André Koch Torres Assis

Archimedes, the Center of Gravity and the Law of the Lever

3rd Edition

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Front cover: An engraving of Archimedes planning the defenses of Syracuse from *Les Vrais Pourtraits et Vies des Hommes Illustres* (Paris 1584) by the French historian André Thévet (1516-1590). The Greek writing on his cap is Αρχιμήδης ὁ γεωμέτρης (Archimedes the geometer).
Source: https://math.nyu.edu/Archimedes/Pictures/ArchimedesPictures.html

Back cover: Photos of a few of the experiments described in this book. A horizontal pasteboard triangle supported at the barycenter by a vertical stick. A rectangle and a plumb line suspended by a needle. An equilibrist upside down supported at the head, with modeling clay on his hands. A lever in equilibrium with different weights on each arm.

ARCHIMEDES, THE CENTER OF GRAVITY AND THE LAW OF THE LEVER

Andre Koch Torres Assis

Homepage: https://www.ifi.unicamp.br/~assis

Third edition

To all those who, down through the centuries, have worked to preserve, translate, interpret, and disseminate the works of Archimedes.

Contents

Preface to the Third Edition

This third edition is an improved and updated version of the book published in 2008 and 2010, in Portuguese and in English.[1]

I inserted a commented translation, from French into English, of Archimedes' work *On the Equilibrium of Plane Figures*. It is based on Mugler's literal translation of the original Greek text.[2]

This third edition has an increased number of references. Some portions of the book have been clarified and better explained.

The words between square brackets, [ ], in the middle of the text have been inserted by myself in order to clarify the meaning of some sentences.

[1] [Ass08b] and [Ass10a].

[2] [Mug71a, pp. 75-100].

Preface to the Second Edition

This second edition is an improved and updated version of the book published in 2008, in English and in Portuguese.[3]

The Figures for the present edition were prepared by Daniel Robson Pinto. The present version has a better division of Chapters, Sections and Subsections. It has also an increased number of references. Misprints have been corrected. Some portions of the book have been clarified and better explained.

[3] [Ass08a] and [Ass08b].

Acknowledgments

The motivation to write this book arose from courses we gave to high school science teachers over the past few years. The exchange of ideas with these teachers and with our collaborators at the University were very rich and stimulating.

The inspiration for the majority of the experiments on equilibrium and the center of gravity (CG) of bodies came from the excellent works of Norberto Ferreira and Alberto Gaspar.[4]

We also thank many friends for suggestions, references and ideas: Norberto Ferreira, Alberto Gaspar, Rui Vieira, Emerson Santos, Dicesar Lass Fernandez, Silvio Seno Chibeni, César José Calderon Filho, Pedro Leopoldo e Silva Lopes, Fábio Miguel de Matos Ravanelli, Juliano Camillo, Lucas Angioni, Hugo Bonette de Carvalho, Ceno P. Magnaghi, Caio Ferrari de Oliveira, J. Len Berggren, Henry Mendell, Steve Hutcheon and Guilherme Silva Mel. We thank as well our students at the Institute of Physics with whom we discussed these ideas. My daughter and Eduardo Meirelles helped with the Figures of the first version in English.[5] The Figures for the present version of this book were prepared by Daniel Robson Pinto, through a fellowship awarded by SAE/UNICAMP, which we thank for this support. Daniel helped also to obtain old figures and references.

Special thanks are due to the Institute of Physics, to the Institute of Mathematics, to the Grupo Gestor de Projetos Educacionais (GGPE) and to FAEPEX of the University of Campinas—UNICAMP, in Brazil, which provided the necessary conditions for the preparation of this book.

Roy Keys, the Editor of Apeiron, has been a supporter for many years. He made an excellent editorial work for this book.

Andre Koch Torres Assis

[4] [Fersd], [Fer06] and [Gas03].

[5] [Ass08a].

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Part I

Introduction

One of the goals of this work is to present the basic phenomena of mechanics through simple experiments performed with inexpensive materials. We present the fundamental experiments on falling bodies, equilibrium and oscillations around equilibrium positions. We also show how the theoretical concepts are formed and modified during this process. The formulation of the basic laws of mechanics have also been tentatively formed, modified and improved with the advancement of science.

We show how more complex phenomena can be explained and clarified by means of elementary experiments. Playful and curious experiments are also presented. They stimulate creativity, critical thinking and a sense of humor in science. They also relate everyday phenomena to the fundamental laws of physics.

The emphasis is placed on experimental activities. After presenting the experiments, we formulate the definitions, concepts, postulates, principles, and laws describing the phenomena. The materials utilized are very simple, easily found at home or in stores, all of them very inexpensive. Even so, we can carry out very precise experiments and construct sensitive scientific instruments. The reader need not depend on any school or research laboratory, as he can build his own equipment and perform all the measurements.

If the experiments presented here are performed in the classroom, each student should ideally perform all the tasks, even when working in a group. Each one should build his own equipment (support, plumb line, lever, etc.), cut out his geometric figures and then take all this personal material home. This procedure is richer in lessons than simple demonstrations of the experiments by a teacher. It is essential that all students get involved, getting hands-on experience.

The book is also rich in historical information, which gives the context in which some laws were discovered, and also different approaches taken in discovering them. Special care is taken regarding the formation of physical concepts and principles, as well as their presentation and formulation. It will be seen, for example, how difficult is to find the correct words to precisely define the center of gravity so that this concept can encompass a whole series of experiments. We distinguish clearly between definitions, postulates, experimental results, and physical laws. We also distinguish explanations from descriptions of phenomena. These aspects illustrate the sociological and human aspects of the formulation of physical laws.

This book is written for students and teachers of science, physics, engineering and mathematics. However, it is not a book of experiments for children. It can be utilized at High Schools or at Universities, depending on the level at which each aspect is analyzed and explored. It has enough experimental and theoretical material to be employed in all levels of teaching. Each teacher should adapt the contents presented here to his own school environment. It can also be utilized in courses on the history and philosophy of science.

The best way to grasp the contents of the book is to perform the majority of the experiments described here in parallel with the reading. There are many philosophical, theoretical, and mathematical approaches relating to physical science. But physics is essentially an experimental science. It is the combination of all these aspects that make it so fascinating. For this reason we strongly recommend that the experiments presented in the book be repeated and improved. We hope that the reader will have the same pleasure in performing these experiments as we had in developing them.

I would like to receive a feedback from readers who have tried to reproduce and develop the experiments described here, or attempted to apply them at their schools and universities. I myself, particularly, would have greatly enjoyed learning physics in this way. That is, instead of learning several formulas by heart and spending most of my time solving mathematical exercises, I would prefer to learn physics in the manner shown here, by having the opportunity to build instruments and perform various experiments, learning in practice how important phenomena were first discovered and interpreted, and reproducing most of these experiments with simple materials. It would also be very interesting to explore different models and theoretical concepts in order to explain these phenomena. This book is our contribution to improving the teaching of physics, in a manner similar to what we did with the basic concepts of electricity with the book *The Experimental and Historical Foundations of Electricity*.[6] We hope that science can thus be presented in a more concrete way, rich in historical context, such that the creativity and critical minds of the students can be stimulated.

I would be happy if this book were translated to other languages. It would be great if teachers of physics might indicate this material to their students, colleagues and social networks. I also hope it will motivate others to try something similar in other areas of science. That is, to teach physics utilizing experiments performed with accessible materials combined with historical

[6] Volume 1: [Ass10b], [Ass10c], [Ass15] and [Ass17]. Volume 2: [Ass18b], [Ass18a] and [Ass19].

information related to the subject.

When necessary we employ the sign $\equiv$ as a symbol of definition. We utilize the International System of Units SI.

Chapter 1

The Life of Archimedes

Figure 1.1: A representation of Archimedes.

Archimedes (circa 287 — circa 212 BC) was an Ancient Greek mathematician, scientist, engineer, astronomer, and inventor from the city of Syracuse in Sicily. Figure 1.1 shows the detail of an engraving by M. Weber of a painting by the Italian artist Niccolò Barabino (1832-1891). The painting is now located in the Modern Art Gallery of the Revoltella Museum in Trieste, Italy.[7]

The account of Archimedes' life given here is drawn essentially from Plutarch, Heath, Dijksterhuis, Netz and Noel.[8]

Archimedes lived from 287 to 212 BC. He was born in Syracuse, on the coast of Sicily, where he spent most of his life. He was the son of Pheidias, an astronomer, who estimated the ratio of the diameters of the Sun and the Moon.

The word "Archimedes" is composed of two parts: *arché*, which means beginning, dominion or original cause; and *mêdos*, which means mind, thinking or intellect. Its meaning is then given by *The Master of Thought* or *The Mind of the Beginning*.[9]

Archimedes spent some time in Egypt. It is possible that he studied at the city of Alexandria, which was then the center of Greek science, with the successors of the mathematician Euclid, who flourished around 300 BC and published the famous book of geometry known as *The Elements*.[10] Many of Archimedes' works were sent to mathematicians who lived in Alexandria or who had been there. The famous Museum in Alexandria, which housed a huge library, one of the largest in antiquity, was founded around 300 BC. It is estimated that it had up to 500,000 papyrus scrolls, with an average of 20,000 words in each scroll. The city was under Roman rule from 30 BC to 400 AD. When Cesar was besieged in the palace of Alexandria in 48 BC, a fire may have reached the book repository, and in 391 AD the library may have been destroyed by decree of Emperor Theodosius I. There are no records of the existence of the library and museum after the fifth century. The Roman Empire was fragmented into two parts, Western and Eastern, in 395. Many works of Archimedes were irremediably lost in the ensuing period.

Archimedes is considered one of the greatest scientists of all time, and the greatest mathematician of antiquity. In modern times only Isaac Newton (1642-1727) is comparable to him, both for producing experimental and theoretical works of great impact, and for his originality and immense influence.

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7 [Rorsd].

8 [Plusd], [Arc02b] and [Hea21], [Dij87] and [NN07].

9 [Hir09, p. 9] and [NN07, pp. 59-60].

10 [Euc56] and [Euc09].

By utilizing the method of exhaustion, Archimedes was able to determine the area, volume, and center of gravity of many important geometrical figures, which had never been accomplished before him. He is considered one of the founders of statics and hydrostatics.

His concentration is well described in this passage from Plutarch (*circa* 46-122):[11]

> And thus it ceases to be incredible that (as is commonly told of him) the charm of his familiar and domestic Siren made him forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil on his body, being in a state of entire preoccupation, and, in the truest sense, divine possession with his love and delight in science.

Archimedes' preoccupation with scientific matters in all aspects of life is also recounted by Vitruvius (circa 90-20 BC) in a famous passage in his book on architecture. It is related to the fundamental principle of hydrostatics, which deals with the upward force exerted upon bodies immersed in fluids. The passage illustrates how Archimedes arrived at this principle, or at least the origin of the initial intuition which led to the discovery. We quote from Mach:[12]

> Though Archimedes discovered many curious matters that evince great intelligence, that which I am about to mention is the most extraordinary. Hiero, when he obtained the regal power in Syracuse, having, on the fortunate turn of his affairs, decreed a votive crown of gold to be placed in a certain temple to the immortal gods, commanded it to be made of great value, and assigned for this purpose an appropriate weight of the metal to the manufacturer. The latter, in due time, presented the work to the king, beautifully wrought; and the weight appeared to correspond with that of the gold which had been assigned for it.
>
> But a report had been circulated, that some of the gold had been abstracted, and that the deficiency thus caused had been supplied by

[11] [Plusd].

[12] [Mac60, pp. 107-108].

> silver, Hiero was indignant at the fraud, and, unacquainted with the method by which the theft might be detected, requested Archimedes would undertake to give it his attention. Charged with this commission, he by chance went to a bath, and on jumping into the tub, perceived that, just in the proportion that his body became immersed, in the same proportion the water ran out of the vessel. Whence, catching at the method to be adopted for the solution of the proposition, he immediately followed it up, leapt out of the vessel in joy, and returning home naked, cried out with a loud voice that he had found that of which he was in search, for he continued exclaiming, in Greek, $\varepsilon\upsilon\rho\eta\chi\alpha$, (I have found it, I have found it!)

An illustration of Archimedes and the golden crown appears in Figure 1.2.[13]

Figure 1.2: Archimedes and the golden crown.

[13] [Rorsd].

Another illustration can be seen in Figure 1.3.[14]

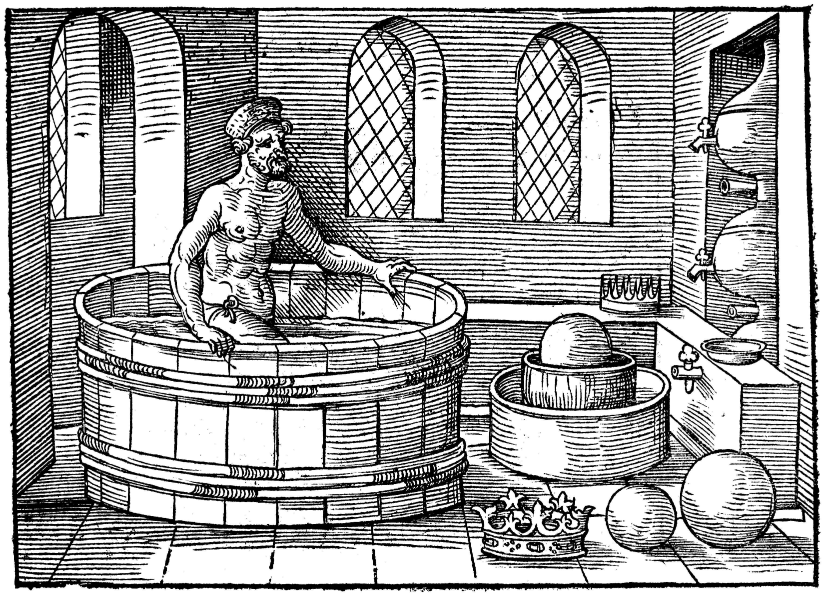

Figure 1.3: Archimedes and the golden crown.

Those works of Archimedes that have survived were addressed to the astronomer Conon of Samos (at that time living in Alexandria), to Conon's disciple Dositheus after the death of Conon, to king Gelon, son of the king Hiero of Syracuse, and to Eratosthenes, librarian of the Library of Alexandria and famous for his precise estimation of the radius of the Earth.

Archimedes would send his works together with some introductory texts. Through these texts we can discover the order of some of his discoveries and a little of his personality. For example, in the introduction of his famous work *The Method of Mechanical Problems*, he stated:[15]

> Archimedes to Eratosthenes greeting.
>
> I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciation and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows. [...] The proofs then

[14] [Rorsd].
[15] [Arc02a, pp. 12-13].

> of these theorems I have written in this book and now send to you. [...]

His habit of sending initially only the statements of some theorems, without demonstrations, may have led some mathematicians plagiarize Archimedes, claiming that his results belonged to them. It is perhaps for this reason that Archimedes on one occasion sent two false results, as he mentioned in the preface of his work *On Spirals*:[16]

> Archimedes to Dositheus greeting.
>
> Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs, the demonstrations are already before you in the books brought to you by Heracleides; and some more are contained in that which I now send you. Do not be surprised at my taking a considerable time before publishing the proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to; otherwise he would have discovered and made manifest all these things, and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics, and that his industry was extraordinary. But, though many years have elapsed since Conon's death, I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation[17] [and which may serve as a warning] how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

Archimedes would often spend years trying to find the proof of a difficult

[16][Arc02b, p. 151].

[17][Note by Heath:] Heiberg reads $\tau\acute{\epsilon}\lambda o\varsigma\ \delta\grave{\epsilon}\ \pi o\theta\epsilon\sigma\acute{o}\mu\epsilon\nu\alpha$, but F has $\tau\acute{\epsilon}\lambda o\upsilon\varsigma$, so that the true reading is perhaps $\tau\acute{\epsilon}\lambda o\upsilon\varsigma\ \delta\grave{\epsilon}\ \pi o\tau\iota\delta\epsilon\acute{o}\mu\epsilon\nu\alpha$. The meaning appears to be simply 'wrong.'

theorem. We can see the perseverance with which he strived to reach his goal in the introduction to *On Conoids and Spheroids*:[18]

> Archimedes to Dositheus greeting.
>
> In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before, and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

Although the works that have come down to us are related to mathematics and theoretical physics, the fame of Archimedes in antiquity is due to his work as an engineer and builder of war machines (catapults, burning mirrors, etc.). One of the inventions attributed to him is a water pumping system known as the cochlias, or Archimedes screw, which is used even to this day. The word cochlias is Greek, meaning snail. It is believed that he invented this hydraulic machine during his stay in Egypt, where it was used for irrigating fields and pumping water.

He built a famous planetarium that had a single hydraulic mechanism which moved several globes simultaneously, reproducing the motions of the stars, the Sun, the Moon, and the planets around the Earth.[19]

He also built a hydraulic organ in which the air fed to the pipes was compressed above water in an air chamber. Also attributed to him are the inventions of the compound pulley, machines for discharging showers of missiles, the Roman balance with unequal arms, etc.

Several authors quote a famous sentence by Archimedes in connection with his mechanical devices and his ability to move great weights with a small force:[20]

> Give me a place to stand on, and I will move the Earth.

An illustration of this statement appeared in the cover of bound Volume II of *Mechanics' Magazine* from 1824.[21]

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[18] [Arc02b, p. 99].

[19] [Wri17].

[20] [Dra63, p. 143], [Dij87, p. 15] and [Arc02b, p. xix].

[21] [WB24] and [Rorsd].

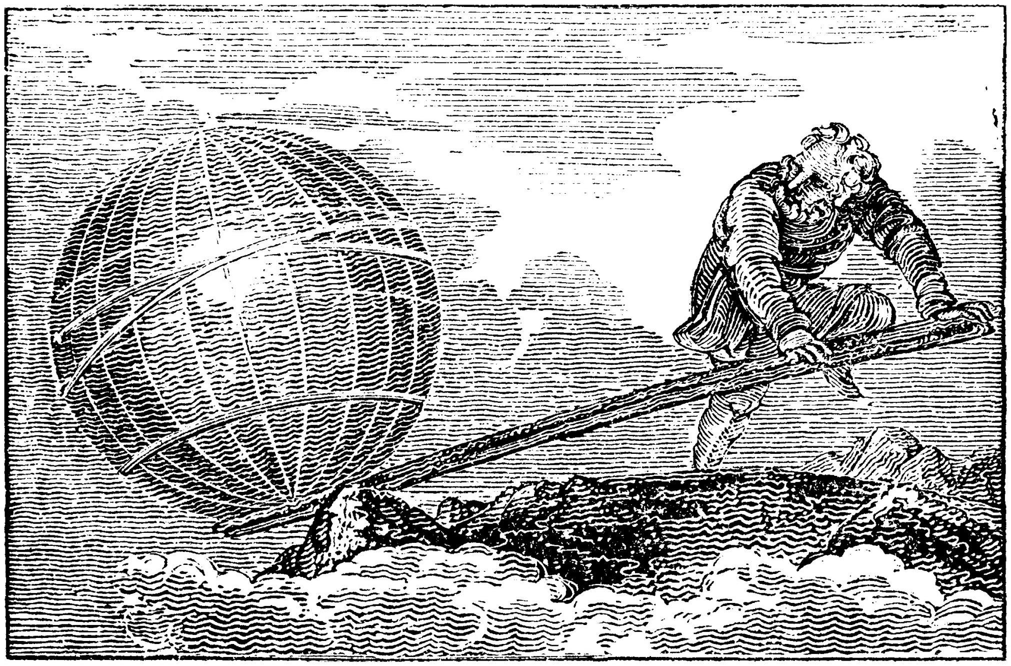

Figure 1.4: "Give me a place to stand on, and I will move the Earth."

This phrase was uttered when he succeeded in accomplishing a task requested by King Hiero to launch a ship weighing many tons into the sea, moving it only with the strength of his hands using a gear composed of a system of pulleys and levers. Plutarch relates this story as follows:[22]

> Archimedes, however, in writing to King Hiero, whose friend and near relation he was, had stated that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. Hiero being struck with amazement at this, and entreating him to make good this problem by actual experiment, and show some great weight moved by a small engine, he fixed accordingly upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labour and many men; and, loading her with many passengers and a full freight,

[22] [Plusd].

> sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.

Hiero was so amazed that he said:[23]

> From that day forth Archimedes was to be believed in everything that he might say.

Plutarch continued:[24]

> The king, astonished at this, and convinced of the power of the art, prevailed upon Archimedes to make him engines accommodated to all the purposes, offensive and defensive, of a siege. These the king himself never made use of, because he spent almost all his life in a profound quiet and the highest affluence. But the apparatus was, in most opportune time, ready at hand for the Syracusans, and with it also the engineer himself.

During the second Punic war between Rome and Carthage, the city of Syracuse was allied with Carthage. Syracuse was attacked by the Romans in 214 BC, under General Marcellus. Many histories about Archimedes have survived in a famous biography of Marcellus written by Plutarch. Marcellus attacked Syracuse by land and sea, heavily armed. According to Plutarch:[25]

> [All machines of Marcellus], however, were, it would seem, but trifles for Archimedes and his machines. These machines he had designed and contrived, not as matters of any importance, but as mere amusements in geometry; in compliance with King Hiero's desire and request, some little time before, that he should reduce to practice some part of his admirable speculation in science, and by accommodating the theoretic truth to sensation and ordinary use, bring it more within the appreciation of the people in general.

Elsewhere, Plutarch writes:[26]

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[23] [Arc02b, p. xix].
[24] [Plusd].
[25] [Plusd].
[26] [Plusd].

> When, therefore, the Romans assaulted the walls in two places at once, fear and consternation stupefied the Syracusans, believing that nothing was able to resist that violence and those forces. But when Archimedes began to ply his engines, he at once shot against the land forces all sorts of missile weapons, and immense masses of stone that came down with incredible noise and violence; against which no man could stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files. In the meantime huge poles thrust out from the walls over the ships sunk some by the great weights which they let down from on high upon them; others they lifted up into the air by an iron hand or beak like a crane's beak and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them. [...] In fine, when such terror had seized upon the Romans, that, if they did but see a little rope or a piece of wood from the wall, instantly crying out, that there it was again, Archimedes was about to let fly some engine at them, they turned their backs and fled, Marcellus desisted from conflicts and assaults, putting all his hope in a long siege.

An illustration of Archimedes' iron hand or claw appears in Figure 1.5. It is a detail of a wall painting in the Stanzino delle Matematiche in the Galleria degli Uffizi (Florence, Italy).[27] It was painted by Giulio Parigi (1571-1635) in the years 1599-1600.

Also connected with the defense of Syracuse is the famous story about burning the Roman ships with mirrors. Archimedes used a great mirror or a system of small mirrors in order to concentrate the sun's rays and focus them on the ships. The two most famous accounts are due to Johannes Tzetzes, a Byzantine scholar, and John Zonaras, both of the twelfth century:

> When Marcellus withdrew them [his ships] a bow-shot, the old man [Archimedes] constructed a kind of hexagonal mirror, and at an interval proportionate to the size of the mirror he set similar small mirrors with four edges, moved by links and by a form of hinge, and made it the centre of the sun's beams–its noon-tide beam, whether

[27] [Rorsd].

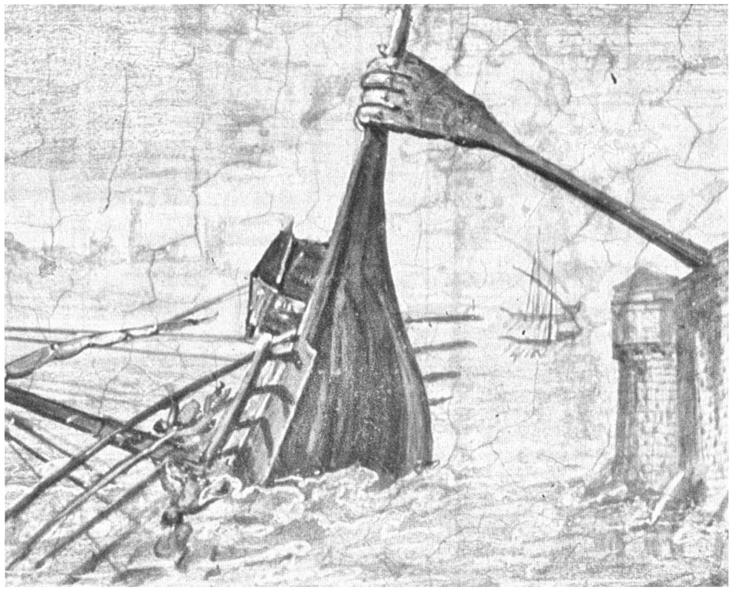

Figure 1.5: Archimedes' iron hand or claw.

> in summer or in mid-winter. Afterwards, when the beams were reflected in the mirror, a fearful kindling of fire was raised in the ships, and at the distance of a bow-shot he turned them into ashes. In this way did the old man prevail over Marcellus with his weapons.[28]

> At last in an incredible manner he [Archimedes] burned up the whole Roman fleet. For by tilting a kind of mirror toward the sun he concentrated the sun's beam upon it; and owing to the thickness and smoothness of the mirror he ignited the air from this beam and kindled a great flame, the whole of which he directed upon the ships that lay at anchor in the path of the fire, until he consumed them all.[29]

An illustration of Archimedes' burning mirror appears in Figure 1.6. It is a wall painting from the Stanzino delle Matematiche in the Galleria degli Uffizi (Florence, Italy), painted by Giulio Parigi (1571-1635) in the years 1599-1600.[30]

[28] J. Tzetzes, as quoted in [Rorsd].
[29] J. Zonaras, as quoted in [Rorsd].
[30] [Rorsd].

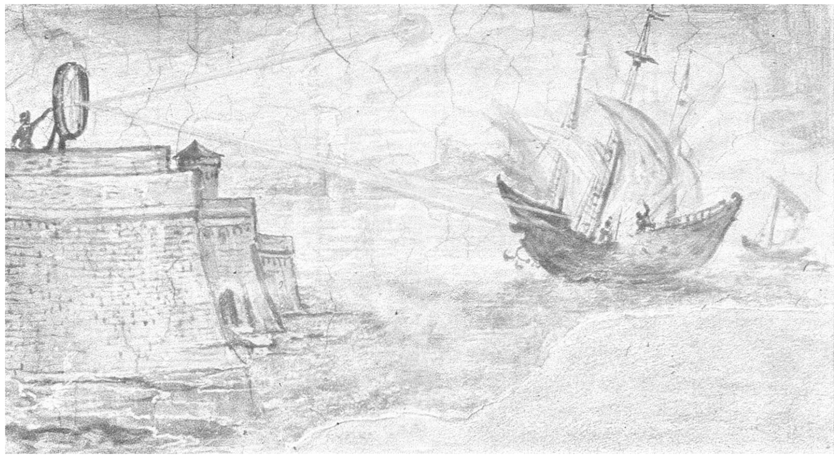

Figure 1.6: Archimedes' burning mirror.

Only after a siege of three years was Marcellus able to conquer Syracuse. Archimedes was killed by a Roman soldier in 212 BC during the capture of the city. Marcellus had given express orders that Archimedes' life should be spared, in recognition of the genius of this enemy who had caused him so many losses. In spite of this, a soldier killed him while he was trying to protect or finish some mathematical discoveries. The last words uttered by Archimedes seem to have been addressed to this soldier:[31]

> Fellow, stand away from my diagram.

Plutarch gives us three different versions of his death:[32]

> But nothing afflicted Marcellus so much as the death of Archimedes, who was then, as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, he never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declining to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through. Others write that a

[31] [Dij87, p. 31].
[32] [Plusd].

> Roman soldier, running upon him with a drawn sword, offered to kill him; and that Archimedes, looking back, earnestly besought him to hold his hand a little while, that he might not leave what he was then at work upon inconclusive and imperfect; but the soldier, nothing moved by his entreaty, instantly killed him. Others again relate that, as Archimedes was carrying to Marcellus mathematical instruments, dials, spheres, and angles, by which the magnitude of the sun might be measured to the sight, some soldiers seeing him, and thinking that he carried gold in a vessel, slew him. Certain it is that his death was very afflicting to Marcellus; and that Marcellus ever after regarded him that killed him as a murderer; and that he sought for his kindred and honoured them with signal favours.

An illustration of his death appears in Figure 1.7. It is a drawing by the French artist Honoré Daumier (1808-79), located in the Szépmüvészeti Museum of Fine Arts, Budapest, Hungary.[33]

During his lifetime, Archimedes expressed the wish that upon his tomb there should be placed a cylinder circumscribing a sphere within it, something like Figure 1.8, together with an inscription giving the ratio between the volumes of these two bodies. We can infer that he regarded the discovery of this ratio as his greatest achievement. This relation appears in Propositions 33 and 34 of the first part of his work *On the Sphere and Cylinder*. These two results are extremely important, and both are due to Archimedes:[34]

> **Proposition 33:** *The surface of any sphere is equal to four times the greatest circle in it.*

That is, in modern language, with A being the area of the surface of a sphere of radius r: $A = 4(\pi r^2)$.

> **Proposition 34:** *Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.*

In modern language, with V_E being the volume of a sphere of radius r, and $V_C = \pi r^2 \cdot (r/3)$ being the volume of a cone of height r and base area

[33] [Rorsd].

[34] [Arc02b, pp. 39 and 41].

Figure 1.7: Archimedes' death.

equal to πr^2, we have $V_E = 4V_C = 4(\pi r^3/3)$. The inscription Archimedes requested for his tomb seems to be related to a Corollary presented at the end of this Proposition:[35]

> From what has been proved it follows that *every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is 3/2 of the sphere, and its surface together with its base is 3/2 of the surface of the sphere.*

In this work *On the Sphere and Cylinder* Archimedes obtained initially the surface of a sphere in Proposition 33. After reaching this result, he obtained the volume of the sphere in Proposition 34. In his other work *The Method of Mechanical Problems* there is a quotation from which we can see that he originally obtained first the volume of the sphere and then, utilizing this result, solved the problem of finding the sphere's surface area.

[35] [Arc02b, p. 43].

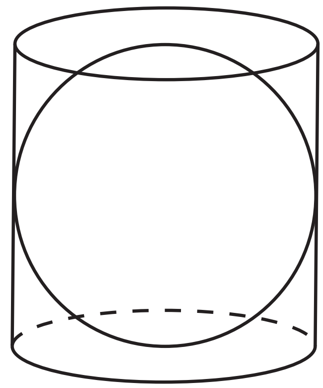

Figure 1.8: A cylinder circumscribing a sphere within it.

Proposition 2 of *The Method of Mechanical Problems* reads as follows:[36]

> (1) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*
>
> (2) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is $1\frac{1}{2}$ times the sphere.*

After demonstrating that the volume of the cylinder is equal to 3/2 the volume of the sphere which it circumscribes, Archimedes said the following:[37]

> From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.[38]

[36] [Arc02a, p. 18].
[37] [Arc02a, pp. 20-21].
[38] [Note by T. L. Heath:]

> That is to say, Archimedes originally solved the problem of finding the solid content

Marcellus saw to it that this wish was fulfilled. Cicero (106-43 BC), the Roman orator, saw this tomb in a neglected state in 75 BC when he was quaestor in Sicily, and restored it. Since then, the tomb has never been found. Cicero wrote the following, as quoted in Rorres:[39]

> But from Dionysius' own city of Syracuse I will summon up from the dust — where his measuring rod once traced its lines — an obscure little man who lived many years later, Archimedes. When I was quaestor in Sicily I managed to track down his grave. The Syracusians knew nothing about it, and indeed denied that any such thing existed. But there it was, completely surrounded and hidden by bushes of brambles and thorns. I remembered having heard of some simple lines of verse which had been inscribed on his tomb, referring to a sphere and cylinder modelled in stone on top of the grave. And so I took a good look round all the numerous tombs that stand beside the Agrigentine Gate. Finally I noted a little column just visible above the scrub: it was surmounted by a sphere and a cylinder. I immediately said to the Syracusans, some of whose leading citizens were with me at the time, that I believed this was the very object I had been looking for. Men were sent in with sickles to clear the site, and when a path to the monument had been opened we walked right up to it. And the verses were still visible, though approximately the second half of each line had been worn away. So one of the most famous cities in the Greek world, and in former days a great centre of learning as well, would have remained in total ignorance of the tomb of the most brilliant citizen it had ever produced, had a man from Arpinum not come and pointed it out!

---

of a sphere before that of finding its surface, and he inferred the result of the latter problem from that of the former. Yet in *On the Sphere and Cylinder I.* the surface is independently found (Prop. 33) and *before* the volume, which is found in Prop. 34: another illustration of the fact that the order of propositions in the treatises of the Greek geometers as finally elaborated does not necessarily follow the order of discovery.

[39] [Rorsd].

Chapter 2

The Works of Archimedes

2.1 Extant Works

The works of Archimedes known to us are available in the original Greek and in Latin.[40] English translations in modern notation have been published.[41] Another version can be found in Dijksterhuis' book.[42] A literal translation from the Greek into French can also be found.[43]

Until one hundred years ago, the oldest and most important manuscripts containing works of Archimedes in Greek (with the exception of *The Method of Mechanical Problems*, which did not appear in any manuscript) were mainly from the 15th and 16th centuries, housed in libraries located in Europe. They had been copied from two other 9th and 10th century Greek manuscripts. One of these manuscripts belonged to the humanist Giorgio Valla, who taught at Venice between 1489 and 1499. This manuscript disappeared between 1544 and 1564. It is not known if it still exists. It contained the following works, in this order:[44] two books of *On the Sphere and Cylinder*, *Measurement of a Circle*, *On Conoids and Spheroids*, *On Spirals*, *On the Equilibrium of Plane Figures*, *The Sand-Reckoner*, *Quadrature of the Parabola*, Eutocius' commentaries of: *On the Sphere and Cylinder*, *Measurement of a Circle*, and *On the Equilibrium of Plane Figures*.

