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Kirchhoff on the Motion of Electricity in Conductors

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We comment on and translate Gustav Kirchhoff's important paper of 1857 entitled "On the motion of electricity in conductors." The significance of this paper is that Kirchhoff proved with action at a distance that electric disturbances travel along wires of negligible resistance with the velocity of light. He accomplished this with the laws of Newtonian electrodynamics (Coulomb, Ampere, F. Neumann and Weber) before Maxwell had formulated his equations.
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This paper presents the first English translation of Kirchhoff's important work *On the motion of electricity in conductors* (*Ueber die Bewegung der Elektricität in Leitern*). It was first published in the *Annalen der Physik* (also known during the last century as *Poggendorff's Annalen*, after Johann Christian Poggendorff of Berlin, who was for a long time its editor), Volume 102, p. 529 (1857). It then appeared in Kirchhoff's collected works: G. Kirchhoff's *Gesammelte Abhandlungen* (Barth, Leipzig, 1882), pp. 154–168, on which we based this translation.

Gustav Kirchhoff (1824–1887) had previously published other papers related to electromagnetism. Two of these have already been translated into English, and here we discuss them briefly due to their relevance to the present paper. In the earlier of the two, first published in 1849, "*On a deduction of Ohm's laws, in connexion with the theory of electrostatics*" (*Philosophical Magazine*, Volume 37, pp. 463–468 (1850)), Kirchhoff for the first time identified Ohm's 'electroscopic force' and the 'tension' in a voltaic cell (battery) with the electrostatic potential. This was conceptually important for establishing a link between electrostatics and electrodynamics. According to him the driving force generating the current at any point of the conductor is due to a difference of electrostatic potential between two adjacent points along its length. This potential is generated by the free electricity (net charge) along the surface of the conductor, which is maintained in a steady state by the voltaic cell. He also corrected Ohm's assumption that in a stationary situation (DC current) there is a uniform distribution of free electricity throughout the body of the conductor. He showed that, for stationary currents, the free electricity can only exist at the surface of the conductor. In the present paper he shows that this is a special case, valid for stationary situations, but that in general there will be free electricity distributed throughout the substance of the conductor.

In the second paper, "On the motion of electricity in wires" (*Philosophical Magazine*, Volume 13, pp. 393–412 (1857)), Kirchhoff develops the theory of propagation of an electrical disturbance along a thin wire, taking into account the self-inductance of the wire. Wilhelm Weber had independently performed a similar investigation shortly before Kirchhoff, but Weber's work was delayed in publication. The remarkable implication of their analyses was that in a circuit of negligible resistivity, oscillating currents could be propagated along the wire with a constant velocity numerically equal to the velocity of light. Moreover, this velocity would be independent of the nature of the conductors, of the cross section of the wire, and of the density of free electricity. This result is even more important if we remember that it came before Maxwell's equations in their complete form (1860–1864). Kirchhoff and Weber's circuit theories were based entirely on the action-at-a-distance laws of Coulomb, Ampere, F. Neumann and Weber. This contradicts the commonly held belief that time delays in the propagation of electrical signals can only be explained with free energy traveling through space. In the present paper Kirchhoff generalizes this theory to three-dimensional conductors of arbitrary shape, which lends importance to the English translation of this remarkable paper.

Before presenting the translation, we would like to make a few comments which may help the understanding of the paper. What Kirchhoff and Weber represent by c would today be written as $\sqrt{2}c$, where this last c has the value 3×10^8 ms^{-1}. The quantity which we represent to-

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day by c (or by $(\mu_o\varepsilon_o)^{-1/2}$ in the International System of Units), is the ratio of electromagnetic to electrostatic units of charge. The value $\sqrt{2}c=\sqrt{2}/\sqrt{\mu_o\varepsilon_o}$ was first determined experimentally by Kohlrausch and Weber in 1856. Moreover, Weber and Kirchhoff usually worked with Fechner's hypothesis (1845), according to which the currents in metallic wires consist of equal and opposite streams of positive and negative electricity. As a result, they customarily wrote $2i$ to denote the current strength. These two facts explain the coefficients 2 and 2×4 which appear in equations (1), (2), (3) and (5) of the present paper.

