. 1. We initially idealize the battery as of negligible thickness. Later on we consider the battery occupying a finite volume. The medium outside the conductor is supposed to be air or vacuum.

Our goal is to find the electric potential ϕ everywhere in space, using the potential at the surface of the conductor as a boundary condition. The problem treated here can be applied to two cases: (a) the toroid is a full homogeneous solid and the battery is a disk; see Fig. 2(a); (b) the toroid is hollow and the battery is a circle; see Fig. 2(b). The symmetry of our problem suggests the approach of toroidal coordinates (η,χ,φ) (Ref. [15], p. 112), defined by

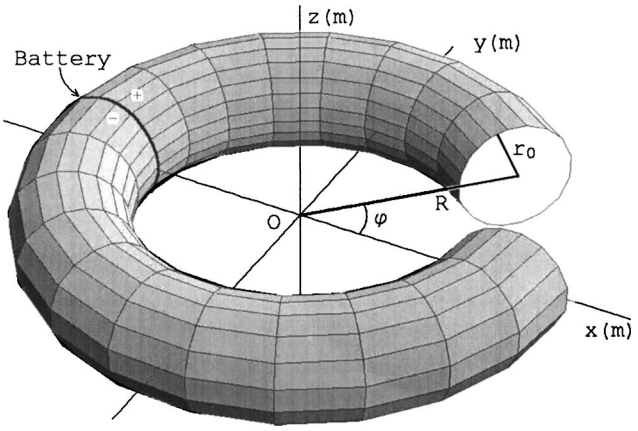

FIG. 1. A toroidal Ohmic conductor with symmetry axis z, smaller radius r_0 (m), and greater radius R (m). A thin battery is located at $\varphi=\pi$ rad, maintaining constant potentials (represented as the "+" and "−" signs) in its extremities. A steady current flows azimuthally in this circuit loop in the clockwise direction, from $\varphi=+\pi$ to $-\pi$ rad.

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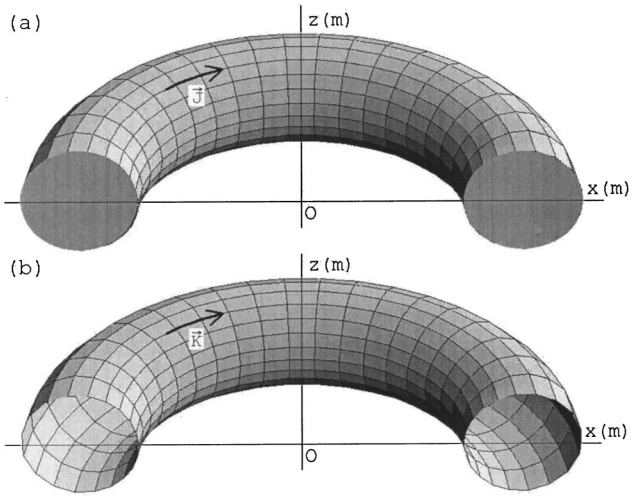

FIG. 2. The two cases being considered here: (a) a full solid resistive toroidal conductor, with an azimuthal volume current density $\vec{J}$ (A/m^2) through the cross section; (b) a hollow resistive toroidal conductor, with an azimuthal surface current density $\vec{K}$ (A/m) through the perimeter $2\pi r_0$ of the hollow toroidal shell.

$$x = a\frac{\sinh\eta\cos\varphi}{\cosh\eta-\cos\chi}, \quad y = a\frac{\sinh\eta\sin\varphi}{\cosh\eta-\cos\chi},$$

$$z = a\frac{\sin\chi}{\cosh\eta-\cos\chi}, \tag{1}$$

where a is a constant that gives the radius of a circle in the $z=0$ plane described by $\eta\to\infty$ (that is, when $\eta\to\infty$ we have $x=a\cos\varphi$, $y=a\sin\varphi$, and $z=0$). The values assumed by the toroidal coordinates are $0\leqslant\eta<\infty$, $-\pi\leqslant\chi\leqslant\pi$ rad, and $-\pi\leqslant\varphi\leqslant\pi$ rad. The inverse transformations are given by

$$\eta = \operatorname{arctanh}\frac{2a\sqrt{x^2+y^2}}{x^2+y^2+z^2+a^2},$$

$$\chi = \arctan\frac{2za}{x^2+y^2+z^2-a^2}, \quad \varphi = \arctan\frac{y}{x}. \tag{2}$$

For the present work, it is convenient to present the expressions for $\cosh\eta$ and for $\cos\chi$:

$$\cosh\eta = \frac{x^2+y^2+z^2+a^2}{\sqrt{(x^2+y^2+z^2-a^2)^2+4a^2z^2}}, \tag{3}$$

$$\cos\chi = \frac{x^2+y^2+z^2-a^2}{\sqrt{(x^2+y^2+z^2-a^2)^2+4a^2z^2}}. \tag{4}$$

The surface of the toroid is described by a constant η_0. The internal (external) region of the toroid is characterized by $\eta>\eta_0$ ($\eta<\eta_0$). The greater radius R and the smaller radius r_0 are related to η_0 and to a by $R=a\cosh\eta_0/\sinh\eta_0$ and $r_0=a/\sinh\eta_0$; see Fig. 1].

Laplace's equation for the electric potential $\nabla^2\phi=0$ can be solved in toroidal coordinates with the method of separation of variables (by a procedure known as R separation), leading to a solution of the form (Ref. [15], p. 112)

$$\phi(\eta,\chi,\varphi) = \sqrt{\cosh\eta-\cos\chi}\,H(\eta)X(\chi)\Phi(\varphi), \tag{5}$$

where the functions H, X, and Φ satisfy the general equations (with $\xi=\cosh\eta$, and p,q constants)

$$(\xi^2-1)H''+2\xi H'-[(p^2-1/4)+q^2/(\xi^2-1)]H=0, \tag{6}$$

$$X''+p^2X=0, \tag{7}$$

$$\Phi''+q^2\Phi=0. \tag{8}$$

III. GENERAL SOLUTION

The solutions of Eqs. (7) and (8) for $p\neq0$ and $q\neq0$ are linear combinations of the general forms $X_p(\chi)=C_{p\chi}\cos(p\chi)+D_{p\chi}\sin(p\chi)$ and $\Phi_q(\varphi)=C_{q\varphi}\cos(q\varphi)+D_{q\varphi}\sin(q\varphi)$, respectively, where $C_{p\chi}$, $D_{p\chi}$, $C_{q\varphi}$, and $D_{q\varphi}$ are constants. When $p=q=0$ the solutions reduce to, respectively, $X_0(\chi)=C_{0\chi}+D_{0\chi}\chi$ and $\Phi_0(\varphi)=C_{0\varphi}+D_{0\varphi}\varphi$. Equation (6) is Legendre's equation, whose solutions are the associated Legendre functions $P^q_{p-1/2}(\cosh\eta)$ and $Q^q_{p-1/2}(\cosh\eta)$, known as toroidal Legendre polynomials (Ref. [16], p. 173).

The solution must be periodic in φ, that is, $\phi(\eta,\chi,\varphi+2\pi)=\phi(\eta,\chi,\varphi)$, and in χ, that is, $\phi(\eta,\chi+2\pi,\varphi)=\phi(\eta,\chi,\varphi)$. This condition implies that $D_{0\varphi}=0$, $D_{0\chi}=0$, $q=1,2,3,\ldots$, and $p=1,2,3,\ldots$.