The last record of the second of the 9th and 10th century manuscripts

40[Hei15].
41[Arc02b].
42[Dij87].
43[Mug70], [Mug71a], [Mug71b] and [Mug72].
44[Arc02b, p. xxiv].

was in the Vatican Library in the years 1295 and 1311. It is not known if this manuscript still exists. It contained the following works, in this order:[45] *On Spirals*, *On the Equilibrium of Plane Figures*, *Quadrature of the Parabola*, *Measurement of a Circle*, *On the Sphere and Cylinder*, Eutocius' commentaries of *On the Sphere and Cylinder*, *On Conoids and Spheroids*, Eutocius' commentaries of *On the Equilibrium of Plane Figures*, *On Floating Bodies*. The latter work on floating bodies, in two parts, was not contained in the first manuscript.

The work *On Floating Bodies* was only known until 1906 from a Latin translation made by the Flemish Dominican Willem van Moerbeke in 1269, based on the second 9th or 10th century manuscript. He produced a Latin translation of all the works of Archimedes to which he had access, which was very important for the dissemination of his ideas. The original manuscript containing Moerbeke's translation was found again in Rome in 1884, and is now at the Vatican Library.

Archimedes wrote in the Doric dialect. In the manuscripts still extant his original language was transformed in some books totally, in others only partially, into the Attic dialect common in Greece. In the 9th century some of his works were translated to Arabic. The first Latin translations of the works of Archimedes and of several scientists and philosophers of Greece were made during the 12th and 13th centuries. Gutenberg invented movable type for the printing press in Europe in the mid-15th century. The publication of Archimedes' works in printed form began in the 16th century, the oldest being from 1503, containing the *Measurement of a Circle* and the *Quadrature of the Parabola*. Printed in 1544, the *Editio Princeps* contained the major known works by Archimedes, in Greek and Latin, with the exception of *On Floating Bodies*. The invention of the press was very important for the spread of his ideas. The first translations of some of his works to a living language, German, were published in 1667 and 1670, by J. C. Sturm.[46] In 1807 the first French translation of all his known works was made by Peyrard. In 1897 and 1912 the first English translation was published by Sir T. L. Heath.

We present here the extant works of Archimedes in the order in which they were written according to Heath.[47] Much controversy surrounds this chronology. Knorr, for example, placed *The Method of Mechanical Problems*

[45] [Dij87, p. 38].
[46] [Stu67].
[47] [Hea21, pp. 22-23] and [Arc02b, p. xxxii].

at the end of his works.[48]

2.1.1 On the Equilibrium of Plane Figures, or The Centers of Gravity of Plane Figures — Book I

Alternative titles: *On the Equilibrium of Planes* or *The Centers of Gravity of Planes. Book I.*

Archimedes derives the law of the lever theoretically utilizing the axiomatic method and calculates the center of gravity of parallelograms, triangles, and trapeziums.[49]

2.1.2 Quadrature of the Parabola

Archimedes found the area of a parabolic segment.[50]

Proposition 24:[51]

> *Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.*

He presented two proofs of this result. In the first, he performed a mechanical quadrature, utilizing the law of the lever. In the second, he performed a geometric quadrature.

2.1.3 On the Equilibrium of Plane Figures, or The Centers of Gravity of Plane Figures — Book II

Alternative titles: *On the Equilibrium of Planes* or *The Centers of Gravity of Planes. Book II.*[52]

Archimedes found the center of gravity of a parabolic segment.

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[48] [Kno79].

[49] In English: [Arc02b, pp. 189-202] and [Dij87, pp. 286-313]. In French: [Arc07] and [Mug71a, pp. 75-100]. In German: [Arc23, pp. 31-46]. In Portuguese: [Ass97] and [Arq08].

[50] In English: [Arc02b, pp. 233-252] and [Dij87, pp. 336-345]. In French: [Arc07] and [Mug71a, pp. 159-203]. In German: [Arc23, pp. 5-29].

[51] [Arc02b, p. 251].

[52] In English: [Arc02b, pp. 203-220] and [Dij87, pp. 346-360]. In French: [Arc07] and [Mug71a, pp. 101-125]. In German: [Arc23, pp. 47-64]. In Portuguese: [Arq04b].

2.1.4 The Method of Treating Mechanical Problems

This work is usually called simply *The Method*.[53]

In this letter to Eratosthenes, Archimedes presented a mechanical method to obtain geometrical results (calculation of areas, volumes and centers of gravity) utilizing the law of the lever and concepts of the theory of the center of gravity. He presented several examples of this heuristic method which he created and employed, illustrating how to apply it. He thus obtained the quadrature of the parabola, the volume and center of gravity of any segment of a sphere, the center of gravity of a semi-circle, the center of gravity of a paraboloid of revolution, and several other results. This work will be discussed in more detail in Section 2.3.

2.1.5 On the Sphere and Cylinder, Books I and II

In this work[54] Archimedes showed that the area of the surface of a sphere is equal to four times the greatest circle passing through the center of the sphere; found the area of any segment of the sphere; showed that the volume of the sphere is equal to two-thirds the volume of the circumscribed cylinder, and that the surface of the sphere is equal to two-thirds the surface of the circumscribed cylinder, including the bases, see Figure 1.8. In the second part of this work, the most important result is how to divide a sphere by a plane in such a way that the ratio of the volumes of the two segments has a given value.

2.1.6 On Spirals

In this work[55] Archimedes defined a spiral through the uniform motion of a point along a straight line, this straight line rotating with a constant angular velocity in a plane. He established the fundamental properties of the spiral, relating the length of the radius vector to the angles of revolution that gen-

[53] In English: [Arc02b, pp. 5-51] and [Dij87, pp. 313-336]. In French: [Mug71b, pp. 78-180]. In German: [Arc63a]. In Portuguese: [Arq04a], [Mag11], [BF17] and [MA19]; see also [AM12], [AM14] and [AM16].

[54] In English: [Arc02b, pp. 1-90] and [Dij87, pp. 141-221]. In French: [Arc07] and [Mug70, pp. 1-134]. In German: [Arc22a]. In Portuguese: [Gru23].

[55] In English: [Arc02b, pp. 151-188] and [Dij87, pp. 264-285]. In French: [Arc07] and [Mug71a, pp. 1-74]. In German: [Arc22b].

erate the spiral. He presented results related to the tangents of the spiral, and showed how to calculate areas of parts of the spiral.

As a curiosity we quote here the first two propositions and the main definition presented in this work. This spiral is represented nowadays in polar coordinates by the relation $\rho = k\varphi$, where k is a constant, ρ is the distance to the z-axis (or from the origin, considering the motion in the xy plane) and φ is the angle of the radius vector relative to the x axis. In this representation in polar coordinates the time does not appear. On the other hand, the historical relevance of the original definition given by Archimedes is the introduction of the time concept in geometry. This was crucial for the later development of classical mechanics.

> **Proposition 1:** *If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.*[56]

> **Proposition 2:** *If each of two points on different lines respectively move along them each at a uniform rate, and if lengths be taken, one on each line, forming pairs, such that each pair are described in equal times, the lengths will be proportionals.*[57]

> **Definition:** If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a **spiral** ($\epsilon\lambda\iota\xi$) in the plane.[58]

2.1.7 On Conoids and Spheroids

In this work[59] Archimedes studied the paraboloids of revolution, the hyperboloids of revolution (conoids) and the ellipsoids (spheroids) obtained by the rotation of an ellipse around one of its axes. The main goal of the work

[56] [Arc02b, p. 155].
[57] [Arc02b, p. 155].
[58] [Arc02b, p. 165].
[59] In English: [Arc02b, pp. 99-150] and [Dij87, pp. 240-263]. In French: [Arc07] and [Mug70, pp. 145-252].

was to investigate the volume of segments of these three-dimensional bodies. He showed, for example, in Propositions 21 and 22, that the volume of a paraboloid of revolution is 3/2 of the volume of the cone which has the same base and the same height:[60]

> **Propositions 21, 22:** *Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis.*

Analogous, but more complex results, are obtained for the hyperboloid of revolution and for the ellipsoid.

2.1.8 On Floating Bodies. Books I and II

In this work[61] Archimedes established the fundamental principles of hydrostatics, giving the weight of a body immersed in a fluid. He also studied the conditions of stability of a spherical segment and a paraboloid of revolution floating in a fluid.

In the first part of this work, Archimedes creates the entire science of hydrostatics. We know of no other author who worked with this subject prior to him. His basic postulate reads as follows:[62]

> Postulate: *Let it be granted that the fluid is of such a nature that of the parts of it which are at the same level and adjacent to one another that which is pressed the less is pushed away by that which is pressed the more, and that each of its parts is pressed by the fluid which is vertically above it, if the fluid is not shut up in anything and is not compressed by anything else.*

Heath's translation of this postulate reads as follows:[63]

> *Postulate* 1: "Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more;

[60][Arc02b, p. 131].

[61]In English: [Arc02b, pp. 253-300] and [Dij87, pp. 373-398]. In French: [Arc07] and [Mug71b, pp. 1-66]. In German: [Arc25]. In Portuguese: [Ass96], [Arq12] and [Arq24].

[62][Dij87, p. 373], see also [Mug71b, p. 6].

[63][Arc02b, p. 253].

> and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else."

Heath's translation, published in 1897, was based on the Latin translation by Moerbeke in 1269, as the original text by Archimedes in Greek had been lost. In 1906, Heiberg found another manuscript containing the original Greek text of this work. Some parts of this manuscript remain undecipherable, and others are missing. In any event it contains this basic postulate, which clarifies the meaning of the last passage. Instead of Heath's "and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else," the correct meaning is that of Dijksterhuis or Mugler, namely, "that each of its parts is pressed by the fluid which is vertically above it, if the fluid is not shut up in anything and is not compressed by anything else."

Beginning with this postulate, Archimedes arrives at an explanation for the spherical shape of the Earth, supposing it to be wholly composed of water. Then he proves the fundamental principle of hydrostatics, known today as Archimedes' principle, in Propositions 5 to 7. When he says that a solid is heavier or lighter than a fluid, he is referring to the relative or specific weight, that is, if the solid is more or less dense than a fluid. Here are the Propositions:[64]

> **Proposition 5:** *Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.*
>
> [...]
>
> **Proposition 6:** *If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.*
>
> [...]
>
> **Proposition 7:** *A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.*

[64] [Arc02b, pp. 257-258].

Based on these propositions, Archimedes determined at the end of the first book the equilibrium conditions of a spherical segment floating in a fluid. In the second part, Archimedes presented a complete investigation of the conditions of equilibrium of a segment of a paraboloid of revolution floating in a fluid. His interest here seems very clear, namely, to study theoretically the stability of ships, although this is not explicitly mentioned. This is a work of applied mathematics, or theoretical engineering.

This is a monumental work which, for some two thousand years, was almost the only text on this topic. It was revived in the renaissance, influencing the works of Stevin (1548-1620) and Galileo (1564-1642).

2.1.9 Measurement of a Circle

This work[65] has not come down to us in its original form. It is probably only a fragment of a larger text. Archimedes showed that the area of a circle is equal to the area of a right-angled triangle whose sides are the radius of the circle and the rectified circumference:[66]

> **Proposition 1:** *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

In modern notation this result can be expressed as follows. Let A_C be the area of a circle of radius r with a circumference $C = 2\pi r$. Let A_T be the area of the triangle described by Archimedes, with its area being given by half its base multiplied by its height. Archimedes' relation is then given by: $A_C = A_T = r \cdot C/2 = \pi r^2$.

He also showed that the exact value of π is between $3\frac{10}{71} \approx 3.1408$ and $3\frac{1}{7} \approx 3.1429$. He obtained this result by circumscribing and inscribing a circle with regular polygons of 96 sides. In his own words:[67]

> **Proposition 3:** *The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.*

[65] In English: [Arc02b, pp. 91-98] and [Dij87, pp. 222-240]. In French: [Arc07] and [Mug70, pp. 135-144]. In German: [Arc63b]. In Portuguese: [GBD21].

[66] [Arc02b, p. 91].

[67] [Arc02b, p. 93].

In the middle of Proposition 3 of the *Measurement of a Circle* he presented precise approximations for the square roots of many numbers, without specifying how he arrived at these results. In modern notation he stated, for example, that $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$, or that, $1.7320261 < \sqrt{3} < 1.7320513$.

2.1.10 The Sand Reckoner

In this work[68] Archimedes deals with the problem of counting the number of grains of sand contained in the sphere of the fixed stars, utilizing estimations by Eudoxus of Cnidus (c. 390 - c. 340 BC), his father Pheidias, and Aristarchus os Samos (c. 310 - c. 230 BC). He proposes a numerical system capable of expressing numbers equivalent to our 8×10^{63}. It is in this work that Archimedes mentioned that the addition of the orders of the numbers (the equivalent of their exponents when the base is 10^8) corresponds to finding the product of these numbers.[69] This is the principle that led to the invention of logarithms many centuries later.

Also in this work, Archimedes mentioned the heliocentric system of Aristarchus. The work of Aristarchus describing his heliocentric system has not been preserved. Here we reproduce the introduction of the *Sand-Reckoner*. This introduction is the oldest and most important evidence concerning the existence of a heliocentric system in antiquity. Due to this extremely important idea, Aristarchus is often called the Copernicus of antiquity,[70] although it would be more appropriate to call Copernicus the Aristarchus of modernity. At the end of the introduction, Archimedes refers to a work called *Principles*, which is probably the title of one of Archimedes' memoirs containing a system of expressing numbers that had been sent to Zeuxippus, and is quoted in the introduction. This work is not extant. Archimedes writes:[71]

> There are some, king Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no

[68] In English: [Arc02b, pp. 221-232] and [Dij87, pp. 360-373]. In French: [Arc07] and [Mug71a, pp. 127-158]. In German: [Stu67]. In Portuguese: [Arqsd].

[69] [Dij87, pp. 360-373].

[70] [Hea20].

[71] [Arc02b, pp. 221-222].

number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognising that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe. Now you are aware that 'universe' is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account ($\pi\acute{\alpha}\ \gamma\rho\alpha\phi\acute{o}\mu\epsilon\nu\alpha$), as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses, in which the premises lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible; for, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this: since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the 'universe' is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind, and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the 'universe.'

> I say then that, even if a sphere were made up of the sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named in the *Principles*[72], some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made.
>
> [...]

2.2 Fragmentary Works

It is also known that Archimedes wrote other works which exist today only in fragments or in references by other writers:

2.2.1 The Stomachion

There are only fragments of the text.[73] It deals with a game like tangram, with 14 pieces which together form a square. Some examples can be seen in Figure 2.1. Archimedes probably tried to find the number of ways in which these 14 pieces can be put together in order to form a square. According to Netz and Noel, this work marks the beginning of geometrical combinatorics.[74]

2.2.2 The Cattle Problem

This work[75] is contained in an epigram communicated by Archimedes to the mathematicians of Alexandria in a letter to Eratosthenes. It is a problem of algebra with 8 unknowns. The complete solution leads to a number with 206,545 digits.

[72][Note by Heath:] $A\rho\chi\alpha\iota$ was apparently the title of the work sent to Zeuxippus. Cf. the note attached to the enumeration of lost works of Archimedes in the Introduction, Chapter II., *ad fin.*

[73]In English: [Dij87, pp. 408-412]. In French: [Mug71b, pp. 67-76]. In Portuguese: [Car21].

[74][NAW04] and [NN07, pp. 329-366].

[75]In English: [Arc02b, pp. 319-326] and [Dij87, pp. 398-401]. In French: [Mug71b, pp. 165-174]. In Portuguese: [CO23].

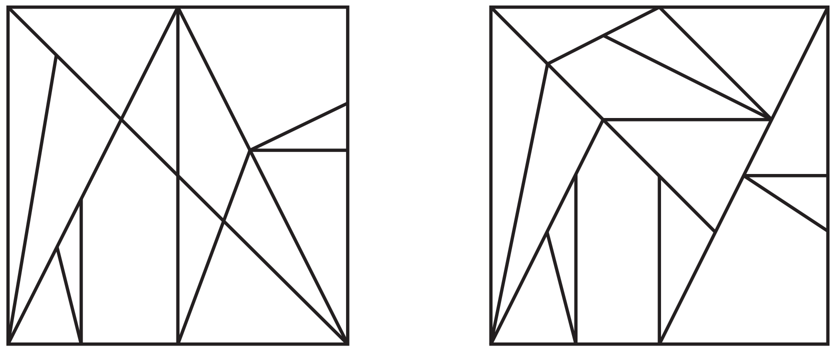


Figure 2.1: Two possible configurations of Archimedes' *Stomachion*.

2.2.3 Book of Lemmas

A collection of important lemmas relating to planimetric figures.[76]

2.2.4 Semi-Regular Polyhedra

The regular polyhedra were known by Plato and are described by Euclid in his book of geometry, *The Elements*.[77] Their faces are composed of regular equal polygons, equilateral and equiangular. There are only 5 regular platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

In this work Archimedes describes the construction of the semi-regular polyhedra which he discovered.[78] Its faces are regular polygons, but with different numbers of sides, such as squares and equilateral triangles. There are only 13 of these solids, all discovered by Archimedes. They are called Archimedian polyhedra.

2.2.5 Area of the Triangle

Some authors consider that Archimedes discovered the expression usually attributed to Heron in the first century AD for the area of a triangle in

[76] In English: [Arc02b, pp. 301-318] and [Dij87, pp. 401-405]. In French: [Arc07] and [Mug71b, pp. 129-164].

[77] [Euc56] and [Euc09].

[78] In English: [Dij87, pp. 405-408].

terms of its sides.[79]

2.2.6 Construction of a Regular Heptagon

Archimedes presented the construction of a heptagon inscribed within a circle.[80]

2.2.7 Lost Works

Other works mentioned by Archimedes or quoted by other authors as being due to Archimedes are not extant. In some cases we know only the title, or have a general idea of their content. The same work may be cited with different names:

Principles, or *Naming of Numbers*.

On how to Express Large Numbers.

On the Centers of Gravity.

Elements of Mechanics.

About the center of gravity of geometric figures and about the law of the lever. Probably his work *On the Equilibrium of Plane Figures* is part of this larger treatise.

On the Center of Gravity and Law of the Lever.

The work *On the Equilibrium of Plane Figures* is probably only a small part of this larger work.

Equilibria.

About the center of gravity of solids.

Book on Columns, or *Book of Supports*.

According to Heron, Archimedes dealt here with bodies supported by two or more columns. He solved the problem of finding which part of the total weight of the body was supported by each pillar.

[79] [Dij87, pp. 412-414].
[80] [Dij87, pp. 414-416].

On Balances, or *On Levers*.

On the center of gravity and the law of the lever.

One work on *Optics*.

Including the law of reflection and studies on refraction.

On Sphere-Making.

A mechanical work describing the construction of a sphere representing the motions of the celestial bodies, probably a description of the famous planetarium built by Archimedes.

On the Calendar.

On the length of the year.

On Circles Touching One Another.

On Parallel Lines.

On Triangles.

On Properties of Right-Angled Triangles.

On the Assumptions for the Elements of Geometry.

Book of Data or Definitions

2.3 The Method of Mechanical Problems

Of all the Archimedes' works known today, the one that has received the greatest attention is *The Method*. One of the few things known about this work until 1906 was its title. Between 1880 and 1881 the Danish scholar J. L. Heiberg (1854-1928), a professor of classical philology at Copenhagen University, published the complete works of Archimedes then known, in Greek and Latin, in three volumes. This book was utilized as the basis for the modern translation of his works into many living languages, such as the English made by T. L. Heath (1861-1940) and published in 1897. When he described the lost works of Archimedes, Heath quoted *The Method* in a single sentence:[81]

[81] [Arc02b, p. xxxviii].

7. ἐφόδιον, a *Method*, noticed by Suidas, who says that Theodosius wrote a commentary on it, but gives no further information about it.

Suidas was a Greek encyclopedist who lived in the 10th century, while Theodosius (circa 160-90 BC) was a mathematician in Anatolia. But in 1899 Heiberg read about a palimpsest of mathematical content found in Constantinople. The word "palimpsest" means "scraped again." Normally it is a parchment that has been used two or three times, after being scraped or washed each time, due to a shortage of parchment or to its high price. This specific parchment contained a Euchologion written in the 12th, 13th, or 14th century, over a mathematical manuscript of the 10th century. From a few specimen lines to which he had access, Heiberg suspected that it contained an Archimedian text. He traveled to Constantinople and examined the manuscript twice, in 1906 and 1908. Fortunately the original text had not been completely washed out and Heiberg was able to decipher much of the contents by inspecting the manuscript and taking photographs. The manuscript contained 185 leaves with Archimedes' works in Greek. Beyond the texts already known, it contained three treasures: (I) fragments of the *Stomachion*, (II) a large part of the Greek text of the work *On Floating Bodies* (until then it was believed to have survived only in the Latin translation made by Willem von Moerbeke in 1269 from a Greek manuscript which is now believed to be lost) and (III) most of *The Method of Mechanical Problems* by Archimedes! A work that had been lost for two thousand years (the last person to study it seems to have been Theodosius), of which we did not know even the contents, appeared out of nowhere, greatly expanding our knowledge about Archimedes. Even the comments on this work by Theodosius are no longer extant. This manuscript contained the following works of Archimedes, in this order: *On the Equilibrium of Plane Figures*, *On Floating Bodies*, *The Method of Mechanical Problems*, *On Spirals*, *On the Sphere and Cylinder*, *Measurement of a Circle* and *Stomachion*.

In 1907 Heiberg published the Greek text of *The Method of Mechanical Problems*, together with a German translation.[82] The commentary was made by Zeuthen. Between 1910 and 1915 Heiberg published a second edition of the complete works of Archimedes, in Greek and Latin, in three volumes. This second edition is much better than the first one, and was republished in 1972.[83] This edition is the basis of all modern translations of Archimedes'

[82] [Hei07], [Hei09] and [Arc63a].

[83] [Hei15].

works. There are now translations of *The Method of Mechanical Problems* in several languages: English,[84] Italian,[85] French,[86] and Portuguese.[87] Heiberg's discovery was featured on the first page of The New York Times in 1907.

But the story does not end here. In the period between 1908 and 1930 the manuscript disappeared, probably having been stolen. Around 1930 a French antiquities collector bought the manuscript, without the knowledge of the external world. In 1991 the collector's family put this manuscript on sale in an auction. Only then was it realized this was the manuscript discovered by Heiberg in 1906 and which was supposed to have been lost. In 1998 it was sold by Christie's, in New York. It was bought for 2 million dollars by an anonymous billionaire and lent to Walters Arts Gallery, of Baltimore, USA. A group of scholars, directed by Nigel Wilson and Reviel Netz, of Stanford University, are working on the restoration, digitization and publication of the manuscript, which contains the only still surviving copy of *The Method of Mechanical Problems*, a work that had been lost for 2,000 years!

The importance of this work lies in the fact that it contains virtually the only account of an ancient mathematician presenting the method that led him to the discovery of his theorems. In all other surviving works we have only the theorems presented in final form, derived with a rigorous logic and with scientifically precise proofs, beginning with axioms and other theorems. This dry presentation conceals the method or the intuition that led to the final result. *The Method of Mechanical Problems* changed all this. In this work Archimedes describes the path he followed in order to discover several significant results on quadrature and cubature (calculation of areas and volumes by integration). He also revealed how he calculated for the first time the center of gravity of several important two- and three-dimensional geometric figures. Here are Archimedes' own words:[88]

> Archimedes to Eratosthenes greeting.
>
> I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows.

[84] [Hei09], [Smi09], [Arc09], [Arc87] and [Arc02a].
[85] [Arc61].
[86] [Arc71].
[87] [Arq04a], [Mag11], [BF17] and [MA19]; see also [AM12], [AM14] and [AM16].
[88] [Arc02a, pp. 12-14].

[...]

The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e., a parabola) is four-thirds of the triangle which has the same base and equal height,

and after this I will give each of the other theorems investigated by

> the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions]...
>
> [I premise the following propositions which I shall use in the course of the work.]
>
> [...]

After this introduction about the life and work of Archimedes, we present several experiments that lead to a precise conceptual definition of the center of gravity of bodies.

Part II

The Center of Gravity (CG)

Chapter 3

Geometry

We begin our work with a little mathematics. We will cut out some plane figures and find their main geometrical properties. Later on we will utilize these figures in experiments. The dimensions we present here are adequate for individual activities. Larger sizes should be used for demonstrations in the classroom, talks, and seminars.

Materials

- Paper board, light cardboard, thick card, or pasteboard. Light plane and rigid sheets (made of wood, plastic, metal or Styrofoam) can also be utilized.
- White sheets of paper.
- Ruler, pen, T-square and protractor.

3.1 Finding the Centers of Circles, Rectangles and Parallelograms

From a pasteboard we draw and cut out a circle 7 or 8 cm in diameter. If the circle is drawn with compasses, the center should be marked with a pen, and indicated with the letter X.

If the circle is drawn with a glass turned upside down, the center can be found by the intersection of two diameters. The diameters can be drawn with a ruler. But it is difficult to be sure if the ruler passes exactly through the center when we do not know exactly where the center is located.

An alternative procedure to find the diameter and center of the circle involves a piece of paper. Later we will perform experiments with the pasteboards, so it is better not to fold them. For this reason the folding we discuss here should be done with similar figures made from sheets of paper. For example, we place the pasteboard circle on a sheet of paper and cut out a similar circle of paper. We then fold the paper circle in two equal halves. We fold it once more so that it is divided into four equal parts, as in Figure 3.1. We can then use a pen to draw the diameters in the paper circle. The center of the circle is the intersection of the diameters. A hole should be made at the center. By placing the paper circle on the pasteboard circle, we can mark the center of the circle on the pasteboard.

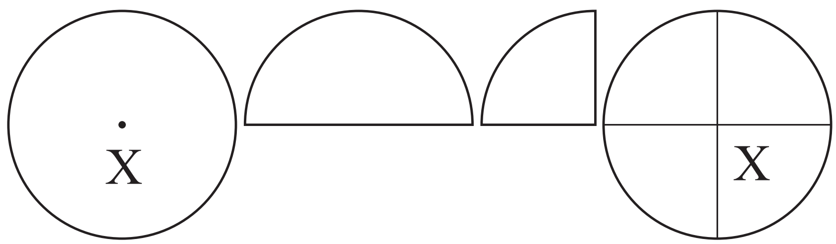


Figure 3.1: Finding the center of a circle by paper folding.

We cut out a pasteboard in the shape of a rectangle with sides of 6 cm and 12 cm. There are two ways to find the center. The simplest one is to connect the opposite vertices. The center of the rectangle is the intersection of these diagonals, marked with the letter X.

The other way is to find (with a ruler or by folding) the central point of each side. We then connect the middle points of opposite sides. The center is the intersection of these straight lines.

The parallelogram is a plane quadrilateral in which the opposite sides are parallel to one another. A parallelogram is cut out from a pasteboard with sides of 6 cm and 12 cm, with the smallest internal angle being 30° (or 45°). The center of this parallelogram can be found by the two methods we used for the rectangle, as in Figure 3.2.

3.2 The Triangle Centers

There are three types of triangle: equilateral (three equal sides), isosceles (only two sides of the same length), and scalene (with three different sides).

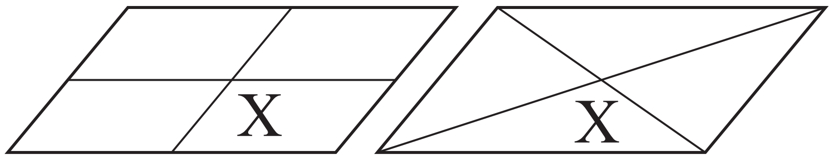


Figure 3.2: Finding the center of a parallelogram.

Every triangle has four special centers: circumcenter (C), barycenter or triangle centroid (B), orthocenter (O), and incenter (I). We will find these four special points in the case of an isosceles triangle with a base of 6 cm and height of 12 cm. With these dimensions each one of the equal sides has a length of 12.37 cm, Figure 3.3.

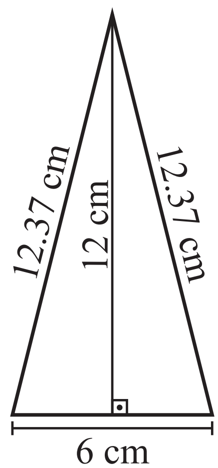


Figure 3.3: Isosceles triangle.

We draw and cut out a triangle of this size from a pasteboard. We also cut out four equal triangles from a sheet of paper. Each one of these four paper triangles will be used to draw the straight lines and locate one of the special points. When necessary, also the folding should be done with these paper triangles.

The circumcenter, C, is the intersection of the perpendicular bisectors. A perpendicular bisector of a straight line AB is a straight line perpendicular to AB and passing through its midpoint M. To find the midpoint of each side we can use a ruler. With a T-square or using the pasteboard rectangle we draw a straight line perpendicular to each side through its midpoint. The intersection of these lines is the circumcenter (C), as in Figure 3.4.

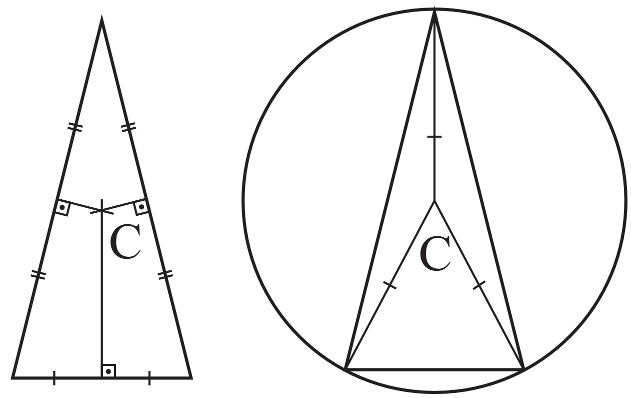


Figure 3.4: The circumcenter and the circumcircle.

Another way of finding the midpoint of each side is by folding. In this case we only need to join the vertices two by two. The folding will be orthogonal to the side, passing through the midpoint.

An important property of the circumcenter is that it is equidistant from the vertices. It is therefore the center of the triangle's circumcircle, as in Figure 3.4.

In every acute triangle (a triangle in which all angles are acute, that is, smaller than 90°), the circumcenter is inside the triangle. In a right-angled triangle the circumcenter is located at the midpoint of the hypotenuse. In every obtuse triangle (a triangle which has an obtuse angle, that is, larger than 90°), the circumcenter is outside the triangle.

The barycenter or triangle centroid (B) is the intersection of the medians, which are the lines connecting the vertices to the midpoints of the opposite sides. It is also called the median center. The midpoint of each side can be found with a ruler or by folding. After finding these midpoints, simply connect them to the opposite vertices. The intersection of these medians is the centroid, as in Figure 3.5. The barycenter is always inside the triangle and has an important property. The distance from the vertex to the centroid is always twice the distance from the centroid to the midpoint of the opposite side.

The orthocenter, O, is the intersection of the altitudes of a triangle, which are the straight lines connecting the vertices to the opposite sides, orthogonal to them. The easiest way to find these lines is to use a T-square or pasteboard

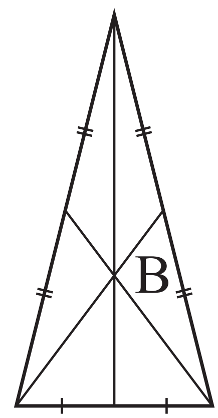


Figure 3.5: The barycenter.

rectangle. We slide the base of the T-square or the rectangle along one side of the triangle until the perpendicular side of the T-square or the rectangle meets the opposite vertex of the triangle. At this moment we draw the perpendicular to the side, connecting it to the opposite vertex, as in Figure 3.6.

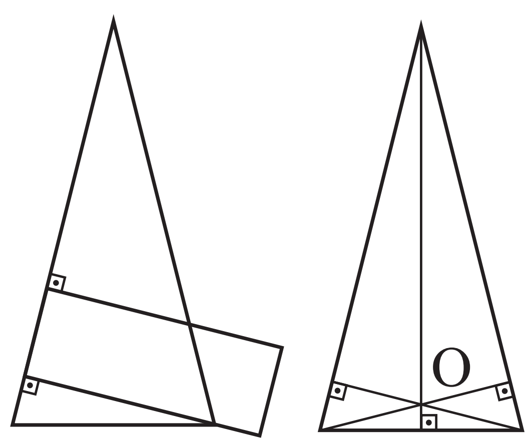


Figure 3.6: The orthocenter.

The orthocenter is the intersection of the altitudes, as in Figure 3.6. The altitudes also represent the smallest distances between the vertices and the opposite sides. Depending upon the dimensions of the triangle, the orthocenter may be inside or outside the triangle.

The incenter, I, is the intersection of the angle bisectors of the triangle, which are the straight lines dividing the vertices into two equal angles. These lines can be obtained with a protractor. But the easiest way is by folding. In this case you only need to join the adjacent sides through the vertex. The folded sides of the paper then divide each vertex into two equal angles. The intersection of the straight lines is the incenter, as in Figure 3.7.

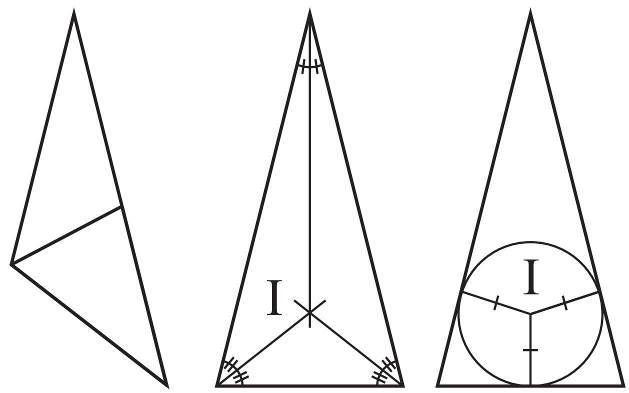


Figure 3.7: The incenter and the incircle.

The incenter is always located inside the triangle. It is equidistant from all sides of the triangle. It is thus also the center of the incircle (the inscribed circle of the triangle, tangent to all three sides), as in Figure 3.7.