Equations (1) to (3) of this paper could be written in modern vectorial notation as

$$\vec{J}=-k\left(\nabla\phi+\frac{\partial\vec{A}}{\partial t}\right)$$

where $\vec{J}$ is the current density, $\vec{J}=(J_x,J_y,J_z)=(u,v,w)$, k is the conductivity of the conductor, ϕ is the electrostatic potential (represented by Kirchhoff as Ω), and $\vec{A}$ is the magnetic vector potential. This is essentially Ohm's law generalized by Kirchhoff to three dimensional conductors and to take into account the effects of self-induction.

Nowadays, we usually utilize F. Neumann's formula (1845) for the vector potential generalized to three dimensions, namely

$$\vec{A}(x,y,z)=\frac{\mu_o}{4\pi}\iiint\vec{J}'\frac{dx'dy'dz'}{r},$$

where $\vec{J}'$ is the current density at the point (x',y',z') and r is the distance between the points (x,y,z) and (x',y',z'). However, in this paper Kirchhoff utilizes Weber's formula for the vector potential, namely

$$\vec{A}(x,y,z)=\frac{\mu_o}{4\pi}\iiint(\vec{J}'\cdot\vec{r})\vec{r}\frac{dx'dy'dz'}{r^3},$$

where $\vec{r}$ is the vector from (x,y,z) to (x',y',z'). Provided we have a closed circuit, both expressions agree with one another. The $\mu_o/4\pi$ does not appear in Kirchhoff's memoir because he utilizes another system of units (the mechanical or absolute system).

Equation (5) would be written today as

$$\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$$

where ρ is the volume density of charge (represented by Kirchhoff as ε). This is the equation of conservation of charge. Kirchhoff also utilizes Poisson's equation (1813) of the electrostatic potential, namely

$$\nabla^2\phi=-4\pi\rho,$$

which he writes as

$$\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+\frac{\partial^2\Omega}{\partial z^2}=-4\pi\varepsilon.$$

We now present the translation.

On the Motion of Electricity in Conductors[1]

G. Kirchhoff

In an earlier paper[2] I developed a theory of the motion of electricity in linear conductors. I will now show how the former considerations can be generalized to conductors of any form.

The Cartesian coordinates x, y, z locate a point in the conductor. The current which at time t flows through this point we resolve along the three coordinate axes to give the current density components u, v, w. These current densities have to be equal to the products of the components of the e.m.f. and electrical conductivity at point (x,y,z) and are assumed to involve one unit of electrical charge. The e.m.f. is partly due to the presence of free electricity, and partly due to induction which arises in all parts of the conductor because of changes in the current. If Ω represents the potential function of the free electricity relative to the point (x,y,z), then the components of the first part of the e.m.f. are

$$-2\frac{\partial\Omega}{\partial x},-2\frac{\partial\Omega}{\partial y},-2\frac{\partial\Omega}{\partial z}$$

In order to derive the components of the second part, I denote the coordinates of a second point of the conductor by x',y',z', while u',v',w' are the values of u, v, w for this point. Let r be the distance between the points (x,y,z) and (x',y',z') and write:

$$U=\iiint\frac{dx'dy'dz'}{r^3}(x-x')[u'(x-x')+v'(y-y')+w'(z-z')]$$
$$V=\iiint\frac{dx'dy'dz'}{r^3}(y-y')[u'(x-x')+v'(y-y')+w'(z-z')]$$
$$W=\iiint\frac{dx'dy'dz'}{r^3}(z-z')[u'(x-x')+v'(y-y')+w'(z-z')]$$

where the integrations extend over all of the volume of the conductor. According to Weber's law of induction, the components of the second part of the e.m.f. under consideration are:

$$-\frac{8}{c^2}\frac{\partial U}{\partial t},-\frac{8}{c^2}\frac{\partial V}{\partial y},-\frac{8}{c^2}\frac{\partial W}{\partial z},$$

1 *Pogg. Annal.* Bd. 102. 1857.

