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The internal and external electric fields for a resistive toroidal conductor carrying a steady poloidal current

J A Hernandes, A J Mania, F R T Luna and A K T Assis

[1] Departamento de Ciências Exatas e Tecnólogicas (DCET), Universidade Estadual de Santa Cruz (UESC), 45662-000 Ilhéus, Bahia, Brazil
[2] Institute of Physics 'Gleb Wataghin', University of Campinas (UNICAMP), 13083-970 Campinas, SP, Brazil

E-mail: jahernandes@gmail.com, mania@uesc.br, tamariz@uesc.br and assis@ifi.unicamp.br

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Abstract
We consider the case of a resistive toroidal conductor carrying a steady current in the poloidal direction. We obtain algebraic expressions for the electric potential, the electric field and the surface charges inside and outside the toroidal shell. We use toroidal coordinates, in which Laplace's equation is R-separable. We analyze the limiting case of a thin toroid, which can be compared with the solution for the ideal straight solenoid.

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1. Introduction

Recently, there has been renewed interest in the electric field inside and outside stationary resistive conductors carrying steady currents. Some geometries have already been analyzed: coaxial cables, cylindrical shells, transmission lines, straight wires, conductor plates, straight strips, spherical shells and a toroidal conductor with azimuthal current [1, pp 125–30], [2, pp 318 and 509–511], [3, 4], [5, pp 334 and 339–40], [6, 7], [8, p 262], [9–14]. A thorough analysis of these cases can be found in [15].

Our goal in this work is to consider a new case of toroidal geometry with the current in the poloidal direction along the toroidal surface. This corresponds to a battery distributed along the inner equator of the toroid, as can be seen in figure 1. The term 'poloidal' is used typically in plasma physics for a toroidal geometry [16, p 284]. The importance of this geometry is to consider a problem in which the current is confined in a finite volume of space. It may also be applied to plasma physics as the geometry is similar to that found in a typical tokamak. In this work, we find the electric potential in the space surrounding the conducting surface, inside and outside the toroid. We also analyze the surface charge distribution along the conductor. These surface charges, maintained by the battery, keep the steady current along the resistive conductor. In long (infinite) straight conductors with a longitudinal current, the surface charges are linear with the longitudinal coordinate [17]. In the case of an idealized infinite cylindrical conductor with azimuthal current (linear potential with the azimuthal coordinate θ, [7]), the surface charges are not linear with the azimuthal coordinate. Instead of this, they vary with $\tan(\theta/2)$. In this work, we also compare our analytical results for the case of a thin toroid with the previously known solution of an ideal cylinder with azimuthal current.

2. Description of the geometry

The toroidal coordinates are (η, χ, φ) [18, p 112]. The orthogonal surfaces correspond to toroidal shells (constant η), spherical bowls (constant χ) and half-planes (constant φ), see figure 2. Their ranges are given by:

$$0 \leqslant \eta < \infty,$$
$$-\pi < \chi \leqslant \pi,$$
$$0 \leqslant \varphi < 2\pi.$$

We also define the following magnitudes:

$$\rho = \sqrt{x^2 + y^2}, \tag{1}$$

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{2}$$

The toroidal and Cartesian coordinates are related by the following equations:

$$x = a\frac{\sinh\eta\cos\varphi}{\cosh\eta - \cos\chi}, \tag{3}$$

$$y = a\frac{\sinh\eta\sin\varphi}{\cosh\eta - \cos\chi}, \tag{4}$$

$$z = a\frac{\sin\chi}{\cosh\eta - \cos\chi}, \tag{5}$$

$$\tanh\eta = \frac{2a\rho}{r^2 + a^2}, \tag{6}$$

$$\tan\chi = \frac{2az}{|r^2 - a^2|}, \tag{7}$$

$$\tan\varphi = \frac{y}{x}. \tag{8}$$

Here, a is a constant that gives the radius of a circle in the $z = 0$ plane described by $\eta \to \infty$ (that is, when $\eta \to \infty$ we have $x = a\cos\varphi$, $y = a\sin\varphi$ and $z = 0$).