The functions $Q^q_{p-1/2}$ are irregular in $\eta=0$ (which corresponds to the z axis, or to great distances from the toroid). For this reason we eliminate them as physical solutions for this problem in the region outside the toroid (that is, $\eta<\eta_0$). Our general solution consists of linear combinations of all possible regular solutions of $P^q_{p-1/2}(\cosh\eta)$, $X_p(\chi)$, and $\Phi_q(\varphi)$:

$$\phi(\eta\leqslant\eta_0,\chi,\varphi) = \sqrt{\cosh\eta-\cos\chi}\sum_{q=0}^{\infty}[C_{q\varphi}\cos(q\varphi) + D_{q\varphi}\sin(q\varphi)]\sum_{p=0}^{\infty}[C_{p\chi}\cos(p\chi) + D_{p\chi}\sin(p\chi)]P^q_{p-1/2}(\cosh\eta). \tag{9}$$

We used the fact that $\sin0=0$ and $\cos0=1$ to sum up from $p=q=0$ to ∞. Here $P^0_{p-1/2}(\cosh\eta)\equiv P_{p-1/2}(\cosh\eta)$ are the Legendre functions (Ref. [17], p. 724).

IV. PARTICULAR SOLUTION FOR A STEADY AZIMUTHAL CURRENT

The surface of the toroid is described by a constant η_0. Here we study the case of a steady current flowing in the azimuthal φ direction along the Ohmic toroid. For this reason we suppose that the potential along the surface of the toroid is linear in φ, $\phi(\eta_0,\chi,\varphi)=A+B\varphi$. This potential

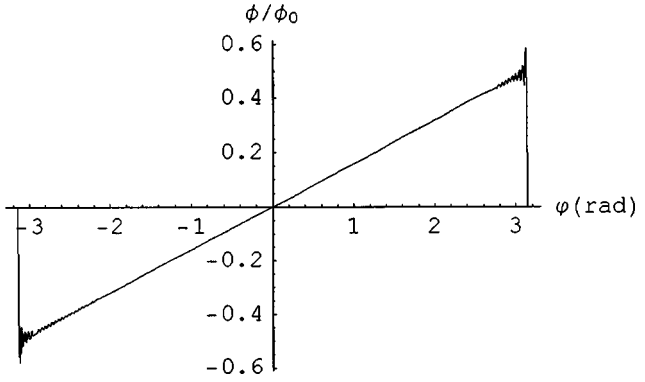

FIG. 3. Fourier expansion of the potential along the conductor surface as a function of the azimuthal angle φ, Eq. (10), with $A=0$ and $B=\phi_0/2\pi$. The overshootings at the extremities are due to the Gibbs phenomenon (Ref. [17], p. 783–787).

can be expanded in a Fourier series in φ:

$$\phi(\eta_0,\chi,\varphi)=A+B\varphi=A+2B\sum_{q=1}^{\infty}\frac{(-1)^{q-1}}{q}\sin(q\varphi). \tag{10}$$

Figure 3 shows the Fourier expansion of the potential along the conductor surface as a function of φ. The oscillations close to $\varphi=\pm\pi$ rad are due to a Fourier series with a finite number of terms. The overshooting is known as the Gibbs phenomenon, a peculiarity of the Fourier series at a simple discontinuity (Ref. [17], pp. 783–7).

We assume that the potential inside the full solid toroidal Ohmic conductor (that is, for $\eta>\eta_0$) [see Fig. 2(a)] is also given by Eq. (10), namely,

$$\phi(\eta>\eta_0,\chi,\varphi)=A+B\varphi. \tag{11}$$

The electric field inside the solid toroid can be expressed in cylindrical coordinates (ρ,φ,z) simply as

$$\vec{E}=-\nabla\phi=-\frac{B}{\rho}\hat{\varphi}. \tag{12}$$

This electric field does not lead to any accumulation of charges inside a full solid conductor because $\nabla\cdot\vec{E}=0$.

These are reasonable results. The potential satisfies Laplace's equation $\nabla^2\phi=0$, as expected. The electric field is inversely proportional to the distance $\rho=\sqrt{x^2+y^2}$ from the z axis. This was to be expected as we are assuming a conductor of uniform resistivity. The difference of potential $\Delta\phi$ created by the battery at $\varphi=\pi$ rad can be related to the azimuthal electric field by a line integral:

$$\Delta\phi=-\int_{\varphi=\pi}^{-\pi}\vec{E}\cdot d\vec{\ell}=-E_{\varphi}2\pi\rho. \tag{13}$$

Here ρ is the radius of a circular path centered on the z axis and located inside or along the surface of the toroid. This shows that E_{φ} should be inversely proportional to ρ, as found in Eq. (12). Comparing Eqs. (12) and (13) yields $B=\Delta\phi/2\pi$.

By Ohm's law $\vec{J}=g\vec{E}$, where g is the uniform conductivity of the wire, we can see that $\vec{J}$ is also inversely proportional to the distance ρ from the z axis inside a full solid homogeneous toroidal conductor.

We now consider the solution outside the conductor, valid for the cases of a solid and a hollow toroid.

We calculate Eq. (9) with $\eta=\eta_0$ and use Eq. (10) as a boundary condition of our problem. As we do not have terms with $\cos(q\varphi)$ in Eq. (10), this means that $C_{q\varphi}=0$ for $q=1,2,3,\ldots$. Comparing the resulting Eq. (9) at $\eta=\eta_0$ with Eq. (10) yields two equations connecting A and B to the C's and D's, namely,

$$A=C_{0\varphi}\sqrt{\cosh\eta_0-\cos\chi}\sum_{p=0}^{\infty}[C_{p\chi}\cos(p\chi)+D_{p\chi}\sin(p\chi)]P_{p-1/2}(\cosh\eta_0), \tag{14}$$

$$B=\frac{qD_{q\varphi}}{2(-1)^{q-1}}\sqrt{\cosh\eta_0-\cos\chi}\sum_{p=0}^{\infty}[C_{p\chi}\cos(p\chi)+D_{p\chi}\sin(p\chi)]P_{p-1/2}^{q}(\cosh\eta_0). \tag{15}$$

We now isolate the term $1/\sqrt{\cosh\eta_0-\cos\chi}$ in Eqs. (14) and (15), expanding it in Fourier series, that is,

$$\frac{1}{\sqrt{\cosh\eta_0-\cos\chi}}=\frac{1}{2\pi}\sum_{p=0}^{\infty}(2-\delta_{0p})\left[\int_{-\pi}^{\pi}\frac{\cos(p\chi')d\chi'}{\sqrt{\cosh\eta_0-\cos\chi'}}\right]\cos(p\chi)=\frac{\sqrt{2}}{\pi}\sum_{p=0}^{\infty}(2-\delta_{0p})Q_{p-1/2}(\cosh\eta_0)\cos(p\chi), \tag{16}$$

where δ_{wp} is the Kronecker delta, which is zero for $w\neq p$ and 1 for $w=p$, and we used an integral representation of $Q_{p-1/2}(\cosh\eta)$ (Ref. [16], p. 156).