2 G. Kirchhoff, *Gesammelte Abhandlungen* (Barth, Leipzig, 1882), p. 131.

where c is the constant velocity with which two electric charges have to move toward each other so that they will not exert a force on each other. If k is the conductivity of the conductor, we have:

$$u = -2k\left(\frac{\partial\Omega}{\partial x} + \frac{4}{c^2}\frac{\partial U}{\partial t}\right) \qquad (1)$$

$$v = -2k\left(\frac{\partial\Omega}{\partial y} + \frac{4}{c^2}\frac{\partial V}{\partial t}\right) \qquad (2)$$

$$w = -2k\left(\frac{\partial\Omega}{\partial z} + \frac{4}{c^2}\frac{\partial W}{\partial t}\right). \qquad (3)$$

It must not be assumed that the free electricity is confined to the surface of the conductor, as in equilibrium cases or at constant current. In fact, it will be shown that, in general, the opposite is true. I denote by ε the density of free electricity at point (x,y,z), by ε' the density at (x',y',z') by e the density in a surface element dS, and by e' the same for a second surface element dS'. Then we have:

$$\Omega = \int \frac{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'}{r}\varepsilon' + \int \frac{\mathrm{d}S'}{r}e', \qquad (4)$$

where the first integration is over the volume, and the second over the surface of the conductor.

To these equations we can add two more which deal with the time changes of the density of free electricity. For every point inside the conductor we have therefore:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{2}\frac{\partial\varepsilon}{\partial t}; \qquad (5)$$

and if we denote the normal to element dS directed inward by N, then further for every point of the surface:

$$u\cos(N,x) + v\cos(N,y) + w\cos(N,z) = -\frac{1}{2}\frac{\partial e}{\partial t}. \qquad (6)$$

* * *

From these equations we can derive a remarkable relationship between ε and Ω. Substituting the values of u, v, w from (1), (2), and (3) into (5), and using:

$$\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2} = -4\pi\varepsilon$$

one finds

$$\frac{\partial\varepsilon}{\partial t} = -16k\left[\pi\varepsilon - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)\right].$$

As the equation for U may be written:

$$U = -\int \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\frac{\partial\frac{1}{r}}{\partial x}\left[u'(x-x') + v'(y-y') + w'(z-z')\right]$$

it follows that:

$$\frac{\partial U}{\partial x} = -\int \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\frac{\partial\frac{1}{r}}{\partial x}u' - \int \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\frac{\partial^2\frac{1}{r}}{\partial x^2}\left[u'(x-x') + v'(y-y') + (z-z')\right].$$

Forming the value of $\partial V/\partial y$ and $\partial W/\partial z$ in a similar manner, one obtains:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = -\int \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\left(u'\frac{\partial\frac{1}{r}}{\partial x} + v'\frac{\partial\frac{1}{r}}{\partial y} + w'\frac{\partial\frac{1}{r}}{\partial z}\right);$$

because of:

$$\frac{\partial^2\frac{1}{r}}{\partial x^2} + \frac{\partial^2\frac{1}{r}}{\partial y^2} + \frac{\partial^2\frac{1}{r}}{\partial z^2} = 0$$

for all points (x',y',z') which do not coincide with point (x,y,z) and extend through the infinitely small volume surrounding point (x,y,z), the integrals of the second parts of $\partial U/\partial x$, $\partial V/\partial y$, $\partial W/\partial z$ are infinitely small. It is easy to convince ourselves of the validity of this last assertion by the method which Gauss used to prove that the contribution to the potential at a point by masses infinitely near to the point is negligible compared to the contribution from continuously distributed matter throughout space.[3] If in the integral on the right side of the equation the differential coefficients with respect to x, y, z, are replaced with the negative coefficients with respect to x',y',z' and the result is divided in three partial differentials with respect to x',y' and z', one obtains:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = -\int \frac{\mathrm{d}S'}{r}\left[u'\cos(N',x) + v'\cos(N',y) + w'\cos(N',z)\right] - \int \frac{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'}{r}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}\right)$$

where N' is the inward directed normal of the surface element dS'. In view of equations (6), (5) and (4), this equation may by written:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{1}{2}\frac{\partial\Omega}{\partial t}$$

From this it follows that:

$$\frac{\partial\varepsilon}{\partial t} = -8k\left(2\pi\varepsilon - \frac{1}{c^2}\frac{\partial^2\Omega}{\partial t^2}\right) \qquad (7)$$

This equation shows clearly that $\varepsilon = 0$ is a special case, and in general we find free electricity inside of conductors. It is probable that the so called mechanical actions of the discharge current of a Leyden jar, as for example in the pul-

[3] *Resultate aus den Beobachtungen des magnetischens Vereins*; 1839, p. 7.

verization of a fine wire, the internal free electricity plays an important role.