Laplace's equation in toroidal coordinates is given by:

$$\begin{aligned}\nabla^2\phi = \frac{(\cosh\eta - \cos\chi)^3}{a^2\sinh\eta}\Bigg[&\frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta - \cos\chi}\frac{\partial\phi}{\partial\eta}\right)\\ &+ \frac{\partial}{\partial\chi}\left(\frac{\sinh\eta}{\cosh\eta - \cos\chi}\frac{\partial\phi}{\partial\chi}\right)\\ &+ \frac{1}{\sinh\eta(\cosh\eta - \cos\chi)}\frac{\partial^2\phi}{\partial\varphi^2}\Bigg].\end{aligned} \tag{9}$$

We want a current along the toroidal surface in the poloidal ($\hat{\chi}$) direction, see figure 1. The intersection of a toroidal surface η_0 with any half-plane φ defines a circle given by the equation $(\rho - a\coth\eta_0)^2 + z^2 = a^2\,\mathrm{cosech}^2\eta_0$. This circle is centered at a distance R_0 from the z-axis and has a radius r_0, as shown in figure 1, given by:

$$r_0 = a\,\mathrm{cosech}\,\eta_0, \tag{10}$$

$$R_0 = a\coth\eta_0. \tag{11}$$

We can see that $R_0 > a$. We define θ as the poloidal angle that parametrizes this circle linearly (see figure 1):

$$\begin{aligned}\theta(\chi) &\equiv \tan^{-1}\left(\frac{z}{\rho - R_0}\right)\\ &= \tan^{-1}\left(\frac{\chi/|\chi|}{\cot|\chi|\coth\eta_0 - \mathrm{cosec}\,|\chi|\,\mathrm{cosech}\,\eta_0}\right).\end{aligned} \tag{12}$$

The angle θ is defined in the range $-\pi < \theta \leqslant \pi$.

We choose a resistivity for the toroidal surface in such a way that our boundary condition for the potential is linear with the poloidal angle θ along the circle defined by the toroidal surface η_0 and any half-plane φ, namely:

$$\phi(\eta_0, \chi, \varphi) = \phi_A + \phi_B\frac{\theta(\chi)}{2\pi}. \tag{13}$$

Here, ϕ_A and ϕ_B are constants, and θ is given by equation (12). The solution for $\phi_B = 0$ (i.e. a charged toroid with constant potential ϕ_A at its surface) has already been treated in [14]. For this reason we set $\phi_A = 0$ in the remainder of this work.

Note that a potential linear with θ is not the same as a potential linear with the toroidal coordinate χ, as can clearly be seen in equation (12). Figure 3 shows the nonlinear relation between θ and χ for a few values of η_0. For $\eta_0 \gg 1$ (which corresponds to a thin toroid) we have:

$$\theta \approx \chi + 2e^{-\eta_0}\sin\chi \approx \chi, \tag{14}$$

as should be expected.

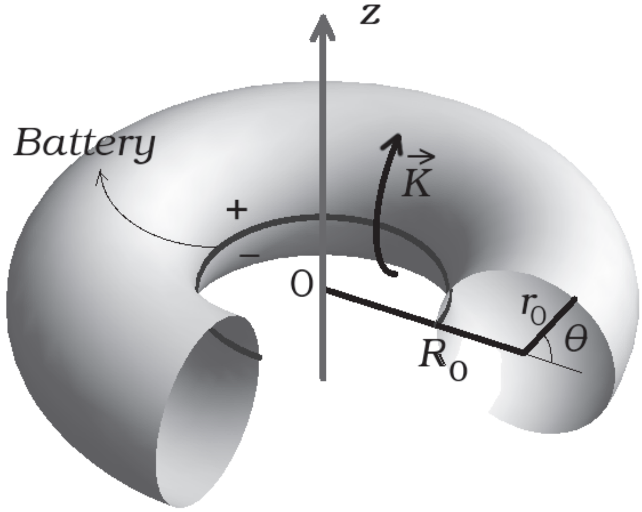

Figure 1. Toroidal conductor, with a surface current $\vec{K}$ along the poloidal direction. The battery is distributed along the inner equator of the toroid. A slice was removed for viewing purposes.