As in Eq. (16) we do not have terms of $\sin(p\chi)$, this means that $D_{p\chi}=0$ in Eqs. (14) and (15). Using Eq. (16) with Eq. (14) yields (for $p=0,1,2,\ldots$)

$$A_p\equiv C_{0\varphi}C_{p\chi}=\frac{A(2-\delta_{0p})}{2\pi P_{p-1/2}(\cosh\eta_0)}\int_{-\pi}^{\pi}\frac{\cos(p\chi')d\chi'}{\sqrt{\cosh\eta_0-\cos\chi'}}=\frac{\sqrt{2}A(2-\delta_{0p})}{\pi}\frac{Q_{p-1/2}(\cosh\eta_0)}{P_{p-1/2}(\cosh\eta_0)}. \tag{17}$$

Using Eq. (16) with Eq. (15) yields

$$B_{pq}\equiv D_{q\varphi}C_{p\chi}=\frac{B(-1)^{q-1}(2-\delta_{0p})}{q\pi P_{p-1/2}^{q}(\cosh\eta_0)}\int_{-\pi}^{\pi}\frac{\cos(p\chi')d\chi'}{\sqrt{\cosh\eta_0-\cos\chi'}}=\frac{2\sqrt{2}B(-1)^{q-1}(2-\delta_{0p})}{q\pi}\frac{Q_{p-1/2}(\cosh\eta_0)}{P_{p-1/2}^{q}(\cosh\eta_0)}. \tag{18}$$

The final solution outside the toroid is given by

$$\phi(\eta \leqslant \eta_0, \chi, \varphi) = \sqrt{\cosh\eta - \cos\chi}\left[\sum_{p=0}^{\infty} A_p \cos(p\chi) P_{p-1/2}(\cosh\eta) + \sum_{q=1}^{\infty} \sin(q\varphi) \sum_{p=0}^{\infty} B_{pq} \cos(p\chi) P^q_{p-1/2}(\cosh\eta)\right], \tag{19}$$

where the coefficients A_p and B_{pq} are given by Eqs. (17) and (18), respectively.

For the region inside the hollow toroid (that is, $\eta > \eta_0$), Fig. 2(b), we have $P^q_{p-1/2}(\cosh\eta \to \infty) \to \infty$, while $Q^q_{p-1/2}(\cosh\eta \to \infty) \to 0$. For this reason we eliminate $P^q_{p-1/2}(\cosh\eta)$ as physical solutions for the region inside the hollow toroid. The potential is then given by

$$\phi(\eta > \eta_0, \chi, \varphi) = A + \sqrt{\cosh\eta - \cos\chi} \sum_{q=1}^{\infty} \sin(q\varphi) \times \sum_{p=0}^{\infty} B'_{pq} \cos(p\chi) Q^q_{p-1/2}(\cosh\eta), \tag{20}$$

where the coefficients B'_{pq} are defined by

$$B'_{pq} \equiv \frac{B(-1)^{q-1}(2-\delta_{0p})}{q\pi Q^q_{p-1/2}(\cosh\eta_0)} \int_{-\pi}^{\pi} \frac{\cos(p\chi')d\chi'}{\sqrt{\cosh\eta_0 - \cos\chi'}} = \frac{2\sqrt{2}B(-1)^{q-1}(2-\delta_{0p})}{q\pi} \frac{Q_{p-1/2}(\cosh\eta_0)}{Q^q_{p-1/2}(\cosh\eta_0)}. \tag{21}$$

Note that the potential inside the solid toroid, Eq. (11), and the potential inside the hollow toroid, Eq. (20), are different. This happens because the discontinuous boundary condition Eq. (10) applies for any $\eta > \eta_0$ inside the solid toroid, particularly for $\varphi \to \pi$ ($\phi \to A + B\pi$) and $\varphi \to -\pi$ ($\phi \to A - B\pi$), where the disk battery is located [see Fig. 2(a)]. This does not happen to the hollow toroid, where the battery is a circle, and the potential must be continuous inside the hollow toroid [see Fig. 2(b)].

We now analyze the potential outside the toroid for the regions far away from the toroid, close to the origin, and along the z axis.

For great distances from the toroid (that is, $r = \sqrt{x^2+y^2+z^2} \gg a$), Eqs. (2)–(4) yield $\eta \approx 2a\sqrt{x^2+y^2}/r^2 \to 0$, $\cosh\eta \approx 1 + 2a^2(x^2+y^2)/r^4 \to 1$, and $\cos\chi \approx 1 - 2a^2z^2/r^4 \to 1$. This means that $\sqrt{\cosh\eta - \cos\chi} \approx a\sqrt{2}/r \to 0$. For $\cosh\eta \approx 1 + \epsilon$, where $0 < \epsilon \ll 1$, we have the following expansion (Ref. [16], pp. 163 and 173):

$$P^q_{p-1/2}(1+\epsilon) \approx \frac{\Gamma(p+q+1/2)}{2^{q/2} q! \Gamma(p-q+1/2)} \epsilon^{q/2} \times \left\{1 + \epsilon\left[\frac{p^2 - 1/4}{2(1+q)} - \frac{q}{4}\right]\right\}. \tag{22}$$

The electric potential, for great distances from the toroid, is given, in spherical coordinates (r, θ, φ), by

$$\phi(r \gg a, \theta, \varphi) \approx \frac{a\sqrt{2}}{r} \sum_{p=0}^{\infty} \cos\left(p \frac{2a\cos\theta}{r}\right) \times \left[A_p + B_{p1}\left(p^2 - \frac{1}{4}\right)\frac{a}{r}\sin\varphi\sin\theta\right], \tag{23}$$

so that $\phi(r \to \infty) \to 0$, as expected.

The potential close to the origin (that is, $r \ll a$) can be calculated in the same manner. We have $\cosh\eta \approx 1 + 2(x^2+y^2)/a^2$ and $\cos\chi \approx -1 + 2z^2/a^2$, so that $\sqrt{\cosh\eta - \cos\chi} \approx \sqrt{2}$. The potential is then given by

$$\phi(r \ll a, \theta, \varphi) \approx \sqrt{2} \sum_{p=0}^{\infty} \cos\left(p \frac{2r\cos\theta}{a}\right) \times \left[A_p + B_{p1}\left(p^2 - \frac{1}{4}\right)\frac{r}{a}\sin\varphi\sin\theta\right]. \tag{24}$$

Along the z axis we have $\sqrt{x^2+y^2} = 0$. From Eqs. (2)–(4) we have $\eta = 0$, $\cosh\eta = 1$, $\cos\chi = (z^2-a^2)/(z^2+a^2)$, and $\sqrt{\cosh\eta - \cos\chi} = a\sqrt{2/(z^2+a^2)}$. The potential along the z axis can be written

$$\phi(r = \sqrt{x^2+y^2+z^2} = |z|, \theta, \varphi) = a\sqrt{\frac{2}{z^2+a^2}} \sum_{p=0}^{\infty} A_p \cos\left(p \arccos\frac{z^2-a^2}{z^2+a^2}\right). \tag{25}$$

We plotted the equipotentials of a full solid toroid on the plane $z = 0$ in Fig. 4 with $A = 0$ and $B = \phi_0/2\pi$. Figure 5 shows a plot of the equipotentials of the full solid toroid in the plane $x = 0$ (perpendicular to the current).