After locating these four centers with the paper triangles, we make holes in the papers at these centers. We then superimpose each of these paper triangles upon the pasteboard triangle and mark these points. The final result in the case of an isosceles triangle with a 6 cm base and 12 cm height is shown in Figure 3.8. We can see that these four points are different from one another, with the orthocenter closer to the base, then the incenter, the barycenter, and the circumcenter. These four points are along a straight line which is the angle bisector, altitude, median, and perpendicular bisector.

For an equilateral triangle these four centers superimpose on one another, as in Figure 3.9 (a).

For an isosceles triangle with 12 cm base and 7 cm height the order of the centers relative to the base is opposite to the order for a 6 cm base and 12 cm height isosceles triangle, as in Figure 3.9 (b).

For a scalene triangle these four centers are not along a single straight line. Moreover, they are not all necessarily inside the triangle. In Figure 3.9

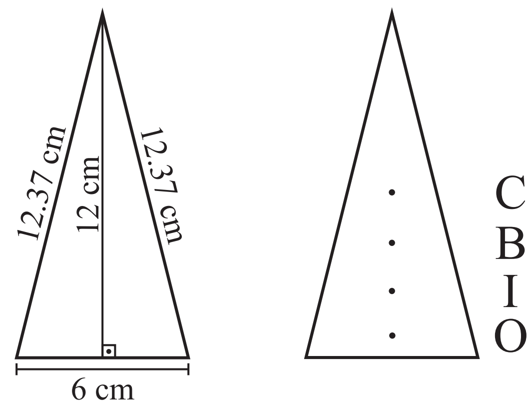


Figure 3.8: Isosceles triangle and its centers.

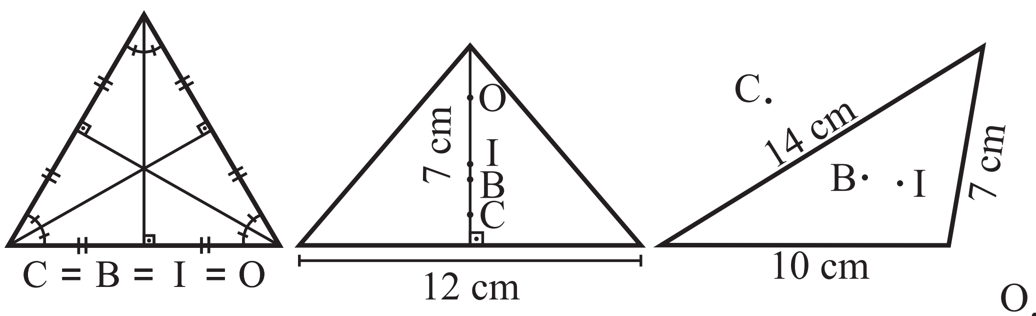


Figure 3.9: The triangle centers in some special cases.

(c) we show an obtuse triangle with sides of 7 cm, 10 cm and 14 cm. We can see that the barycenter and the incenter are inside the triangle, while the circumcenter and orthocenter are outside it.

Chapter 4

Experiments and Definitions of the Center of Gravity (CG)

Thus far we have dealt only with geometry. Now we will begin to perform experiments. The majority of the experiments described here were inspired by the excellent works of Ferreira and Gaspar, highly recommended.[89]

We will use a few primitive concepts, that is, concepts that we cannot define without avoiding vicious circles. These are: body, relative orientation of bodies (body B located between bodies A and C, for instance), distance between bodies, change of position between bodies, and time between physical events.

Experiment 4.1

We hold a coin and release it at a certain height above the ground. We observe that it falls to the ground, Figure 4.1 (a). The same happens with the pasteboard circles, rectangles and triangles.

This is one of the simplest and most important experiments of mechanics. Not all bodies fall to the ground when released in air. A helium filled balloon, for example, rises when released in air, moving away from the surface of the Earth. The direction of a thread holding a stationary party balloon filled with helium in a windless location coincides with the direction of free fall of a coin, Figure 4.1 (b).

[89] [Fersd], [Fer06] and [Gas03].

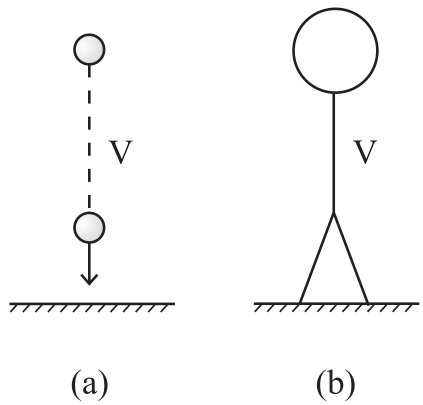


Figure 4.1: (a) A vertical straight line V is defined as the direction of free fall of a heavy body moving towards the Earth after being released at rest. (b) The vertical line V can also be defined as the direction of a thread holding a stationary party balloon filled with helium in a windless location.

4.1 Definitions

We now define a few concepts that will be employed throughout this work.

Rigid body: Any body whose parts do not change their relative orientations and distances when this body moves relative to other bodies. The triangle pasteboard, for instance, can be considered as a rigid body for the purposes of this book. Even when the triangle falls and rotates relative the surface of the Earth, the parts of the triangle remain fixed relative to one another (the distance between any two points belonging to the triangle remain constant in time). On the other hand, a cat walking on the sidewalk cannot be considered a rigid body. The distance between its feet, or between a foot and the tip of the tail, does not remain constant in time. Most experiments in the first part of this book will be performed with rigid bodies. When we say "body," normally we refer to a "rigid body," unless specified otherwise.

Motion and rest: We say that two bodies A and B are in relative motion (or rest), when the distance between any particle i of body A and any particle j of body B does (does not) change with the passage of time. In this work we will often speak of the motion and rest of a body relative to the Earth. When we say simply that a body is at rest or in motion, we normally mean that it is at rest or in motion relative to the surface of the Earth. The same should be understood for all the parts of a body in relation to all the parts of the Earth.

Equilibrium: In general, we will refer to the equilibrium of a body as its state of rest in relation to the Earth. That is, when we say that a body is in equilibrium, we mean that all of its parts remain at rest relative to the ground with the passage of time. When a triangle is in our hands, with our hands at rest relative to the ground, we say that it is in equilibrium. When the triangle is falling to the ground, it is no longer in equilibrium.

Gravity: Name given to the property which makes the bodies fall toward the surface of the Earth when released at rest. This can also be expressed by saying that gravity is the tendency of bodies to be attracted toward the center of the Earth.

Going down and going up: When we say that a body is going down (going up), we mean that it is moving toward (away from) the surface of the Earth. Instead of these verbs, we can also employ analogous terms, like descend and ascend, or fall and rise.

Above and below, superior and inferior, upper and lower, on top of and below: When we say that body A is above body B, we mean that B is between the Earth and body A. When we say that body A is below body B, we mean that A is between the Earth and body B. When we refer to the top (bottom) part of a body, we mean its part farthest (closest) to the surface of the Earth.

Vertical: Straight line defined by the direction followed by a small dense body (like a metal coin) when it falls toward the Earth due to the action of gravity, beginning at rest. It is also the straight line followed by a body which moves upward when released at rest (like a helium balloon, in a region without wind). That is, the vertical V is not an arbitrary straight line. It is a very specific straight line connected with the Earth's gravity. Here we are neglecting the influence of wind. In order to decrease the influence of air and wind, it is best to perform this experiment of free fall with small and heavy bodies, like coins, see Figure 4.1.

Horizontal: Any straight line or plane orthogonal to the vertical line.

It should be stressed that all these concepts are connected to the Earth, indicating physical properties related to the gravitational interaction of the bodies with the Earth. That is, they are not abstract or purely mathematical concepts. They are defined beginning from mechanical experiments performed at the surface of the Earth.

It is important to introduce all these concepts explicitly because they will be utilized throughout this book. Nevertheless, it should be stressed that they are idealizations which are never found exactly in nature as they were defined here. For example, no body is perfectly rigid. Even when a book is resting above a table, its molecules are vibrating. In this sense, no body is actually in equilibrium according to the previous definition of "equilibrium," as parts of this body will always be moving relative to the surface of the Earth, even when the body as a whole, macroscopically, is at rest. When we support a body from below with a stick, the body will suffer a small curvature, even if it is a metal plate. However, for phenomena at a macroscopic scale, these details (the vibration of the molecules, or the small curvature of the body) are not easily observable, or may be irrelevant for the case under consideration. For this reason the concepts already defined make sense at the macroscopic scale and should be understood as such.

4.2 Support for the Experiments

After these definitions we can go on with the experiments. We concentrate on the phenomena leading to the definition of center of gravity. To this end we will need a stand to support the plane pasteboard figures we made earlier. There are several ways to make a rigid support:

A bamboo barbecue skewer: We use children's modeling clay as a base and fix the bamboo skewer vertically, with the tip down, as in Figure 4.2. It is important to stress that the tip should be pointing toward the Earth; otherwise it will be very difficult to perform the equilibrium experiments. The bamboo skewer can also be fixed in an eraser or on another appropriate base.

A pencil as a support: We fix a pencil vertically in a sharpener, with its tip down.

A bottle as a support: If the pasteboard figures are large (with lengths of approximately 20 cm or 40 cm, which is an appropriate size for demonstrations in the classroom), you can use a glass or plastic bottle as a support, with the pasteboard resting above the cover. If the bottle is made of plastic, it will be convenient to fill it with water so that it does not tip over while we are conducting the experiments, as in Figure 4.2.

A rigid wire as a support: Another interesting possibility is to utilize a thick, solid vertical wire with a spiral base, as in Figure 4.2. If the wire is rigid but thin, it may be difficult to balance the figures horizontally above it. Moreover, the wire could pierce a hole in Styrofoam sheets, etc. As a result, a thick, rigid wire is preferred.

A nail as a support: In this case we only need a nail fixed in a cork, rubber, wooden board, or other convenient base. The head of the nail should be horizontal, with the tip of the nail fixed in the base.

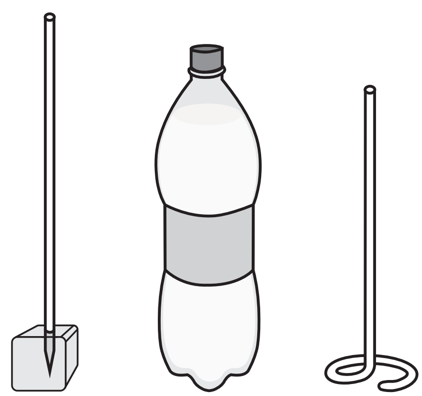


Figure 4.2: Different supports for the experiments.

There are many other possibilities. The important aspects to highlight are that the support should be rigid, firmly attached to its base. The support must be vertical, its upper end should be flat and remain in a horizontal plane. Moreover, the size of the upper end should be small compared with the dimensions of the figures to be balanced on it. However, the upper end cannot be too small (like a skewer, pin, needle, or nail with its tips facing upwards). If this happens, it becomes difficult to balance the figures, and the experiments may fail. The upper end must be small enough to ensure the body's point of equilibrium is well located, but it should not be too small, otherwise it will render many experiments unfeasible. With a little practice, it is easy to find the appropriate dimensions.

4.3 First Experimental Procedure to Find the *CG*

Experiment 4.2

We try to balance the circle, rectangle, and parallelogram pasteboards in a horizontal plane by supporting them on a vertical stand. We take the circle, for example, lay it horizontal, and place it with the support under one of its points, releasing the circle at rest. We observe that it always falls to the ground, except when the support is under the center of the circle. In all these plane figures we have analyzed, it is observed that there is a single point that must lie on the vertical support for the figure to remain horizontally stationary when released at rest. Experience teaches us that for the rectangle and parallelogram, this special point is also the center of these figures, as happened with the circle, Figure 4.3.

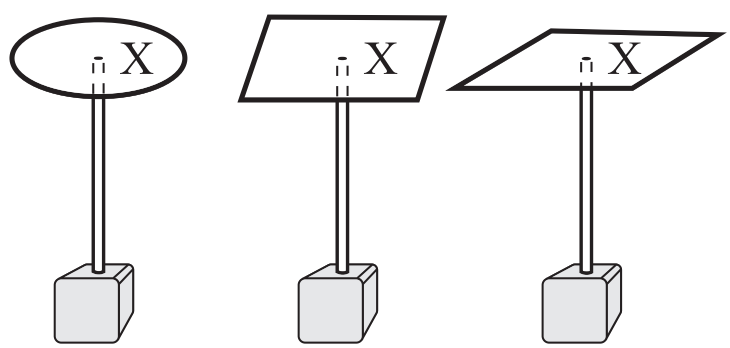


Figure 4.3: The circle, the rectangle and the parallelogram only remain at rest when the supports are below their centers.

As an historical curiosity, it is worth noting that Archimedes was the first to prove theoretically that the center of gravity of a circle is its center, and that the center of gravity of a parallelogram is the intersection of its diagonals (rectangles and squares are particular cases of parallelograms). Lemma 6 of *The Method of Mechanical Problems* stated the following:[90]

> The center of gravity of a circle is the point that is also the center [of the circle].

Proposition 9 of his work *On the Equilibrium of Plane Figures* stated:[91]

> *The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides.*

And finally, Proposition 10 of the same work stated:[92]

> The centre of gravity of a parallelogram is the point of intersection of its diagonals.

4.3.1 Provisional Definition $CG1$

These bodies only remain in equilibrium after being released at rest when the upper end of the support is under their centers. This equilibrium is connected with the Earth's gravity. Our first reaction would be to call the centers of these bodies their "centers of gravity." However, from the result of the next experiment and its analysis, we will see that this definition has to be modified. In any event, for the time being, we can say from the experiments performed thus far that only when these specific bodies are supported by their centers do they remain in equilibrium when released at rest. We thus give a first provisional definition.

Provisional Definition $CG1$: We call the center of gravity of a body its geometric center. This point will be represented by the letters CG.

Experiment 4.3

We now balance an arbitrary triangle (equilateral, isosceles or scalene) on a horizontal plane above a vertical support. As a concrete example we will consider the pasteboard isosceles triangle of base $a = 6$ cm and height $b =$

90[Arc02a, p. 15].
91[Arc02b, p. 194].
92[Arc02b, p. 195].

12 cm. This triangle has its four special centers (orthocenter, circumcenter, barycenter and incenter) well separated from one another. We utilize now a barbecue bamboo skewer as the vertical support. In this way we can locate clearly the equilibrium point of the triangle. That is, the point below which the bamboo skewer should be placed in such a way that the triangle remains in equilibrium, after placed in a horizontal plane and released at rest. Experiment teaches that the triangle always falls to the ground, except when supported by the barycenter, as in Figure 4.4. When the triangle is supported by its circumcenter, orthocenter, incenter, or by any other point (except the barycenter), it always falls to the ground after being released at rest.

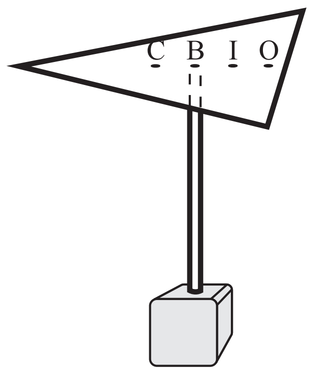


Figure 4.4: We can only balance a horizontal triangle by supporting it through its barycenter.

Once more Archimedes was the first to prove theoretically that the center of gravity of any triangle coincides with the intersection of the medians. Propositions 13 and 14 of his work *On the Equilibrium of Plane Figures* read:[93]

> **Proposition 13:** *In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side.*
>
> [...]
>
> **Proposition 14:** *It follows at once from the last proposition that the centre of gravity of any triangle is the intersection of the lines*

[93] [Arc02b, pp. 198 and 201].

drawn from any two angles to the middle points of the opposite sides respectively.

Can we say that the barycenter of a triangle is its geometric center? Does every triangle have a geometric center? In order to answer these questions we need to know what we mean by "geometric center." Intuitively we think of a geometric center as a point of symmetry of the body. In order to quantify this qualitative idea of symmetry, we can think of the center X of a rectangle. Let us consider a straight line AXB passing through X, making an angle θ with the base, and dividing the rectangle into two parts of areas A_1 and A_2, as in Figure 4.5.

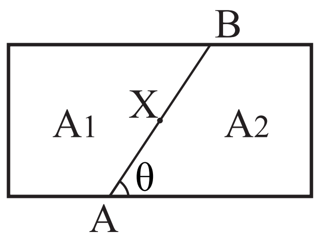


Figure 4.5: The geometric center X of a rectangle: The segment $AX = XB$ and the area $A_1 = A_2$ for any angle θ.

There are two criteria by which we can say that X is the geometric center of the rectangle. (I) The straight line AXB is always divided in two equal segments by X. That is, $AX = XB$ for every angle θ. (II) The straight line AXB always divides the rectangle into two equal areas. That is, $A_1 = A_2$ for any angle θ. These two properties will not be valid for any other point of the rectangle, only for its center X. Let P be another point of the rectangle. A straight line APB may be divided in two equal segments $AP = PB$ when it is inclined by a specific angle θ_I relative to the base of the rectangle, but this will no longer be valid when we change the angle θ_I. Another straight line CPD may divide the rectangle into two equal areas when it is inclined by an angle θ_{II} relative to the base of the rectangle. But once again, this will not be valid when we change the angle θ_{II}. We then conclude that the rectangle has only a single geometric center, the same being true of a circle and some other symmetric figures, such as a parallelogram or an ellipse.

On the other hand, criteria (I) and (II) in the previous paragraph will not be true for any point P of a given triangle. That is, given an arbitrary triangle, there is no point P_I belonging to it such that all straight lines passing through P_I will satisfy criterion (I). Moreover, there is no point P_{II} belonging to it such that all straight lines passing through P_{II} will satisfy criterion (II). In this sense we can say that no triangle has a geometric center. On the other hand, every triangle has four special centers (circumcenter, barycenter, orthocenter, and incenter).

In order to illustrate this fact we consider the isosceles triangle $V_1V_2V_3$ with base a and height b, Figure 4.6. The area of this triangle is $ab/2$. The median connecting the center of the base to the vertex V_2 is divided into two parts of equal lengths by a point P located at a distance $b/2$ from the base and from the vertex V_2. A straight line parallel to the base, passing through P and limited by the sides of the triangle is also divided into two parts of equal lengths by P. On the other hand, the straight line V_1PQ (where the point Q is the intersection of the side V_2V_3 with the extended line V_1P) is not divided into two parts of equal lengths by P. That is, criterion (I) is not satisfied by P. The same is true for any other point on the triangle.

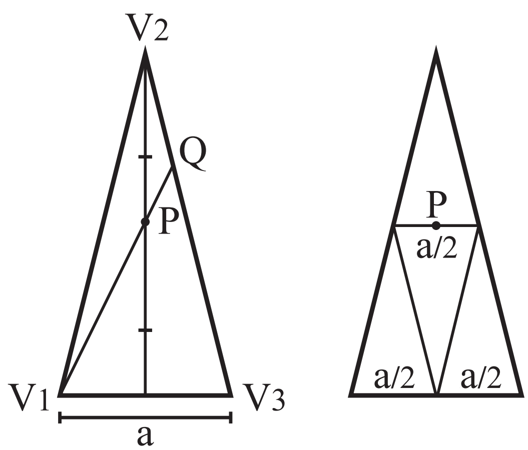


Figure 4.6: Criteria (I) and (II) will not be true for any point P of a triangle.

Nor is criterion (II) satisfied by P. The straight line passing through V_2 and P divides the triangle into two parts of equal area. On the other hand, the straight line parallel to the base and passing through P does not divide the triangle into two parts of equal areas, as in Figure 4.6. The upper triangle has only a quarter of the total area, while the lower trapezium has three quarters of the total area.

The barycenter B is located at a distance $b/3$ from the midpoint of the base and at a distance $2b/3$ from the upper vertex. This shows at once that it does not satisfy the previous criterion (I). The extended straight lines connecting B to any one of the vertices divide the triangle in two parts of equal areas. But this will not be the case, for example, for a straight line parallel to the base and passing through B, as in Figure 4.7.

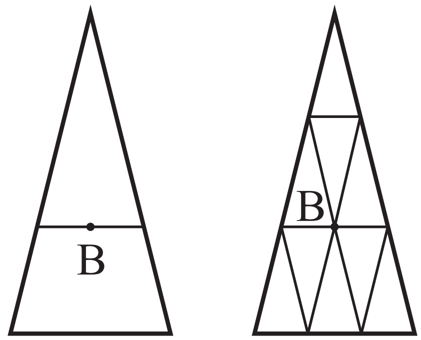


Figure 4.7: A straight line parallel to the base and passing through the barycenter divides the triangle into two figures having different areas.

In this case the area of the upper triangle is equal to 4/9 of the total area, while the area of the lower trapezium is equal to 5/9 of the total area. In order to confirm this without performing the calculations, all we need to do is to cut out nine equal isosceles triangles, each with a base of $a/3$ and height of $b/3$ (area of $ab/18$). We can fill the superior triangle with four of these small triangles, and the inferior trapezium with five of these small triangles, as in Figure 4.7.

Even the most symmetrical triangle, the equilateral triangle, has no geometric center that satisfies criterion (I) or criterion (II). In this case the four special centers coincide at the barycenter B of the triangle. And we just saw that the barycenter of an isosceles triangle does not satisfy any of these criteria. As the equilateral triangle is a particular case of an isosceles triangle, it follows automatically that the barycenter of an equilateral triangle will not satisfy any of these criteria. Nevertheless, we can say that the equilateral triangle has a center of symmetry given by $C = B = O = I$. Although this point does not satisfy criteria (I) and (II), there is symmetry of rotation around an axis orthogonal to the plane of the triangle and passing through this point. That is, any characteristic of the triangle is repeated with a rotation of 120° around this point. For this reason it is possible to say that the barycenter of an equilateral triangle is its center of symmetry, yet not its geometric center.

We then conclude that a triangle has no geometric center defined according to criteria (I) and (II). Nevertheless, experience teaches us that every triangle can be balanced horizontally when supported by a thin vertical rigid stand placed under its barycenter, but not when we place the support under any other point of the horizontal triangle.

4.3.2 Provisional Definition $CG2$

The discussion of Subsection 4.3.1 suggests that we should change our previous definition of the center of gravity. We now give a second, more precise provisional definition of a CG.

Provisional Definition $CG2$: The center of gravity is a certain point in the body such that if the body is supported by this point and released at rest, it remains in equilibrium relative to the Earth.

Later on we will need to change this definition yet again for a more general concept. But for the time being it is a suitable definition. From the experiments performed thus far it follows that any body has only a single point satisfying this definition. If the body is released at rest when supported by any other point, it does not remain in equilibrium, but falls to the ground. For circles and parallelograms the CG is the center of these bodies, while for the triangles it coincides with the barycenter.

Another way of looking at the CG has to do with the weight of the bodies. Later on in this book we will quantify this concept and show how it can be measured. But we all have an intuitive notion of the weight of a body as a quantitative measure of the gravitational force. We say that body A is heavier than body B when it is more difficult to keep A at a certain height from the ground than it is to keep B at rest at the same height. This difficulty can be indicated by our sweat, by the fatigue we feel in an outstretched arm, or by the deformation created by bodies A and B upon the body supporting them (in the case of a flexible support like a spring, for example).

4.3.3 Provisional Definition $CG3$

In the previous Figures we saw that the whole weight of the circle, rectangle, parallelogram or triangle was supported by the bamboo skewer placed at a single point below each of these bodies. We can then give a new provisional definition of the CG.

Provisional Definition $CG3$**:** The center of gravity of a body is the point of application of the gravitational force acting upon it. That is, it is the point of the body upon which all the gravity acts when it is at rest, the point where the weight of the body is located or concentrated. It can also be called the center of weight of a body.

As we have seen, a triangle has no geometric center. This leads to an important conclusion which will be explored in the following experiment.

Experiment 4.4

We have seen that not all straight lines passing through the barycenter of a triangle divide it into two equal areas. As we are dealing with homogeneous plane figures, the weight of any part is proportional to its area. This fact suggests a very interesting experiment. We cut out a pasteboard triangle with base $a = 6$ cm and height $b = 12$ cm. The barycenter is located along the median connecting the superior vertex to the midpoint of the base, at a distance $2b/3$ from the superior vertex. We can then cut this triangle in two parts with a pair of scissors, cutting a straight line parallel to the base and passing through the barycenter. We then connect the two parts only by the central region around the old barycenter utilizing a small piece of pasteboard. Alternatively, we can remove two narrow strips parallel to the base on either side of the barycenter, keeping only a small region around the barycenter, as in Figure 4.8.

We then try to balance this figure horizontally above a vertical support. We observe that the body only remains balanced horizontally when the support is placed below the barycenter. That is, although the area and weight of the trapezium are larger than the area and weight of the small triangle (which goes from the superior vertex to the straight line passing through the barycenter), the system as a whole remains in equilibrium. If these two parts were not rigidly connected, each one of them would fall to the ground after being released. We then conclude that the CG is not, necessarily, the point which divides the body in two equal areas or in two equal weights. We will discuss this aspect in more detail in Section 8.2.

Experiment 4.5

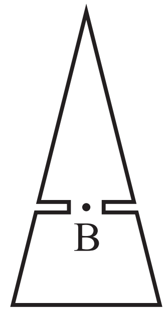


Figure 4.8: Although the upper triangle and the lower trapezium have different areas and weights, the figure can be kept in horizontal equilibrium by a support placed below the barycenter.

There is another way to perform this experiment without cutting the larger triangle. We take the original triangle of base a and height b and balance it horizontally by placing the triangle on the edge of a ruler in the vertical plane. The edge of the vertical ruler should be parallel to the base of the triangle, placed below its barycenter. The extended vertical plane passing through the ruler divides the triangle into two different areas, that is, into two different weights. Nevertheless, the triangle remains in equilibrium when supported by this ruler, as in Figure 4.9.

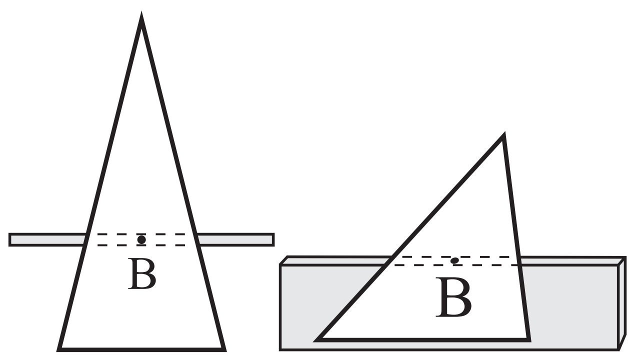


Figure 4.9: The horizontal triangle remains in equilibrium above a horizontal ruler placed below its barycenter.

4.4 Experiments with Concave Bodies or Bodies with Holes

We now cut out some concave figures, such as a letter C, a first quarter Moon, a boomerang, etc. Some pierced bodies should also be prepared, such as a washer (a metal washer is easily found). To cut the interior circumference of a pasteboard washer, a radial cut can be made between the exterior and interior circles. But with a pair of pointed scissors this is unnecessary. The outside diameters of these figures can be 8 cm or 10 cm, for instance. The interior diameters can be of the order of 4 cm or 6 cm. But these dimensions are not so important. For the following experiments you will need to cut out at least two equal figures of each model (two letters C of the same size and shape, two Moons, two washers, etc.). One set of these figures will be used in the next experiment, while the other set of identical figures will be used in later experiments (with sewing threads attached to these figures with adhesive tape).

Experiment 4.6

We try to balance these figures in a horizontal plane by placing them above a vertical support, as we did with the triangle. We observe that we cannot balance any of them. They always fall to the ground, no matter where we place the support, as in Figure 4.10 (a).

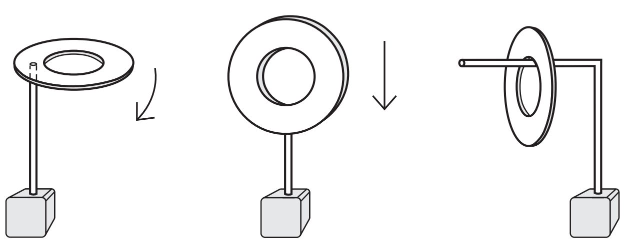


Figure 4.10: (a) The washer falls to the ground when we try to support it horizontally by placing its metal part above a vertical stand. (b) It also falls to the ground when we try to support it vertically by an edge. (c) We can support it by a horizontal bamboo skewer passing through its hole.

Even when we try to balance them on an edge, by placing the figures in a vertical plane above the support, we do not succeed; they always fall to the ground, as in Figure 4.10 (b).

The only way to balance them is to hold the bamboo skewer horizontally and the figures in a vertical plane, with the bamboo skewer passing through a hole in the bodies, or supporting the concave part of the figures, as in Figure 4.10 (c).

There are different ways to analyze this experiment. The first is to conclude that some concave bodies or bodies with holes do not have a specific center of gravity, but do have an entire line of gravity. The washer, for instance, can remain balanced in a vertical plane when supported by any point belonging to its interior circumference. On the other hand it cannot be balanced when the vertical bamboo skewer is placed exactly at the empty center of the washer, which is its geometric center. If we follow definition $CG2$ rigorously, we must say that the washer has a line of gravity, namely, its interior circumference. In this case we cannot say that it has a specific CG located at a single point.

The same thing can be said of definition $CG3$. After all, the horizontal bamboo skewer in Figure 4.10 (c) is holding the whole weight of the vertical washer supported at a point on the interior circumference of the washer. But a vertical bamboo skewer cannot support a horizontal washer when the end of the bamboo skewer is located at the empty center of the washer. If we follow definition $CG3$ rigorously, we should say that the washer has a line of weight or a line of gravity, but does not have a specific point that could be called its CG.

Another way to analyze this experiment is to say that the CG does not need to be located "in the body." That is, we may say that it does not need to be located at any point coinciding with a material part of the body. The CG might then be located in empty space, at a point in some definite spatial relation with the body (like the geometric center of the washer, for instance), even though not physically connected to the body.

If we adopt this second alternative option, we will need to change our definition $CG2$. We will also need to find another way of experimentally locating the CG in these special cases, as presented in the next experimental procedure.

Experiment 4.7

We attach two taught sewing threads to the washer with adhesive tape, as if they were two diameters intersecting each other at the center of the washer. Now we can balance the washer in a horizontal plane by placing the vertical support under the intersection of the threads, as in Figure 4.11.

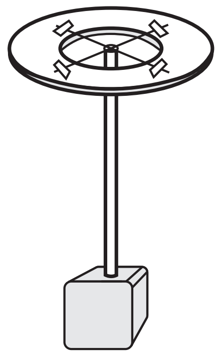


Figure 4.11: The washer can be supported by its center utilizing two stretched sewing threads.

We can also find a similar point for the Moon or for letter C, by trial and error, that is, the intersection of two taught threads attached to the figures so that they remain in equilibrium horizontally when the support is placed vertically under this intersection.

4.4.1 Provisional Definition $CG4$

If we accept this second alternative, we need to generalize our definition $CG2$ to include these special cases. A more general definition is the following:

Provisional Definition $CG4$**:** We call the center of gravity of a body a point in the body or outside it such that, if the body is supported by this point and released at rest, it remains in equilibrium relative to the Earth. When this point is located outside the body, a rigid connection must be made between this point and the body in order for the body to remain in equilibrium when released at rest.

This definition has a problem. After all, when we make this material rigid connection (like the taught threads attached with adhesive tape) the original body has been modified. But provided the weight of this material connection is small in comparison with the weight of the body, this is a reasonable procedure.

Even so there is another problem with this definition, as we will see in the next experiments.

Experiment 4.8

We now attach two loose threads of the same length to the washer with adhesive tape. The length of each thread should be greater than the external diameter of the washer. They are attached as in the previous experiment, at the same locations, that is, the straight line joining the two pieces of adhesive tape attaching each thread passes through the center of the washer. The only difference is the length of the threads. In this case we can also balance the whole system on a support. But now the intersection of the two threads touching the support is along the axis of symmetry of the washer, as in Figure 4.12. It is no longer located at its geometric center.

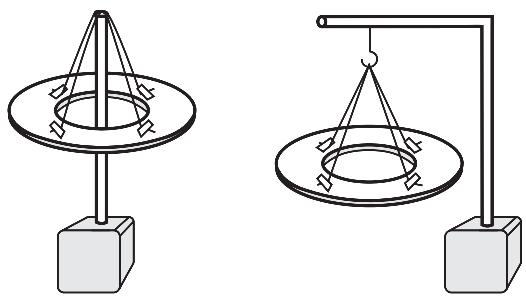


Figure 4.12: The washer can also be supported along its axis of symmetry utilizing loose threads.

If we follow the second alternative discussed previously (where the CG does not need to be located in the body, and could be located in empty space), we must conclude that the washer has not just one center of gravity, but an infinite set of them located along its axis of symmetry. That is, the whole axis of symmetry of the washer might be called its "axis or line of gravity." This would be true not only according to definition $CG3$, but also according to definition $CG4$.

Experiment 4.9

Definition $CG3$ has also problems with concave bodies or bodies with holes. According to this definition, the CG is the point of application of the gravitational force, that is, the point where gravity acts. The problem with this definition is that we normally consider gravity to be a force acting upon material bodies due to an interaction between the body and the Earth. It would be difficult to say that the point of application of the gravitational force on a washer was acting on the empty space where its geometric center is located. The force of the Earth cannot act on empty space. As a result, definition $CG3$ will have to be modified.

One way to illustrate this fact that the weight does not in fact acts on empty space, only on the bodies themselves, is shown in the next experiment.

Experiment 4.10

We now support the washer from above, as in Figure 4.13. We pass a skewer through its center and no force acts upon the skewer. That is, when the skewer reaches the center of the washer, no force will act upon it. And no force will act upon the skewer when it passes through the center of the washer.

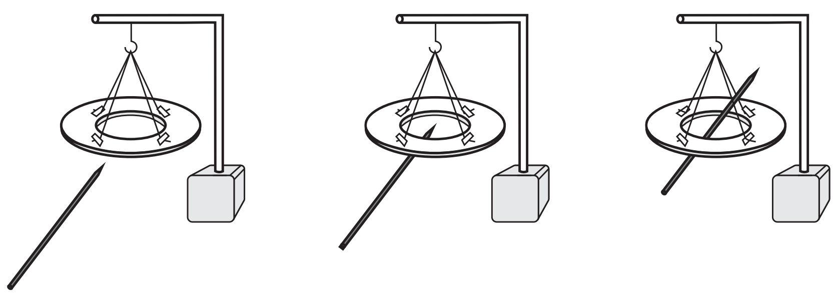


Figure 4.13: No force acts upon a skewer when it passes through the center of a washer supported from above.