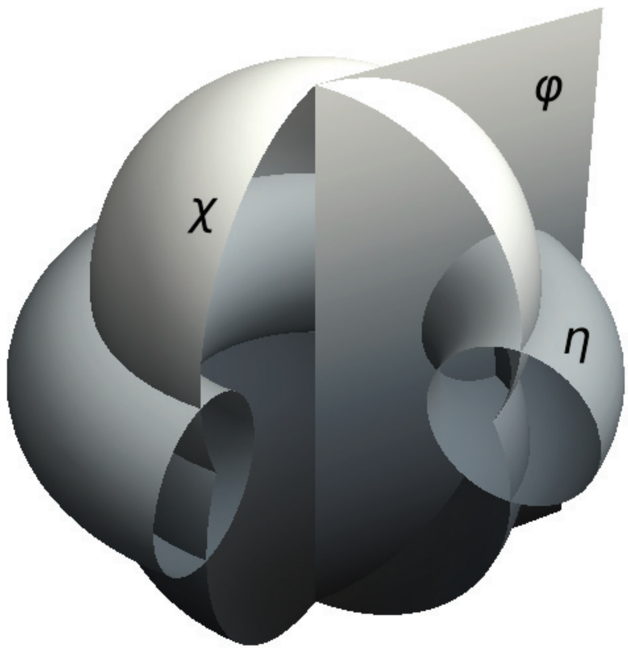

Figure 2. Orthogonal toroidal surfaces, corresponding to toroidal shells (constant η), spherical bowls (constant χ) and half-planes (constant φ). A slice was removed for viewing purposes.

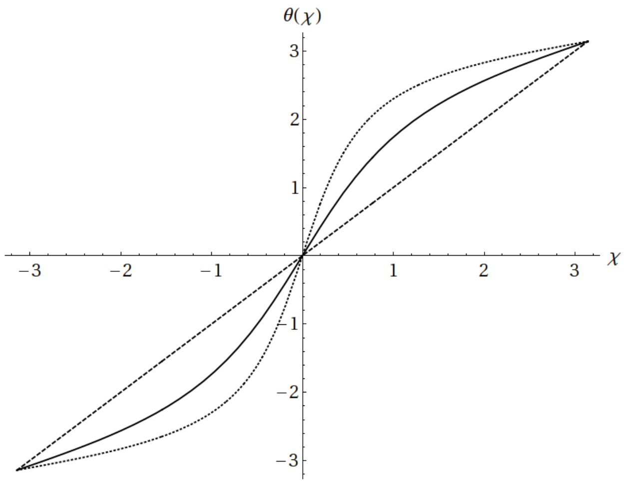

Figure 3. Poloidal angle θ given by equation (12) as a function of χ. The various curves correspond to $\eta_0 = 0.5$ (dotted line), $\eta_0 = 1.0$ (thin line) and $\eta_0 = 10$ (dashed line).

3. Solution of Laplace's equation

In toroidal coordinates, Laplace's equation is R-separable in the form:

$$\phi(\eta, \chi, \varphi) = \sqrt{\cosh\eta - \cos\chi}\, H(\eta) X(\chi) \Phi(\varphi). \tag{15}$$

The functions $H(\eta)$, $X(\chi)$ and $\Phi(\varphi)$ must obey the equations:

$$(\cosh^2\eta - 1)H'' + 2\cosh\eta H' - \left[\left(p^2 - \frac{1}{4}\right) + \frac{q^2}{\cosh^2\eta - 1}\right] H = 0, \tag{16}$$

$$X'' + p^2 X = 0, \tag{17}$$

$$\Phi'' + q^2 \Phi = 0. \tag{18}$$

The boundary condition, equation (13), states that the potential on the conductor surface does not depend on the coordinate φ. We should expect the same for the potential in all space, i.e. $\partial\phi/\partial\varphi = 0$. Applying this condition into equation (18) yields $q = 0$. We can set the constant function $\Phi(\varphi) = \text{constant}$ as unity, $\Phi(\varphi) = 1$. The electric potential must also be periodic with the coordinate χ, with period 2π: $\phi(\chi + 2\pi) = \phi(\chi)$. From equation (17) this can only be obtained with integer p and the functions $\sin(p\chi)$ and $\cos(p\chi)$.

Equation (16) is the toroidal Legendre equation, whose solutions are the toroidal Legendre functions $P_{p-\frac{1}{2}}(\cosh\eta)$ and $Q_{p-\frac{1}{2}}(\cosh\eta)$ [19, p 173]. Functions $Q_{p-\frac{1}{2}}$ must not appear in the potential outside the toroid ($\eta < \eta_0$) as they diverge for $\eta \to 0$, which corresponds to the z-axis and also for infinite distances from the origin. Functions $P_{p-\frac{1}{2}}$ should not appear in the potential inside the toroid ($\eta > \eta_0$) as they diverge for $\eta \to \infty$, which corresponds to the circle of radius a inside the toroid. The potential must also be continuous.