V. ELECTRIC FIELD AND SURFACE CHARGES

In toroidal coordinates the gradient is written as

$$\nabla\phi = \frac{1}{a}(\cosh\eta - \cos\chi)\left(\hat{\eta}\frac{\partial\phi}{\partial\eta} + \hat{\chi}\frac{\partial\phi}{\partial\chi} + \frac{\hat{\varphi}}{\sinh\eta}\frac{\partial\phi}{\partial\varphi}\right). \tag{26}$$

The electric field can then be calculated by $\vec{E} = -\nabla\phi$, whose components for the region outside the toroid ($\eta < \eta_0$) are given by

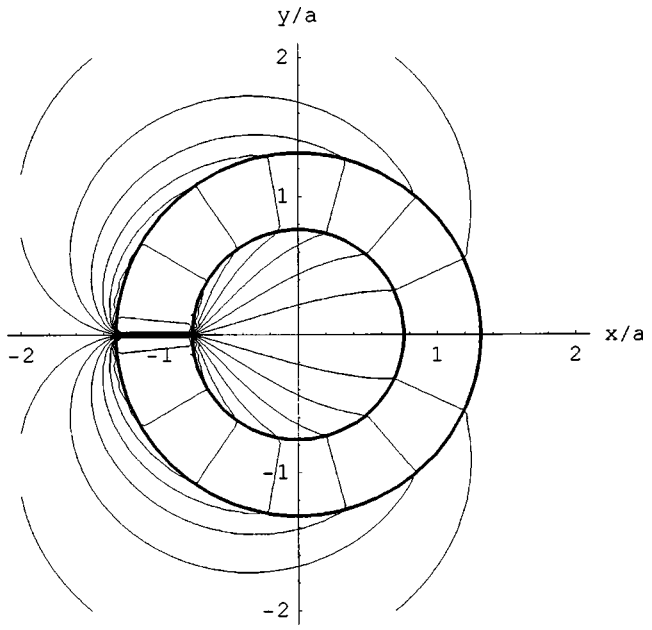

FIG. 4. Equipotentials for a resistive full solid toroidal conductor in the plane $z=0$. The bold circles represent the borders of the toroid. The current runs in the azimuthal direction, from $\varphi=+\pi$ to $-\pi$ rad. The thin battery is on the left ($\varphi=\pi$ rad). We have used $\eta_0=2.187$.

$$E_\eta=-\frac{\sinh\eta\sqrt{\cosh\eta-\cos\chi}}{a}\times\sum_{p=0}^{\infty}\cos(p\chi)\left\{A_p\left[\frac{1}{2}P_{p-1/2}(\cosh\eta)+(\cosh\eta-\cos\chi)P'_{p-1/2}(\cosh\eta)\right]+\sum_{q=1}^{\infty}\sin(q\varphi)B_{pq}\left[\frac{1}{2}P^q_{p-1/2}(\cosh\eta)+(\cosh\eta-\cos\chi)P^{q\,\prime}_{p-1/2}(\cosh\eta)\right]\right\},\tag{27}$$

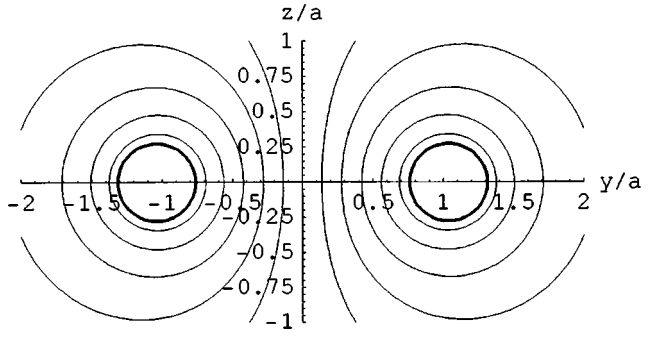

FIG. 5. Equipotentials in the plane $x=0$ for a resistive full solid toroidal conductor carrying a steady azimuthal current, Eq. (19), with $A=0$ and $B=\phi_0/2\pi$. The bold circles represent the conductor surface. We have used $\eta_0=2.187$.

$$E_\chi=-\frac{\sqrt{\cosh\eta-\cos\chi}}{a}\sum_{p=0}^{\infty}\left[\frac{\sin\chi\cos(p\chi)}{2}-p(\cosh\eta-\cos\chi)\sin(p\chi)\right]\left[A_pP_{p-1/2}(\cosh\eta)+\sum_{q=1}^{\infty}\sin(q\varphi)B_{pq}P^q_{p-1/2}(\cosh\eta)\right],\tag{28}$$

$$E_\varphi=-\frac{(\cosh\eta-\cos\chi)^{3/2}}{a\sinh\eta}\sum_{q=1}^{\infty}q\cos(q\varphi)\times\sum_{p=0}^{\infty}B_{pq}\cos(p\chi)P^q_{p-1/2}(\cosh\eta),\tag{29}$$

where $P^{q\,\prime}_{p-1/2}(\cosh\eta)$ are the derivatives of $P^q_{p-1/2}(\cosh\eta)$ relative to $\cosh\eta$. The electric field inside the full solid toroid ($\eta>\eta_0$) is given simply by

$$E_\eta=0,\quad E_\chi=0,\quad E_\varphi=-\frac{\cosh\eta-\cos\chi}{a\sinh\eta}B=-\frac{B}{\sqrt{x^2+y^2}}.\tag{30}$$

The surface charge distribution that creates the electric field inside (and outside) the conductor, keeping the current flowing, can be obtained by Gauss's law (by choosing a Gaussian surface involving a small portion of the conductor surface) for the full solid toroid, Fig. 2(a):

$$\sigma(\eta_0,\chi,\varphi)=\varepsilon_0[\vec{E}(\eta<\eta_0)\cdot(-\hat{\eta})+\vec{E}(\eta>\eta_0)\cdot\hat{\eta}]_{\eta_0}=\frac{\varepsilon_0\sinh\eta_0}{a}\left\{\frac{A+B\varphi}{2}+(\cosh\eta_0-\cos\chi)^{3/2}\times\sum_{p=0}^{\infty}\cos(p\chi)\left[A_pP'_{p-1/2}(\cosh\eta_0)+\sum_{q=1}^{\infty}\sin(q\varphi)B_{pq}P^{q\,\prime}_{p-1/2}(\cosh\eta_0)\right]\right\}.\tag{31}$$

VI. THIN TOROID APPROXIMATION

Suppose that the toroid is very thin, with its radii described by an outer radius $R=a\cosh\eta_0/\sinh\eta_0\approx a$ and an inner radius $r_0=a/\sinh\eta_0$, such that $r_0\ll R$ (see Fig. 1). The surface of the toroid is described by $\eta_0\gg1$, and consequently $\cosh\eta_0\gg1$. The Legendre function of the second kind, $Q_{p-1/2}(\cosh\eta_0)$, that appears in Eqs. (17) and (18) for the coefficients A_p and B_{pq} can be approximated utilizing (Ref. [16], p. 164)

$$Q_{p-1/2}(\cosh\eta_0)\approx\frac{\sqrt{\pi}\Gamma(p+1/2)}{2^{p+1/2}p!\cosh^{p+1/2}\eta_0},\tag{32}$$

where Γ is the gamma function (Ref. [17], p. 591).