Instead of utilizing a skewer, we can also pass a spring through the center of the washer. We observe that no force will acts upon the spring. That is, the spring will not be compressed nor stretched when passing through the center of the washer in the situation for which the washer is supported from above, Figure 4.14.

This experiment shows that, in reality, the whole weight of the washer is not acting at its geometric center. The fact that the weight of a body only acts on its material portions can also be seen in the next experiments.

Experiment 4.11

On the other hand, when a vertical skewer moving upwards touches a metal portion of a horizontal washer suspended from above, as in Figure

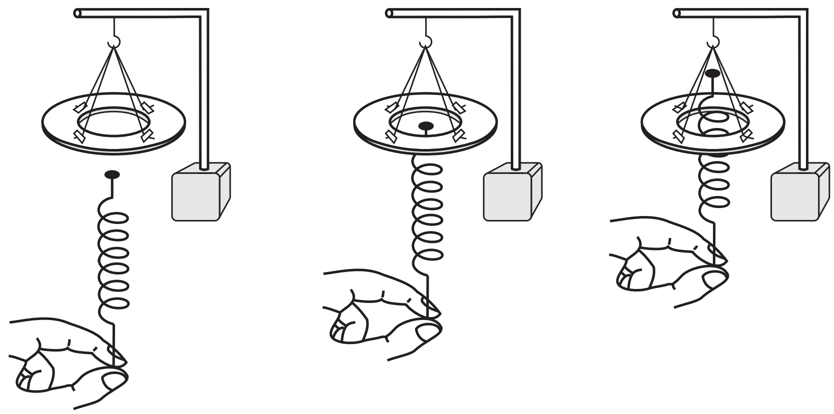


Figure 4.14: No force acts upon a spring when it passes through the center of a washer supported from above.

4.15, the washer and the skewer act on one another. This portion of the washer moves upwards, while the washer exerts a downward force on the skewer.

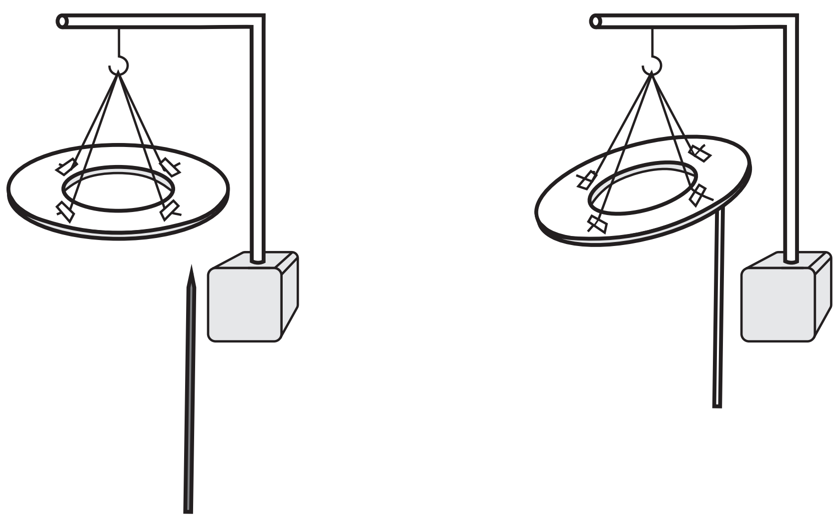


Figure 4.15: A downward force acts upon a skewer (or upon a spring, compressing it) when the skewer moves upwards a metal portion of a washer supported from above.

If we replace the skewer by a vertical spring, we observe that the spring is compressed after touching a metal portion of the washer, pressing it upwards.

Experiments 4.10 and 4.11 show that there is no real force acting on the empty center of a washer, while the real weight of the washer due to its gravitational interaction with the Earth acts on its material parts, that is, upon its metal components.

Another problem with definition $CG3$ appears in the next experiment.

Experiment 4.12

As we will see along this work, the center of gravity of a washer is its geometric center. In the present experiment we release the washer horizontally. Below the center of the washer we place a vertical skewer aligned with the washer's axis of symmetry. No force is exerted upon the skewer even when the center of the washer passes through the upper extremity of the skewer. That is, even when the CG of the washer passes through the skewer, the skewer is not compressed by the washer, Figure 4.16.

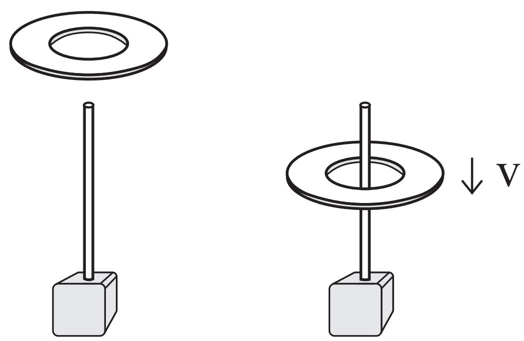


Figure 4.16: A vertical skewer is not compressed when the center of a horizontal washer passes through the upper extremity of the skewer. Here V is the velocity of the washer.

The same happens when we utilize a vertical spring instead of a vertical skewer, Figure 4.17. That is, the spring is not compressed when the washer passes through it.

Experiment 4.13

Suppose that now we have 3 vertical skewers placed side by side. We place them in such a way that their vertical projections coincide with the metal parts of the horizontal washer, as in Figure 4.18. We then release

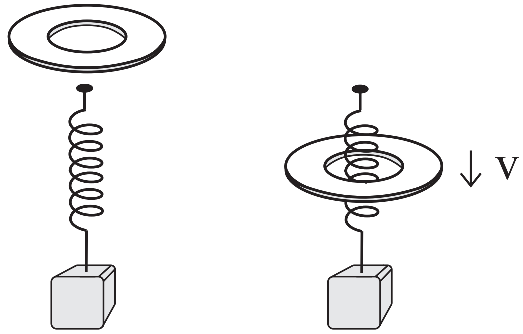


Figure 4.17: A vertical spring is not compressed when the center of a horizontal washer passes through the upper extremity of the spring. Here V is the velocity of the washer.

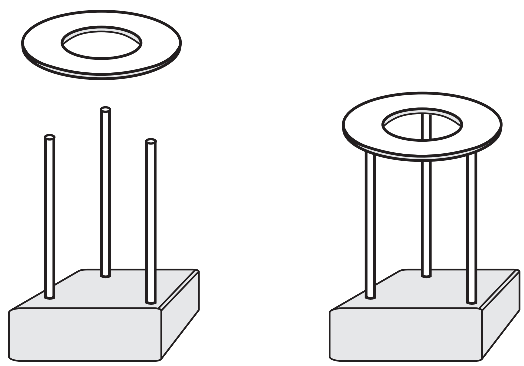


Figure 4.18: The three vertical skewers are compressed when the washer touches them, decreasing the downward velocity of the washer. The skewers remain compressed while the washer remains at rest above them.

the horizontal washer above the skewers. It is observed that they are compressed when the washer touches them, decreasing its downward velocity. They remain compressed as long as the washer is stationary on them.

The same happens utilizing a system of 3 vertical springs instead of 3 vertical skewers, Figure 4.19. That is, the springs are compressed when the horizontal washer touches them, decreasing its downward velocity. They remain compressed while the washer is stationary above them.

One of the possible interpretations of these experiments is that in reality there is no effective weight acting at the empty center of the washer in free fall, although this empty center is in fact its center of gravity, as will be seen

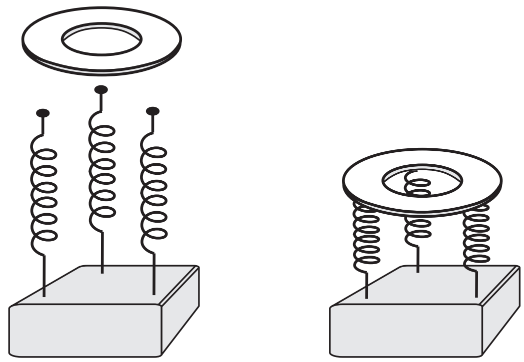


Figure 4.19: The three vertical springs are compressed when the washer touches them, decreasing the downward velocity of the washer. The springs remain compressed while the washer remains at rest above them.

in this work. This interpretation of these experiments contradicts definition $CG3$. According to this interpretation, the weight would be effectively acting at the metal parts of the washer. That is, the weight only acts at the metal components of the washer, but not at its empty center.

4.4.2 Provisional Definition $CG5$

For the reasons discussed in Subsection 4.4.1, we need to replace definition $CG3$. The new definition might, for instance, go something like the following.

Provisional Definition $CG5$**:** The center of gravity is one point inside or outside the body which behaves as if all gravitational force were acting at this point. When this point is located outside the body, there must be a rigid connection between this point and the body in order to perceive or measure the entire gravitational force acting at this point.

This is a very reasonable definition. The greatest difficulty we encounter when dealing with it is how to locate this point. Let us consider, for instance, the washer with loose threads, Figure 4.12. It is supported by four pieces of adhesive tape attached to it. These pieces of tape are supported by two taught threads. These threads, meanwhile, are supported at their intersection by a base or hook. That is, the washer behaves as if all its weight were supported along its axis of symmetry, at the intersection of the threads, away from the geometric center of the washer, provided we utilize threads attached to the washer. So it would make more sense to talk of a line of gravity, or a line of weight, instead of a center of gravity or a center of weight. In the present case this line of gravity, or this line of weight, would be the straight line coinciding with the axis of symmetry of the washer.

Analogous difficulties with definition $CG5$ happen in Experiments 4.9 and 4.12.

In the next experiments we will see another problem that arises even with the more general definitions $CG4$ and $CG5$.

4.5 Experiments with Three-Dimensional Bodies

Thus far we have performed experiments with "plane" figures, or two-dimensional bodies. However, every material body is three-dimensional. When we say that a figure is plane or two-dimensional, we mean that its thickness is much smaller than the other dimensions involved in the problem. The thickness d of the pasteboard rectangle, for example, is much smaller than the length of its sides a and b, that is $d \ll a$ and $d \ll b$. We now perform experiments with bodies in which all three dimensions are of the same order of magnitude.

The bodies we will consider are a cube or die with plane faces, a sphere, a hexagonal metal screw-nut and an egg. For lighter bodies we use children's modeling clay and the barbecue bamboo skewer as support. For the egg (and other heavy spheres) we can use the table as a support, since it only touches the table in a small region due to its convex shape at all points of the egg.

Experiment 4.14

We release these bodies upon a horizontal support and observe the points at which they remain in equilibrium. In the case of the cube we find six points of equilibrium, namely, the centers of the faces, as in Figure 4.20 (a).

For the metallic screw-nut we also find six points of equilibrium, the centers of the six external faces. Moreover, using the procedure involving the intersection of sewing threads (which we used previously with the washer), it can be shown that all points along the axis of symmetry are also points of equilibrium of the screw-nut. It also remains balanced by any point along

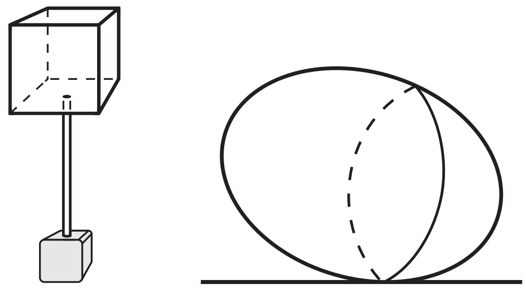


Figure 4.20: A cube and an egg.

the internal circumference or cylinder surface if the barbecue bamboo skewer is fixed in a horizontal position, as was the case with the washer shown in Figure 4.10 (c).

The sphere remains in equilibrium at all points of its surface. Therefore it has an infinite number of equilibrium points.

The most interesting case is that of the egg, which has a whole line of equilibrium. This line forms a circumference on the shell, such that the plane of this circumference is orthogonal to the axis of symmetry of the egg, as in Figure 4.20 (b).

From this experiment we conclude that many geometric bodies have more than one center of gravity if we follow definitions $CG2$, $CG3$, $CG4$ or $CG5$. The cube, for instance, would have six centers of gravity, the egg would have a whole line and the sphere its entire surface. The screw-nut would have six of these centers, the centers of the external faces, in addition to its internal circumference and to all points along its axis of symmetry. In order to be consistent with this discovery, we should talk of points, lines or surfaces of gravity, instead of speaking of a single "center" of gravity for each body.

4.6 Plumb Line, Vertical and Horizontal

Fortunately there is another experimental procedure involving gravity with which we can find a single and specific point in each rigid body related to its condition of equilibrium relative to the Earth. By utilizing the second experimental procedure which will be described in Section 4.7 we can obtain another definition of the CG which avoids the previous problems and which has a relevant physical meaning. As this new procedure employs a plumb line, we first explain this instrument.

We begin presenting some definitions.

Plumb line: A plumb line is a string with a weight (or plumb bob), used to provide a vertical reference line. This is the name given to any thread fixed at its upper end (this end remains at rest relative to the Earth) and which has a heavy body fixed at its lower end. The plumb line must be free to oscillate around the extreme upper end, Figure 4.21.

The point of suspension, represented in some Figures by the letters PS: Point at which the body is hanging or suspended from above, as we will see in the next experiments. Often in our experiments the point of suspension will coincide with the location of the pin or needle holding the body and the plumb line.

The lower support or auxiliary point, represented in some Figures by the letters PA: This is the upper end of a support on which the body rests, being supported from below. It can be, for instance, the upper end of the bamboo skewer used as a support below the bodies in some of the previous experiments.

The upper end of the plumb line can be held by our fingers, or tied to a bar or a hook, etc. In our experiments we will fix this top end to a rigid support at rest relative to the Earth. We stick a pin or needle into the upper part of our bamboo skewer placed in a vertical position, as in the experiments performed earlier. On the pin we will hang pierced pasteboard figures and also a plumb line. The plumb line will be a sewing thread with a weight at the bottom. We could simply tie or fasten it to the pin, but we will need to remove and replace the plumb line on the pin several times. Therefore it is best to make a small loop at the top of the thread. At the bottom of the thread we tie a plumb or a piece of modeling clay. The device to be used in the experiments is shown in Figure 4.21.

One of the advantages of this device is that it allows us to repeat the previous experiments in which we supported pasteboard figures horizontally on a vertical bamboo skewer. In order to avoid the hindrance of the pin (touching the figure placed horizontally above it), the pin should be stuck in a little below the end of the bamboo skewer.

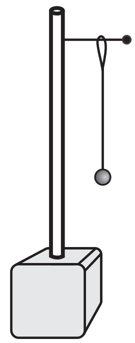


Figure 4.21: Plumb line.

Sometimes we will also support some plane figures vertically on the pin, through holes in these figures. For this reason the pin should not be perfectly horizontal, but inclined with its head a little above the point stuck in the bamboo skewer, in order to prevent the pasteboard figures from sliding off.

If we wish to perform experiments only with the plumb line, a bamboo skewer should be tied to it, in a horizontal position. In this way we avoid the pins, which can be dangerous if we perform these experiments with children. The bamboo skewer is laid on a table, half of it extending beyond the table. The part on the table is kept in its place by a book or other weight on it. The plumb line hangs from the part of the bamboo skewer outside the table, free to oscillate, as shown in Figure 4.22. The pierced pasteboard figure will also hang from the bamboo skewer, instead of being suspended by the pin.

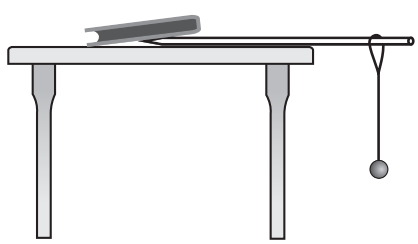


Figure 4.22: Plumb line.

Another practical alternative is to use a thread or string tied to a bar or broomstick fixed in a horizontal position.[94] At the bottom of the thread we attach a hook on which the flat figure will be hung (passing the hook through

[94] [Gas03, p. 138].

the hole made in the cardboard figure). The plumb line will also be hung on the hook, as in Figure 4.23.

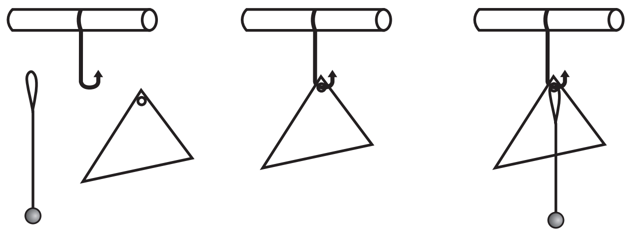


Figure 4.23: Support with hook, plumb line and pierced pasteboard figure.

Experiment 4.15

We hang the plumb line from the support and wait until it reaches equilibrium. Then we release a coin at rest close to the plumb line. We observe that the direction of fall is parallel to the plumb line, as in Figure 4.24.

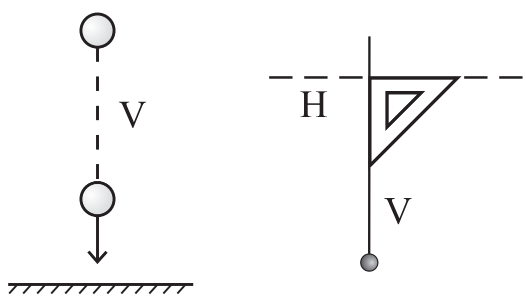


Figure 4.24: Finding the vertical (V) and the horizontal (H) with a falling body or with a plumb line.

This is the main function of a plumb line. When it is at rest relative to the Earth, it indicates the vertical direction. In this sense it is a better indicator than a falling body, as it has a visible line, permanent and stable (when there is no wind blowing, etc.). Bricklayers often use a plumb line (a small weight tied to a string) to determine if a wall being built is vertical or not. To do this, they place the plumb line next to the wall and check if the plane of the wall is parallel to the plumb line.

There are three principal methods to find the horizontal direction.

A) We first find the vertical, V. This can be obtained by the free fall of a heavy body, or with a plumb line. Then we place a large T-square parallel to the plumb line. The direction orthogonal to the line indicated by the T-square is then, by definition, the horizontal direction, H, as in Figure 4.24.

B) With a spirit level. Usually it has the shape of a small parallelepiped with an internal cylindrical transparent vessel containing a liquid with a bubble. There are two straight marking lines along the axis of the cylinder, symmetrically located relative to the center. The spirit level is placed on a surface. When the bubble remains in the middle of the two marks, the surface is horizontal, as in Figure 4.25 (a).

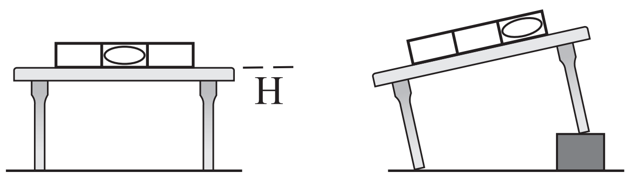


Figure 4.25: Finding the horizontal with a spirit level.

When the bubble remains at one of the ends of the vessel, the surface is not horizontal, Figure 4.25 (b). The side where the bubble is located is higher than the opposite side. The spirit level works due to the action of gravity and the upward thrust exerted in a fluid (the principle of Archimedes).

C) We use a transparent hose open at both ends and partially filled with a liquid, such as water. The hose is kept stationary relative to the ground, and one waits for the liquid to also comes to rest. The straight line connecting the two free surfaces of liquid indicates the horizontal direction, as in Figure 4.26. The operation of this hose is based on the equilibrium of liquids under the action of gravity.

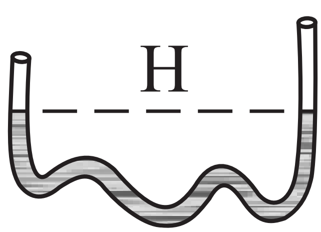


Figure 4.26: Finding the horizontal with a transparent hose open at both ends.

As a curiosity it is worth mentioning here how bricklayers build orthogonal walls. After finishing a wall, they mark two points on it 4 m apart horizontally, A and B. The first point, A, is at the end of the wall where the other wall is to be built. Next they find a third point C such that the distance between A and C is 3 m and the distance between B and C is 5 m. The straight line connecting AC is then orthogonal to the straight line connecting AB, as in Figure 4.27. Instead of these specific distances, any multiple of them can be used (30 cm, 40 cm and 50 cm, for instance). The principle behind this method is the theorem of Pythagoras. That is, in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other sides. And a triangle with sides 3 m, 4 m and 5 m satisfies this theorem. The same holds for a triangle with sides of lengths proportional to these numbers.

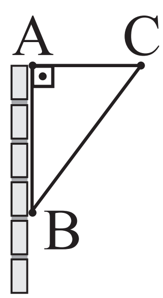


Figure 4.27: A practical procedure to build orthogonal walls.

4.7 Second Experimental Procedure to Find the CG

The first method to find the center of gravity was described in the previous experiments of balancing circles, parallelograms and triangles horizontally above a vertical bamboo skewer. This is the simplest and most intuitive way to understand the meaning of the center of gravity. With this procedure we can also perceive that it is a single point for each body. The experiments show that these bodies only remain in equilibrium when supported by a single point called the CG. But there were conceptual problems with this approach, as we saw before. We return to these geometric figures and perform another set of experiments.

We now present the second method to find the CG of these figures, which avoids the problems already presented. We use plane pasteboard figures of the same shape and size as before. But now we make two or three holes in each figure with nails or a single-hole punch-pliers, Figure 4.28.

Figure 4.28: A single hole punch-pliers.

The diameters of the holes should be small compared with the dimensions of the figures (so that they will not change the weight of the figures appreciably), but large enough for these figures to hang freely on the pin or hook. That is, the friction between the pin and the figures should be very small, such that the figures can oscillate freely around the pin. Single-hole punch-pliers are very practical and work very well with pasteboard figures with dimensions larger than 5 cm. The circular holes they make allow the figures to swing freely when they hang by a pin or even on a horizontal barbecue bamboo skewer.

Experiment 4.16

We make a small hole in a pasteboard circle equal to the one used before. The hole should be made in an arbitrary position which does not coincide with the center of the circle. We then hang this circle on a pin stuck into a vertical bamboo skewer. That is, with a horizontal pin the plane of the circle will be vertical. The location of the pin will be represented in the next Figures by the letters PS, indicating that it is the point of suspension. The plumb line is also placed on the pin. We wait until the plumb line reaches equilibrium, remaining at rest relative to the Earth. The circle is then released at rest. Experience shows that it does not remain at rest in all positions from which it is released. It only remains at rest after being released when the circle is in a special orientation which will be called here a *preferential position* or *preferred configuration*. In this preferential position the center X of the circle is *vertically below* the pin, as indicated by the plumb line, Figure 4.29 (a). That is, if the circle is released at rest from this position, it remains in equilibrium.

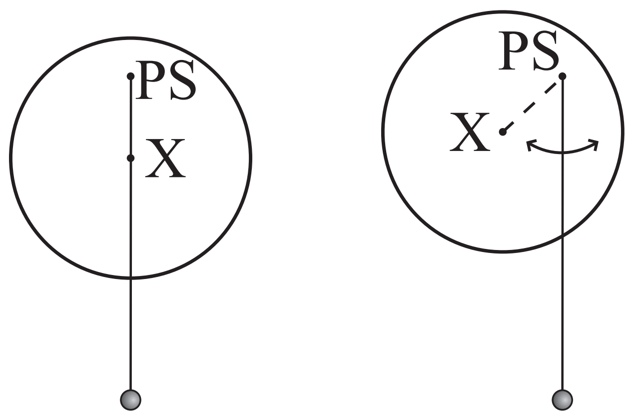


Figure 4.29: (a) The circle remains at rest when released in the preferential position. (b) When it is not released in the preferential position, its center will oscillate around the vertical passing through the point of support PS, until it stops in the preferential position due to friction.

If the circle is released at rest with its center outside the vertical passing through the pin, we observe that the center of the circle swings around this vertical, Figure 4.29 (b). After a few oscillations, the circle stops in the preferential position due to friction.

When the circle stops its oscillations, it is observed that its center X remains *vertically below* the pin.

Instead of hanging the circle on the pin, we could also tie the pierced circle with a thread passing through its hole. The upper end of the thread is then attached to a support above the circle. We again observe the same phenomena as before, provided the circle is free to oscillate around the thread. That is, the downward extended vertical along the thread will pass through the center of the circle when it reaches equilibrium.

We can now present the second experimental procedure to find the CG.

We consider a pasteboard circle with two or three small holes pierced in arbitrary locations. We hang it with the pin passing through one of its holes, and release the circle at rest. Normally it oscillates and reaches equilibrium.

We hang a plumb line by the pin and wait for the whole system to reach equilibrium. We then use a pencil to draw a straight line in the circle coinciding with the vertical indicated by the plumb line. We represent this straight line by the letters PS_1E_1, where PS_1 is the point of suspension indicated by the pin (these letters should be written at the side of the first hole) and E_1 is the bottom end of the body along this vertical, as in Figure 4.30 (a).

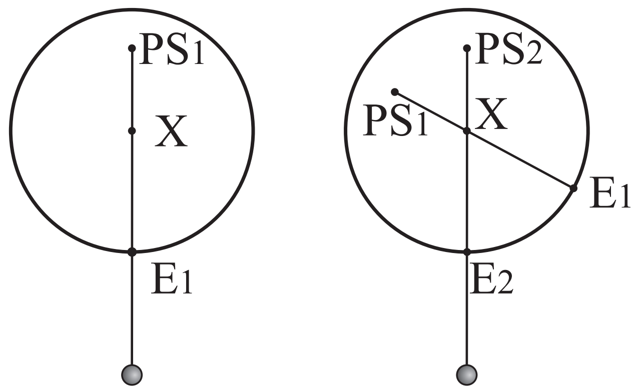


Figure 4.30: Second experimental procedure to find the CG of a circle.

The plumb line and the circle are removed from the pin. We now repeat the procedure, this time hanging the circle by a second hole PS_2. This second hole must be outside the straight line PS_1E_1. We hang the plumb line, wait for the system to reach equilibrium and draw a second vertical PS_2E_2, as in Figure 4.30 (b).

Experience shows that the two straight lines PS_1E_1 and PS_2E_2 intersect at a point which coincides with the center of the circle. If we repeat the procedure by hanging the circle and plumb line by a third hole PS_3, the third vertical PS_3E_3 will also pass through the center of the circle. It is convenient to draw three or more lines like this in order to find the point of intersection with greater precision. This procedure also shows that all verticals intersect at a single point.

But this coincidence of all points of intersection is not always perfect. One reason for this fact is the friction that always exists between the circle and the plumb line while the system is oscillating, before reaching equilibrium. Sometimes this friction prevents the plumb line from reaching a vertical direction when at rest, as the line can stick on irregularities in the pasteboard. But the main reason for a lack of coincidence of all points is the difficulty in drawing the verticals upon the figure to coincide with the plumb line. We have to attach the thread with our fingers in order to draw the lines. At this moment we could change the real direction indicated by the plumb line very slightly.

But with a little practice and patience we can optimize this procedure. Then we can say with certainty that all verticals intersect at the center of the circle. Remember that we are considering holes of small diameters as compared with the size of the figure. This means that these holes do not significantly disturb the weight of the figure.

Experiment 4.17

This procedure is repeated with a rectangle and a parallelogram, by piercing two or three small holes in each figure. The verticals are drawn and we observe that their intersections coincide with the centers of these figures, as in Figure 4.31 (a) and (b).

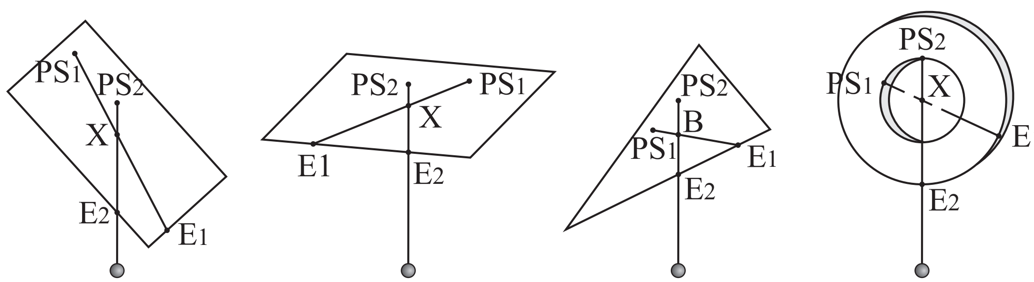


Figure 4.31: Second experimental procedure to find the *CG* of a rectangle, of a parallelogram, of a triangle and of a washer.

By repeating the same procedure with a triangle we find that the intersection of the verticals coincides with the barycenter of the triangle, as in Figure 4.31 (c).

Experiment 4.18

We can repeat the procedure with a pasteboard washer by piercing two or three holes in it and hanging it by a pin. Alternatively the washer can be hung by its interior circumference, keeping the washer in a vertical plane. We then hang the plumb line by the pin and draw the first vertical line. By repeating the procedure with another point along the internal circumference, we find that the intersection of the verticals coincides with the center of the washer, as in Figure 4.31 (d). This agrees with the intersection of the two stretched threads performed before, Experiment 4.7.

We can also compare the present experiment with the one in which we used two loose threads, Experiment 4.8. In this case the vertical passing through the intersection of the loose threads coincides with the direction of the vertical bamboo skewer placed below them or with the downward projection passing through the hook holding the threads from above, as in Figure 4.12. That is, this vertical coincides with the axis of symmetry of the washer. And this axis of symmetry also passes through the geometric center of the washer. This means that all verticals drawn in this experiment intersect at the center of the washer.

Experiment 4.19

We now repeat this procedure utilizing a pasteboard Moon in first quarter or with a pasteboard letter C. The pasteboard figures remain balanced on a vertical plane.

Once more we observe that the intersection of all verticals coincides with the previous experiment performed with stretched threads, Experiment 4.7. In Experiment 4.7 we had stretched horizontal threads supported at their intersection above a vertical stick.

Experiment 4.20

We cut out a plane pasteboard figure of arbitrary shape, devoid of any symmetry. Two or three small holes are pierced in the figure. We then localize its CG by the first procedure. That is, we try to find the specific point at which the vertical support must be placed in order for the figure to remain in equilibrium in a horizontal plane when released at rest. We mark this point with a pen in both sides of the pasteboard.

Then we use the second procedure to locate the CG. That is, we hang the figure in a vertical plane by a horizontal pin passing through one of its holes and wait for equilibrium. We then draw a vertical line with the help of a plumb line. We observe that it passes through the CG obtained with the first procedure, although the figure lacks symmetry. The same happens when we hang the figure by the second or third hole.

The essence of these experiments can be stated as follows. A rigid body hangs by a point of suspension PS_1, such that it is free to rotate in all directions around this point. For each PS there will be a preferential position such that the body will remain in equilibrium when released at rest. If it is not released at rest in this preferential position, it will oscillate around the vertical passing through PS, until it stops due to friction. After the body reaches equilibrium, a vertical is drawn passing through PS_1. Choose a second point PS_2 outside this vertical. The body is suspended by PS_2 and the procedure is repeated. Experience shows that the two verticals obtained in this way intersect at a single point. The same happens when the body is suspended by any other point PS. That is, all verticals passing through the points of suspension intersect at a single point.

4.7.1 Practical Definition $CG6$

These facts lead us to a more general definition of the CG:

Practical Definition $CG6$**:** The center of gravity of a body is the meeting point of all vertical lines passing through the suspension points of the body when it is in equilibrium while suspended by these points, with freedom to rotate around them.

The detailed procedure to find the CG by drawing the verticals through the points of suspension has already been presented. It is illustrated in Figure 4.32 for a body of arbitrary shape.

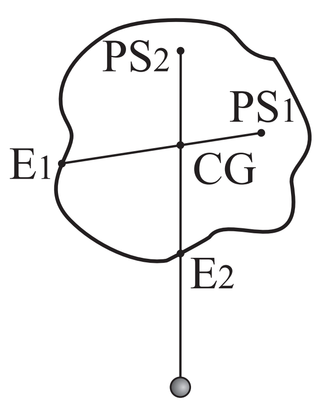


Figure 4.32: Second experimental procedure to find the CG of a figure with an arbitrary shape.

Experience teaches us that the CG is unique for each body. Moreover, it does not need to coincide with any material part of the body, as we have seen with concave figures or figures with holes. It is important to emphasize two points in this practical definition. (A) The body must be free to rotate around the point of suspension. We can keep a homogeneous ruler in equilibrium horizontally, for instance, by holding it at one end with our fingers, provided we press our fingers together to prevent the ruler from rotating. In this case we should not draw the vertical line through the point of suspension because the figure is not free to rotate. If we let the ruler oscillate around our fingers, it will not remain in this position when released at rest. Instead, it will oscillate, stopping with its larger axis in the vertical direction. (B) We should only draw the verticals in order to find the CG after the body has reached equilibrium, that is, when all its parts are at rest relative to the Earth. No vertical should be drawn while it is oscillating around the equilibrium position.

This last definition of the CG is much more abstract than $CG2$. Definition $CG2$ is more intuitive and clearly indicates the existence of a single, specific point in each body, such that it remains in equilibrium under the action of gravity when supported by this point. But definition $CG2$ has problems when dealing with concave or volumetric bodies, as we saw before. Definition $CG6$ is more general and can be applied to all cases considered here (including concave figures, figures with holes, three-dimensional bodies of arbitrary shape, hollow bodies, etc.).

A three-dimensional body must be suspended by a thread attached to one of its external points PS_1. We wait until the body reaches equilibrium. Then we must imagine the vertical extended downward through PS_1 until it reaches the end E_1 of the body. We then suspend the body by the thread attached to another external point PS_2. We wait until the body reaches equilibrium and imagine the vertical extended downward through PS_2, reaching another external point E_2 of the body. The intersection of these two verticals is the CG of the body. This procedure is illustrated for the case of a cube in Figure 4.33.