* * *

I would like to apply the theory developed here to the case considered in the initially mentioned paper, *i.e.* the case in which the conductor is an infinitely thin wire with no electrical bodies in its vicinity. I will show that the theory furnishes the same results which I obtained previously, and in addition it supplies answers to questions which so far have remained unanswered.

To begin with I will simplify the general equation by the assumption that the conductor is cylindrical of circular cross-section, and that the current, as well the distribution of free electricity, is symmetrical about the axis. I take the axis as the x-direction, and for y and z I introduce the new coordinates ρ and φ, so that:

$$y=\rho\cos\varphi,\ z=\rho\sin\varphi$$

and correspondingly:

$$y'=\rho'\cos\varphi',\ z'=\rho'\sin\varphi'$$

Furthermore, I denote the current density, perpendicular to the current along the axis—positive for the progressive direction of the axis—at point (x,ρ,φ) by σ, and at point (x',ρ',φ') by σ'. We then have:

$$v=\sigma\cos\varphi,\ w=\sigma\sin\varphi,$$

$$v'=\sigma'\cos\varphi',\ w'=\sigma'\sin\varphi'$$

Hence:

$$u=-2k\left(\frac{\partial\Omega}{\partial x}+\frac{4}{c^2}\frac{\partial U}{\partial t}\right), \qquad (8)$$

where:

$$U=\int\frac{dx'\rho'd\rho'd\varphi'}{r^3}(x-x')\left[u'(x-x')+\sigma'\left(\rho\cos(\varphi-\varphi')-\rho'\right)\right] \qquad (9)$$

If we ignore the action of the free electricity on the end-faces of the cylinder, then, with α being the radius of the cylinder, equation (4) may be written:

$$\Omega=\int\frac{dx'\rho'd\rho'd\varphi'}{r}\varepsilon'+\alpha\int\frac{dx'd\varphi'}{r}e'. \qquad (10)$$

Equation (5) becomes:

$$\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial\rho\sigma}{\partial\rho}=-\frac{1}{2}\frac{\partial\varepsilon}{\partial t}; \qquad (11)$$

and equation (6), which refers to the surface, becomes:

$$\sigma=\frac{1}{2}\frac{\partial\varepsilon}{\partial t}. \qquad (12)$$

The expressions for Ω and U are greatly simplified if it is assumed that the cross-section of the cylinder is infinitely small, while the wire is of finite length. I call this length l, and the origin of the coordinates is taken to be the middle of the cylinder. The limits of the integrations in the x'-direction are then $-l/2$ and $+l/2$. For brevity I will take:

$$x'-x=\xi;$$

for dx' the integrand may then be written $d\xi$. The integration along ξ then has the limits $-l/2-x$ and $l/2-x$, of which the first one is always negative and the second one is always positive. The quantity r of the integrals is determined by the equation:

$$r^2=\xi^2+\beta^2$$

where:

$$\beta^2=\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi').$$

For the transformation of the second part of Ω in the integral:

$$\int_{-\frac{l}{2}-x}^{\frac{l}{2}-x}\frac{d\xi e'}{\sqrt{\beta^2+\xi^2}}$$

I will develop e' according to Taylor's theorem in powers of ξ, that is:

$$e'=e+\frac{\partial e}{\partial x}\xi+\frac{\partial^2 e}{\partial x^2}\frac{\xi^2}{1\cdot 2}+\ldots;$$

the individual terms into which the integral has been split then take the form:

$$\frac{1}{1\cdot 2\cdots n}\frac{\partial^n e}{\partial x^n}\int\frac{\xi^n d\xi}{\sqrt{\beta^2+\xi^2}}$$