Accordingly, the general solutions for the potential outside ($\eta \leqslant \eta_0$) and inside ($\eta \geqslant \eta_0$) the toroid, respectively, must be given by:

$$\phi(\eta \leqslant \eta_0, \chi, \varphi) = \sqrt{\cosh\eta - \cos\chi}\left\{\sum_{p=0}^{\infty}[A_p\cos(p\chi) + B_p\sin(p\chi)]P_{p-\frac{1}{2}}(\cosh\eta)\right\}, \tag{19}$$

$$\phi(\eta \geqslant \eta_0, \chi, \varphi) = \sqrt{\cosh\eta - \cos\chi}\left\{\sum_{p=0}^{\infty}[C_p\cos(p\chi) + D_p\sin(p\chi)]Q_{p-\frac{1}{2}}(\cosh\eta)\right\}. \tag{20}$$

The constant coefficients A_p, B_p, C_p and D_p must be determined from the boundary condition, equation (13).

Calculating equations (19) and (20) for $\eta = \eta_0$ and comparing them to equation (13) yields:

$$\phi_B\frac{\theta(\chi)}{2\pi} = \sqrt{\cosh\eta_0 - \cos\chi}\left\{\sum_{p=0}^{\infty}[A_p\cos(p\chi) + B_p\sin(p\chi)]P_{p-\frac{1}{2}}(\cosh\eta_0)\right\}, \tag{21}$$

$$\phi_B\frac{\theta(\chi)}{2\pi} = \sqrt{\cosh\eta_0 - \cos\chi}\left\{\sum_{p=0}^{\infty}[C_p\cos(p\chi) + D_p\sin(p\chi)]Q_{p-\frac{1}{2}}(\cosh\eta_0)\right\}. \tag{22}$$

We define $a_p \equiv A_p P_{p-\frac{1}{2}}(\cosh\eta_0)$, $b_p \equiv B_p P_{p-\frac{1}{2}}(\cosh\eta_0)$, $c_p \equiv C_p Q_{p-\frac{1}{2}}(\cosh\eta_0)$ and $d_p \equiv D_p Q_{p-\frac{1}{2}}(\cosh\eta_0)$. With these definitions, equations (21) and (22) can be rewritten as:

$$\frac{\phi_B(\theta/2\pi)}{\sqrt{\cosh\eta_0 - \cos\chi}} = \sum_{p=0}^{\infty}[a_p\cos(p\chi) + b_p\sin(p\chi)] = \sum_{p=0}^{\infty}[c_p\cos(p\chi) + d_p\sin(p\chi)]. \tag{23}$$

This equation resembles a Fourier expansion of the left-hand side. Applying the Fourier series theory [20, chapter 14], we can find the coefficients A_p, B_p, C_p and D_p:

$$A_p = 0, \tag{24}$$

$$B_p = \frac{\phi_B}{2\pi^2 P_{p-\frac{1}{2}}(\cosh\eta_0)}\int_{-\pi}^{\pi}\frac{\theta(\chi)\sin(p\chi)\,\mathrm{d}\chi}{\sqrt{\cosh\eta_0 - \cos\chi}}, \tag{25}$$

$$C_p = 0, \tag{26}$$

$$D_p = \frac{\phi_B}{2\pi^2 Q_{p-\frac{1}{2}}(\cosh\eta_0)}\int_{-\pi}^{\pi}\frac{\theta(\chi)\sin(p\chi)\,\mathrm{d}\chi}{\sqrt{\cosh\eta_0 - \cos\chi}}. \tag{27}$$

Here θ is given by equation (12). The equipotential lines for any given half-plane φ are shown in figure 4.

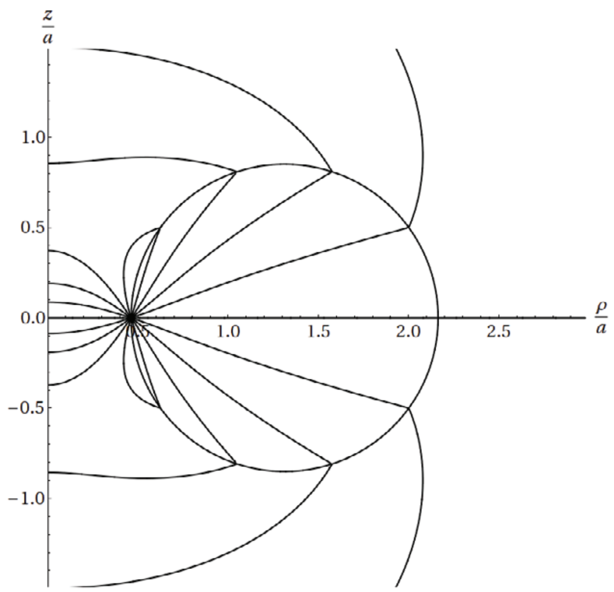

Figure 4. Equipotential lines of the conducting toroid with poloidal current —clockwise along the central circle depicted in this figure. We used $N = 100$ terms in equations (19) and (20), with $\eta_0 = 1$ and $\phi_A = 0$.