Because Eq. (32) has a factor of $\cosh^{-p-1/2}\eta_0\ll1$, we can neglect all terms with $p>0$ in Eq. (19) compared with the

term with $p=0$. The potential inside the full solid thin toroid is given by Eq. (11), while the potential outside the full solid thin toroid ($\eta_0 \gg 1$) is

$$\phi(\eta \leqslant \eta_0, \chi, \varphi) = \sqrt{\frac{\cosh\eta - \cos\chi}{\cosh\eta_0}} \left[A \frac{P_{-1/2}(\cosh\eta)}{P_{-1/2}(\cosh\eta_0)} + 2B \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{q} \sin(q\varphi) \frac{P^q_{-1/2}(\cosh\eta)}{P^q_{-1/2}(\cosh\eta_0)} \right]. \quad (33)$$

It is interesting to find the expressions for the potential and electric field outside but in the vicinity of the conductor (that is, $\eta_0 > \eta \gg 1$). A series expansion of the functions $P^q_{-1/2}(\xi)$ and $P^{q\prime}_{-1/2}(\xi)$ around $\xi \to \infty$ gives as the most relevant terms (Ref. [16], p. 173)

$$P^q_{-1/2}(\xi) \approx \frac{\sqrt{2/\pi}}{\Gamma(1/2-q)} \frac{\ln(2\xi) - \psi(1/2-q) - \gamma}{\sqrt{\xi}},$$

$$P^{q\prime}_{-1/2}(\xi) \approx \frac{\sqrt{2/\pi}}{\Gamma(1/2-q)} \frac{1}{\xi^{3/2}} \left[1 - \frac{\ln(2\xi) - \psi(1/2-q) - \gamma}{2} \right], \quad (34)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, and $\gamma \approx 0.577\,216$ is the Euler gamma. The potential just outside the thin toroid, Eq. (19), can then be written in this approximation far from the battery as

$$\phi(\eta_0 \geqslant \eta \gg 1, \chi, \varphi) = (A + B\varphi) \frac{\ln(8\cosh\eta)}{\ln(8\cosh\eta_0)}. \quad (35)$$

The potential inside the hollow thin toroid, Eq. (20), can be written in this approximation far from the battery as

$$\phi(\eta \geqslant \eta_0 \gg 1, \chi, \varphi) = A + B\varphi. \quad (36)$$

The surface charge distribution in this thin toroid approximation is given by

$$\sigma(\eta_0 \gg 1, \chi, \varphi) \approx \frac{\varepsilon_0 \sinh\eta_0}{a \ln(8\cosh\eta_0)} (A + B\varphi) = \frac{\epsilon_0 (A + B\varphi)}{r_0 \ln(8a/r_0)} \equiv \sigma_A + \sigma_B \varphi. \quad (37)$$

We defined σ_A and σ_B by this last equality. We obtained that the surface charge density far from the battery is a linear function of the azimuthal angle φ in the case of a thin toroid.

We can calculate the total charge q_A of the thin toroid as a function of the constant electric potential A. For this, we integrate the surface charge density σ in χ and φ (in the approximation $\cosh\eta_0 \gg 1$):

$$q_A = \int_{-\pi}^{\pi} h_\chi \, d\chi \int_{-\pi}^{\pi} h_\varphi \, d\varphi \, \sigma(\chi, \varphi) = \frac{4\pi^2 \varepsilon_0 A a}{\ln(8\cosh\eta_0)} = \frac{4\pi^2 \varepsilon_0 A a}{\ln(8a/r_0)}, \quad (38)$$

where $h_\eta = h_\chi = a/(\cosh\eta - \cos\chi)$ and $h_\varphi = a\sinh\eta/(\cosh\eta - \cos\chi)$ are the scale factors in toroidal coordinates [18]. Notice that from Eq. (38) we can obtain the capacitance of the thin toroid (Ref. [19], p. 127):

$$C = \frac{q_A}{A} = \frac{4\pi^2 \varepsilon_0 a}{\ln(8\cosh\eta_0)} = \frac{4\pi^2 \varepsilon_0 a}{\ln(8a/r_0)}. \quad (39)$$

The potential along the z axis is given, from Eq. (25) in the thin toroid approximation, by

$$\phi(r = \sqrt{x^2+y^2+z^2} = |z|, \theta, \varphi) = \frac{q_A}{4\pi\varepsilon_0} \frac{1}{\sqrt{z^2+a^2}}. \quad (40)$$

Equation (40) coincides with the Coulombic result for a charged thin toroid of radius a in the $z=0$ plane and with total charge q_A.

It is useful to define a new coordinate system:

$$\lambda' = a\varphi, \quad \rho' = \sqrt{(\sqrt{x^2+y^2} - a)^2 + z^2}. \quad (41)$$

We can interpret λ' as a distance along the toroid surface in the φ direction, and ρ' as the shortest distance from the circle $x^2+y^2=a^2$ located in the plane $z=0$. When $\eta_0 > \eta \gg 1$ (that is, $r_0 < \rho' \ll a$), Eqs. (41) and (3) result in $\cosh\eta \approx a/\rho'$ and $\cosh\eta_0 \approx a/r_0$, so that the potential just outside the thin toroid, Eq. (35), can be expressed as

$$\phi = \left(A + B\frac{\lambda'}{a} \right) \frac{\ln(8a/\rho')}{\ln(8a/r_0)}. \quad (42)$$

Equation (42) can be written in a slightly different form. Consider a certain piece of the toroid between the angles φ_0 and $-\varphi_0$, with potentials in these extremities given by $\phi_R = A + B\varphi_0$ and $\phi_L = A - B\varphi_0$, respectively. This piece has a length of $\ell = 2a\varphi_0$. The potential can then be written as

$$\phi = \left(A + \frac{2B\varphi_0}{\ell}\lambda' \right) \frac{\ln(\ell/\rho') - \ln(\ell/8a)}{\ln(\ell/r_0) - \ln(\ell/8a)} \approx \left(\frac{\phi_R + \phi_L}{2} + \frac{\phi_R - \phi_L}{\ell}\lambda' \right) \frac{\ln(\ell/\rho')}{\ln(\ell/r_0)}, \quad (43)$$

where in the last approximation we neglected the term $\ln(\ell/a)$, utilizing the approximation $r_0 < \rho' \ll a$ (so that $\ell/r_0 > \ell/\rho' \gg \ell/8a$). The electric field can be expressed in this approximation as

$$\vec{E} = -\left(\frac{\phi_R + \phi_L}{2} + \frac{\phi_R - \phi_L}{\ell}\lambda' \right) \frac{\hat{\eta}}{\rho' \ln(\ell/r_0)} - \frac{\phi_R - \phi_L}{\ell} \frac{\ln(\ell/\rho')}{\ln(\ell/r_0)} \hat{\varphi}. \quad (44)$$

Equations (43) and (44) can be compared to Eqs. (12) and (13) of Assis, Rodrigues, and Mania [10], reproduced below as Eqs. (45) and (46), respectively. They studied the case of a long straight cylindrical conductor of radius r_0 carrying a constant current, in cylindrical coordinates (ρ',φ,z) (note that the conversions from toroidal to cylindrical coordinates in this approximation are $\hat{\eta}\approx-\hat{\rho}'$ and $\hat{\varphi}\approx\hat{z}$). In their case, the cylinder has a length ℓ and radius $r_0\ll\ell$, with potentials ϕ_L and ϕ_R in the extremities of the conductor, and $RI=\phi_L-\phi_R$:

$$\phi(r\geqslant a)=\left(\frac{\phi_R+\phi_L}{2}+\frac{\phi_R-\phi_L}{\ell}z\right)\frac{\ln(\ell/\rho')}{\ln(\ell/r_0)}, \tag{45}$$

$$\vec{E}(\rho'\geqslant a)=\left(\frac{\phi_R+\phi_L}{2}+\frac{\phi_R-\phi_L}{\ell}z\right)\frac{\hat{\rho}'}{\hat{\rho}'\ln(\ell/r_0)} - \frac{\phi_R-\phi_L}{\ell}\frac{\ln(\ell/\rho')}{\ln(\ell/r_0)}\hat{z}. \tag{46}$$

Our result for the potential in the region close to the thin toroid coincides with the cylindrical solution, as expected.