But we have:

$$\int\frac{\xi^n d\xi}{\sqrt{\beta^2+\xi^2}}=\frac{1}{n}\xi^{n-1}\sqrt{\beta^2+\xi^2}-\frac{n-1}{n}\beta^2\int\frac{\xi^{n-2}d\xi}{\sqrt{\beta^2+\xi^2}}$$

and

$$\int\frac{d\xi}{\sqrt{\beta^2+\xi^2}}=\log\left(\xi+\sqrt{\beta^2+\xi^2}\right)$$

$$\int\frac{\xi d\xi}{\sqrt{\beta^2+\xi^2}}=\sqrt{\beta^2+\xi^2}$$

When β is infinitely small, which occurs when α is infinitely small, the first—and only the first—term becomes infinitely large. One may therefore neglect all following terms compared to the first one, and write:

$$\int\frac{e'd\xi}{\sqrt{\beta^2+\xi^2}}=2e\log\frac{\sqrt{l^2-4x^2}}{\beta}$$

or also, by neglecting finite terms compared to the infinite term:

$$= 2e \log \frac{\ell}{\beta}.$$

Furthermore:

$$\int_0^{2\pi} \log \beta \mathrm{d}\varphi' = 2\pi \log \rho', \text{ when } \rho' > \rho.$$

In the second part of Ω we have $\rho' = \alpha$. The second part therefore is:

$$\alpha \int \frac{\mathrm{d}x' \mathrm{d}\varphi'}{r} e' = 4\pi\alpha e \log \frac{1}{\alpha}.$$

Similar considerations may be applied to the first part of Ω. Denoting the value of ε at the point (x, ρ', φ') by ε_o', then these considerations lead to:

$$\int \frac{\varepsilon' \mathrm{d}x'}{r} = 2\varepsilon_o' \log \frac{\ell}{\beta}$$

Furthermore:

$$\int \log \beta \mathrm{d}\varphi' = 2\pi \log \rho', \text{ when } \rho' > \rho$$
$$= 2\pi \log \rho, \text{ when } \rho > \rho'.$$

For both of these expressions we may write $2\pi \log \alpha$ when ignoring finite quantities compared to infinite quantities. Therefore:

$$\int \frac{\mathrm{d}x' \rho' \mathrm{d}\rho' \mathrm{d}\varphi'}{\mathrm{r}} \varepsilon' = 4\pi \log \frac{\ell}{\alpha} \int_0^\alpha \rho' \mathrm{d}\rho' \varepsilon_o'.$$

Let:

$$2\pi\alpha e + 2\pi \int_0^\alpha \rho' \mathrm{d}\rho' \varepsilon_o' = E,$$

that is, if $E\mathrm{d}x$ is the amount of free electricity contained in the element $\mathrm{d}x$ of the wire,[4] then we find:

$$\Omega = 2E \log \frac{\ell}{\alpha}. \qquad (13)$$

The expression of U in equation (9) can be treated in the same way. In this expression I am thinking of u' and σ' to be developed in powers of ξ, and the values of u and σ at the point (x, ρ', φ') to be denoted by u_o' and σ_o'. In the parts into which the expression can be split we find integrals of the form:

$$\int \frac{\xi^n \mathrm{d}\xi}{\left(\beta^2 + \xi^2\right)^{3/2}}.$$

We have:

$$\int \frac{\xi^n \mathrm{d}\xi}{\left(\beta^2 + \xi^2\right)^{3/2}} = \frac{1}{n-2} \frac{\xi^{n-1}}{\sqrt{\beta^2 + \xi^2}} - \frac{n-1}{n-2} \beta^2 \int \frac{\xi^{n-2} \mathrm{d}\xi}{\left(\beta^2 + \xi^2\right)^{3/2}},$$

$$\int \frac{\xi \mathrm{d}\xi}{\left(\beta^2 + \xi^2\right)^{3/2}} = -\frac{1}{\sqrt{\beta^2 + \xi^2}}$$

$$\int \frac{\xi^2 \mathrm{d}\xi}{\left(\beta^2 + \xi^2\right)^{3/2}} = -\frac{\xi}{\sqrt{\beta^2 + \xi^2}} + \log\left(\xi + \sqrt{\beta^2 + \xi^2}\right).$$