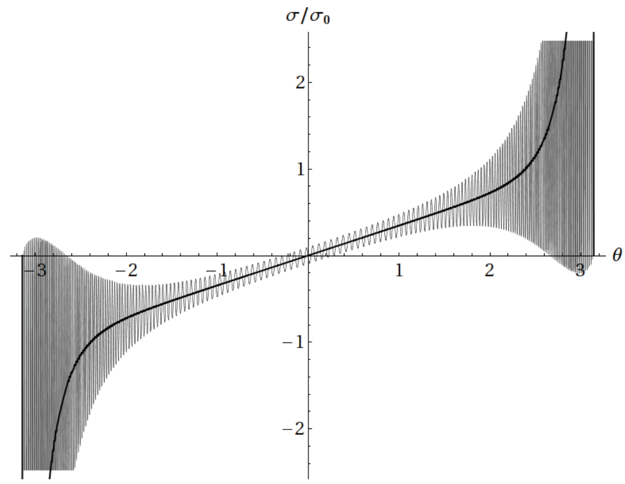

Figure 5. Surface charge density σ given by equation (35) as a function of θ. The continuous line represents $\bar{\sigma}$ given by equation (37). Both surface charge densities are normalized by $\sigma_0 = \varepsilon_0\phi_B/a$. We used $\phi_A = 0$ and $\eta_0 = 1$.

4. Electric field and surface charges

The electric field can be obtained from the potential in toroidal coordinates as given by:

$$\vec{E} = -\nabla\phi = -\frac{\cosh\eta - \cos\chi}{a}\left(\hat{\eta}\frac{\partial\phi}{\partial\eta} + \hat{\chi}\frac{\partial\phi}{\partial\chi} + \frac{\hat{\varphi}}{\sinh\eta}\frac{\partial\phi}{\partial\varphi}\right). \tag{28}$$

The components of the electric field outside (o) and inside (i) the toroidal surface are given by, respectively:

$$E_\eta^{(o)} = -\frac{\sinh\eta\sqrt{\cosh\eta - \cos\chi}}{a} \times \left\{\sum_{p=0}^{\infty} B_p \sin(p\chi)\left[\frac{1}{2}P_{p-\frac{1}{2}}(\cosh\eta) + (\cosh\eta - \cos\chi)P'_{p-\frac{1}{2}}(\cosh\eta)\right]\right\}, \tag{29}$$

$$E_\chi^{(o)} = -\frac{\sqrt{\cosh\eta - \cos\chi}}{a}\left\{\sum_{p=1}^{\infty} B_p\left[\frac{\sin\chi\,\sin(p\chi)}{2} + p\cos(p\chi)(\cosh\eta - \cos\chi)\right]\right\}P_{p-\frac{1}{2}}(\cosh\eta), \tag{30}$$

$$E_\varphi^{(o)} = 0, \tag{31}$$

$$E_\eta^{(i)} = -\frac{\sinh\eta\sqrt{\cosh\eta - \cos\chi}}{a} \times \left\{\sum_{p=0}^{\infty} D_p \sin(p\chi)\left[\frac{1}{2}Q_{p-\frac{1}{2}}(\cosh\eta) + (\cosh\eta - \cos\chi)Q'_{p-\frac{1}{2}}(\cosh\eta)\right]\right\}, \tag{32}$$

$$E_\chi^{(i)} = -\frac{\sqrt{\cosh\eta - \cos\chi}}{a} \times \left\{\sum_{p=1}^{\infty} D_p\left[\frac{\sin\chi\,\sin(p\chi)}{2} + p\cos(p\chi)(\cosh\eta - \cos\chi)\right]\right\}Q_{p-\frac{1}{2}}(\cosh\eta), \tag{33}$$

$$E_\varphi^{(i)} = 0. \tag{34}$$

In the above expressions, we have used $P'_{p-\frac{1}{2}}(z) \equiv \mathrm{d}P_{p-\frac{1}{2}}(z)/\mathrm{d}z$ and $Q'_{p-\frac{1}{2}}(z) \equiv \mathrm{d}Q_{p-\frac{1}{2}}(z)/\mathrm{d}z$.

