VIRIAL THEOREM FOR WEBER'S LAW

R. S. Mendes[1], L. C. Malacarne[1] and A. K. T. Assis[2]
[1] *Departamento de Física Universidade Estadual de Maringá, Av. Colombo 5790, 87020-900 Maringa, PR, Brasil*
[1] rsmendes@dfi.uem.br and lcmala@dfi.uem.br
[2] *Instituto de Física "Gleb Wataghin", Universidade Estadual de Campinas, - Unicamp, 13083-970 Campinas, Sao Paulo, Brasil*
[2] assis@ifi.unicamp.br and http://www.ifi.unicamp.br/∼assis

We derive the virial theorem for Weber's law applied to gravitation and electromagnetism.

Recently there has been a renewed interest in Weber's law as applied to gravitation and electromagnetism (for discussion and references see, for instance, [1 - 7]). In the case of gravitation, for instance, it has been shown that Weber's law gives a complete mathematical implementation of Mach's principle (the inertia of any body being due to its gravitational interaction with the distant universe). There have been many people working along this line including E. Schrödinger, [8] and [9].

Most theoretical treatments related to Weber's law up to now have been restricted to a single body or to only two bodies. Our goal in this work is to begin the extension of these approaches to an statistical treatment of a many-body interaction. We concentrate here on the virial theorem.

Suppose two particles i and j located at $\vec{r}_i$ and $\vec{r}_j$ relative to the origin of an inertial frame of reference O. Their velocities and accelerations relative to this frame are represented by, respectively: $\vec{v}_i = d\vec{r}_i/dt$, $\vec{v}_j = d\vec{r}_j/dt$, $\vec{a}_i = d^2\vec{r}_i/dt^2$ and $\vec{a}_j = d^2\vec{r}_j/dt^2$. They are separated by a distance $r_{ij} \equiv |\vec{r}_i - \vec{r}_j| \equiv |\vec{r}_{ij}|$,moving with radial relative velocity and acceleration given by $\dot{r}_{ij} = dr_{ij}/dt$ and $\ddot{r}_{ij} = d^2r_{ij}/dt^2$. According to Weber's law their energy of interaction is given by ([3, Chapter 3]):

$$U_{ij} = \frac{k_{ij}}{r_{ij}} \left(1 - \frac{\xi}{2c^2} \dot{r}_{ij}^2 \right) \ , \tag{1}$$

where $c = 3 \times 10^8 m/s$. For gravitation $k_{ij} = -Gm_i m_j$ and $\xi = 6$ ($G = 6.67 \times 10^{-11}\ Nm^2kg^{-2}$ is the constant of gravitation), while $k_{ij} = q_i q_j / 4\pi\varepsilon_o$ and $\xi = 1$ for electromagnetism ($\varepsilon_o = 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}$ is the permittivity of free space). There are two ways of deriving the force exerted by j on i. The first is to utilize $\vec{F}_{ji} = -\hat{r}_{ij} dU_{ij}/dr_{ij}$, where $\hat{r}_{ij} \equiv \vec{r}_{ij}/|\vec{r}_{ij}| = (\vec{r}_i - \vec{r}_j)/r_{ij}$ is the unit vector pointing from j to i:

$$\vec{F}_{ji} = k_{ij} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[1 + \frac{\xi}{c^2} \left(r_{ij} \ddot{r}_{ij} - \frac{\dot{r}_{ij}^2}{2} \right) \right] \ . \tag{2}$$

The second form is to utilize the function S defined by ([10, p. 203]):

$$S_{ij} = \frac{k_{ij}}{r_{ij}} \left(1 + \frac{\xi}{2c^2} \dot{r}_{ij}^2 \right) \tag{3}$$

The central force is then given by the standard procedure:

$$\vec{F}_{ji} = \hat{r}_{ij} \left(\frac{d}{dt} \frac{\partial S}{\partial \dot{r}_{ij}} - \frac{\partial S}{\partial r_{ij}} \right) \ . \tag{4}$$