Of the specified integrals taken from a negative to a positive finite limit, only for $n = 2$ do we obtain an infinity, provided β is infinitely small. All other integrals can be neglected compared with this, and the finite part of the infinite term can also be neglected. A factor of it is:

$$u_o' - \frac{\partial \sigma_o'}{\partial x}\left(\rho \cos(\varphi - \varphi') - \rho'\right)$$

but, because of the smallness of ρ and ρ', we can replace this by u_o'. Using the same method used before for the calculation of Ω, we obtain:

$$U = 4\pi \log \frac{\ell}{\alpha} \int \rho' \mathrm{d}\rho' u_o'.$$

If we denote by i the quantity of electricity which in unit time passes through the cross-section of the wire, *i.e.* the current intensity, the equation can be simplified to:

$$U = 2i \log \frac{\ell}{\alpha}.$$

Substituting this value of U and the value of Ω from (13) into the equation (8), we obtain:

$$u = -4 \log \frac{\ell}{\alpha} k \left(\frac{\partial E}{\partial x} + \frac{4}{c^2} \frac{\partial i}{\partial t}\right).$$

The right-hand side of this equation is independent of ρ, and since u is independent of ρ we have:

$$i = \pi\alpha^2 u;$$

hence:

$$i = -4\pi\alpha^2 k \log \frac{\ell}{\alpha} \left(\frac{\partial E}{\partial x} + \frac{4}{c^2} \frac{\partial i}{\partial t}\right). \qquad (14)$$

A second equation between the quantities E and i can be derived from equations (11) and (12). If one multiplies the first one with $\rho \mathrm{d}\rho \mathrm{d}\varphi$, then integrates it over the cross-section of the wire, and subtracts from the result the second equation, after having multiplied it by $2\pi\alpha$, one obtains:

$$\frac{\partial i}{\partial x} = -\frac{1}{2} \frac{\partial E}{\partial t}. \qquad (15)$$

The derivation of equations (14) and (15) presupposes that the wire is straight. But since these equations show

4 E is here the same quantity which in the former paper was denoted by e.

that the electrical state at a point inside the wire is independent of the electrical state at all other points at a finite distance from the former, the equations will also be valid for bent wires. The radius of curvature, however, has to be everywhere finite, so that the distance between two points, with a finite piece of wire between them, cannot be infinitely close to each other. Equations (14) and (15) are the very same equations which I derived for the same case in the earlier paper. The more general theory developed here, therefore, leads to the same results obtained before, but it leads to further consequences. If, for example, (14) and (15) are used to determine E and (13) to determine Ω, it is possible to calculate ε from (7), *i.e.* the density of free electricity inside the wire, so long as ε is given for zero time. If the initial value of ε is independent of ρ, then ε remains independent of it, that is the density of electricity is the same at all points of the cross-section, for according to (13) Ω is independent of ρ, and ρ does not appear in equation (7). After calculating ε one can find e. If the initial value of ε is independent of ρ, as has been assumed, we make use of the equation:

$$E = 2\pi\alpha e + \pi\alpha^2\varepsilon$$

With the same assumption it is easy to calculate σ from ε because:

$$\sigma = \frac{1}{2}\frac{\rho}{\alpha}\frac{\partial e}{\partial t}.$$

That this equation is valid for $\rho = \alpha$ we learn from equation (12), and that σ is proportional to ρ from equation (11). If one multiplies it by $\rho d\rho$ and integrates, remembering that u and ε are independent of ρ, one finds:

$$\sigma = -\frac{\rho}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial \varepsilon}{\partial t}\right) + \frac{\text{Constant}}{\rho}.$$

The constant of integration has to be zero, because, for $\rho = 0$, σ must not be infinite. In fact the opposite is true; it has to disappear, because along the axis of the wire the current has to be in the direction of the axis.