Now that we have a clear and general practical definition of the CG, we can clarify the concepts related to the support and suspension of a body with two definitions.

> **Point of Support or Auxiliary Point PA:** We say that a body in equilibrium is supported by a point (by a small region or by a small

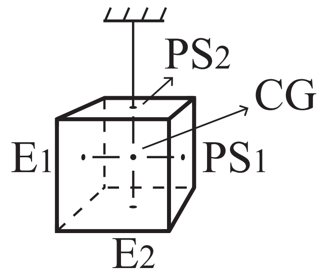


Figure 4.33: Finding the CG of a cube by the second experimental procedure.

surface) when this point of support is below the CG of the body. This auxiliary point or point of support will be represented by the letters PA.

Point of Suspension PS**:** We say that a body in equilibrium is suspended by a point when this point of suspension is above the CG of the body. This point of suspension will be represented by the letters PS.

After these definitions we can continue with the experiments.

4.8 Third Experimental Procedure to Find the CG

We now analyze the experiments performed earlier with three-dimensional bodies. The cube or die remained in equilibrium when the vertical bamboo skewer was placed under the center of each one of its sides. By extending these six verticals upward from the auxiliary point PA (the center of each face), we find that they intersect at the center of the cube. The same happens with the verticals extended upward from the centers of the six external faces of the hexagonal screw-nut: they intersect at the center of symmetry of the nut. The sphere remains in equilibrium when supported by any point on a flat table. The verticals extended upwards from these points of support all meet at the center of the sphere. The egg remained in equilibrium on a horizontal table when supported by any point along a specific circumference of its shell. By supporting the egg by two or three of these points belonging to this specific circumference and extending the verticals upward through these points of support, we can see that they all meet at a specific point inside the egg.

We first support the body by a point of support PA_1. We extend the vertical passing through PA_1 upward to E_1, where E_1 is the upper end of the body along this vertical line. We then support the body by another point of support PA_2 which is not along the first vertical line. We extend the second vertical passing through PA_2 upward to E_2, where E_2 is the upper end of the body along this second vertical line. The intersection of these two verticals is the CG of the body, as in Figure 4.34.

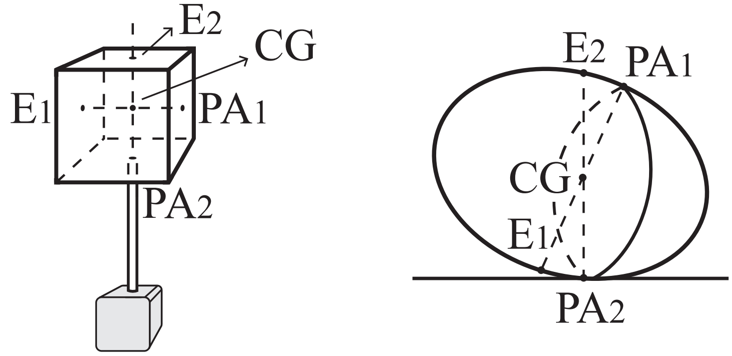


Figure 4.34: Finding the CG of a cube and an egg by the third experimental procedure.

That is, it is possible to locate the CG of a body both by the intersection of the downward verticals drawn from the points of suspension, PS, and by the upward verticals drawn from the points of support or auxiliary points, PA.

4.8.1 Practical Definition $CG7$

This suggests another practical procedure of finding the CG:

Practical Definition $CG7$**:** The center of gravity of a body is the meeting point of all vertical lines passing through the auxiliary points of support of the body when it is in equilibrium on these points, with freedom to rotate around them.

In definition $CG6$, the verticals are extended downward from the suspension points, PS, when the body is in equilibrium and free to rotate around these points. On the other hand, in definition $CG7$, the verticals are extended upward from the auxiliary points, PA, when the body is in equilibrium and free to rotate around these points.

The center of gravity obtained by practical definition $CG6$ always coincides with the center of gravity obtained by practical definition $CG7$. This can be seen, for instance, by hanging any of these three-dimensional bodies by threads connected to a rigid support. The thread can be tied to the bodies if they have holes, or attached to them with chewing-gum or with a piece of modeling clay.

Let us suppose, for instance, that we tie the upper end of a thread to a fixed support and attach the lower end to a sphere with chewing-gum. We release the system and wait until it reaches equilibrium. In this case the point of suspension (where the chewing-gum touches the sphere) will be vertically above the center of the sphere. The same is true for other bodies.

4.9 Conditions of Equilibrium for Supported Bodies

We now conclude this initial Section with a few more experiments. They are very simple but extremely important. We will work with bodies for which the centers of gravity have already been determined experimentally. Some of these new experiments (or parts of them) were performed previously. Here we will establish the conditions of equilibrium and motion for bodies supported from below, that is, for which the CG is above the PA.

Experiment 4.21

We will work with a triangle, but the experiment can be performed with any plane figure for which the CG coincides with one of its material points. We first use a pen to mark the CG (barycenter) of the triangle. We then try to balance it horizontally by placing it on several supports and releasing it at rest. We begin with a vertical bottle. Equilibrium occurs whenever the CG of the triangle is above the bottle cap. If the vertical through the CG falls outside the bottle cap, the triangle falls down, its CG approaching the surface of the Earth. Next, a pencil is held vertically with the tip facing downwards inside a sharpener. We observe again that we can balance the triangle whenever the center of gravity is located anywhere on the flat upper end of the pencil. We now utilize a vertical bamboo skewer with its tip stuck in a clump of modeling clay. Once more we can balance the triangle horizontally as before, but there is not much freedom left here. That is, any small horizontal motion of the CG which removes it from the upper end of the bamboo skewer makes the triangle fall to the ground. When we use a vertical bamboo skewer with its tip pointed upward as a support, it is very difficult to balance the triangle. Any shaking of our hands when we release the triangle is enough to unbalance it and cause it to fall. The same happens with any leaning or quivering of the bamboo skewer due to wind or some other factor. Finally, it is extremely difficult to balance the triangle on the tip of a vertical pin or needle. Sometimes we can only succeed if we stick the pin in the pasteboard (finishing with the experiment) or deform the triangle a little. Many people never succeed in balancing the triangle horizontally on the tip of a vertical needle, no matter how long they try.

Other examples of this fact can be found in one of the previous experiments in which a cube or a metal screw-nut was balanced on a vertical bamboo skewer with its tip downward. Equilibrium was achieved only when its CG (the center of symmetry of the cube or nut) was placed vertically above the upper horizontal surface of the bamboo skewer.

We conclude that a body can only remain in equilibrium if its CG is vertically above the region of support. Moreover, it is extremely difficult to balance a body when its CG is vertically above the support in cases where the area of the upper end of the support tends to zero, approaching a mathematical point. This can be shown clearly in the next experiment.

Experiment 4.22

We make a small circular hole in the pasteboard triangle of the previous experiment. We hang it on a pin stuck in a vertical bamboo skewer. The horizontal pin passes through the hole and the plane of the triangle is vertical. We turn the triangle in such a way that its CG and the pin are aligned vertically, with the CG above the pin. We release the triangle at rest, holding the base of the bamboo skewer firmly. Experience shows that the triangle does not remain in this position. Its CG begins to swing widely around the vertical extended downward through the pin, until the triangle reaches equilibrium, as in Figure 4.35. In the final position, the pin and the CG are vertical, but with the CG vertically below the pin.

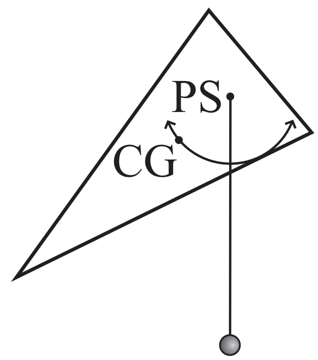


Figure 4.35: When the triangle is released at rest in a vertical plane with its CG vertically above the pin, it does not remain at rest. It begins to oscillate around the vertical passing through the pin. The oscillations decrease their amplitude due to friction. The triangle reaches equilibrium with its CG vertically below the pin, remaining at rest relative to the Earth.

Experiment 4.23

We now consider a homogeneous sphere on a horizontal table. We can release it at rest in any position, and it remains in equilibrium. If we give it a small horizontal motion, it rolls until it stops due to friction.

Experiment 4.24

An analogous experiment can be performed with any cylindrical homogeneous container with its CG along the axis of symmetry (a cylindrical metal can or plastic bottle, for instance). It remains in equilibrium when released at rest in any position. If it is given a small horizontal motion so that it begins to roll around the line of support, it moves until it stops due to friction.

We now perform a series of three experiments analogous to what we did with the egg earlier, but with a slightly different symmetry which shows more clearly what is happening. We will deal with a cylindrical shampoo bottle with an elliptical cross section (for which b is half the large diameter or major axis and a is half the small diameter or minor axis, with $b > a$). The center of gravity is along the axis of symmetry of the bottle, passing through the center of the two elliptical bases.

Experiment 4.25

The shampoo bottle is set down on a horizontal surface and released at rest. We observe that it only remains in equilibrium when released in such a way that the line of support is along the end of the minor axis $2a$, as in Figure 4.36 (a). In this position the CG is vertically above this line of support. By definition we will call this configuration the *preferential position* of the vessel or its *preferred configuration*.

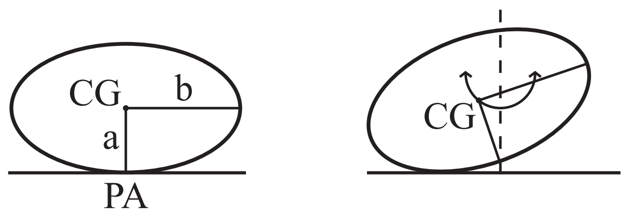


Figure 4.36: (a) Preferential position of the vessel. (b) When the bottle is released at rest outside the preferential position, the CG oscillates around the vertical through PA.

Experiment 4.26

If we turn the vessel slightly in the clockwise or in the anticlockwise direction around PA and release the vessel, it does not remain at rest. Instead, the straight line connecting the centers of the ellipses will begin to oscillate around the previous vertical line passing through the PA, as shown in Figure 4.36 (b), until the container reaches equilibrium after stopping due to friction. The final position it reaches is the preferential position. This experiment is analogous to what happens with a rocking chair.

We can see in Figure 4.37 that when we rotate the container in the clockwise or in the anticlockwise direction around the point PA in the preferential position, the CG will no longer be along the vertical line passing through the new point or line of contact. Moreover, the CG will be higher in this new position than it was in the preferential position. When the container is released at rest in this new position, the initial direction of motion (that is, the side toward which the vessel will turn) is such that the CG will approach the surface of the Earth. The final position reached by the container, which coincides with the preferential position, is the configuration for which the CG is in the lowest possible position.

Experiment 4.27

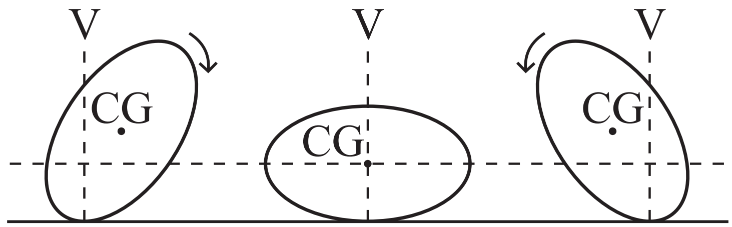


Figure 4.37: When a body is released at rest, the direction of motion is such that the CG moves downwards. The central position is that of stable equilibrium.

The container is now released at rest in a position for which the CG is vertically above the lower end of the major axis $2b$. It is practically impossible to balance the container in this position if the floor is flat and smooth. The container always falls toward one or the other side. To find out the side toward which it will fall, we only need to release it at rest with the CG slightly away from the previous vertical line. In this case the initial direction of motion always causes the CG move closer to the ground, as in Figure 4.38. The final position of equilibrium is once again the preferential configuration with the CG vertically above the lower extremity of the smaller semi-axis a.

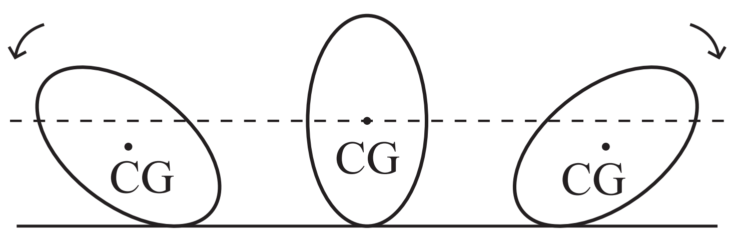


Figure 4.38: The central position is that of unstable equilibrium.

These and other analogous experiments can be summarized as follows. Suppose a rigid body is placed on flat horizontal surface and released at rest. It will remain in equilibrium only if its CG is vertically above the surface of contact. If the downward projection of the CG lies outside the region of contact, the body will not remain at rest. The initial direction of motion in this case is such that its CG will approach the ground.

4.9.1 Definitions of Stable, Unstable and Neutral Equilibrium

These experiments suggest the following definitions:

Stable Equilibrium: This occurs when the CG is vertically above the region of contact and, moreover, any perturbation in the position of the body increases the height of its CG. We call this configuration the *preferential position* of the body or its *preferred configuration*.

It is observed experimentally in these cases that any perturbation in the body will cause the CG to oscillate around the vertical passing through the region of support in the preferential configuration, with the body swinging until it reaches equilibrium, because friction will decrease the amplitude of oscillation. In the final position, the body returns to the initial configuration of stable equilibrium.

Neutral Equilibrium or Indifferent Equilibrium: This occurs when the CG is vertically above the region of support and, moreover, any perturbation in the position of the body does not change the height of its CG relative to the ground.

In these cases it is observed that the body remains in equilibrium for any position in which it is released at rest. If the body is given a small push and begins to move, it will continue to move in this direction until it stops due to friction.

Unstable Equilibrium: This occurs when the CG is vertically above the region of support and, moreover, any perturbation in the position of the body decreases the height of its CG relative to the ground.

In this case it is observed that any perturbation in the position of the body will move its CG away from the initial position. Moreover, the body does not return to the initial position.

4.10 Stability of a Body

Yet another property connected with the equilibrium of a body supported from below can be derived from these conditions of stable and unstable equilibrium. This property can also be verified experimentally.

To do so, we use a rectangular parallelepiped of sides a, b and c. It can be a brick, a homogeneous wood block, a match or shoe box, etc. We will always work with the surface bc in a vertical position. From symmetry considerations, and also experimentally, it is easy to verify that the CG of the homogeneous parallelepiped is located at its center. We place a plumb line at the center of the face bc. If the body is a homogeneous wood block, the simplest procedure is to put a nail at the center of the surface and tie the thread attached to a plumb onto it. If the parallelepiped is a shoe box, we can pass a bamboo skewer through the centers of both parallel faces of sides b and c. We then tie a plumb line onto it. For a match box we can pass a pin or needle through the centers of both faces, and then tie a plumb line to it. To prevent the parallelepiped from falling to the ground due to the weight of the plumb line, it is important for the weight of the plumb line to be much smaller than the weight of the parallelepiped. The experiment does not work as well if the parallelepiped is very thin, that is, if side a is much smaller than sides b and c (as is the case with a pasteboard rectangle, where the thickness of the rectangle is much smaller than its sides). In these cases it is difficult to balance the body with surface bc in a vertical plane. After everything has been prepared we begin the experiments.

Experiment 4.28

We begin with the parallelepiped at rest above a horizontal table, with side c vertical and side b horizontal. Surface ab is horizontal, together with its four vertices V_1, V_2, V_3 and V_4, as in Figure 4.39 (a).

We define rotation in the vertical plane around the horizontal axis V_1V_2 when V_5V_6 moves down and V_3V_4 moves up as indicating a positive angle, as in Figure 4.39 (b).

If we rotate the parallelepiped around the axis V_1V_2 of an angle θ and release it at rest, its initial motion is such that its CG falls, as we saw in the conditions for stable and unstable equilibrium. It is easy to see that there will be a *critical angle* for which the straight line passing through V_1V_2 and by the CG will be vertical, coinciding with the direction of the plumb line, as in Figure 4.39 (c). This critical angle is represented by θ_c. In this situation the CG is in its highest position.

If the parallelepiped is released at rest from an initial angle smaller than the critical angle, it will tend to return to the position with side c vertical and side b horizontal, $\theta = 0^o$. A rotation in this sense will lower the CG.

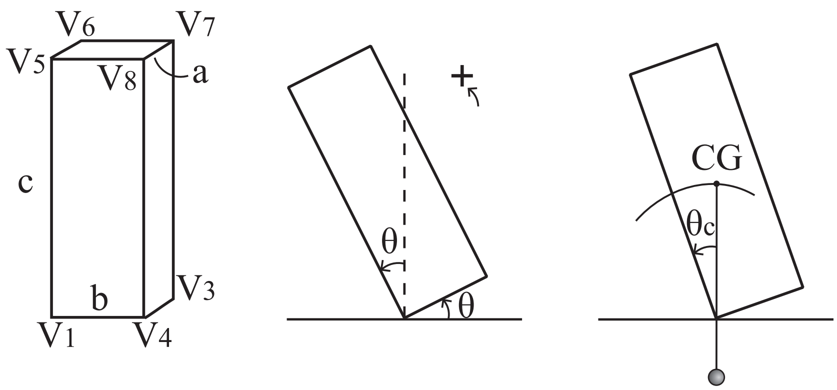


Figure 4.39: (a) A brick. (b) Rotation of an angle θ. (c) The critical angle θ_c for which the CG is in its highest position.

If the initial angle is higher than the critical angle, the body will tend to move away from the initial position, falling toward the side where c tends to a horizontal position and b tends to a vertical position, $\theta = 90^o$. A rotation in this sense will also lower the CG.

The position of the critical angle is one of unstable equilibrium, Figure 4.39 (c).

Figure 4.40 relates the angles α and θ_c.

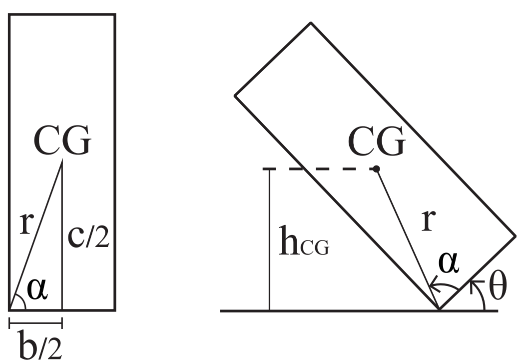


Figure 4.40: Geometrical properties of a brick.

From Figure 4.40 (a) we can see that the tangent of the angle α between the base V_1V_4 and the straight line connecting the vertex V_1 to the CG is given by c/b:

$$\tan\alpha = \frac{c}{b} \ . \tag{4.1}$$

From Figures 4.39 and 4.40 we can see that the critical angle θ_c is given by

$$\theta_c = 90^o - \alpha \ . \tag{4.2}$$

This means that

$$\tan\alpha = \tan(90^o - \theta_c) = \frac{c}{b} \ . \tag{4.3}$$

From Figure 4.40 (b) we can see that in general the value of the height of the CG, represented by h_{CG}, is given by:

$$h_{CG} = r\sin(\alpha + \theta) \ , \tag{4.4}$$

where r is given by

$$r = \frac{\sqrt{c^2 + b^2}}{2} \ . \tag{4.5}$$

When $\theta = 0^o$ we have $h_{CG} = c/2$. When $\theta = 90^o$ we have $h_{CG} = b/2$. The highest value acquired by the CG relative to the ground happens for $\alpha + \theta = 90^o$, when $h_{CG} = r$.

When $c = b$ we have $\alpha = \theta_c = 45^o$. In this case the smallest value for the height of the CG is given by $h_{CG} = b/2 = c/2 = 0.5c$. The highest value is given by $h_{CG} = \sqrt{2}c/2 \approx 0.7c$.

If $c = 3b$, $\alpha = 71.6^o$ and $\theta_c = 18.4^o$. In this case we have $h_{CG} = c/2 = 0.50c$ when $\theta = 0^o$, $h_{CG} = 10^{1/2}c/6 \approx 0.53c$ when $\theta = \theta_c$, and $h_{CG} = c/6 \approx 0.17c$ when $\theta = 90^o$.

In the case for which $c = b/3$ we have $\alpha = 18.4^o$, $\theta_c = 71.6^o$, $h_{CG} = c/2 = 0.50c$ when $\theta = 0^o$, $h_{CG} = 10^{1/2}c/2 \approx 1.6c$, when $\theta = \theta_c$ and $h_{CG} = 3c/2 = 1.5c$ when $\theta = 90^o$.

From these conditions we conclude that the stability of a body supported from below *in stable equilibrium* increases when the height of its CG decreases. That is, the critical angle increases when we decrease the height of the CG.

We can control this experiment by working with a body of the same weight and external shape, but for which we can change the position of its

CG. The idea here is to use a hollow rectangular box of sides a, b and c which has the CG at the center of the box. We will suppose that the side bc is always vertical. We then place another body inside the box, suspended at a height h from the base, as in Figure 4.41.

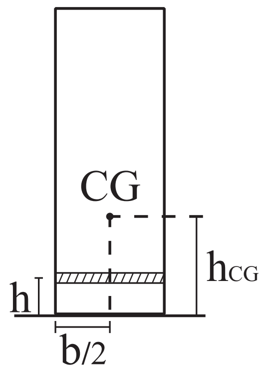


Figure 4.41: A box with an internal heavy body.

What is important now is that we can control this height. For a match box, for instance, we can attach a number of fishing sinkers with modeling clay to the lower or upper part of the box. We can then check that the CG of the system box-sinkers is located at some intermediate point between the center of the box and the center of the sinkers. Let us suppose that it is at a height h_{CG} from the base of the box over a horizontal surface, situated along the axis of symmetry of the lower base b of the box, as in Figure 4.41.

Experiment 4.29

We place sinkers inside a match box, along the bottom side, and place the match box on a horizontal surface. We rotate the system around one of the axes of the base, releasing it at rest. We observe that for some angles the system returns to the position for which $\theta = 0^o$, while for angles greater than a certain critical angle θ_{cI} the box falls to the other side, towards $\theta = 90^o$, moving away from the initial position. We now invert the position of the shots, in such a way that they remain attached internally to the match box, but on its top side. We repeat the same procedure, and now obtain another critical angle θ_{cS}. It is found experimentally that this new critical angle is much smaller than the previous critical angle, $\theta_{cS} < \theta_{cI}$.

By the previous definition of equilibrium we conclude that the match box is in a position of stable equilibrium whether the sinkers are below or above. The reason for this is that any small perturbation of this position, for clockwise and for anticlockwise rotation with initial angles smaller than the critical angle, the box returns to the initial position when released at rest. Nevertheless, we can say that the box with the sinkers on the bottom is more stable than the box with the sinkers on the top, as the critical angle in the first case is much larger than the critical angle in the second case.

Definition of the stability of a system: The size of this critical angle can then be considered the degree of stability of the system. That is, *for two systems which are in stable equilibrium*, it is defined that the system which has a greater critical angle has a larger stability than the system which has a smaller critical angle.

We now want to know the value of the critical angle θ_c for this system. When the box rotates around the axis V_1V_2 of an angle θ, as in the previous experiment, it returns to the position for which $\theta = 0^o$ when released at rest if $\theta < \theta_c$. If $\theta > \theta_c$, the box does not return to the position for which $\theta = 0^o$ when released at rest, but falls to the opposite side, towards $\theta = 90^o$. Let α be the angle between the horizontal base b and the straight line connecting V_1V_2 to the CG. We then have the result given by Equation (4.6), see Figure 4.42.

$$\tan\alpha = \frac{h_{CG}}{b/2} = \frac{2h_{CG}}{b} \ , \tag{4.6}$$

At the critical angle we have

$$\alpha + \theta_c = 90^o \ . \tag{4.7}$$

Therefore,

$$\theta_c = 90^o - \alpha = 90^o - \arctan\frac{2h_{CG}}{b} \ . \tag{4.8}$$

If the height of the CG, namely, h_{CG}, is very small, much smaller than b, the critical angle will be very high, close to 90°, which indicates high stability of the body. If h_{CG} is much larger than b, the critical angle will be very small, close to 0°. Any perturbation in the system will make it fall, moving away from the initial position. From this last equation we conclude that, in order to increase the stability of a system, we must decrease the ratio h_{CG}/b. There

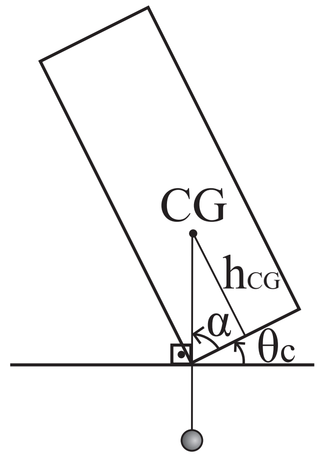


Figure 4.42: Conditions of stability for a body.

are two basic ways to do this, namely: (A) Decrease the height of the CG (as we saw for the match box with the sinkers in the lower side). (B) Increase the base around which the system rotates.

There is still another criterion in order to define the stability of a system. This second criterion will not be discussed in this book. We present here only an example illustrating it. Consider an empty can and another can of the same shape and size but completely filled with a liquid or solid substance. The centers of gravity of both systems are located at the same height relative to the ground. These two cans have the same critical angle, as they have the same shape and size. By the previous definition we conclude that they have the same stability. On the other hand, it is necessary a larger energy to make the can filled with a liquid or solid fall to the ground, than the energy required to make the empty can fall to the ground. External perturbations throw to the ground more easily an empty can than a heavy one. Examples of these external perturbations are a trembling ground, small objects colliding with the cans, etc. For this reason we say that a filled can is more stable than an empty can of the same shape and size.[95] This is related with a *dynamic equilibrium*, involving the masses of objects, a topic which will not be discussed here.

[95] [Wal07, Chapter 1, Section 1.149: Stability of a pop can].

4.11 Conditions of Equilibrium for Suspended Bodies

We now consider the main conditions of equilibrium and motion for bodies suspended from above. That is, with the point of suspension PS above the CG of the body. We will consider convex bodies or bodies with holes so that they can hang from a pin passing through a hole or by a thread tied to a hole. Once again we will consider bodies whose centers of gravity were already determined, and those where the hole does not coincide with the CG. Some of these new experiments, or parts of them, have already been performed. But they will be repeated in order to clearly establish the conditions of equilibrium and motion of suspended bodies. We will work with a triangle, but similar experiments can be performed with other bodies.

Experiment 4.30

We hang a triangle by a pin passing through one of its holes, releasing it at rest. We observe that it only remains in equilibrium if its CG is vertically below the PS. This configuration is called the *preferential position* of the suspended body or its *preferred configuration*.

Experiment 4.31

We now rotate the triangle around the PS by a certain angle, such that the CG and the pin are no longer along a vertical line. The triangle is released at rest. We observe that the CG oscillates around the vertical passing through the PS, as shown in Figure 4.43. The amplitude of oscillation decreases due to friction. When the triangle stops, it returns to the preferential position with the PS and CG along a vertical line. Moreover, in equilibrium the CG is below the PS.

From Figure 4.43 we can see that the preferential position has the CG in the lowest position. Any clockwise or anticlockwise rotation of the triangle around the PS increases the height of the CG.

Experiment 4.32

We begin with a symmetrical bicycle wheel (that is, one with the CG at the center of the wheel), at rest, suspended by a horizontal axis. The wheel

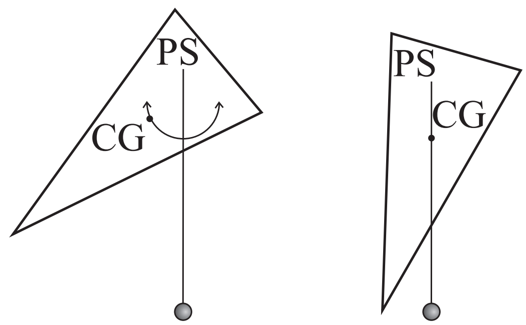


Figure 4.43: (a) When released at rest, the CG oscillates around the vertical passing through the PS. (b) The body stops in equilibrium with the CG vertically below the PS.

is attached to the axis by a ball-bearing, in such a way that it is not loose on the axis. We could also use a pasteboard disc pierced at the center. We pass a nail through the center of the disc, with a diameter a little smaller than the diameter of the hole, in such a way as to leave little room between them, only enough space for the disc to rotate around the axis. The plane of the disc or bicycle wheel should be vertical, with the axis horizontal. We can release the wheel or disc at rest in any position and it remains in equilibrium. If we rotate the wheel or disc slowly one way, giving it a small angular rotation, it continues to rotate in this sense, its angular velocity decreasing due to friction, until it stops.

In these cases the wheel and the disc are suspended by the upper part of the axis, which is a little above the CG of the bodies (located at the center of the wheel or disc). Nevertheless, any rotation of the wheel or disc around the axis does not change the height of the CG relative to the ground.

4.11.1 Stable and Neutral Equilibrium

These experiments suggest the following definitions:

> **Stable Equilibrium:** When the CG is vertically below the PS and, moreover, when any perturbation of this position moves the CG upwards. The configuration for which the CG is vertically below the PS is called the *preferential position* or *preferred configuration.*
>
> It is observed that if the body is released at rest in its preferred position, it will remain in equilibrium. If it is disturbed, it will oscillate around

its preferred position, decreasing its amplitude of oscillation due to friction, until it returns to its preferred position. For this reason, this situation is called stable equilibrium.

Neutral Equilibrium: This occurs when the CG is vertically below the PS and when any perturbation in this position does not change the height of the CG relative to the ground.

In this case the body remains in equilibrium in any position where it is released at rest. If it receives a small impulse and begins to rotate around the PS, it will continue to turn in this direction until it stops due to friction.

Experiment 4.33

Before we move on, another experiment is worth performing. We cut out a pasteboard figure in the shape of the letter T. The length from the tip of one arm of the T to the tip of the other arm should be 15 cm. The height of the T should be 15 cm or 20 cm. The width of the arms and body of the T should be 2 cm. We make 10 or 11 holes along the axis of symmetry of the T with a single-hole punch-pliers. We call them F_1 to F_{11}, with F_1 at the intersection of the arms and with F_{11} at the end of the figure. We can also make holes at the hands of the two arms, as in Figure 4.44.

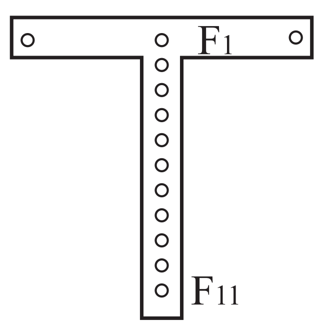


Figure 4.44: A pasteboard T with holes.

To begin with, we locate the CG of the T. This can be done, for instance, by hanging it by the holes at the hands and drawing the verticals when the figure is in equilibrium. The CG is the intersection of these two verticals, which should be along the axis of symmetry of the T, closer to F_1 than to

F_{11}. The T will then be released at rest, hung by a hole along its axis of symmetry, with the arms horizontal and its body below the arms (that is, with F_1 above F_{11}). When it is suspended by holes above the CG, such as F_1 or F_2, it remains in equilibrium at the position in which it was released. On the other hand, when it is released at rest by holes below the CG, such as F_{10} or F_{11}, it turns to one side or the other, swings a few times with decreasing amplitude, and stops upside down with F_{11} above F_1.

This experiment again illustrates that the configuration with the CG vertically above the PS is unstable if the area of support is very small, like a point. On the other hand, the configuration with the CG vertically below the PS is stable, even when the area of support is very small, like a point. Although the explanation for this experiment is based on principles we have already seen, it is very interesting. After all, all holes have the same diameter and allow the same rotation around the PS. But only in certain cases will the body rotate when released at rest, inverting the position of the arms relative to the body of the T.

4.12 Cases in which the CG Coincides with the PS

It may be impossible to do a real experiment in which a body is suspended or supported by a point which passes exactly through its CG, and is free to rotate around this point. Even when we try to approximate this situation from below, the CG will be always slightly above the auxiliary point of support PA. This is the case, for instance, where the pasteboard triangle is horizontal and supported by a vertical skewer placed below the barycenter of the triangle, Experiment 4.3. Here, the point of contact between the skewer and the cardboard is slightly below the CG of the triangle. The center of gravity of the triangle is located at a point in the center of the thickness of the cardboard. When we try to approximate this situation from above, the CG will always be slightly below the PS. This is the case, for instance, of a triangle in the vertical plane suspended by a horizontal pin passing through a hole made at the barycenter of the triangle. The diameter of the hole must be slightly larger than the diameter of the pin, in order to allow free rotation of the triangle. In this case the PS will be the point of contact between the pin and the upper part of the hole, while the CG will be at the center of the hole.

Another difficulty arises for three-dimensional bodies. For instance, if we have a solid parallelepiped, we can only support it by a stick touching its outside surface, or by a thread tied to its outside surface. Its CG is located in the middle of the parallelepiped, inside the brick. In order to suspend or support the body by this point, we need to make a hole in the parallelepiped. This hole would change the distribution of matter of the parallelepiped. But if the width of this hole is very small compared with the sides of the parallelepiped, we can neglect this modification to the matter of the brick. But even after making this hole, it is difficult to imagine a real system allowing freedom of rotation of the brick around its CG.

From what we have already seen, we can imagine what would happen if we could perform this experiment. We have already seen that the CG of any rigid body released at rest tends to move toward the surface of the Earth. If the body were held exactly by the CG, with freedom to rotate around this point, any rotational movement it makes would not change the height of the CG in relation to the Earth. In this case, the body would remain in equilibrium in all positions in which it was placed and released at rest, regardless of its orientation in relation to the ground.

Let us consider an idealized experiment with a triangle in a horizontal plane suspended from above exactly at its barycenter by a vertical string, or supported by a vertical stick under it. The straight segment CGV_1 connects the CG with the vertex V_1 of the triangle, while the segment CGE connects the CG with the East-West direction. Let α be the angle between CGV_1 and CGE. If released at rest in a horizontal plane supported by its CG, the triangle will remain in this position for any value of α, Figure 4.45.