It is also possible to obtain directly the force components, see [11, pp. 525-535]. For instance, the x component of the force is given by (similarly for the y and z components):

$$F_{ji}^x = \frac{d}{dt} \frac{\partial S}{\partial \dot{x}_i} - \frac{\partial S}{\partial x_i} \ . \tag{5}$$

It should be observed that there is a sign difference in front of the velocity dependent terms as regards U and S. If two bodies interact with one another according to Weber's law the conserved energy of the system (when there are no other forces present) is given by $T + U$, where $T = \sum m_i \vec{v}_i \cdot \vec{v}_i / 2$ is the kinetic energy of the particles. On the other hand the Lagrangian L of the system is given by $L = T - S$. The same change of sign in the velocity terms happens in classical electromagnetism with Darwin's Lagrangian, [12, pp. 593-5] and [3, pp. 177-9]. In order to derive the virial theorem we first define the usual time average of an arbitrary variable $A(t)$ over a time interval τ, namely, $< A > \equiv (1/\tau) \int_0^\tau A(t) dt$. Accordingly the time average of the time derivative of A is given by $< dA/dt >= [A(\tau) - A(0)]/\tau$. If the motion is periodic so that all coordinates repeat after a certain time then this last equation vanishes if we choose τ as this period. Alternatively it will also vanish for non periodic motions if we choose τ sufficiently large and assume that $A(t)$ remains finite for all time. As one or the other of these conditions are usually valid for all physical systems, [13, p. 83], we will assume them here, namely, $< dA/dt >= 0$.

Consider now a system of N particles interacting only through Weber's law (2). Newton's law of motion applied to particle i reads $\sum_{j\neq i}\vec{F}_{ji} = d\vec{p}_i/dt$, where $\vec{p}_i = m_i\vec{v}_i$ is the linear momentum of the particle of mass m_i and the summation goes from $j = 1$ to N, excepting the case $j = i$.

Making a dot product of both sides of this equation with $\vec{r}_i$, adding for all particles and performing a time average yields

$$< \sum_{i=1}^{N}\sum_{j\neq i}\vec{r}_i\cdot\vec{F}_{ji} > = < \sum_{i=1}^{N} m_i\vec{r}_i\cdot\vec{a}_i > , \tag{6}$$

where we are assuming constant masses. The right hand side can be written as $< \sum_{i=1}^{N} d(m_i\vec{r}_i\cdot\vec{v}_i)/dt > - < \sum_{i=1}^{N} m_i\vec{v}_i\cdot\vec{v}_i >$. With $< dA/dt >= 0$ the first term goes to zero. Defining the kinetic energy of the system of N particles by the usual expression $T \equiv \sum_{i=1}^{N} m_i\vec{v}_i\cdot\vec{v}_i/2$ we conclude that the right hand side of Eq. (6) is given by $-2 < T >$. We can express the left hand side as (utilizing that $\vec{F}_{ji} = -\vec{F}_{ij}$, as is the case for Weber's law):

$$< \sum_{i=1}^{N}\sum_{j\neq i}\vec{r}_i\cdot\vec{F}_{ji} > = < \vec{r}_1\cdot(\vec{F}_{21}+\vec{F}_{31}+...+\vec{F}_{N1})$$

$$+ \vec{r}_2\cdot(\vec{F}_{12}+\vec{F}_{32}+...+\vec{F}_{N2}) + ... + \vec{r}_N\cdot(\vec{F}_{1N}+\vec{F}_{2N}+...+\vec{F}_{N-1,\ N}) >$$

$$= < (\vec{r}_{12}\cdot\vec{F}_{21}+\vec{r}_{13}\cdot\vec{F}_{31}+...+\vec{r}_{1N}\cdot\vec{F}_{N1}) + (\vec{r}_{23}\cdot\vec{F}_{32}+\vec{r}_{24}\cdot\vec{F}_{42}+...$$

$$+ \vec{r}_{2N}\cdot\vec{F}_{N2}) + ... + (\vec{r}_{N-1,\ N}\cdot\vec{F}_{N,\ N-1}) > = < \sum_{j>i}\vec{r}_{ij}\cdot\vec{F}_{ji} > , \tag{7}$$

where in the last sum i and j go from 1 to N, with the condition that $j > i$. Up to now the algebra is standard and applicable to any force satisfying Newton's law of action and reaction.