* * *

In the previous paper I discussed the solution of equations (14) and (15) for the special case which is approached the smaller the resistance of the wire is made. I proved that in this case the electricity in the wire progresses like a wave in a taught string with the velocity of light in empty space. It is of interest to consider the opposite case which is approached the greater the resistance of the wire is made. I will do this here on the assumption that the two ends of the wire are connected with each other.

As in the previous paper, I let the resistance of the wire be r, and write:

$$\log\frac{\ell}{\alpha} = \gamma;$$

then the solution of the differential equations (14) and (15), whatever the value of r, is as follows:

$$E = \sum\left(C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}\right)\sin nx + \left(C_1' e^{-\lambda_1 t} + C_2' e^{-\lambda_2 t}\right)\cos nx,$$

$$i = \sum -\frac{1}{2n}\left(\lambda_1 C_1 e^{-\lambda_1 t} + \lambda_2 C_2 e^{-\lambda_2 t}\right)\cos nx + \frac{1}{2n}\left(\lambda_1 C_1' e^{-\lambda_1 t} + \lambda_2 C_2' e^{-\lambda_2 t}\right)\sin nx$$

where n is a multiple of $2\pi/l$, and λ_1 and λ_2 have the values:

$$\frac{c^2 r}{32\gamma l}\left[1 \pm \sqrt{1 - \left(\frac{32\gamma}{cr\sqrt{2}}nl\right)^2}\right]$$

and C_1, C_2, C_1', and C_2' are arbitrary constants. The summation is over all values of n. The C-constants are easily determined if E and i are given for $t = 0$. If the functions of x, which must transform to E and i for $t = 0$, have the form:

$$\sum\left(E_n \sin nx + E_n' \cos nx\right)$$

and

$$\sum\left(-i_n \cos nx + i_n' \sin nx\right)$$

one obtains the equations:

$$E_n = C_1 + C_2$$
$$i_n = \frac{1}{2n}(\lambda_1 C_1 + \lambda_2 C_2);$$

and

$$E_n' = C_1' + C_2'$$
$$i_n' = \frac{1}{2n}(\lambda_1 C_1' + \lambda_2 C_2');$$

their solutions are:

$$C_1 = \frac{\lambda_2 E_n - 2n i_n}{\lambda_2 - \lambda_1}$$
$$C_2 = \frac{-\lambda_1 E_n - 2n i_n}{\lambda_2 - \lambda_1}$$
$$C_1' = \frac{\lambda_2 E_n' - 2n i_n'}{\lambda_2 - \lambda_1}$$
$$C_2' = \frac{-\lambda_1 E_n' - 2n i_n'}{\lambda_2 - \lambda_1}.$$

In the earlier paper we examined the case in which:

$$\frac{32\gamma}{cr\sqrt{2}}$$

can be treated as infinitely large. It will now be assumed that this quantity is infinitely small. The two roots λ_1 and λ_2 are then real. If λ_2 is the greater root, so by ignoring terms of lower order:

From this it follows:

$$\frac{\lambda_1}{\lambda_2}=\left(\frac{16\gamma}{cr\sqrt{2}}nl\right)^2;$$

this expression is infinitely small, because nl is a multiple of 2π, which is finite. The expressions of the C-coefficients may then be written:

$$C_1=E_n-\frac{2n}{\lambda_2}i_n,\ C_1'=E_n'-\frac{2n}{\lambda_2}i_n',$$

$$C_2=-\frac{\lambda_1}{\lambda_2}E_n+\frac{2n}{\lambda_2}i_n,\ C_2'=-\frac{\lambda_1}{\lambda_2}E_n'+\frac{2n}{\lambda_2}i_n.$$

The coefficient of $\sin nx$ in the expression of E is therefore:

$$E_n\left(e^{-\lambda_1 t}-\frac{\lambda_1}{\lambda_2}e^{-\lambda_2 t}\right)-\frac{2n}{\lambda_2}i_n\left(e^{-\lambda_1 t}-e^{-\lambda_2 t}\right)$$

or

$$E_ne^{-\lambda_1 t}-\frac{2n}{\lambda_2}i_n\left(e^{-\lambda_1 t}-e^{-\lambda_2 t}\right),$$

and the coefficient of $-\cos nx$ in the expression of i:

$$E_n\frac{\lambda_1}{2n}\left(e^{-\lambda_1 t}-e^{-\lambda_2 t}\right)-i_n\left(\frac{\lambda_1}{\lambda_2}e^{-\lambda_1 t}-e^{-\lambda_2 t}\right)$$