The density of surface charges, $\sigma(\chi)$, can be obtained from Gauss's law applied to a small closed surface involving a piece of the conducting surface [21, section I.5]. It is related to the normal ($\hat{\eta}$) components of the electric field inside and outside the toroid. The total surface charge density is then given by:

$$\sigma(\chi) = \frac{\varepsilon_0\sinh\eta_0}{a}(\cosh\eta_0 - \cos\chi)^{3/2} \times \left\{\sum_{p=0}^{\infty}\sin(p\chi)\left[B_p P'_{p-\frac{1}{2}}(\cosh\eta_0) - D_p Q'_{p-\frac{1}{2}}(\cosh\eta_0)\right]\right\}. \tag{35}$$

This function is represented in figure 5. The numerous oscillations are probably due to the fact that this series does not converge in this form. We had already observed this behavior in the case of an infinite straight cylinder with azimuthal current [7] and [15, pp. 141–6]. We had also observed this oscillatory behavior in cases with finite geometry, like the toroid and the sphere with azimuthal currents, [13] and [14], respectively. The density of surface charges σ in the first case (straight cylinder) could be put in the closed form of a convergent function of the azimuthal coordinate φ:

$$\sigma_{\text{Heald}}(\varphi) = \frac{\varepsilon_0 \phi_B}{\pi a} \tan\frac{\varphi}{2} = \frac{2\varepsilon_0 \phi_B}{\pi a}\left[\sum_{q=1}^{\infty}(-1)^{q-1}\sin(q\varphi)\right]. \tag{36}$$

The convergence does not occur in the solution of this case expressed by Fourier series.

In the other cases (toroid and sphere) there was no such solution in terms of closed form functions. In the cases of the toroid and sphere with azimuthal currents we adopted the following procedure in order to handle this series, equation (35) [13, 14]. We calculated the mean value for a complete oscillation of the surface charge density. This mean value is represented by $\bar{\sigma}(\chi)$. The typical interval for each oscillation goes from $\chi - \pi/N$ to $\chi + \pi/N$, where N is the number of terms used in the series, [13, equation (34)]:

$$\bar{\sigma}(\chi) \equiv \frac{N}{2\pi}\int_{\chi-\pi/N}^{\chi+\pi/N} \sigma(\chi')\,\mathrm{d}\chi'. \tag{37}$$

This procedure was used in equation (36) using the closed form as validation for the method. The continuous line in figure 5 is the function $\bar{\sigma}$ given by this last equation. The averaged function $\bar{\sigma}(\chi)$ is not linear as regards the toroidal coordinate χ. This nonlinear behavior of the density of surface charges along the current path had already been observed before for the case of curved conductors in other geometries [7, 13, 14].

The surfaces at $z > 0$ and for $z < 0$ have opposite charges. They form dipoles along the $\hat{k}$-direction (here $\hat{k}$ is the unit vector along the z-axis), with the charges separated by a distance $2z$. Using the scale factors $h_\chi = a/(\cosh\eta - \cos\chi)$ and $h_\varphi = a\sinh\eta/(\cosh\eta - \cos\chi)$, the total electric dipole of the toroid is given by:

$$\begin{aligned}\vec{p} &= \hat{k}\int_{-\pi}^{\pi}\mathrm{d}\chi\int_{-\pi}^{\pi} 2z h_\chi h_\varphi \sigma(\chi)\mathrm{d}\varphi \\ &= \frac{2a^2\varepsilon_0\phi_B \sinh^2\eta_0}{\pi}\hat{k} \\ &\times\left\{\sum_{p=1}^{\infty}\left[\frac{P'_{p-1/2}(\cosh\eta_0)}{P_{p-1/2}(\cosh\eta_0)} - \frac{Q'_{p-1/2}(\cosh\eta_0)}{Q_{p-1/2}(\cosh\eta_0)}\right]\right. \\ &\times\left.\int_{-\pi}^{\pi}\frac{\chi\sin p\chi\,\mathrm{d}\chi}{\sqrt{\cosh\eta_0 - \cos\chi}}\int_0^{\pi}\frac{\sin\chi\sin p\chi\,\mathrm{d}\chi}{(\cosh\eta_0 - \cos\chi)^{3/2}}\right\}. \end{aligned} \tag{38}$$

5. Special cases

We can make some approximations in the expressions for the electric potential and electric field corresponding to three special cases of interest: (i) regions far away from the conductor, (ii) regions close to the origin and (iii) the potential due to a thin toroid.