We now make a direct use of Weber's law. With $< dA/dt >= 0$ and Eqs. (2) and (1) this can be written as (utilizing that $\vec{r}_{ij}\cdot\hat{r}_{ij} = r_{ij}$):

$$< \sum_{j>i}\frac{k_{ij}}{r_{ij}}\left[1+\frac{\xi}{c^2}\left(r_{ij}\ddot{r}_{ij}-\frac{\dot{r}_{ij}^2}{2}\right)\right] >$$

$$= < \sum_{j>i}\frac{k_{ij}}{r_{ij}}\left(1-\frac{\xi}{2c^2}\dot{r}_{ij}^2\right) > = < \sum_{j>i}U_{ij} > = < U > , \tag{8}$$

where U is Weber's potential energy for the whole system. To arrive at this result we utilized $< dA/dt >= 0$ in the term $k_{ij}\xi\ddot{r}_{ij}/c^2$, which can also be written as $d(k_{ij}\xi\dot{r}_{ij}/c^2)/dt$. We then derived the virial theorem for Weber's law: $< U >= -2 < T >$. The difference as regards the usual newtonian inverse square law lies in the fact that now the potential energy is the weberian one, U_{ij} given by Eq. (1).

Although specific applications of the virial theorem are beyond the scope of this letter, a few possibilities should be outlined. This theorem creates, for instance, the possibility of a full statistical treatment of plasma physics based on Weber's electrodynamics. It can also be applied in astrophysics in order to deal with problems related with the formation or stability of the solar system or of galaxies utilizing Weber's law applied to gravitation. This letter represents a first step in this direction.

Acknowledgments:

One of the authors (AKTA) wishes to thank the Alexander von Humboldt Foundation of Germany for a research fellowship during which this work was completed.

References

1. A. G. Sokol'skii and A. A. Sadovnikov. Lagrangian solutions for Weber's law of attraction. *Soviet Astronomy (Astronomical Journal)*, 31:90–93, 1987.
2. J. P. Wesley. Weber electrodynamics, Part III. Mechanics, gravitation. *Foundations of Physics Letters*, 3:581–605, 1990.
3. A. K. T. Assis. *Weber's Electrodynamics*. Kluwer Academic Publishers, Dordrecht, 1994. ISBN: 0-7923-3137-0.
4. E. T. Kinzer and J. Fukai. Weber's force and Maxwell's equations. *Foundations of Physics Letters*, 9:457–461, 1996.
5. A. K. T. Assis. *Relational Mechanics*. Apeiron, Montreal, 1999. ISBN: 0-9683689-2-1.
6. J. Guala-Valverde. *Inercia y Gravitacion*. Fundacion Julio Palacios, Neuquen, Argentina, 1999. In collaboration with J. Tramaglia and R. Rapacioli.
7. M. d. A. Bueno and A. K. T. Assis. *Inductance and Force Calculations in Electrical Circuits*. Nova Science Publishers, Huntington, New York, 2001. ISBN: 1-56072-917-1.
8. E. Schrödinger. Die Erfüllbarkeit der Relativitätsforderung in der klassischen Mechanik. *Annalen der Physik*, 77:325–336, 1925.
9. E. Schrödinger. The possibility of fulfillment of the relativity requirement in classical mechanics. In J. B. Barbour and H. Pfister, editors, *Mach's Principle - From Newton's Bucket to Quantum Gravity*, pages 147–158, Boston, 1995. Birkhäuser. Translated by J. B. Barbour.
10. E. T. Whittaker. *A History of the Theories of Aether and Electricity*, volume 1: *The Classical Theories*. Humanities Press, New York, 1973.
11. A. O'Rahilly. *Electromagnetic Theory – A Critical Examination of Fundamentals*. Dover, New York, 1965.
12. J. D. Jackson. *Classical Electrodynamics*. John Wiley, New York, second edition, 1975.
13. H. Goldstein. *Classical Mechanics*. Addison-Wesley, Reading, 2nd edition, 1980.