By setting E_n' and i_n' for E_n and i_n, one obtains the coefficients of $\cos nx$ in E, and of $\sin nx$ in i. Excluding the case when the initial value of i is infinitely large, compared to the value which i assumes for constant initial values of E, the expression can be simplified when the initial value of $i=0$. It can be seen that when $i=0$ for $t=0$, that is when $i_n=0$, the value of i is of the order of $E\lambda_1/2n$. Under the same circumstances i_n is of the order of $E_n\lambda_1/2n$. The coefficients of $\sin nx$ in E and of $-\cos nx$ in i may be written

$$E_ne^{-\lambda_1 t}$$

and

$$E_n\frac{\lambda_1}{2n}e^{-\lambda_1 t}+\left(i_n-E_n\frac{\lambda_1}{2n}\right)e^{-\lambda_2 t}.$$

If one excludes from these considerations the values of t which are so small that $\lambda_1 t$ becomes infinitely small, then $\lambda_2 t$ becomes infinitely large. Hence, the second term in the second expression can be neglected compared with the first one. As the same considerations with respect to the coefficients of $\cos nx$ and $\sin nx$ are valid in the expressions of E and i, then, substituting for λ_1 the previously obtained value, we have:

$$E=\sum\left(E_n\sin nx+E_n'\cos nx\right)e^{-\frac{8\gamma l}{r}n^2 t} \qquad (16)$$

$$i=\frac{4\gamma l}{r}\sum n\left(-E_n\cos nx+E_n'\sin nx\right)e^{-\frac{8\gamma l}{r}n^2 t}. \qquad (17)$$

These expressions are independent of c. When c is infinitely large, the solutions of the differential equations (14) and (15) become:

$$i=-\frac{4\gamma l}{r}\frac{\partial E}{\partial x}$$

$$\frac{\partial i}{\partial x}=-\frac{1}{2}\frac{\partial E}{\partial t}.$$

Eliminating i, one obtains:

$$\frac{\partial E}{\partial t}=\frac{8\gamma l}{r}\frac{\partial^2 E}{\partial x^2},$$

which is an equation of the same form as the one which determines the conduction of heat in the conductor. Therefore, in the case considered here, the electricity propagates through the metal like heat does.

With the assumptions made with regard to the resistance r' in equations (16) and (17), it is easily proved *a posteriori* that (16) and (17) are real solutions of (14) and (15). It is possible to convince oneself without difficulty that $(4/c^2)(\partial i/\partial t)$ is infinitely small compared with $\partial E/\partial x$ when i and E are taken from (17) and (16).

The case in which the ends of the wire are separated from each other, and are subject to two potential values, can be treated in a similar manner as the case where the wire forms a closed loop. In the open circuit, and provided the resistance of the wire is large enough, one finds the same analogy between the conduction of electricity and heat.

With Jacobi's resistance standard, a copper wire of 7.62 m length, 0.333 mm diameter, as shown in the previous paper, is:

$$\frac{32\gamma}{rc\sqrt{2}}=2070$$

For a wire of the same material, the same cross-section, and a length of 1000 km this quantity is 0.034, By way of an approximation, it can be treated as infinitely large in the first case, and as infinitely small in the second case. In the first case the electricity propagates like a wave in a taught string, and in the second case it travels like heat.

Thomson has examined the motion of electricity in an underwater telegraph wire. He assumed—without checking the reliability of this assumption—that induction makes no significant contribution to the phenomena. For this case he showed that electricity propagates like heat. The present considerations have proved that this conclusion is also justified in the case of a simple wire, provided it is long enough. It will be all the more correct in the underwater telegraph wire, in which the motion of the electricity is considerably slowed down on account of conduction in the seawater.