VII. CHARGED TOROID WITHOUT CURRENT

Consider a toroid described by η_0, without current but charged to a constant potential ϕ_0. Using $A=\phi_0$ and $B=0$ in Eqs. (19), (11), and (20), we have the potential inside and outside the toroid, respectively:

$$\phi(\eta\geqslant\eta_0,\chi,\varphi)=A=\phi_0, \tag{47}$$

$$\phi(\eta\leqslant\eta_0,\chi,\varphi)=\sqrt{\cosh\eta-\cos\chi}\sum_{p=0}^{\infty}A_p\cos(p\chi) \times P_{p-1/2}(\cosh\eta), \tag{48}$$

where $P_{p-1/2}(\cosh\eta_0)$ are the Legendre functions, and the coefficients A_p are given by Eq. (17). This solution is already known in the literature (Ref. [20], p. 239), (Ref. [21], p. 1304).

It is also possible to obtain the capacitance of the toroid, by comparing the electrostatic potential at a distance r far from the origin, Eq. (23), with the potential given by a point charge q, $\phi(r\geqslant a)\approx q/4\pi\varepsilon_0 r$:

$$\phi(r\geqslant a,\theta,\varphi)\approx\frac{a\sqrt{2}}{r}\sum_{p=0}^{\infty}\frac{\sqrt{2}\phi_0(2-\delta_{0p})}{\pi}\frac{Q_{p-1/2}(\cosh\eta_0)}{P_{p-1/2}(\cosh\eta_0)} =\frac{q}{4\pi\epsilon_0 r}. \tag{49}$$

The capacitance of the toroid with its surface at a constant potential ϕ_0 can be written as $C=q/\phi_0$. From Eq. (49) this yields (Ref. [20], p. 239), (Ref. [22], pp. 5–13), (Ref. [23], p. 9), (Ref. [24], p. 375)

$$C=8\varepsilon_0 a\sum_{p=0}^{\infty}(2-\delta_{0p})\frac{Q_{p-1/2}(\cosh\eta_0)}{P_{p-1/2}(\cosh\eta_0)}. \tag{50}$$

Utilizing the thin toroid approximation $\eta_0\gg1$, one can obtain the capacitance of a circular ring, Eq. (39).

Another case of interest is that of a charged circular wire discussed below, which is the particular case of a toroid with $r_0\to0$. With $\eta_0\gg1$ and $\cosh\eta_0\gg1$ we have $R\approx a$. Keeping only the term with $p=0$ in Eqs. (17) and (48) yields

$$\phi(\eta\leqslant\eta_0,\chi,\varphi)=\phi_0\sqrt{\frac{\cosh\eta-\cos\chi}{\cosh\eta_0}}\frac{P_{-1/2}(\cosh\eta)}{P_{-1/2}(\cosh\eta_0)} =\frac{q_A}{4\pi\sqrt{2}\varepsilon_0 a}\sqrt{\cosh\eta-\cos\chi} \times P_{-1/2}(\cosh\eta), \tag{51}$$

where in the last equation we combined Eq. (38). Expressed in spherical coordinates (r,θ,φ), the potential for the thin toroid becomes

$$\phi(r,\theta,\varphi)=\frac{q_A}{4\pi\varepsilon_0}\frac{1}{[(r^2-a^2)^2+4a^2r^2\cos^2\theta]^{1/4}} \times P_{-1/2}\left(\frac{r^2+a^2}{\sqrt{(r^2-a^2)^2+4a^2r^2\cos^2\theta}}\right). \tag{52}$$

From Eqs. (47) and (38) we can see that the constant electrostatic potential along the thin toroid expressed in terms of its total charge q_A is given by

$$\phi(r_0\ll R,\theta,\varphi)=\frac{q_A/2\pi a}{2\pi\varepsilon_0}\ln\frac{8a}{r_0}. \tag{53}$$

Even when the linear charge density $q_A/2\pi a$ remains constant, we can see from this expression that the potential diverges logarithmically when $a/r_0\to\infty$.

We can expand Eq. (52) on $r_</r_>$, where $r_<$ $(r_>)$ is the lesser (greater) of a and $r=\sqrt{x^2+y^2+z^2}$. We present the first three terms:

$$\phi(r,\theta,\varphi)\approx\frac{q_A}{4\pi\varepsilon_0}\left\{\frac{1}{r_>}-\frac{1+3\cos(2\theta)}{8}\frac{r_<^2}{r_>^3} + \frac{3}{512}[9+20\cos(2\theta)+35\cos(4\theta)]\frac{r_<^4}{r_>^5}\right\}. \tag{54}$$

Equations (51)–(54) can be compared with the solution given by Jackson (Ref. [25], p. 93). Jackson gives the exact electrostatic solution of the problem of a charged circular wire (that is, a toroid with radius $r_0=0$), in spherical coordinates (r,θ,φ):

$$\phi(r,\theta,\varphi)=\frac{q_A}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{r_<^{2n}}{r_>^{2n+1}}\frac{(-1)^n(2n-1)!!}{2^n n!}P_{2n}(\cos\theta), \tag{55}$$

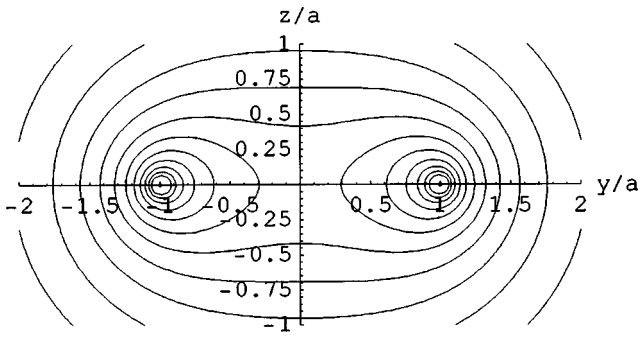

FIG. 6. Equipotential lines on the plane $x=0$ (perpendicular to the toroid) for the charged thin wire without current. Both Eqs. (51) and (55) coincide with one another. We utilized $\eta_0=38(\cosh\eta_0=1.6\times10^{16})$ and $a=1$. Notice the difference between this figure and Fig. 5: the left and right sides of the conductor here possess the same charge signs, while in Fig. 5 they have opposite signs.

where q_A is the total charge of the wire. Equation (55) expanded to $n=2$ yields exactly Eq. (54). We have checked that Eqs. (52) and (55) are the same for at least $n=30$.