Let us now suppose another idealized situation. The triangle is released at rest in a vertical plane supported exactly at its barycenter. We assume that it has complete freedom to rotate around any horizontal axis passing through the CG. Let β be the angle between the segment CGV_1 and the vertical indicated by a plumb line. After being released at rest, the triangle will remain in its initial orientation, no matter the value of β, Figure 4.46.

We now consider a third idealized situation. The normal to the plane of the triangle is initially inclined through an angle γ in relation to the vertical indicated by a plumb line. We suppose that the triangle is supported exactly at its barycenter, being free to turn around any axis passing through its CG. If released at rest, it will remain in equilibrium for any value of γ, Figure 4.47.

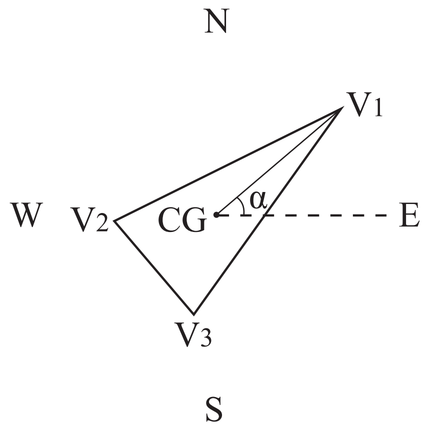


Figure 4.45: The horizontal triangle supported at its barycenter remains in equilibrium for any angle α.

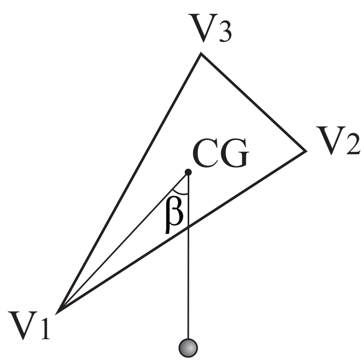


Figure 4.46: The vertical triangle supported at its barycenter will remain in equilibrium after being released at rest for any angle β.

We have seen from the previous real experiments that the CG of any rigid body tends to move closer to the surface of the Earth when the body is released at rest. Therefore, if the body is suspended exactly by the CG, being released at rest and free to rotate in any direction around this point, the body will not move, regardless of its initial orientation relative to the ground. After all, if it did begin to rotate in any direction, the CG would remain at the same height. As the tendencies to rotate in opposite directions cancel one another, it will not begin to rotate when released at rest.

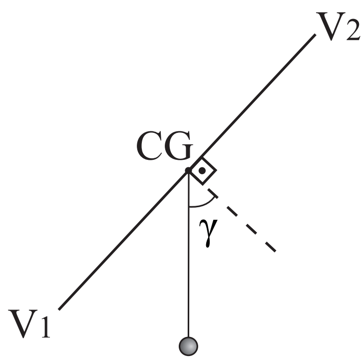


Figure 4.47: The inclined triangle supported by its barycenter remains in equilibrium for any angle γ.

4.12.1 Definitive Definition $CG8$

This leads to a new and definitive definition for the CG.

Definitive Definition $CG8$**:** The center of gravity of a rigid body is a point such that, if the body is conceived to be suspended by this point, having freedom to rotate in all directions around this point, the body thus supported, when released at rest, will remain stationary and preserve its original position, whatever its initial orientation relative to the Earth.

If this point is located in empty space, as for concave figures or figures with holes, we have to imagine a rigid structure connecting the body to this point, so that the body can be suspended from the point.

Later on we will see that Archimedes seems to have defined the CG in this way, that is, with a definition similar to $CG8$.

The main difference between definition $CG8$ and definition $CG4$ is that in $CG8$ we say that the body will remain in equilibrium when released at rest *no matter what the initial orientation of the body relative to the ground.*

Let us consider a washer, for example. It can remain in equilibrium when released at rest in a vertical plane, suspended by any point on its internal circumference, as in Figure 4.48 (a). In this case the axis of the washer makes an angle $\theta = 90^o$ with the vertical line. We define the angle θ as the smaller angle between the axis of the washer and the vertical line. According to definition $CG4$, this point of suspension PS along the internal circumference could be considered a center of gravity for the washer.

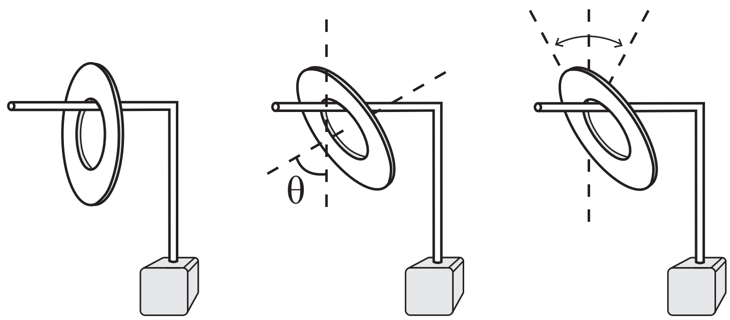


Figure 4.48: (a) A washer can remain at rest when suspended by its internal circumference. However, it does not remain in equilibrium for all orientations of release. (b) Let us suppose it is released at rest with $\theta \neq 90^o$. (c) In this case the center of the washer will oscillate around the vertical passing through the point of support after it is released.

We now suppose that the washer is released at rest when suspended by the same point, but with its axis no longer orthogonal to the vertical line. This means that the body will be released at rest with $\theta \neq 90^o$, as in Figure 4.48 (b). In this case the washer does not remain in equilibrium. After being released, the plane of the washer will oscillate around the vertical line passing through the PS, as in Figure 4.48 (c). The amplitude of oscillation decreases due to friction. The washer finally stops in the orientation for which $\theta = 90^o$. This is the preferential position of the washer, Figure 4.48 (a).

According to definition $CG8$, this point of support along the internal circumference cannot be considered the CG of the washer. We have already seen in the practical procedure given by $CG6$ that the real CG of the washer is its center of symmetry at the center of the washer. When the washer hangs by a point PS located on the internal circumference, the CG will be at its lowest position when it is vertically below this PS, where $\theta = 90^o$. This is a position of stable equilibrium. When we decrease the angle θ, the CG moves upward. If it is released at rest in this new position, gravity will cause the CG to move downward.

Suppose now we attach some spokes in the washer, like the spokes of a bicycle wheel. This can be done with taught threads, or we can consider a real bicycle wheel. Let us suppose that the washer or wheel is suspended by its center and is free to rotate in any direction around this point. If it is released at rest with its axis making an angle θ with the vertical line, it will remain in equilibrium for any θ, as in Figure 4.49.

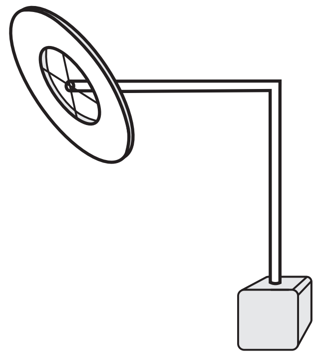


Figure 4.49: When supported exactly by its CG, a body would remain in equilibrium after being released at rest, no matter the orientation of the body relative to the ground.

By definition $CG8$, we can now conclude that this is the real CG of the washer. The reason why it remains in equilibrium for any value of θ, when suspended by its center, is that the height of the CG is independent of θ. And this is the main property of the neutral equilibrium.

We called definition $CG8$ as definitive. The word "definitive" should be understood as between quotation marks. The reason is that it is strictly valid only in regions of uniform gravitational forces. These are the regions in which a certain test body is always under the action of the same gravitational force (in intensity and direction) at all points of the region. This is true for small bodies in the vicinity of the Earth. The gravitational forces acting upon each particle of the test body can be considered parallel to one another, and all vertical.

But there are situations in which this is not valid. Let us present a specific example in which we make several assumptions: (A) The body exerting the gravitational force is like the Earth, but shaped like an apple, with the greatest distance between any two particles of this apple-Earth given by d_E; (B) the body under the action of the gravitational force is like the Moon, but shaped like a banana, with the greatest distance between any two particles of this banana-Moon given by d_M; (C) the distance between an arbitrary particle i of this apple-Earth and another arbitrary particle j of the banana-Moon being given by $d_{ij} = d_E + d_M + e_{ij}$, with $0 < e_{ij} \ll d_E + d_M$. In this case a unique center of gravity will not exist. Depending upon the relative orientation between the banana-Moon and the apple-Earth, there will be distinct lines of equilibrium. In cases like this, the concept of a center of gravity looses its meaning.

In any event, definition $CG8$ may be used for a test body of small dimensions when compared with the radius of the Earth.

4.13 Cases in which the CG does Not Change Its Height by Rotating the Body, Although the CG is Above the Auxiliary Point PA

It may be impossible to perform an experiment for which the rigid body is supported exactly at its CG, while having freedom to rotate in all directions around this point. Despite this fact, there are some real experiments which can be performed illustrating definition $CG8$.

The situation of Figure 4.45 can be simulated by Experiment 4.3. In this experiment a triangle remains at rest in a horizontal plane while being supported above a vertical skewer. The vertical projection of the skewer passes through the CG of the triangle. The straight line connecting a specific vertex of the triangle to its CG can make any angle α with the East-West direction. The triangle remains at rest in the horizontal plane after being released at rest, no matter the value of α.

This situation is not exactly the situation described in definition $CG8$. After all, the triangle has a certain thickness, even when it is very thin. Therefore, the portion of the pasteboard which is in contact with the skewer is not exactly the CG of the triangle, as this CG is located in the center of the thickness of the pasteboard. In any event, this experiment indicates a neutral equilibrium as regards the rotation around a vertical axis passing through the CG. After all, we can change the angle α without changing the height of the CG relative to the ground.

In the next experiments we show how to make something analogous to the situations of Figures 4.46 and 4.47.

Experiment 4.34

We pass a skewer through the plane of a pasteboard triangle. The skewer remains fixed relative to the pasteboard, orthogonal to the plane of the triangle. The diameter of the hole should be the same as the diameter of the skewer, so that the skewer and the triangle may be considered as a single rigid body. When the triangle rotates relative to an axis, the same must happen with the skewer. This will be indicated in the next Figures by a black semi-circle marked at the end of the skewer, in its cross section. That is, by the orientation of this semi-circle it is possible to detect the rotation of the skewer relative to the ground.

Initially we suppose that the skewer passes through a hole in the pasteboard triangle which does not coincide with its CG. We support the skewer horizontally by two vertical rectangular poles. The plane of the triangle will always be vertical. The preferential position is that for which the CG of the triangle is located vertically below the skewer, Figure 4.50 (a). When the triangle is released at rest in this preferential position, it remains stationary relative to the ground.

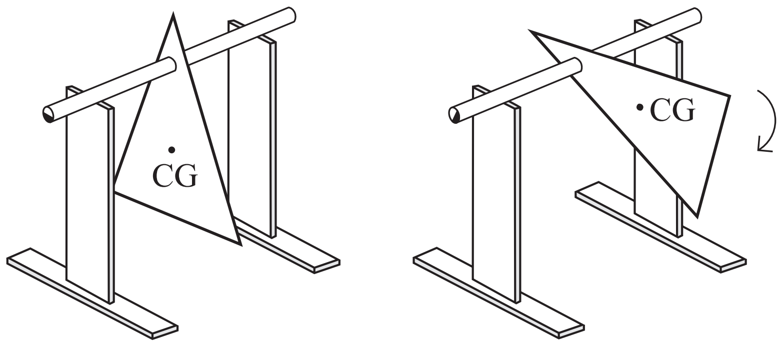


Figure 4.50: (a) The preferential configuration with the CG vertically below the horizontal skewer. The triangle remains at rest if released in this position. (b) When the triangle is not released in the preferential configuration, the CG will oscillate around the vertical plane passing through the skewer. The amplitude of oscillation will decrease due to friction. The triangle will stop in the preferential configuration.

Let us now consider that it is released at rest with the CG of the triangle outside the vertical plane passing through the skewer, as in Figure 4.50 (b). In this case the triangle does not remain at rest. The CG of the triangle will oscillate around the vertical plane passing through the skewer. The amplitude of oscillation will decrease due to friction. The triangle will stop in the preferential position.

Experiment 4.35

We suppose once again a skewer fixed orthogonally to the plane of a pasteboard triangle. But now we assume that the axis of symmetry of the skewer passes exactly though the CG of the triangle. The skewer is supported horizontally above two vertical rectangular poles, with the plane of the triangle in the vertical orientation. In this case the triangle can be released at rest in any orientation that it remains stationary relative to the ground, Figure 4.51.

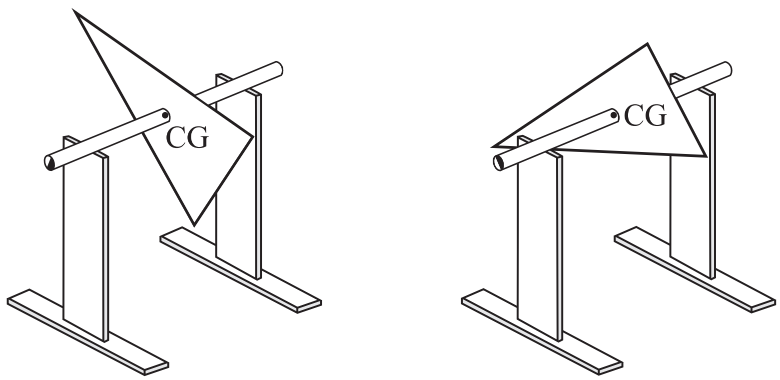


Figure 4.51: Situation for which the axis of symmetry of the skewer passes exactly through the CG of the triangle. In this case the triangle remains in equilibrium no matter its initial orientation relative to the ground.

This situation is not exactly that described in definition $CG8$. In the present case the skewer is supported by the lower portions of its cross section which are in contact with the rectangular poles. This means that it is not supported exactly by its axis of symmetry which passes through the center of the skewer. Therefore the fulcrum or axis of support does not pass through the CG of the triangle, being in fact below this CG. In any event, we can rotate the skewer around its axis of symmetry, rolling it above the rectangular poles, without changing the height of the CG relative to the ground. When we rotate the skewer, the triangle rotates together with it, as both bodies constitute a single rigid unity. When we rotate the skewer, we change the portions of its cross section which are in contact with the rectangular poles below it. But the height of the CG of this system does not change. Therefore we have a situation of neutral equilibrium relative to rotations around the skewer. This experiment simulates the case described in Figure 4.46.

Experiment 4.36

We now make a slit in a wood barbecue skewer, so that we can pass a pasteboard triangle through it, Figure 4.52. The skewer and the triangle must form a single rigid body, with the triangle fixed in the skewer. When the triangle rotates, the same must happen with the skewer.

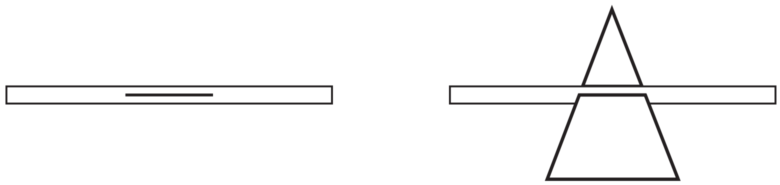


Figure 4.52: A pasteboard triangle passes through a slit in the skewer, being fixed relative to it.

Initially we suppose that the CG of the triangle is outside the slit, as in Figure 4.53. The skewer is supported horizontally above two vertical rectangular poles. The preferential position is that for which the CG is vertically below the horizontal skewer, as in Figure 4.53 (a). When it is released at rest in this position, it remains in equilibrium.

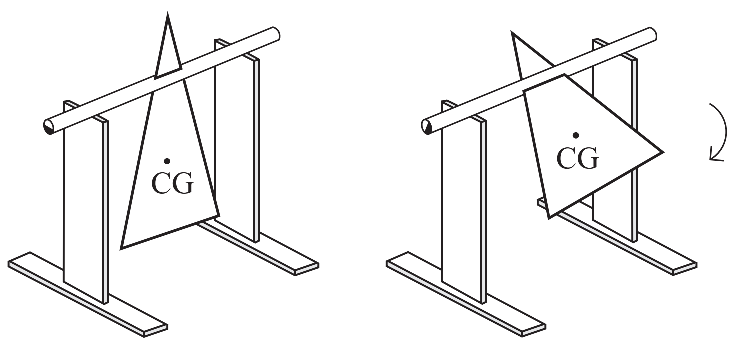


Figure 4.53: (a) The preferential configuration of the triangle with its CG vertically below the horizontal skewer. The triangle remains at rest after being released in this position. (b) When the triangle is not released in the preferential position, the CG oscillates around the vertical plane passing thought the skewer. The amplitude of oscillation will decrease due to friction. The triangle will stop in the preferential configuration.

Let us now suppose that it is released at rest with the CG outside the vertical plane passing through the skewer. In this case the system does not remain in equilibrium. The CG begins to oscillate around the vertical plane passing through the skewer, with the amplitude of oscillation decreasing due to friction, Figure 4.53 (b). It stops in the preferential configuration.

Experiment 4.37

We now assume that the CG of the triangle is located exactly at the axis of symmetry of the skewer, Figure 4.54.

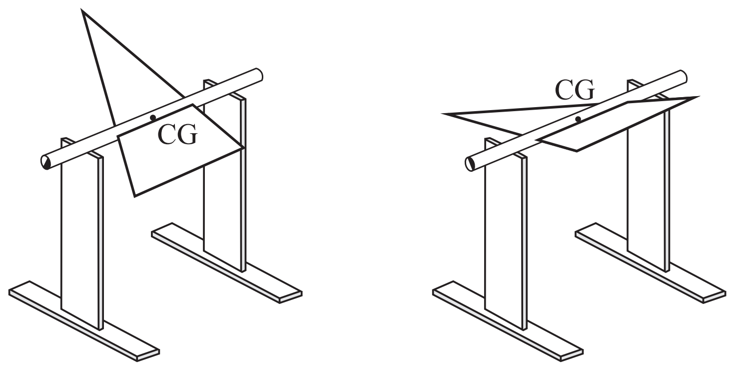


Figure 4.54: In this case the CG of the triangle is located along the axis of symmetry of the skewer. The system remains in equilibrium no matter its initial orientation relative to the ground.

The horizontal skewer is supported by two vertical rectangular poles. In this case the system remains in equilibrium after being released at rest, no matter its orientation relative to the ground.

Once more this situation is not exactly the situation described by definition $CG8$. Now the skewer is supported by the lower portions of its cross section which are in contact with the rectangular poles. On the other hand, the CG of the triangle is located exactly along the axis of symmetry of the skewer. And this axis of symmetry passes through the center of the skewer. Therefore, when the skewer rolls above the rectangular poles, the CG does not change its height relative to the ground. We have a neutral equilibrium as regards this kind of motion. It simulates the situation described in Figure 4.47.

4.14 The Definitive Definition of the CG Is Extremely Abstract

It is important to emphasize that the definitive definition of the center of gravity of a body given by $CG8$ in Subsection 4.12.1 is extremely abstract. This can be perceived by the conditional expressions "if the body is conceived to be suspended from this point". As we will see in Section 6.1, Archimedes himself seems to have defined the center of gravity in this abstract way.

In reality, it is impossible to suspend a rigid body exactly by its center of gravity in such a way that it is free to rotate in all directions around the CG. We only arrive at this definition of the CG by extrapolating the cases in which the body, when suspended from above by different points of suspension PS, remains in equilibrium when released at rest. The CG can then be found in practice by the intersection of all the vertical lines passing through these different points of suspension PS. The center of gravity can also be found when the body is supported from below by different auxiliary points PA, remaining in equilibrium. The CG is then found by the intersection of all the vertical lines passing through these different auxiliary points PA.

Although we cannot in practice locate the CG utilizing definition $CG8$, this abstract point can be located for any rigid body utilizing the practical procedures $CG6$ and $CG7$.

4.15 Summary

We now present the main conclusions we have reached thus far.

Definitions:

1. Equilibrium of a body is when the body and its parts do not move relative to the Earth.
2. The vertical is the straight line traced by a body in free fall at the surface of the Earth, beginning at rest. It is also indicated by the direction of a plumb line at rest relative to the ground. The vertical line is also indicated by the thread holding a balloon filled with helium, with the balloon at rest relative to the ground.
3. The horizontal is any straight line or plane orthogonal to the vertical.

4. The center of gravity of a rigid body is a point such that, if the body is conceived to be suspended by this point, having freedom to rotate in all directions around this point, the body thus supported, when released at rest, will remain stationary and preserve its original position, whatever its initial orientation relative to the Earth.

Experimental results:

1. The CG is unique for each rigid body.
2. Free bodies fall to the ground when released at rest.
3. The direction of free fall coincides with the direction of a plumb line in equilibrium.
4. The CG can be found in practice by the intersection of all the vertical lines passing through the points of suspension of the body when it remains in equilibrium, being free to rotate around these points of suspension.
5. Any body can remain in equilibrium after being released at rest, provided it is supported from below with its CG located vertically above the surface of contact between the body and the support.
6. Any body can also remain in equilibrium after being released at rest if it is suspended by a point PS around which the body is free to rotate, provided the CG is vertically below the PS.
7. Equilibrium can be stable, unstable, or neutral.
   (a) Equilibrium will be stable when any perturbation in the position of equilibrium of the body increases the height of the CG relative to the ground.
   (b) Equilibrium will be unstable when any perturbation in the position of equilibrium of the body decreases the height of the CG relative to the ground.
   (c) Equilibrium will be neutral or indifferent when any perturbation in the position of equilibrium does not change the height of the CG relative to the ground.
8. When there is stable equilibrium, any perturbation in the position of the body will cause it to oscillate around the position of equilibrium, with decreasing amplitude of oscillation due to friction, until it stops at the position of stable equilibrium.

9. When there is unstable equilibrium, any perturbation in the position of the body will move it away from this position and the body will not return to the original position.
10. The initial direction of motion for the perturbed body released at rest will be such that its CG moves downward from its initial height in the position of unstable equilibrium.

Until now we have only described these facts. We are not explaining the experimental data. We are merely summarizing the main results. We will now utilize these basic experimental facts to explain other phenomena that are more complex, but that can be derived from these observations.

Chapter 5

Exploring the Properties of the Center of Gravity

5.1 Fun Activities with the Equilibrist

One of the most interesting classroom activities utilizes a pasteboard equilibrist. It makes the students assimilate and incorporate all the concepts we have seen thus far. It is also fun, especially if performed with several people simultaneously. The idea is to give a problem to the students and to let them solve it by themselves. The teacher should not tell them how to solve the problem and should not explain the causes of the phenomena observed. Only the sequence of tasks needs to be given. This activity should be performed after the students have performed the main experiments already described. Each student should prepare his own equipment (bamboo skewer, plumb line, a pasteboard equilibrist, etc.), and also perform all the procedures described here. At the end of the activity the students should keep their apparatus as a gift.

Materials: A support with a plumb line. A pasteboard equilibrist, as in Figure 5.1, with the dimensions in centimeters. Some modeling clay. Single-hole punch-pliers.

The support with the plumb line could be a vertical barbecue bamboo skewer with the tip stuck in modeling clay, while a pin or needle is stuck horizontally in the top of the bamboo skewer. The plumb line can be made with sewing thread and a plumb or modeling clay at the bottom, as before. When the equilibrist becomes too heavy with the clay, so that the pin or

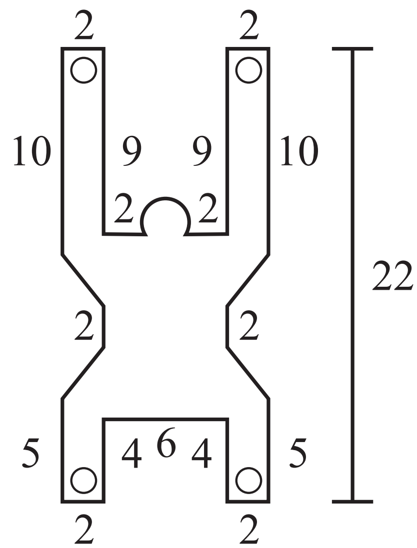


Figure 5.1: An equilibrist with dimensions in centimeters. There are holes in the hands and feet.

needle slips out of the bamboo skewer, or the plumb slips off the needle, we can support the equilibrist with a horizontal bamboo skewer on a table, sticking out from the table, with the plumb line tied to it. In this case the equilibrist will hang by the bamboo skewer itself passing through a hole in the pasteboard, instead of being suspended by the pin, as in the previous case.

The exact dimensions of the equilibrist are not so relevant. The most important aspect for the time being is that it should by symmetric around the body's axis, with the arms pointing up and the legs down, as in Figure 5.1. The arms should be longer than the legs, as in most situations the equilibrist will be upside down. The dimensions shown in Figure 5.1 are appropriate for the activities to be described, in which the paper puppet is balanced in the hands of the students.

Another very important property of the equilibrist is that it should be rigid, non-deformable. If we put a large amount of clay on it, a pasteboard equilibrist could bend. In order to prevent this deformation, the pasteboard should be rigid. We can even have an equilibrist made of stiff plastic, which is not so difficult to obtain. If the equilibrist is bent by the clay used in this experiment, what is described will not be observed in some cases.

Initially several identical equilibrists should be cut out from a pasteboard, so that each student receives one of them. The students should pierce the hands and feet of the equilibrist with single-hole punch-pliers, as shown in Figure 5.1. After this has been done, the first task is to locate the CG of the puppet using the two procedures the students have already learned:

(I) We find the point on which the equilibrist should be supported so that it remains in equilibrium horizontally above a vertical stand after being released at rest, Figure 5.2 (a).

(II) We suspend the equilibrist with a needle passing through the holes in the hands or feet. Nest, in each case, a vertical line is drawn using a plumb line.

The CG should be marked on the pasteboard, preferably on the front and back sides, as in Figure 5.2 (b).

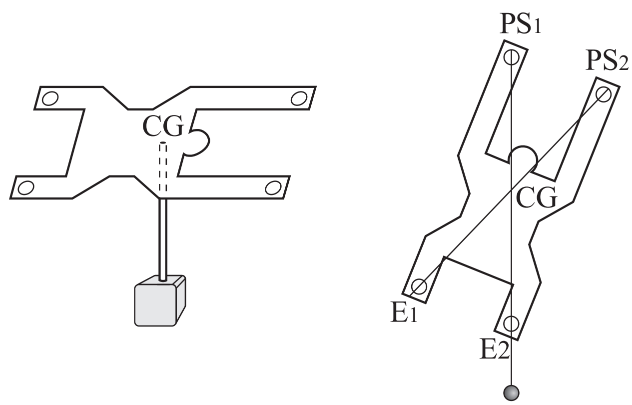


Figure 5.2: Finding the CG of the equilibrist by the first and second experimental procedures.

We then begin with the most interesting part of the activities. We ask the students to try to balance the puppet upside down, by placing it above the pointing finger. The finger should be extended horizontally, below the head of the equilibrist. After a few minutes of trials, no one succeeds. Some think that the problem is the curved shape of the head.

We then ask the students to try to balance the puppet with the head upwards and the pointing finger extended horizontally, as if the puppet was sitting on the finger. After several trials, no one succeeds, although now the line of contact is straight and horizontal. For the time being we should not try to explain why they have failed. The idea is to go on with the game.

We now ask them to balance the puppet in a horizontal position, placing the pointing finger vertically below it. Now all of them succeed. They easily observe that the puppet's CG is above the finger.

After this first activity, we again ask them to balance the puppet horizontally, but now with the pointing finger placed vertically below the head of the puppet. Once more no one succeeds, Figure 5.3 (a).

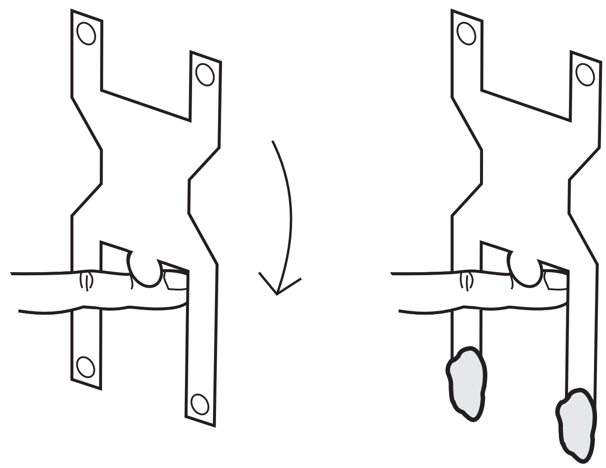


Figure 5.3: (a) We can not keep an equilibrist upside down in our finger. (b) However, we can keep an equilibrist upside down in our finger by placing enough modeling clay on both of the puppet's hands.

Now comes the most stimulating part of the game. We give a piece of modeling clay to each student. We again ask them to try to balance the puppet upside down, on the horizontal pointing finger placed below the puppet's head. We tell them that they can now attach the clay anywhere on the puppet, except on the "hair" of the puppet, that is, on the lower part of the head, to prevent it from sticking to the finger. They can put it on the CG, on the hands, on the legs or wherever they wish. We also tell them that the clay can be attached to the equilibrist as a single whole, or in two or more pieces. The idea here is to encourage the students to experiment, without giving recipes for the solution to the problem. They are shy and leery about what to do at first. But little by little they begin to relax and play the game. After a few minutes, one or two students succeed in balancing the puppet upside down on their fingers. The others begin to see what they have done, and in a short time all of them succeed. The secret of success is to place enough clay on both of the puppet's hands, until it remains upside down balanced on our horizontal pointing finger, as in Figure 5.3 (b).

When an equilibrist does not stay exactly on the vertical, all we have to do is move the clay away from the head (placing it at the tips of the hands, or even hanging from the hands), or increase the amount of clay on the hands. Eventually it will hang vertical and upside down.

After all the students have managed to do the experiment, we ask them to remove the clay and put it somewhere else on the puppet until it remains at rest vertically with the head on top, sitting upon the horizontally extended pointing finger. One or two students will managed this more quickly than before. The others see what they have done and sooner or later all have managed to get the puppet vertical. The secret of success is to place the clay on the feet of the equilibrist, as in Figure 5.4 (a).

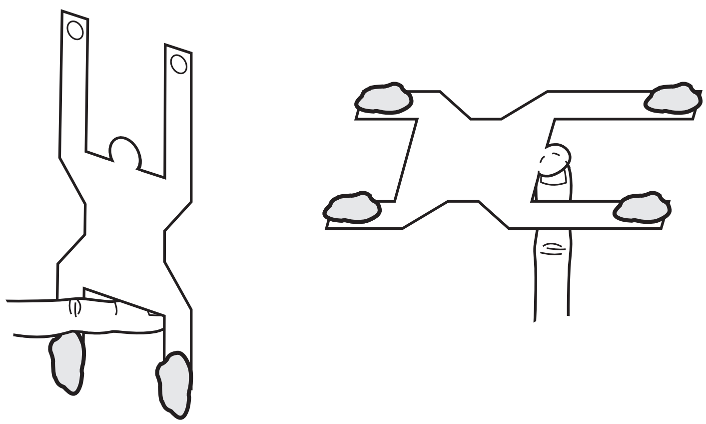


Figure 5.4: (a) Equilibrating the puppet in a vertical plane with the head on top, sitting upon the horizontally extended pointing finger, by placing enough modeling clay on both of the puppet's feet. (b) Equilibrating the puppet in a horizontal plane by placing the pointing finger vertically below the head. In both cases the trick is where to put the modeling clay and how much to use.

We then ask the students to change the position of the clay again until the puppet remains balanced horizontally, supported on the pointing finger extended vertically, placed under the head of the puppet. We ask them to avoid placing clay on the head of the puppet, to prevent it from sticking to the pointing finger. After some effort, all of them succeed. (Some students need to see what the others have done before they pick up the trick.) In this case they can attain the final result in several ways, as there is more than one possibility. A common technique is to place clay on the hands and feet in the right amounts until the equilibrist remains horizontal, as in Figure 5.4 (b).

After this part of the game, we again ask them to change the location of the modeling clay until the puppet remains upside down vertically, supported on the pointing finger extended horizontally under the head of the puppet. Now they will all quickly place enough clay on the hands of the puppet until it reaches the desired position, as in Figure 5.3 (b). To show that the equilibrium in this position is very stable, we ask them to rock or blow the equilibrist gently. We can also ask them to balance it over the flat tip of the bamboo skewer, then raise everything with their hands until the arms are stretched. We can even balance the puppet upside down supported on the horizontal needle attached to the bamboo skewer! Even in this case they can rock or blow air on the puppet gently, and it will not fall, but only oscillate around the equilibrium situation, always returning to its vertical position upside down. Everyone admires this. This is a remarkable and striking experiment that makes a deep impression on everyone. The stability achieved by this puppet is truly admirable.

We then ask the students about the location of the center of gravity in this new situation (equilibrist upside down with clay on the hands). A few of them may think it is located in the same place as before (in the middle of the chest), but the majority will believe that it is located in the head of the puppet, more specifically in its hair. In other words, they believe it is located at the point where the head touches the finger.

Without giving the correct answer, we ask them to locate exactly the CG using the second method mentioned previously. That is, to locate the CG by suspending the equilibrist (with clay on his hands) through the needle in the support. We first suspend it by the hole in one foot and draw the vertical line with the help of a plumb line, as in Figure 5.5 (a).

Then we suspend it by the hole in the other foot and draw the second vertical. We must tell the students that it is important to locate the CG precisely. This should be done carefully. When they try to do this, some of them say that the method "does not work," because the vertical lines do not seem to intersect (that is, they do not intersect where they expected). Despite this initial reaction, we ask them to continue with the experiment. The final result, when the verticals are carefully drawn, is something like the result shown in Figure 5.5 (b).