5.1. Solution far from the conductor

Far away from the conductor we have $r = \sqrt{x^2 + y^2 + z^2} \gg r_0$ and $r \gg R_0$. We can then use the following approximations [19, p 163]:

$$\eta \approx \frac{2a\rho}{r^2} \ll 1, \tag{39}$$

$$\cosh\eta \approx 1 + \frac{2a^2\rho^2}{r^4}, \tag{40}$$

$$\chi \approx \frac{2za}{r^2}, \tag{41}$$

$$\sqrt{\cosh\eta - \cos\chi} \approx \frac{a\sqrt{2}}{r}, \tag{42}$$

$$P_{p-\frac{1}{2}}(1 + 2a^2\rho^2/r^4) \approx 1 + \frac{2a^2\rho^2}{r^4}\left(\frac{p^2}{2} - \frac{1}{8}\right). \tag{43}$$

The electric potential given by equation (19) can then be approximated in this region by the following expression:

$$\phi(r, z) \approx \frac{2\sqrt{2}za^2 B_1}{r^3}. \tag{44}$$

This solution corresponds to the electric potential of an electric dipole $\vec{p}$, as given by:

$$\vec{p} = 8\pi\varepsilon_0 a^2\sqrt{2}B_1\hat{k}. \tag{45}$$

Here ε_0 is the permittivity of free space, $\hat{k}$ is the unity vector along the z-axis, and the coefficient B_1 can be obtained from equation (25). Equation (45) is consistent with equation (38) in this approximation.

5.2. Solution close to the origin

In the region very close to the origin of the coordinate system, we have $r \ll a$. This region surrounds the center of the toroid. In this case the following approximations are valid:

$$\eta \approx \frac{2a}{\rho}, \tag{46}$$

$$\cosh\eta \approx 1 + \frac{2\rho^2}{a^2}, \tag{47}$$

$$\chi \approx \frac{2z}{a}, \tag{48}$$

$$\sqrt{\cosh\eta - \cos\chi} \approx \sqrt{2}. \tag{49}$$

The potential given by equation (19) close to the origin can then be approximated as:

$$\phi(r, z) \approx B_1\frac{2\sqrt{2}z}{a}. \tag{50}$$

5.3. Thin toroid

A thin toroid can be described by the surface $\eta_0 \gg 1$. In this case equations (10) and (11) yield $r_0 \approx 2ae^{-\eta_0}$ and $R_0 \approx a$. We can then use the following formulas to approximate the integral that appears on equations (25) and (27), and also the following expression for the toroidal Legendre functions valid for large arguments [19, pp 164 and 173]:

$$\int_{-\pi}^{\pi}\frac{\theta(\chi)\sin(p\chi)\mathrm{d}\chi}{\sqrt{\cosh\eta_0 - \cos\chi}} \approx \frac{2\pi(-1)^{p-1}}{p\sqrt{\cosh\eta_0}}, \quad p > 0, \tag{51}$$

$$P_{-1/2}(\cosh\eta \gg 1) \approx \frac{\sqrt{2}}{\pi}\frac{\ln(8\cosh\eta)}{\sqrt{\cosh\eta}}, \tag{52}$$

$$P_{p-\frac{1}{2}}(\cosh\eta \gg 1) \approx \frac{2^{p-1/2}(p-1)!\cosh^{p-1/2}\eta}{\sqrt{\pi}\,\Gamma(p+1/2)}, \quad p>0, \tag{53}$$

$$Q_{p-\frac{1}{2}}(\cosh\eta \gg 1) \approx \frac{\sqrt{\pi}\,\Gamma(p+1/2)}{2^{p+1/2}p!\cosh^{p+1/2}\eta}, \quad \forall p. \tag{54}$$