We plotted both Eqs. (51) and (55) in Fig. 6. They yield the same result, as expected. It is worthwhile to note that in spherical coordinates we have an infinite sum, Eq. (55), while in toroidal coordinates the solution is given by a single term, Eq. (51). The agreement shows that Eqs. (51) and (55) are the same solution only expressed in different forms.

Figure 7 shows the potential as function of ρ (in cylindrical coordinates) in the plane $z=0$. Equations (51) and (55) give the same result.

VIII. DISCUSSION AND CONCLUSION

Figure 4 can be compared with the experimental result found by Jefimenko (Ref. [14], Fig. 3), reproduced here in Fig. 8 with Fig. 4 overlaid on it. Jefimenko painted a circular conducting strip on a glass plate utilizing a transparent conducting ink. A steady current flowed in the strip by connecting its extremities with a battery. By spreading grass seeds on the glass plate he was able to map the electric field lines inside and outside the strip (in analogy with iron filings mapping the magnetic field lines). The equipotential lines ob-

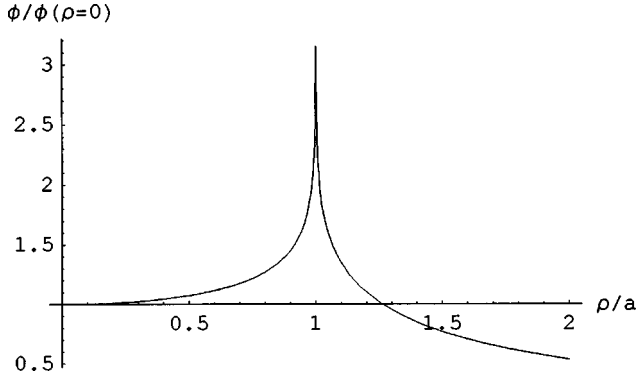

FIG. 7. Normalized potential as function of ρ (distance from z axis) on the plane $z=0$. Equations (51) and (55) give the same result. We utilize $\eta_0=38(\cosh\eta_0=1.6\times10^{16})$ and $a=1$.

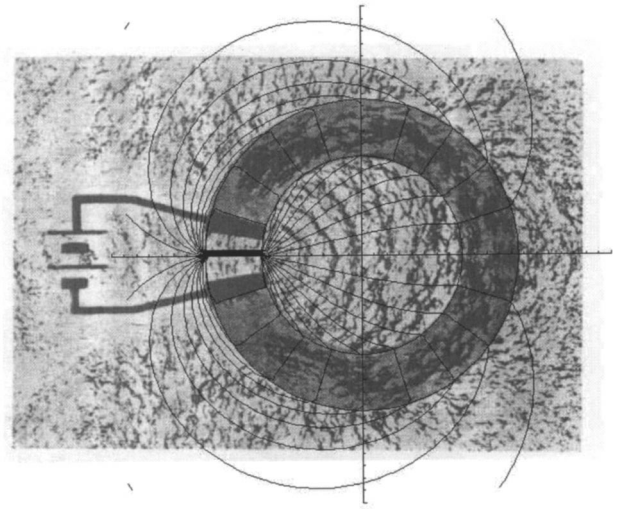

FIG. 8. Jefimenko's experiment (Ref. [14], Fig. 3) in which the lines of electric field were mapped using grass seeds spread over a glass plate. There is a circular conducting strip carrying a steady current. Figure 4 has been overlaid on it—the equipotential lines are orthogonal to the electric field lines.

tained here are orthogonal to the electric field lines. There is a very reasonable agreement between our theoretical result and the experiment.

In order to have a better fit to his data we should consider an extended battery. As we can see from his experiment, he painted two sections of his strip with a conducting ink of much smaller resistivity than the remainder of the strip. These sections located at $-\varphi_j<\varphi<-\varphi_i$ and $\varphi_i<\varphi<\varphi_j$ were charged to opposite potentials. Considering these sections as of zero resistivity, we can model analytically the potential inside and along the surface of the toroid as

$$\phi(\eta\geqslant\eta_0,\chi,\varphi)=\begin{cases}-B\dfrac{\varphi_i(\pi+\varphi)}{(\pi-\varphi_j)}, & -\pi<\varphi<-\varphi_j,\\ -B\varphi_i, & -\varphi_j<\varphi<-\varphi_i,\\ B\varphi, & -\varphi_i<\varphi<\varphi_i,\\ B\varphi_i, & \varphi_i<\varphi<\varphi_j,\\ B\dfrac{\varphi_i(\pi-\varphi)}{(\pi-\varphi_j)}, & \varphi_j<\varphi<\pi.\end{cases} \tag{56}$$

Notice that the potential described by Eq. (56) no longer has a discontinuity at $\varphi=\pi$ rad. The potential is linear between $\varphi=-\varphi_i$ and $\varphi=\varphi_i$, constant for $-\varphi_j<\varphi<-\varphi_i$ and $\varphi_i<\varphi<\varphi_j$, and linear for $-\pi$ rad$<\varphi<-\varphi_j$ and for $\varphi_j<\varphi<\pi$ rad. The boundary condition Eq. (10) is now replaced by

$$\phi(\eta_0,\chi,\varphi)=2B\sum_{q=1}^{\infty}\frac{\sin(q\varphi)}{q^2}\left[\frac{\sin(q\varphi_j)}{\pi-\varphi_j}+\frac{\sin(q\varphi_i)}{\varphi_i}\right]. \tag{57}$$

The potential from Eq. (57) is represented in Fig. 9 with the values $\varphi_i=9\pi/10=2.83$ rad and $\varphi_j=17\pi/18=2.97$ rad.

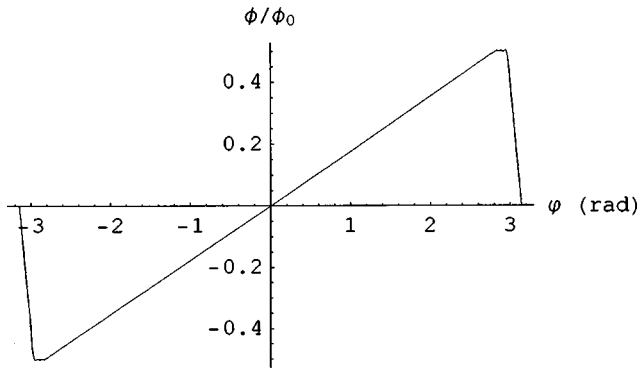

FIG. 9. Fourier expansion of the potential along the conductor surface as a function of the azimuthal angle φ, Eq. (57), with $B=\phi_0/2\varphi_i$. Comparing this figure with Fig. 3 we can observe that the oscillations, as well as the overshooting, no longer appear, as the potential is now continuous for $0\leqslant\varphi\leqslant 2\pi$ rad. We have used $\varphi_i=9\pi/10=2.83$ rad and $\varphi_j=17\pi/18=2.97$ rad.

The equipotentials in the plane $z=0$ are plotted in Fig. 10. Figure 11 represents Jefimenko's experiment with Fig. 10 overlaid on it. The agreement is now even better than in Fig. 8.