If we extend these two verticals, we see that they intersect outside the

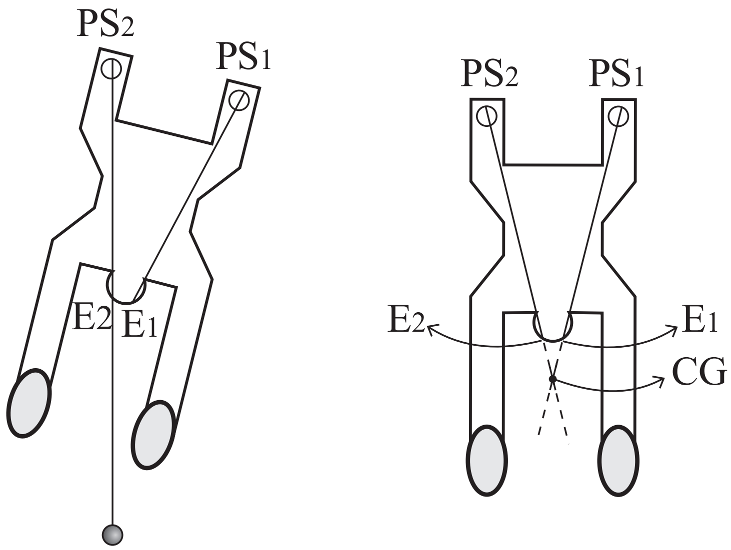


Figure 5.5: Finding the CG of the equilibrist with modeling clay in both hands.

head, at a point along the axis of symmetry of the puppet, between the head and the hands (or between the hair and the lower part of the clay), Figure 5.5 (b). It is interesting to ask the students to make a drawing like this in their notebooks, full size, utilizing their own puppet with clay on the hands as a model.

To find the exact location of the CG of the doll with modeling clay in its hands, students are asked to balance the doll on its side, in a vertical plane, resting some point of the arm on the horizontal pin, until the axis of the body is parallel to the horizontal. The center of gravity is located at the intersection of the body's axis of symmetry with the vertical line passing through the pin, obtained with the aid of a plumb line, Figure 5.6.

Only after the students have performed all these activities should the teacher explain what has happened. The explanation is that in the cases without clay it was not possible to balance the puppet upside down, nor seated upon the finger with the head at the top, because the CG located at the chest of the equilibrist was always above the auxiliary point of support PA. And these are conditions of unstable equilibrium. Any shaking of the finger or puppet causes the equilibrist to fall to the ground, because the tendency of the CG is always to approach the surface of the Earth, as in Figure 5.7.

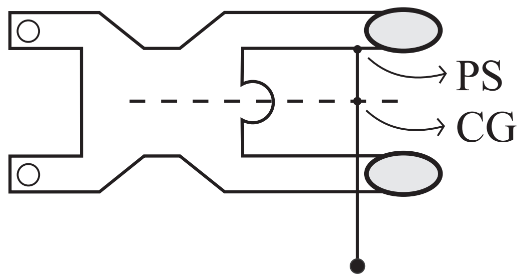


Figure 5.6: Another way of finding the CG of the equilibrist with modeling clay in both hands.

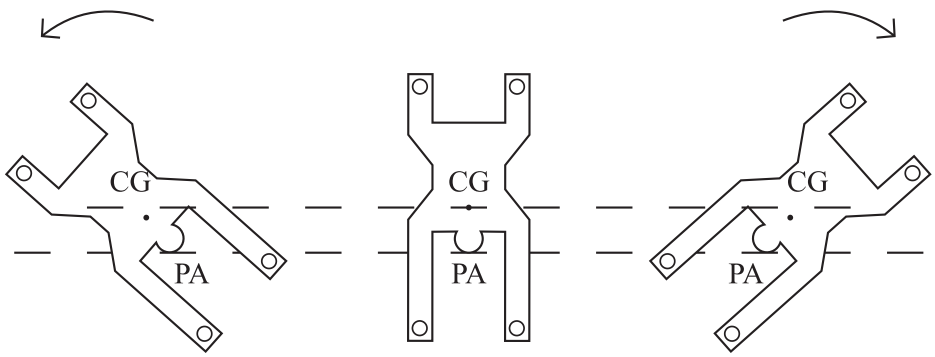


Figure 5.7: Unstable equilibrium.

By the same token, it was not possible to balance the puppet horizontally with the vertical finger under its head, because there was no support below the CG located in the middle of the chest. Therefore, when the puppet was released, the CG always fell to the ground.

On the other hand, when clay was placed on the hands of the puppet and it was balanced upside down on a finger placed under its head, the CG was located below our finger. That is, below the point of suspension, PS. This is a configuration of stable equilibrium. If we turn the puppet clockwise or anticlockwise, we raise the CG in relation to the original height of the CG in the position of equilibrium, as in Figure 5.8.

The same happens when we lean the puppet forward or backward, that is, with the nose or the back of the neck approaching the ground. Again we are raising the CG. This means that any rotation of the puppet around the point of suspension PS increases the height of the CG. As the tendency of the CG is always to fall due to gravity, the puppet tends to return to

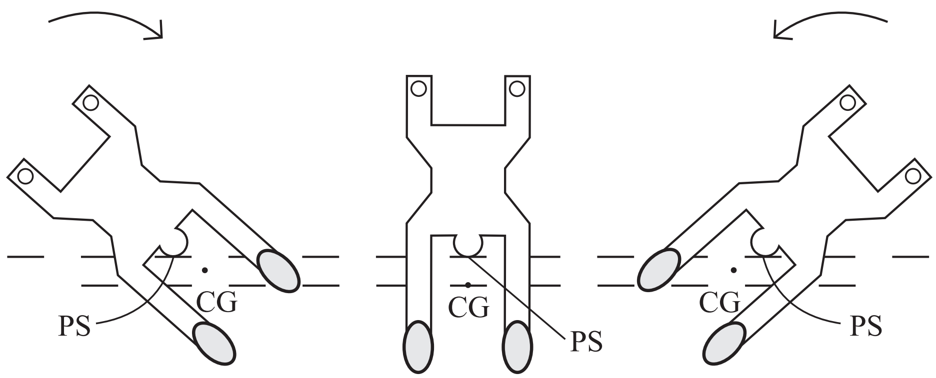


Figure 5.8: Stable equilibrium.

the position of stable equilibrium after it is released. In this upside down configuration the CG is in its lowest possible position.

When the puppet is sitting on our finger with modeling clay on the feet, the CG is again located between the bottom of the clay and the point of suspension PS (point of contact between our finger and the puppet), as in Figure 5.9.

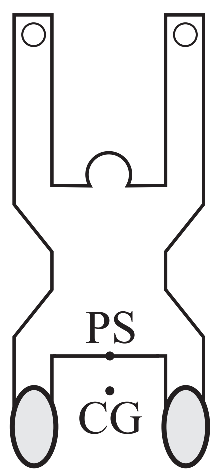


Figure 5.9: Another situation of stable equilibrium.

Any rotation of the puppet around the PS raises the CG. Gravity causes the CG to fall to the ground. After a few oscillations, the equilibrist remains seated on our finger at rest.

When we put clay on the hands and feet of the puppet, so that it stays lying down in a horizontal position, supported by our vertical pointing finger under its head, the CG is also located vertically below the point of suspension. In this case it is difficult to locate the CG exactly. But in Figure 5.10 we show a deformed puppet in order to illustrate the location of the CG.

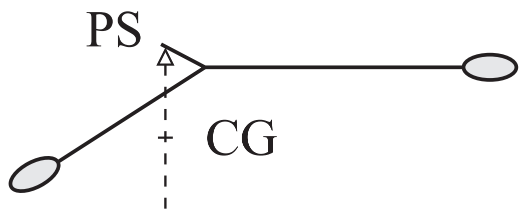


Figure 5.10: A horizontal puppet in stable equilibrium with modeling clay in both hands and feet.

The body is horizontal, the head is raised a little, the arms are inclined downward a little, and the clay is placed on the hands and feet. The point of suspension PS is represented by a small triangle placed below the head. The CG is located vertically below the PS.

All the phenomena observed with the equilibrist can be explained with the basic experimental data and properties of the CG we have already presented. But it is extremely important that all students perform these experiments themselves in the classroom, each with his own equilibrist and clay, because this creates a deep impression upon them. The feelings of mystery and awe stimulated by this experiment are really remarkable. With this playful experiment they learn the main concepts relating to the CG.

5.2 Equilibrium Toys

In addition to a male equilibrist, we can also make a female equilibrist, as in Figure 5.11. Instead of using modeling clay on the hands and feet, we can also use fishing sinkers or other appropriate material. For a more durable figure, it is best to use thin sheets of wood or plastic instead of the pasteboard.

Other symmetric figures can also be made, such as a butterfly, a parrot or a frog. The black points in these Figures represent extra weights (modeling clay, for example), as in Figure 5.12.[96]

[96] [Gas03, p. 141].

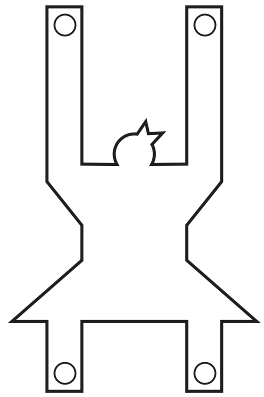


Figure 5.11: A female equilibrist.

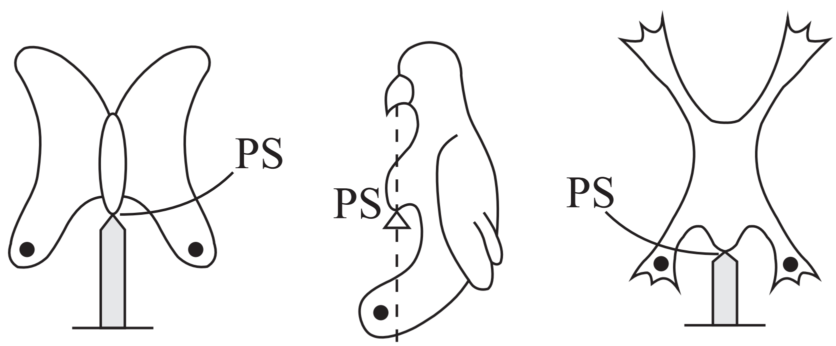


Figure 5.12: A butterfly, a parrot and a frog.

Some shops sell the bird equilibrist supported by its beak. Normally it is made of plastic, with shot hidden in the wings, and sometimes in the tail. It can also be made of pasteboard, as in Figure 5.13.

In this case we put clay or small shot in the wings and tail until it remains in equilibrium horizontally, supported on a vertical stand under the beak. Most people believe that in this case the CG is on the end of the beak, where it touches the vertical support. But as we have already seen, in a situation of stable equilibrium the CG is not exactly at the beak, but a little below it, between the beak and the lower part of the shot in the wings. When we remove the bird from this equilibrium position (by raising or lowering one of its wings, or by lifting or bringing down its tail), releasing it at rest, it oscillates around the equilibrium position, its amplitude of oscillation

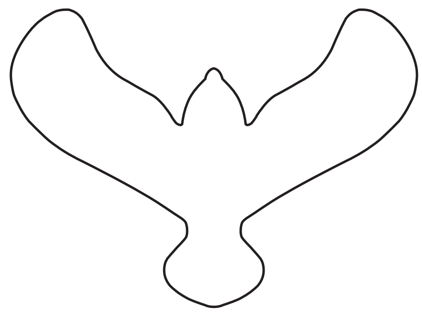


Figure 5.13: A bird equilibrist.

decreasing due to friction, until it returns to the horizontal position. In this stable position the CG is in its lowest possible location.

The pasteboard equilibrist works exactly like this bird when the puppet is balanced horizontally with the pointing finger placed vertically under its head. The appropriate weights placed at the hands and feet of the puppet lower the CG of the system, so that in horizontal equilibrium the CG is vertically below the head. The advantage of the pasteboard equilibrist as compared with the bird bought in shops is that by changing the amount and location of the clay we can use the equilibrist both horizontally, like the bird, and sitting on our hands with the head at the top, or upside down balanced vertically on our finger.

There are also equilibrium figures made of homogeneous sheets of wood or plastic which do not require any additional weights. Some of the most interesting examples are the macaw and toucan, as in Figure 5.14.[97]

These figures can also be made of rigid pasteboard. The foot can be a toothpick or a needle. In the toucan of Figure 5.14, the foot is only the pasteboard in the shape of a triangle. The important thing is that the macaw and toucan should have a large tail, such that the CG is located in the empty space between the foot and the tail. When this happens, the toucan remains balanced vertically supported by the tip of its foot. Any perturbation causes it to oscillate around this situation of stable equilibrium in which the CG is in its lowest position.

Another toy that is known to everyone is the roly-poly doll.[98] It is based on the same principles that we have seen thus far. To build a roly-poly we

[97] [Fersd].
[98] [Gas03, pp. 148-150].

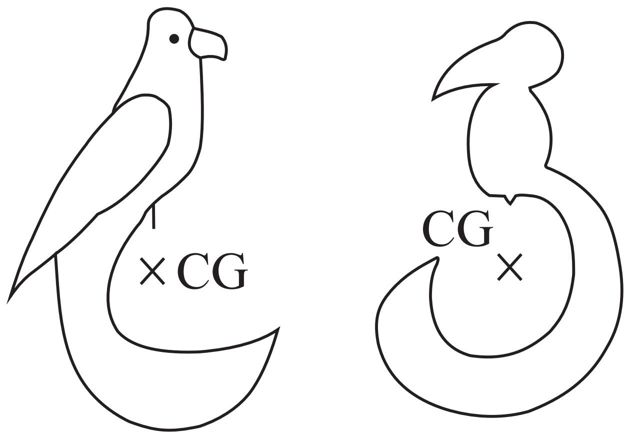


Figure 5.14: The macaw and the toucan.

need only two hemispheres or Styrofoam spherical shells, plus some shot, modeling clay, or another weight. The CG of the homogeneous sphere is located at the center of the sphere. The CG of the extra weight is located at the center of the extra weight, assuming it is spherical in shape. When we place the shot at the bottom of one of the hemispheres, the CG of the whole system is located between the shot and the center of the sphere, as in Figure 5.15.

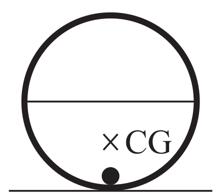


Figure 5.15: The roly-poly doll.

This is the position of stable equilibrium for the roly-poly doll, as the CG for the whole system is at its lowest position. By tipping the roly-poly clockwise or anticlockwise, we shift its CG away from the vertical passing through the new point of support, raising the CG. Gravity returns the doll to its stable position, as in Figure 5.16.

The flip-flop turtle is another interesting toy.[99] It is a different type of

[99][Gas03, pp. 151-153].

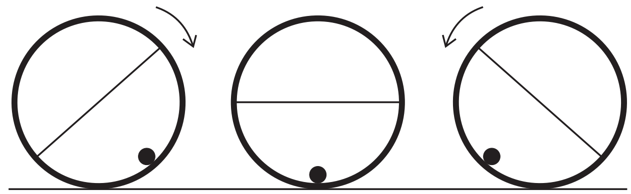


Figure 5.16: Stable equilibrium of the roly-poly doll.

roly-poly in which the extra weight is placed asymmetrically relative to the equator of one hemisphere, as in Figure 5.17.

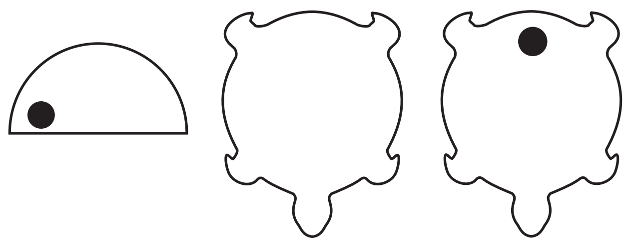


Figure 5.17: The flip-flop turtle.

In this case we use only one hemisphere, the extra weight and a plane pasteboard figure having the same diameter as the hemisphere, but with four legs and a head simulating the shape of a turtle. The weight should be placed opposite to the head. We can hold the turtle upside down with its legs in a horizontal plane, pressing it by its chin. When we release the turtle in this position, it turns over, returning to upright position, as in Figure 5.18.

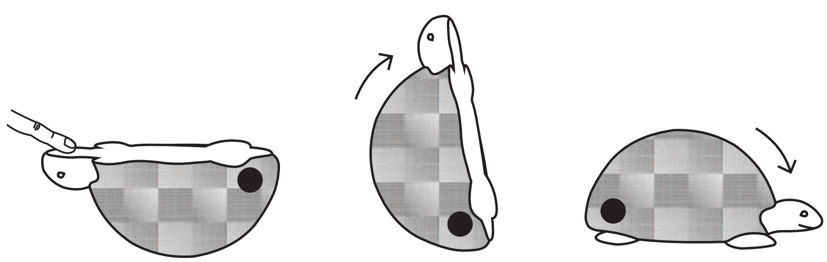


Figure 5.18: The flip-flop turtle in action.

The reason for this behavior is that the initial position is not a situation of equilibrium, because the CG is not in its lowest location. In the position of stable equilibrium the plane of the base and legs remains inclined relative to the vertical line. Small perturbations around this position cause the turtle to wobble around it. When we place the turtle upside down with its base horizontal and release it at rest, it begins to move, lowering its CG. But as it acquires enough kinetic energy and we have only one hemisphere (unlike the roly-poly doll which has an external spherical or symmetrical shape relative to the position of equilibrium), the turtle turns over when the plane of the base and legs go beyond the vertical line.

5.3 Equilibrium Games in the Pub

Equilibrium games are often found in pubs and bars. All of them can be explained by the principles presented here. Despite this, these games have surprising effects and leave a deep impression on everyone.

One of the most common is a needle or toothpick passing through the axis of a cork.[100] We then stick two metal forks in the cork, inclined downward toward the tip of the needle. The whole system can be balanced by placing the tip of the needle above a bottle, as in Figure 5.19 (a).

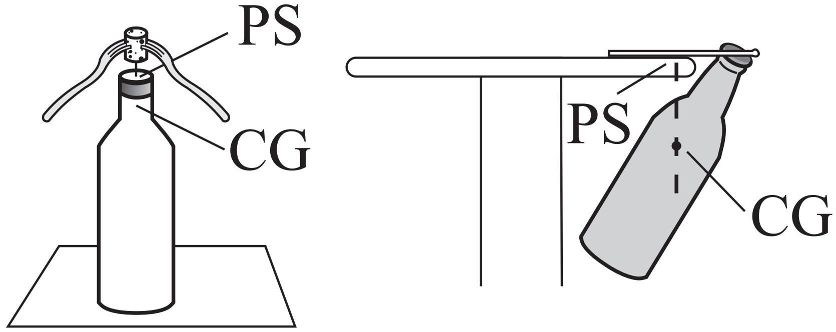


Figure 5.19: Two interesting equilibrium situations.

Most people think that the CG is at the tip of the needle. However, as a matter of fact, the tip of the needle is only the point of suspension PS of the system. In stable equilibrium, as we saw before, the CG is located vertically below the PS. In order to show that this is a condition of stable equilibrium, we can blow lightly on one of the forks so that the system turns around the

[100][Gas03, p. 144].

vertical axis. It is also possible to blow vertically from top to bottom on one of the forks (or to lower it a little with our finger, releasing it at rest). The system will oscillate around the horizontal plane, finally stopping at the equilibrium configuration.

Another interesting situation is a full bottle, with cap, supported at the edge of a thin table by a bottle opener, as in Figure 5.19 (b).[101] The PS along the plane of the bottle opener is once again vertically above the CG along the axis of symmetry of the bottle. To try this experiment, it is wise to place a pillow or cushion below the bottle. This will prevent it from breaking if it falls to the ground while we are performing the experiment.

One of the most remarkable experiments utilizes a metal fork with its teeth connected to a spoon. A toothpick is passed partly through the teeth of the fork. At this moment we can balance the system by placing our pointing finger vertically under the toothpick, as in Figure 5.20.

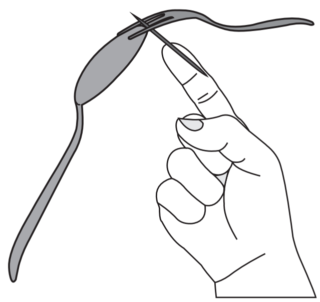


Figure 5.20: Another curious equilibrium situation.

This localizes the appropriate PS for the toothpick. We can then go on with the game. We now support a second toothpick in the mouth of an open bottle. Supporting the bottle firmly with our hands, we place the first toothpick with the PS above the upper tip of the second toothpick. With a little practice we can then finally release the system so that it remains in equilibrium, as shown in Figure 5.21.

Once more the CG of the system is located vertically below the PS. The amazing fact about this game is that the PS is supported by a single point, namely, the upper tip of the second toothpick. Many people are surprised by this equilibrium, because they incorrectly believe that the CG is exactly at the point of contact of the two toothpicks. Moreover, this is a highly

[101] [Gas03, p. 144].

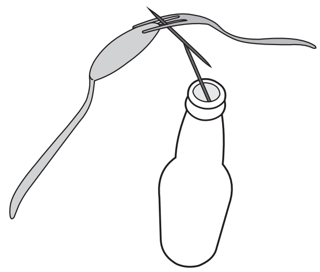


Figure 5.21: The first toothpick is supported above the upper tip of the second toothpick, supported in the mouth of an open bottle.

stable equilibrium. In order to show this, we only need to blow on the spoon horizontally, so that the system turns around the vertical direction passing through the PS. It is also possible to blow on the spoon from above (or lower it slightly with our finger, releasing it at rest). In this case the system oscillates around the horizontal plane, coming back to the original position of equilibrium.

5.4 Equilibrium of the Human Body

Several interesting experiments can be made related to the equilibrium of the human body.[102] The legs and arms of a person can move independently from the rest of the body, moving forwards, backwards, upwards or downwards. All these movements change the location of the person's CG.

Let us consider initially the situation in which a person is standing above a flat surface. The CG is located above the ground. As we have seen before, equilibrium is only possible in this case when the CG is vertically above the region of support. When a person is standing, the CG is approximately at the center of the chest. The person can then remain in equilibrium in this situation provided the vertical projection of the CG falls inside the region bounded by the feet, as in Figure 5.22 (a).

When the person stands with feet spread apart (wide support base), this region expands, as in Figure 5.22 (b). By this procedure, we increase the stability of this kind of equilibrium, as was seen in Equation (4.8).

In the first game we ask the person to bend at the waist, keeping knees straight, and touch the toes. Once this has been done, we ask the person

[102] [dCsd].

Figure 5.22: Region of equilibrium for a standing person.

to repeat the procedure, but this time, standing with the back up against a wall, with buttocks and heels touching the wall. This time it cannot be done. In order to understand what happens, it is best to ask the student to stand at the side of the classroom, in profile. The situation can be depicted on the blackboard. When the person is standing, the downward vertical projection of the CG passes through the feet. The person can only touch the toes by moving the buttocks backward and the head forward, in such a way that the projection of the CG continues to fall through the region enclosed by the feet, as in Figure 5.23 (a).

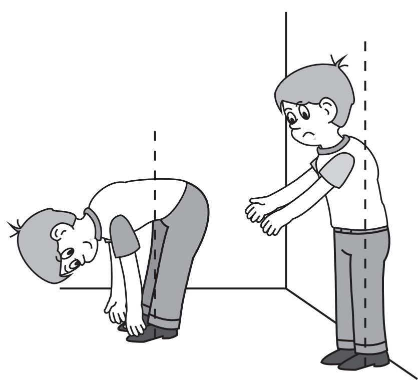


Figure 5.23: Equilibrium of the human body.

Now suppose the person stands with the back up against a wall. The person can no longer bend fully at the waist. When the arms and waist are lowered, the vertical projection of the CG falls outside the region between the feet, because the wall prevents the buttocks from moving backwards, as in Figure 5.23 (b). The person looses equilibrium and falls forward.

Another game involves raising the left foot to the side while standing on the right foot. Everyone can do this. We then ask for the person to repeat the procedure, but now standing with the right shoulder and right foot against the wall. No one can raise the left foot and stand on the right foot for a few seconds in this new situation, as in Figure 5.24 (a).

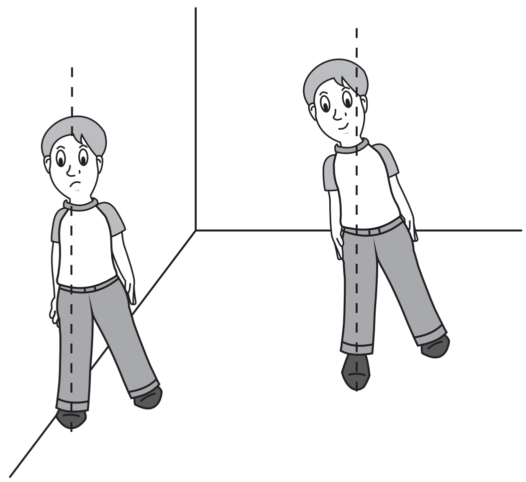


Figure 5.24: Another situation of equilibrium of the human body.

The explanation is the same as in the previous case. When the person is standing with both feet on the ground, the vertical projection of the CG falls between the feet. The person can only balance on the right foot while raising the left foot to the one side by leaning to the opposite side, in such a way that the vertical projection of the CG falls over the foot which is on the ground, Figure 5.24 (b). Now let us consider the case in which the person is standing with the right shoulder and right foot against the wall. When the person lifts the left foot, the body has a tendency to move to the opposite side, in order to maintain balance. But the rigid wall prevents the upper part of the body from moving. The vertical projection of the CG when the left foot is raised to the side now falls outside the region of the right foot, Figure 5.24 (a). The CG then starts moving toward the ground, the person looses balance, and cannot complete the movement.

A third game is based on the same principle. We ask the person to stand on the toes. Everyone can do this, Figure 5.25 (a).

We then ask the person to repeat the procedure, but now standing facing a wall, keeping the nose and toes touching the wall. Now the person cannot remain in equilibrium while standing on the toes, as in Figure 5.25 (b). The explanation is the same as in the other cases, but now with movements of

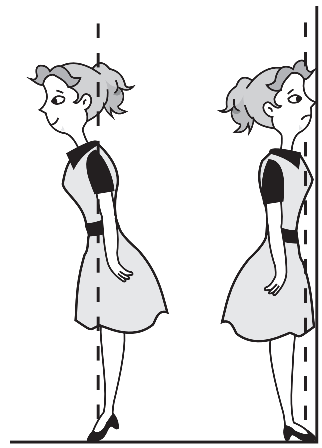


Figure 5.25: Equilibrium by standing on the toes.

smaller magnitude. That is, the wall prevents the forward motion of the body. When the person stands on the toes, the vertical projection of the CG falls behind the toes. The person looses equilibrium and can no longer stand on the toes for a few seconds.

One of the most interesting experiments of this kind shows a distinction in the location of the CG for men and women of the same height. As women have larger hips than men, their CG is a little lower than the CG of men of the same height and weight. We ask a woman to kneel down and touch the elbows with the knees, with the hands on the ground, as if praying. We place a match box on the ground, touching the tip of her fingers. We ask the woman to place her hands at the back and to try and knock down the match box with her nose, and then to come back to her initial position without touching the ground with her hands, as in Figure 5.26.

Figure 5.26: Equilibrium by kneeling down.

The majority of women can do this after a few trials. But men cannot normally do this. Let us consider the situation where the woman touches the match box with her nose. The vertical projection of her CG falls over the region occupied by her knees and feet. The CG of standing men is normally higher than the CG of standing women of the same height. If we suppose a man touching the match box with his nose, the vertical projection of his CG falls outside the region occupied by his knees and feet, and inside the region between the knees and the match box. As the tendency of the CG is to fall when there is no support below it, the man looses balance and cannot knock the match box down. If he tries to do this, he will fall to the ground and will not come back to his original position with his hands at his back, without first touching the ground with his hands.

Other situations of equilibrium occur when the CG is below the point of suspension PS. The most interesting example is a toy representing an acrobat on a tightrope in a circus. The CG of a person is normally in the middle of the chest. If the person is standing above a tight rope, it is difficult to keep the projection of the CG falling exactly above the small region occupied by the feet. Normally this is done by a continuous deformation of the body in order to achieve balance.

An alternative procedure in order to be in equilibrium above the tight rope is to hold a long curved stick with weights at the tips, as in Figure 5.27.

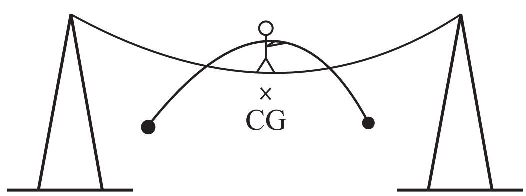


Figure 5.27: Acrobat on a tightrope.

The goal of this curved stick is to lower the CG of the system (person plus stick) below the feet. Any disturbance in the person's position will raise the CG. This will happen not only for clockwise and anticlockwise rotations, but also when the person leans forward or backward. As the tendency of the CG is to fall, the equilibrist ends in stable equilibrium, in which the acrobat stands vertically above the CG. This is a configuration of stable equilibrium. This is an idealized situation of equilibrium for rigid bodies. As an example we can have an equilibrist and the curved stick made of metal and rigidly connected to one another, as it happens in some toys.

The fun game we played with the pasteboard equilibrist presents a situation analogous to this one for a rigid body. Normally we cannot keep the pasteboard in balance seated on our finger. But when we place enough clay on the feet of the equilibrist, we can keep it balanced on our finger, with the body of the equilibrist in a vertical plane. No matter which direction it wobbles, it always returns to the position of stable equilibrium. In this condition the CG is in the lowest position, vertically below the PS.

This is the ideal situation of equilibrium for rigid bodies, as in some toys. For a real acrobat in a circus, the stick is sometimes straight and the CG of the system may be located above the feet of the tumbler. The person tends to fall after any disturbance. In order to maintain balance, the acrobat needs to be constantly in motion, bending and stretching his body in order to keep changing all the time the position of his CG. When the person is falling to one side, he moves the stick to the other side. The person and stick need to stay constantly in motion.[103]

5.5 The Extra-Terrestrial, ET

Another curious toy is the extra-terrestrial, also known as ET.[104] It can be made with two corks, two toothpicks, and four bamboo barbecue skewers, pieces of pasteboard for the hands and feet, plus a vertical stand to support it. Instead of the toothpicks we can also employ nails or needles.

The ET has two independent parts. If one of the corks is smaller than the other, it should be utilized in the upper part. We pass a toothpick, nail or needle through the axis of the cork. The bamboo barbecue skewers will form the arms of the ET, when inserted into the cork. They should be inclined downward, to the same side where the toothpick is pointing outward. This will also be the general shape of the body and legs of the ET, as in Figure 5.28 (a).

On the outer tips of the bamboo skewers we attach pieces of pasteboard in the shape of hands. After finishing the upper part of the ET, we try to balance it on our finger placed under the toothpick. If it falls to one side, we can increase the weight or size of the hands, or we can change the inclination of the bamboo skewers by placing them closer to the vertical direction (in order to lower the CG of the upper part). It is important to note that the CG of the upper part should be below the lower tip of the toothpick in order to achieve stable equilibrium, Figure 5.28 (b).

[103] [Wal07, Chapter 1, Section 1.91: Tightrope walk].
[104] [Fer06].

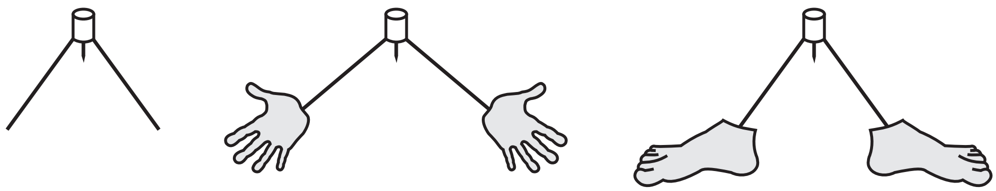


Figure 5.28: Making the two parts of the ET.

The lower part of the ET is made similarly. We may have to increase the weight of the feet relative to the weight of the hands in order to significantly lower the CG of the whole system. Once more, the lower part should be well balanced in a vertical plane before we proceed with the experiment, Figure 5.28 (c).

We can then support the upper part of the ET on the lower part, by balancing the upper toothpick on the lower cork. We next support the lower toothpick on a rigid support. The final setup should be similar to Figure 5.29.

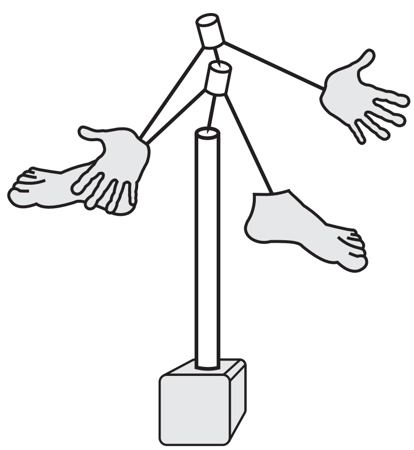


Figure 5.29: The complete ET.

This puppet is not a rigid body, as its two parts are free to wobble and turn independently of one another. Nevertheless each part of the ET can be considered, separately, a rigid body. By rocking or blowing the ET we produce some very curious motion.

Each one of these parts will only be balanced if its CG is below the tip of its toothpick. Moreover, the CG of the whole ET must be located below the tip of the lower toothpick. Nevertheless, there are two possible alternatives. In the first case, the CG of the upper part is below the tip of the lower toothpick. In the second case, the CG of the upper part is above the tip of the lower toothpick.

This is an amusing toy that can raise many questions from the students.

Chapter 6

Historical Aspects of the Center of Gravity

Figure 6.1: An engraving of Archimedes planning the defenses of Syracuse from *Les Vrais Pourtraits et Vies des Hommes Illustres* (Paris, 1584) by the French historian André Thévet (1516-1590). The Greek writing on his cap is *Archimedes the geometer*, [The84, p. 46] and [Rorsd].

6.1 Definition and Comments of Archimedes, Heron, Pappus, Eutocius and Simplicius on the Center of Gravity

In this Section we will see several definitions of the CG that have been presented through the centuries. We will see that it has always been difficult to find appropriate words to define the CG in a general way. Several important authors have dealt with this subject. In Chapter 9 we will deal with the theoretical calculation of the CG. For the time being it is important to keep in mind the general definition $CG8$, Subsection 4.12.1, and the practical procedures to locate the CG given by $CG6$ and $CG7$, Subsections 4.7.1 and 4.8.1, respectively.