In this case, we can write the electric potential utilizing three different expressions depending on the region of interest: (i) outside the toroid, (ii) outside the toroid but close to the toroidal surface and (iii) inside the thin toroid. The solutions for these three regions are given by, respectively:

$$\phi(\eta \leqslant \eta_0, \chi) \approx \frac{\phi_{\mathrm{B}}}{\pi}\sqrt{\frac{\cosh\eta-\cos\chi}{\cosh\eta_0}} \times \left[\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\frac{P_{p-1/2}(\cosh\eta)}{P_{p-1/2}(\cosh\eta_0)}\sin(p\chi)\right], \tag{55}$$

$$\phi(1 \ll \eta \leqslant \eta_0, \chi) \approx \frac{\phi_{\mathrm{B}}}{\pi}\left[\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\frac{\cosh\eta}{\cosh\eta_0}\right)^p\sin(p\chi)\right], \tag{56}$$

$$\phi(\eta \geqslant \eta_0 \gg 1, \chi) \approx \frac{\phi_{\mathrm{B}}}{\pi}\left[\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\frac{\cosh\eta_0}{\cosh\eta}\right)^p\sin(p\chi)\right]. \tag{57}$$

Here, we can make a convenient change of variables. From equations (1) and (5), we can define the distance from the internal axis of the toroid (the circle of radius a) as:

$$r' = a\,\mathrm{sech}\,\eta. \tag{58}$$

This means that:

$$\rho \approx a + r'\cos\chi, \tag{59}$$

$$z \approx r'\sin\chi. \tag{60}$$

Note that from equation (10) we must have $r'(\eta_0) \approx r_0$. With this substitution and using equation (14), we can rewrite equations (56) and (57) by the following expressions:

$$\phi(\eta_0 \geqslant \eta \gg 1, \chi) \approx \frac{\phi_{\mathrm{B}}}{\pi}\left[\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\frac{r_0}{r'}\right)^p\sin(p\theta)\right], \tag{61}$$

$$\phi(\eta \geqslant \eta_0 \gg 1, \chi) \approx \frac{\phi_{\mathrm{B}}}{\pi}\left[\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\frac{r'}{r_0}\right)^p\sin(p\theta)\right]. \tag{62}$$

These last solutions correspond to the electric potential outside and inside a conducting cylinder with azimuthal current [7, equations (3) and (4)].

The potential far away from the toroid, given by equation (44), is the potential of a small toroid (a thin toroid, as seen from far away). By using equations (51)–(54), and:

$$\int_0^{\pi}\frac{\sin p\chi\,\mathrm{d}\chi}{\sqrt{\cosh\eta_0-\cos\chi}} \approx \frac{1-(-1)^p}{p\sqrt{\cosh\eta_0}}, \tag{63}$$

$$\int_0^{\pi}\frac{\sin\chi\sin p\chi\,\mathrm{d}\chi}{(\cosh\eta_0-\cos\chi)^{3/2}} \approx \frac{\pi}{2\cosh^{3/2}\eta_0}\delta_{1p}, \tag{64}$$

we can approximate the electric dipole $\vec{p}$, equation (38), by the following expression:

$$\vec{p} \approx \frac{4\pi a^2\varepsilon_0\phi_{\mathrm{B}}}{\cosh\eta_0}\hat{k}. \tag{65}$$

The above equation is exactly the same as the one obtained from equation (45), when the coefficient B_1 is approximated using equations (51)–(54) for a thin toroid.

6. Conclusion

We considered a toroidal resistive conductor carrying a steady poloidal current. The steady current is kept by a thin circular battery which maintains a surface charge distribution along the conductor surface. These charges generate the electric field that drives the current along the toroidal surface. They also generate electric fields in the internal and external regions around the conductor surface. These electric fields have normal ($\hat{\eta}$) and tangential ($\hat{\chi}$) components of the same order of magnitude. These components of the electric field cannot be neglected in any experiment that uses this geometry. Our results should also be valid for slowly varying currents, i.e. with current wavelengths much greater than the toroidal dimensions.

We have obtained the exact algebraic expressions for the electric potential, equations (19) and (20), internal and external electric fields, equations (29)–(33), and the surface charge distribution, equation (35). These results were obtained through the solution of Laplace's equation in toroidal coordinates by means of the technique of separation of variables. We have also obtained approximate expressions valid for three special cases: (i) regions far from the toroid, equation (44), (ii) regions close to the origin, equation (50), and (iii) for a thin toroid, equations (61) and (62). This last case could then be compared to an infinite conducting cylinder carrying a steady azimuthal current. This yielded the same result, as expected. The main advantage of our approach is that our solutions are valid for a current flowing along a finite region of space.

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