Despite this agreement, it should be mentioned that Jefimenko's experiment has a conducting strip painted on a glass plate. On the other hand, our theoretical results presented in Figs. 4 and 10 represent an equatorial slice through a three-dimensional toroid. In another experiment Jefimenko, Barnett, and Kelly succeeded in measuring directly the equipotential lines inside and outside a hollow rectangular conductor carrying a steady current [26] and (Ref. [2], pp. 300–301). They utilized a radioactive α source to ionize the air at the point where the field was to be measured. The α source acquired the same potential as the field at that point and the potential was measured with an electrometer connected to the α source. If one day a similar experiment is performed with a toroid, it will be possible to obtain a better comparison with our theoretical results.

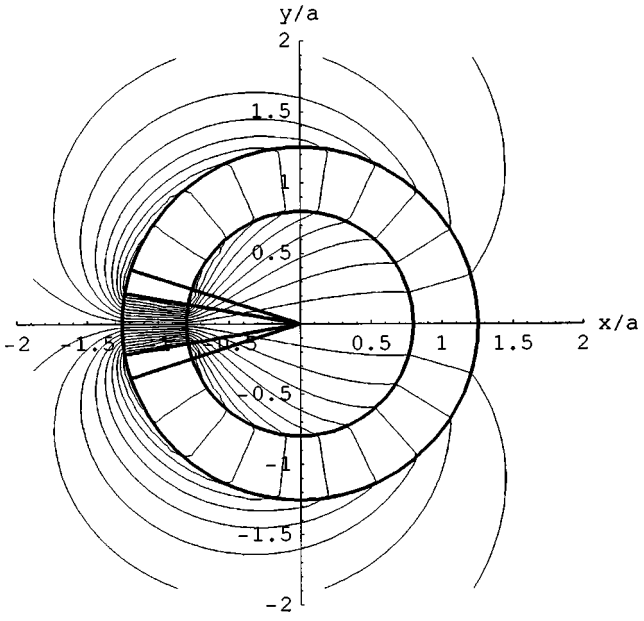

FIG. 10. Equipotentials in the plane $x=0$ for a resistive toroidal conductor carrying a steady azimuthal current, using Eq. (57) as boundary condition and $B=\phi_0/2\varphi_i$. The bold circles represent the conductor surface and the bold straight lines represent the angles $\varphi=\pm\varphi_i=\pm 9\pi/10=2.83$ rad and $\varphi=\pm\varphi_j=\pm 17\pi/18=2.97$ rad. We have used $\eta_0=2.187$.

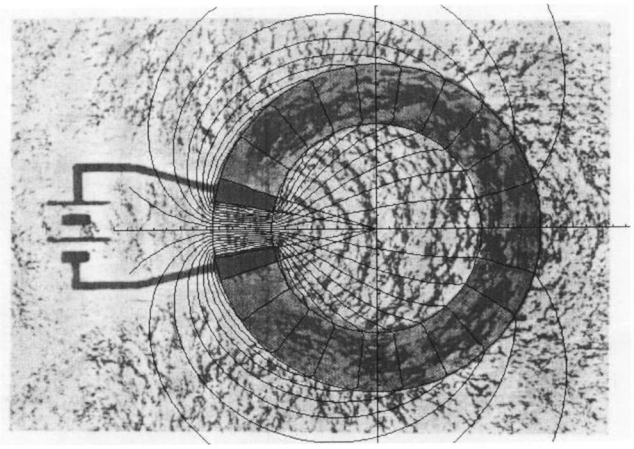

FIG. 11. Jefimenko's experiment (Ref. [14], Fig. 3) with Fig. 10 overlaid on it—the equipotential lines are orthogonal to the electric field lines.

Our solution inside and along the surface of the full solid toroid yields only an azimuthal electric field, namely, $|E_\varphi|=\Delta\phi/2\pi\rho$. But even for a steady current we must have a component of $\vec{E}$ pointing away from the z axis, E_ρ, due to the curvature of the wire. Here we are neglecting this component due to its extremely small order of magnitude compared with the azimuthal component E_φ. To show this, consider a conducting electron of charge $-e$ and mass m moving azimuthally with drift velocity v_d in a circle of radius ρ around the z axis. In a steady state situation there will be a redistribution of charges along the cross section of the toroid, creating an electric field E_ρ, which will exert a centripetal force on the conduction electrons. By Newton's second law of motion the force eE_ρ results in a centripetal acceleration such that $eE_\rho=mv_d^2/\rho$. Suppose we have a 14-gauge copper wire ($r_0=8.14\times10^{-4}$ m) of 1 m length bent in a circle of radius $R=\rho=(1/2\pi)=1.59\times10^{-1}$ m carrying a current of 1 A. The drift velocity is given by $v_d=3.55\times10^{-5}$ m/s, the resistance of the wire is $8.13\times10^{-3}\ \Omega$, and the potential difference created by the battery is $\Delta\phi=8.13\times10^{-3}$ V. This yields $E_\varphi=8.13\times10^{-3}$ V/m and $E_\rho=4.5\times10^{-20}$ V/m. That is, $E_\rho\ll E_\varphi$, which justifies neglecting the E_ρ component of the electric field.

It has been pointed out elsewhere [10] that a stationary conductor carrying a steady current uniform over its cross section generates a charge distribution inside the conductor. This charge distribution creates a radial electric field inside the conductor. There is then an electric force on the conduction electrons that counteracts the radial magnetic force that arises due to the movement of the conduction electrons. This is known as the radial Hall effect. However, this electric field is rather small (10^{-5} smaller than the electric field that main-

tains the current flowing, on a typical copper conductor with 1 mm diameter and 4×10^{-3} m/s drift velocity; see [10]) and has been neglected in the present work.

The problem of a stationary toroidal conductor with a steady current has never been solved in the literature. Here, we have obtained the electric potential, the electric field, and the surface charges, respectively, Eqs. (19) and (27)–(31). The beautiful experimental result of Jefimenko showing the electric field outside the conductor is complemented by this present theoretical work, with excellent agreement, Figs. 8 and 11. The electric potential and electric field of the thin toroid approximation with a steady current, respectively, Eqs. (43) and (44), agree with the known case of a long straight cylindrical conductor carrying a steady current, Eqs. (12) and (13) of [10]. The electric potential of the thin toroid approximation without current agrees with the known result for a charged wire (Ref. [25], p. 93).

The other theoretical solutions known in the literature usually considered an infinite or very long straight conductor or coaxial cable (Ref. [1], pp. 125–33), (Ref. [5], pp. 336–7), [10,6,11]. The only cases of an analytical solution in a curved conductor are those of Jefimenko and Heald, who considered a solenoid of infinite length carrying a steady azimuthal current (Ref. [2], p. 318) and [7]. The only known solution for the potential due to a closed steady current flowing in a bounded volume is that of Jackson [3], who considered a coaxial cable of finite length. However, as he considered the external return conductor of zero resistivity, the electric field outside his cable was found to be zero. Here, on the other hand, we obtained a theoretical solution for the potential due to a steady azimuthal current flowing in a toroidal resistive conductor which yielded an electric field not only inside the toroid but also in the space surrounding it. Our solution showed a reasonable agreement with Jefimenko's experiment, which proved the existence of this external electric field due to a resistive steady current.

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