We now discuss a few historical aspects of the concept of center of gravity, CG. In particular, we will analyze how the concept was defined and how it was obtained experimentally. We are interested in the period in which this concept originated and was established. The information here is drawn essentially from the original treatises of Archimedes, Heron and Pappus; and from the works of Heath, Duhem, Drachmann and Dijksterhuis.[105]

The observation that bodies fall to the ground when released at rest above the Earth is extremely old. The same can be said of the fact that rigid bodies can remain in equilibrium after being released at rest when they are supported by a rigid stand placed below some specific point. It is probable that all ancient civilizations knew this. Nevertheless, the systematic and scientific treatment of the conditions which determine the equilibrium of bodies upon the surface of the Earth originated in Greece. At least Greece is the origin of the oldest documents dealing with the CG that give theoretical results on the subject.

Archimedes is the main person who investigated this concept in ancient Greece. The CG is also called barycenter. The prefix "bary" is a Greek root meaning weight or heavy. The literal meaning of the word barycenter is "center of weight." The simplest way to understand this expression and the concept behind it is to observe the experiment where we supported a pasteboard triangle in a horizontal plane, standing on a vertical support placed under its centroid. The triangle only remains in equilibrium after

[105] [Arc02b], [Her79], [Her88], [Pap82], [Hea21], [Dra63], [Dra68], [Dij87], [Duh05], [Duh06] and [Duh91].

being released at rest when supported by this point. The whole weight of the figure is supported by this point, as if it were concentrated in it. It is then natural to call this specific point the center of weight, or barycenter, of the triangle.

The oldest extant work of Archimedes is called *On the Equilibrium of Plane Figures*, or *On the Center of Gravity of Plane Figures*.[106] The center of gravity appears in postulates 4 to 7, without any prior definition:[107]

> Postulate 4: When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide.
>
> Postulate 5: In figures which are unequal but similar the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.
>
> Postulate 6: If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.
>
> Postulate 7: In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure.

In all likelihood the CG had been defined by Archimedes in one of his other works on mechanics that is no longer extant, namely, *On the Centers of Gravity*, *Elements of Mechanics*, *On Equilibria*, *On Balances* or *On Levers*, and *Book of Supports*.

In Proposition 6 of his work *Quadrature of the Parabola*, Archimedes wrote:[108]

> Every suspended body — no matter what its point of suspension — assumes an equilibrium state when the point of suspension and the center of gravity are on the same vertical line. This has been demonstrated.

This shows that Archimedes' knew the practical procedure $CG6$ of how to find the CG experimentally. That is, we suspend the rigid body by a

[106] [Arc02b, p. 189] and [Dij87, p. 286].
[107] [Arc02b, pp. 189-190].
[108] [Duh91, p. 463], [Duh06, p. 307] and [Mug71a, p. 171].

point of suspension PS_1, wait until the body reaches equilibrium and draw the vertical passing through the PS_1 with the help of a plumb line. We suspend the body by another point of suspension PS_2 which is not along the first vertical, wait until it reaches equilibrium, and draw a second vertical through PS_2. The intersection of the two verticals is the CG of the body. But it is important to emphasize that to Archimedes this was not a definition of the CG. Instead, he proved this result theoretically utilizing a previous definition of the CG of a body, as well as some postulates that are now unknown.

The crucial sentence in the previous paragraph, mentioning that this Proposition has been demonstrated for *every body*, does not appear in its full generality in Heath's translation of Archimedes' work. Heath's work is a paraphrase, that is, it conserves Archimedes original ideas, but rephrases them in modern notation and omits parts of the text which he did not consider essential. Here is Heath's presentation of Archimedes' key Propositions 6 and 7 of his work *Quadrature of the Parabola*.[109] In these Propositions the expression $\triangle BCD$ means the area of the triangle BCD, which is supposed to have uniform density. That is, its weight is proportional to its area, the same holding for the area P of the rectangle, which he will use in this Proposition.

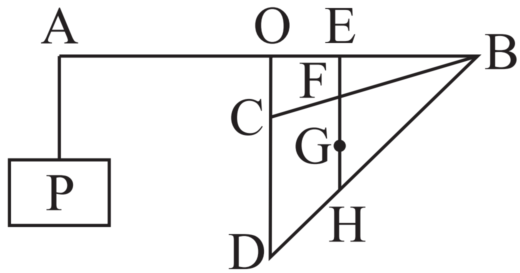
> **"Propositions 6, 7**[110].
>
> *Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD in which the angle C is right or obtuse be suspended from B and O, so that C is attached to O and CD is in the same vertical line with O. Then, if P be such an area as, when suspended from A, will keep the system in equilibrium,*
>
> $$P = \frac{1}{3}\triangle BCD \ .$$

[109] [Arc02b, p. 238].

[110] [Note by Heath:]

> In Prop. 6 Archimedes takes the separate case in which the angle BCD of the triangle is a right angle so that C coincides with O in the figure and F with E. He then proves, in Prop. 7, the same property for the triangle in which BCD is an obtuse angle, by treating the triangle as the difference between two right-angled triangles BOD, BOC and using the result of Prop. 6. I have combined the two propositions in one proof, for the sake of brevity. The same remark applies to the propositions following Prop. 6, 7.

Take a point E on OB such that $BE = 2OE$, and draw EFH parallel to OCD meeting BC, BD in F, H respectively. Let G be the middle point of FH.



Then G is the centre of gravity of the triangle BCD.

Hence, if the angular points B, C be set free and the triangle be suspended by attaching F to E, the triangle will hang in the same position as before, because EFG is a vertical straight line. "For this is proved[111]."

Therefore, as before, there will be equilibrium.

Thus

$$P : \triangle BCD = OE : AO = 1 : 3 ,$$

or

$$P = \frac{1}{3}\triangle BCD .''$$

Eutocius of Ascalon (480-540) wrote commentaries on three works by Archimedes: *Measurement of a Circle*, *On the Sphere and Cylinder*, and *On the Equilibrium of Plane Figures*. Apparently he did not know the other works. In his comments on Book I of *On the Equilibrium of Plane Figures*, Eutocius clarifies a few points regarding the CG. These ideas are from Eutocius, not Archimedes, but are interesting nevertheless. We translate them from the French version published by Charles Mugler in 1972, which is a literal translation from the Greek:[112]

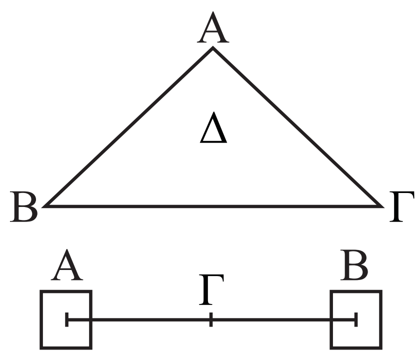
[111] [Note by Heath:]

Doubtless in the lost book $\pi\epsilon\rho\acute{\iota}\ \zeta\upsilon\gamma\hat{\omega}\nu$. Cf. the Introduction, Chapter II., *ad fin.* [Compare the Introduction, Chapter II, near the end.]

[112] [Mug72, pp. 166-167].

Commentaries of Eutocius relative to Book I of Archimedes' work *On the Equilibrium of Plane Figures*.

Introduction to book I. [...] In this work Archimedes defines the center of motion of a plane figure as the point such that, when we suspend the figure by this point, it remains parallel to the horizon, and defines the center of motion or of gravity of two or of several plane figures as the point such that, when we suspend the figures by this point, the beam (sc.[113] connecting the figures) remains parallel to the horizon.



Let, for instance, $AB\Gamma$ be the triangle and inside it the point $\triangle$, such that when the triangle is suspended by this point, it remains parallel to the horizon. Therefore, it is clear that the parts B and Γ of the triangle balance one another and that none of them inclines more than the other relative to the horizon. In the same way, let AB be the beam of a balance and A and B the magnitudes suspended by it. If the beam, being suspended by Γ, keeps A and B in equilibrium, and remains parallel to the horizon, Γ will be the point of suspension of the magnitudes A and B.

These are clear and intuitive definitions, as we saw in the experiments presented earlier. But they are limited, because they do not deal with concave figures or figures with holes, for which the CG is located in empty space. Moreover, they do not apply to three-dimensional bodies. In spite of these facts, these definitions illustrate many important aspects of the CG. It is

[113]The abbreviation "sc." stands for *scilicet*, a Latin adverb meaning "namely," "that is to say," or "it is permitted to know." It is sometimes used like here to indicate that a word (or a group of words) is missing from a text.

also interesting to see the alternative expressions utilized for the CG: center of motion and point of suspension.

To form an idea of how the concept of CG might had been defined by Archimedes, we quote here a few passages from the work *Mechanics* by the mathematician Heron (first century AD), from the *Mathematical Collection* by the mathematician Pappus (fourth century AD), and from the *Commentaries* of the philosopher Simplicius (sixth century AD), about *On the Heavens*, of Aristotle (384-322 BC). These authors discussed Archimedes' works, quote some of his works no longer extant and, probably, followed his concepts and lines of reasoning when dealing with the barycentric theory.

There is much controversy about the period in which Heron of Alexandria lived, but nowadays it is agreed that he flourished in the first century AD. There are only fragments of his book *Mechanics*, in three parts, in Greek. But a complete Arabic translation of this work has been preserved. Translations have been made to other modern languages (French, in 1893, and German, in 1900) from this Arabic version.

Heron presented a definition of the CG as given by the stoic Poseidonius (or Posidonius):[114]

> The center of gravity or of inclination is a point such that when the weight is suspended from that point, it (the weight) is divided into two equal parts.

Heath translates this sentence as:[115]

> The centre of gravity or of inclination is a point such that, if the body is hung up at it, the body is divided into two equal parts.

This definition is vague and problematic. In the first place it is difficult to know how a point, or even a vertical line passing through this point (if we interpret Poseidonius sentence thus) can divide a three-dimensional body into two parts. Even if the body is a plane figure, a point will not divide it into two parts. And a straight line will only divide a plane figure into two parts if it lies in the same plane as the figure. Therefore, we would need to imagine a triangle, for instance, suspended in a vertical plane. But even in this case not all verticals passing through the CG will divide the triangle into two parts

[114] [Her79, p. 42] and [Her88, Chapter 24, p. 93].

[115] [Hea21, p. 350].

with equal area or with equal weight. Let us suppose a homogeneous triangle suspended in a vertical plane. We have seen in Section 4.3 that a straight line passing through the CG and through a vertex divides the triangle into two parts with the same area and the same weight. On the other hand, a straight line parallel to the base of the triangle and passing through the CG does not divide the triangle into two equal parts, see Figure 4.7. Despite this fact, a triangle hanging in a vertical plane will remain in equilibrium after being released at rest if it hangs by the CG or by any other point which is vertically above the CG. The same problem will arise with Poseidonius' definition even if we interpret it as saying that the CG is a point such that, if the body is suspended from it, the body is divided by any vertical plane through the point of suspension into two equal parts. In this case we can imagine a triangle balanced on a horizontal plane, supported on a vertical plane placed below it (as a matter of fact the support needs to have a small thickness, like the edge of a ruler). If the vertical plane passes through a vertex and the CG, the body will remain in equilibrium and the upward projection of this plane will divide the triangle into two equal areas or into two equal weights. On the other hand, if the vertical plane is parallel to the base of the triangle and passes through the CG, its upward projection will not divide the triangle into two equal areas nor into two equal weights. Yet the triangle will also remain in equilibrium after being released at rest, as was seen in Experiment 4.5.

Another expression utilized by Heron to designate the CG, apart from "center of gravity," is "center of inclination" or "center of fall." This expression was probably also utilized in ancient Greece. This is a very interesting and instructive expression. We saw that any body denser than air tends to fall toward the ground when released at rest. If the body is suspended by a point of suspension PS and released at rest, so that it can turn around this point, the initial motion of the CG (supposing that it does not coincide with the PS) is toward the ground. Therefore, it behaves as if the tendency to fall were concentrated at the CG of the body.

Heron then says that Archimedes distinguished between the "center of inclination" and the "point of suspension" (or between the "center of gravity" and the "point of support"). Heron continues:[116]

> **As for the point of support, it is the point on a body or an incorporeal figure such that when the object is**

[116][Her79, p. 42] and [Her88, Chapter 24, p. 93].

suspended from that point, its parts are in equilibrium. By this I mean, it is neither depressed nor elevated.

The expression "incorporeal figure" may mean the cases in which the CG is located in empty space, like the CG of a ring.

Heath translated this sentence as follows:[117]

> **The point of suspension is a point on the body such that, if the body is hung up at it, all the parts of the body remain in equilibrium and do not oscillate or incline in any direction.**

What is called the "point of suspension" here and the definition Heron gave of it may have been how Archimedes defined the center of gravity. Later on we will see an analogous definition by Pappus.

Heron also writes:[118]

> The center of inclination in each body is one single point to which converge all the vertical lines through the points of suspension. The center of gravity in certain bodies is outside the substance of these bodies; this is what happens, for instance, in arches and in bracelets. All the lines following the projections of the ropes converge at a common point.

He appears to have described the practical procedure $CG6$ here, that is, to find the CG through the intersection of all verticals passing through the points of suspension when the body is in equilibrium, at rest relative to the Earth. This is the most important practical procedure to locate the CG. Heron mentioned that this CG may be located in empty space, outside the substance of the bodies, as is the case for rings or wheels.

Heron mentioned that Archimedes solved problems like the following in his book *On Columns* or *On Supports*:[119] A heavy beam or a wall supported on a number of pillars, equidistant or not, even or not even in number, and projecting or not projecting beyond one or both of the extreme pillars, finding how much of the weight is supported on each pillar. Heron also says that the

[117] [Hea21, p. 350].

[118] [Her88, Chapter 24, p. 95].

[119] [Her88, Chapters 25-31] and [Hea21, p. 350].

same principles can be applied when the body (beam or wall) is suspended by cables. In another part of his book Heron considered the problem of a triangle of uniform thickness, with its plane horizontal, supported by a pillar under each vertex. He then found the weight supported by each pillar in several cases: (a) when they support only the triangle, (b) when they support the triangle plus a given weight placed at any location over the triangle. Then Heron found the CG of the system when known weights were placed over the vertices of the triangle. He then extended his analysis to polygons.

Heron mentioned also the following:[120]

> So Archimedes says weights will not incline upon a line or upon a point.

That is, we can prevent a body from falling towards the Earth supporting it along a straight line, or supporting it above a fixed point. Related to this aspect, Pappus considered a body supported in a single point by a vertical stick placed below the body. He mentioned that:[121]

> If the body is in equilibrium, the line of the stick produced upwards must pass through the centre of gravity.

This is analogous to our practical definition $CG7$ on Subsection 4.8.1.

Pappus presented an explicit definition of the CG, namely:[122]

> **We say that the center of gravity of any body is a point within that body which is such that, if the body be conceived to be suspended from that point, the weight carried thereby remains at rest and preserves its original position.**

Heath presented this definition of the CG as:[123]

> **The point within a body which is such that, if the weight be conceived to be suspended from the point, it will remain at rest in any position in which it is put.**

[120] [Her79, p. 43] and [Her88, pp. 93-94].
[121] [Hea21, p. 350], see also [Pap82, pp. 817-818].
[122] [Pap82, Book VIII, p. 815] and [Dij87, p. 299].
[123] [Hea21, p. 430].

Drachmann presented this definition as follows:[124]

> **We say that the centre of gravity of any body is a point situated within, and such that if the body is imagined to be suspended from it, the weight will be at rest as it hangs and will keep its original position.**

In another context Pappus wrote:[125]

> **It is also clear that, if the grave is suspended by thought at this center, it will not turn around and will remain at rest, preserving the initial position it took in its solicitation;[126] for, all the planes extended by this center divide the grave into balanced parts; so that it will not be affected by any cause of revolution, [its parts, formed on either side of the point, are balanced in all positions].[127]**

Solicitation here means tendency to fall toward the Earth due to gravity. Simplicius quotes a similar definition by Archimedes:[128]

> **The centre of gravity is a certain point in the body such that, if the body is hung up by a string attached to that point, it will remain in its position without inclining in any direction.**

We can illustrate this definition in Figure 6.2, which is a combination of Figures 4.45, 4.46 and 4.47. This is a thought experiment, as it may be impossible to suspend a body exactly by its CG, with the body free to rotate in any direction around this point. In any event, the idea is that if we could conceive an experiment like this, what would happen is that the body would remain in equilibrium, no matter its initial orientation relative to the ground.

---

[124] [Dra63, p. 102].

[125] [Pap82, Book VIII, p. 818].

[126] [Note by Paul Ver Eecke:] $\varepsilon\nu\ \tau\eta\ \varphi o\rho\alpha$, in (its) solicitation (towards the center of the earth), expression that Hultsch renders by the neo-Latin word created by Newton "gravitatio", gravitation (Cfr. *loc. cit.*, vol. III, p. 1033, l. 26).

[127] [Note by Paul Ver Eecke:] The phrase we place in brackets is considered by Hultsch to be an interpolation (cfr. *loc. cit.*, vol. III, p. 1032, ll. 32-33).

[128] [Hea21, pp. 24 and 350].

A horizontal triangle, for instance, will remain in equilibrium no matter the value of the angle α between the segment CGV_1 and the segment CGE. Here CGV_1 is the segment connecting the triangle's CG with one of its vertices V_1, while CGE is the segment connecting the triangle's CG with the East-West direction, Figure 6.2 (a). A vertical triangle will remain in equilibrium for all angles β between the segment CGV_1 and the vertical indicated by a plumb line, Figure 6.2 (b). Consider now a triangle in which the normal to its plane makes an angle γ with the vertical, Figure 6.2 (c). When it is supported exactly at its CG, the triangle will remain in equilibrium no matter the value of γ.

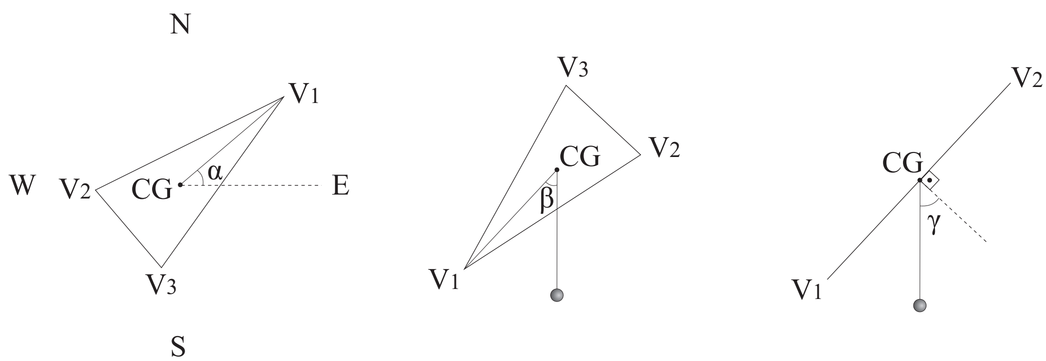


Figure 6.2: A body suspended exactly at its center of gravity remains in equilibrium for all orientations it may have relative to the Earth.

In a real experiment in which the body is suspended by a point of suspension PS which does not coincide with the CG, being free to rotate around the PS, the body will only remain in equilibrium after being released at rest if it is in the preferential position in which the PS and the CG are along a vertical line, with the PS vertically above the CG. If this is not the case, the body will turn around the PS after being released, in such a way that the CG approaches the ground. In equilibrium the PS and the CG will be along a vertical line, with the CG below the PS.

Pappus described a practical procedure for locating the CG.[129] He imagined a rectangular vertical plane over which a body will be suspended, balanced on the upper horizontal edge of the plane. The plane extended upwards divides the body into two parts which balance one another. Next the body was supported over the same upper horizontal edge of the plane again, but

[129] [Pap82, Book 8, pp. 816-818].

this time with the body in a different orientation relative to the ground. The plane extended upward again divides the body into two parts which balance one another. These two planes extended upward meet at a single vertical line. The body also remained in equilibrium when supported by a vertical line extended upward, as if it were supported at a point by a vertical stick. He repeated the procedure of balancing the body over the vertical rectangle in two new orientations of the body and obtains another vertical line (the intersection of these two new planes extended upwards). The intersection of both vertical lines is the CG of the body. Pappus than said the following:

> It is also clear that, if the grave is suspended by thought at this center, it will not turn around and will remain at rest, preserving the initial position it took in its solicitation; for all the planes extended by this center divide the grave into balanced parts; so that it will not be affected by any cause of revolution.

According to Pappus, this is the most essential part of the barycentric theory. Moreover, Pappus said that the elements that are demonstrated by means of this doctrine are taught in the books *On Equilibria* by Archimedes and in Heron's *Mechanics*.

The procedure described by Pappus is analogous to our practical definition $CG7$. In other words, if a rigid body is supported at a point PA_1 by a vertical stick, the line of the stick extended upward (that is, the vertical V_1 passing through PA_1) must pass through the CG. We now imagine that the body with a new orientation relative to the ground is balanced at another point PA_2 by the same stick. The line of the stick extended upward is another vertical V_2 passing through PA_2. The intersection of these two verticals is the CG. This is analogous to the intersection of two verticals extended downward by two points of suspension, described by the practical procedure $CG6$.

Everything we have seen so far suggests that Heron, Pappus and Simplicius directly consulted certain treatises by Archimedes that are no longer extant. The definitions in boldface in this Section presented by Heron, Pappus and Simplicius are analogous to our definition $CG8$, Subsection 4.12.1. These authors also proposed practical procedures for locating the CG analogous to our $CG6$ and $CG7$, Subsections 4.7.1 and 4.8.1, respectively.

6.2 Theoretical Values of Center of Gravity Obtained by Archimedes

Here we cite the theoretical values obtained by Archimedes for the centers of gravity of discrete bodies and several one-, two- and three-dimensional figures. There are proofs of most of these results in the known works of Archimedes. In some cases, such as the CG of the cone, Archimedes gives only the results, stating that they had been proved previously. However, the calculations are not to be found in any of his extant works. It is presumed that he calculated them in another work which has been lost during the last two thousand years. Recently we calculated the center of gravity of the cone utilizing Archimedes' original method.[130]

6.2.1 Discrete Bodies

Proposition 4 of *On the Equilibrium of Plane Figures*:[131]

> If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity.

Proposition 5 of the same work:

> If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude.

Corollary 1 of the same work:

> The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal, while the distances between their centres of gravity are equal.

Corollary 2 of the same work:

[130] [MA12] and [AM16, Appendix].
[131] [Arc02b, p. 191].

> If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones.

Proposition 6 of the same work:

> Suppose the magnitudes A, B to be commensurable, and the points A, B to be their centres of gravity. Let DE be a straight line so divided at C that
>
> $$A : B = DC : CE \ .$$
>
> We have then to prove that, if A be placed at E and B at D, C is the centre of gravity of the two taken together.

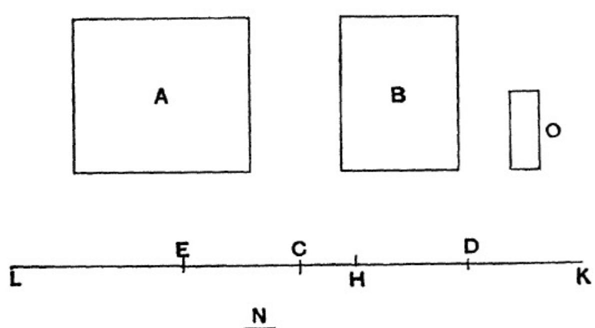


Figure 6.3: Proposition 6 of *On the Equilibrium of Plane Figures*, Book I.

Proposition 7 of the same work:

> Suppose the magnitudes to be incommensurable, and let them be $(A+\alpha)$ and B respectively. Let DE be a line divided at C so that
>
> $$(A+\alpha) : B = DC : CE \ .$$
>
> [...] Hence $(A+\alpha)$, B taken together have their centre of gravity at C.

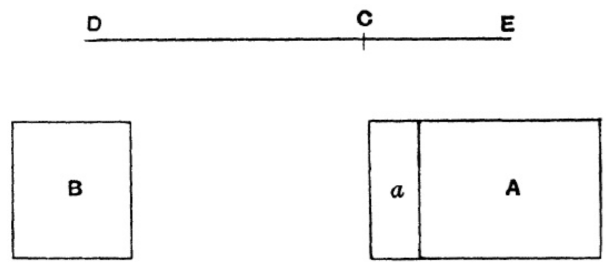


Figure 6.4: Proposition 7 of *On the Equilibrium of Plane Figures*, Book I.

6.2.2 One-dimensional Figures

Center of gravity of a straight line:

The Method of Mechanical Problems:[132]

> The centre of gravity of any straight line is the point of bisection of the straight line.

In Heath this is Lemma 3, while in Mugler it is Lemma 4.

6.2.3 Two-dimensional Figures

Center of gravity of a parallelogram:

In *On the Equilibrium of Plane Figures*, Book I, Proposition 10:[133]

> The centre of gravity of a parallelogram is the point of intersection of its diagonals.

In *The Method of Mechanical Problems*:[134]

> The centre of gravity of any parallelogram is the point in which the diagonals meet.

In Heath this is Lemma 5, while in Mugler it is Lemma 6.

[132] [Arc02a, p. 14] and [Mug71b, p. 85].
[133] [Arc02b, p. 195].
[134] [Arc02a, p. 14] and [Mug71b, p. 85].

Center of gravity of a triangle:

In *On the Equilibrium of Plane Figures*, Book I, Proposition 14:[135]

> The centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.

In *The Method of Mechanical Problems*:[136]

> The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another.

In Heath this is Lemma 4, while in Mugler it is Lemma 5.

Center of gravity of a trapezium:

In *On the Equilibrium of Plane Figures*, Book I, Proposition 15:[137]

> In any trapezium having two parallel sides the centre of gravity lies on the straight line joining the middle points of the parallel sides, in such a way that the segment of it having the middle point of the smaller of the parallel sides for extremity is to the remaining segment as the sum of double the greater plus the smaller is to the sum of double the smaller plus the greater of the parallel sides.

Heath presented this proposition as:[138]

> If AD, BC be the two parallel sides of a trapezium $ABCD$, AD being the smaller, and if AD, BC be bisected at E, F respectively, then the centre of gravity of the trapezium is at a point G on EF such that $GE : GF = (2BC + AD) : (2AD + BC)$.

[135] [Arc02b, p. 201].
[136] [Arc02a, p. 14] and [Mug71b, p. 85].
[137] [Dij87, p. 312].
[138] [Arc02b, p. 201].

Center of gravity of a circle:

In *The Method of Mechanical Problems*:[139]

> The centre of gravity of a circle is the point which is also the centre [of the circle].

In Heath this is Lemma 6, while in Mugler it is Lemma 7.

Center of gravity of a semicircle:

In Proposition 12 of *The Method of Mechanical Problems*, Archimedes found the center of gravity of half a cylinder, that is, of a cylinder divided into two equal parts by a plane passing through the center of the cylinder. This result is analogous to obtaining the CG of a semicircle. See the discussion by Heath.[140]

Center of gravity of a parabola:

In *On the Equilibrium of Plane Figures*, Book II, Proposition 8:[141]

> The centre of gravity of any segment comprehended by a straight line and an orthotome [parabola] divides the diameter of the segment in such a way that the part towards the vertex of the segment is half as large again as the part towards the base.

Heath presented this Proposition as follows:[142]

> If AO be the diameter of a parabolic segment, and G its centre of gravity, then $AG = (3/2)GO$.

Here A is the vertex of the parabolic segment, Figure 6.5.

[139] [Arc02a, p. 15] and [Mug71b, p. 85].
[140] [Arc02a, pp. 38-40].
[141] [Dij87, p. 353].
[142] [Arc02b, p. 214].

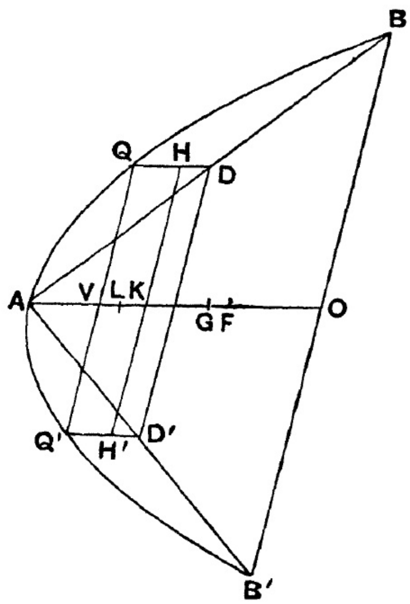


Figure 6.5: Proposition 8 of *On the Equilibrium of Plane Figures*, Book II.

6.2.4 Three-dimensional Figures

Center of gravity of a cylinder:

In *The Method of Mechanical Problems*:[143]

> The centre of gravity of any cylinder is the point of bisection of the axis.

In Heath this is Lemma 7, while in Mugler it is Lemma 8.

Center of gravity of a prism:

In *The Method of Mechanical Problems*:[144]

> In any prism the center of gravity is the point of bisection of the axis.

In Mugler this is Lemma 9. This lemma does not appear in Heath.[145]

[143] [Arc02a, p. 15] and [Mug71b, p. 85].
[144] [Mug71b, p. 85].
[145] [Arc02a].

The "axis" here refers to the line segment joining the centers of gravity of the two bases, as appears from the application of this lemma in Proposition 13 of *The Method of Mechanical Problems*.[146] A prism is a solid figure with similar, equal and parallel ends, and with sides which are parallelograms.

Center of gravity of a cone:

In *The Method of Mechanical Problems*:[147]

> The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base].

In Heath this is Lemma 8, while in Mugler[148] it is Lemma 10.

Center of gravity of a paraboloid of revolution:

In *On Floating Bodies*, Book II, Proposition 2:[149]

> Let the axis of the segment of the paraboloid [of revolution] be AN [...] Let C be the centre of gravity of the paraboloid BAB' [...] Then, since $AN = (3/2)AC$ [...].

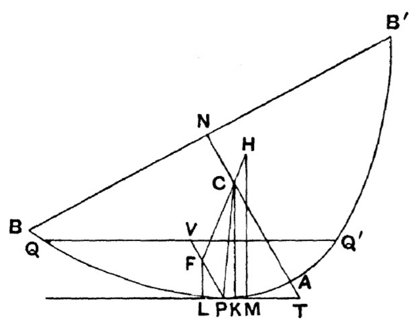


Figure 6.6: Proposition 2 of *On Floating Bodies*, Book II.

In *The Method of Mechanical Problems*, Proposition 5:[150]

[146] [Dij87, p. 316, Note 1].
[147] [Arc02a, p. 15].
[148] [Mug71b, p. 85].
[149] [Arc02b, pp. 264-5].
[150] [Arc02a, p. 25].

> The centre of gravity of a segment of a right-angled conoid (i.e., a paraboloid of revolution) cut by a plane at right angles to the axis is on the straight line which is the axis of the segment, and divides the said straight line in such a way that the portion of it adjacent to the vertex is double of the remaining portion.

This is also discussed by Dijksterhuis.[151] That is, if the paraboloid of revolution has an axis of symmetry AN, with A being the vertex, the center of gravity C is located along AN in such a way that $AC = 2CN$, or $AN/AC = 3/2$.

Center of gravity of a hemisphere:

In *The Method of Mechanical Problems*, Proposition 6:[152]

> The centre of gravity of any hemisphere [is on the straight line which] is its axis, and divides the said straight line in such a way that the portion of it adjacent to the surface of the hemisphere has to the remaining portion the ratio which 5 has to 3.

That is, if the hemisphere has a radius R and its plane face is along the xy plane, centered at the origin, the center of gravity will be along the z-axis (axis of symmetry) at a point z_{CG} such that $z_{CG} = 3R/8$.

Center of gravity of a segment of a sphere:

In *The Method of Mechanical Problems*, Proposition 9:[153]

> The centre of gravity of any segment of a sphere is on the straight line which is the axis of the segment, and divides this straight line in such a way that the part of it adjacent to the vertex of the segment has to the remaining part the ratio which the sum of the axis of the segment and four times the axis of the complementary segment has to the sum of the axis of the segment and double the axis of the complementary segment.

[151] [Dij87, p. 326].
[152] [Arc02a, p. 27].
[153] [Arc02a, p. 35].

Center of gravity of a segment of an ellipsoid:

Archimedes found in Proposition 10 of *The Method of Mechanical Problems* the center of gravity of any segment of an ellipsoid.

Center of gravity of a segment of a hyperboloid of revolution:

Archimedes found in Proposition 11 of *The Method of Mechanical Problems* the center of gravity of any segment of a hyperboloid of revolution.

General Comments on These Results:

It should be emphasized here that all of these results were derived theoretically by Archimedes, beginning from his postulates. In other words, they were derived mathematically. In an earlier Section of this book we saw how to obtain some of these results (such as the CG of a circle, rectangle, and triangle) experimentally. At the end of the book we will discuss how Archimedes calculated the CG of the triangle, as well as a modern mathematical definition of the CG.

These values of the CGs of different geometrical figures obtained by Archimedes (circa 287-212 BC) remained essentially for 2,000 years as the only values known to science. Only during the XVIIth century physicists such as Stevin (1548-1620) and Galileo (1564-1642) began to obtain new results for the CG of different bodies applying different theoretical techniques. To this end it was crucial their careful study of Archimedes' original works.

Part III

Balances, Levers, and the Oldest Law of Mechanics

By now we have arrived at a definition of the CG given by $CG8$, Subsection 4.12.1, and two practical procedures of finding it experimentally, $CG6$ and $CG7$, Subsections 4.7.1 and 4.8.1, respectively. But these formulations do not enable us to calculate theoretically the CG of any discrete nor continuous distributions of matter. In order to perform these calculations we will need the concept of weight, a procedure to measure it, and also the law of the lever. This is our goal here.

We have seen in the experiments with the triangle in equilibrium, and in the geometrical analysis following it, that not all straight lines passing through the CG of a plane homogeneous figure divide it into two equal areas. An example can be seen in Figures 4.7 and 4.9. In the experiments with the pasteboard equilibrist we saw that by changing the location of the modeling clay attached to the equilibrist we could change the position of the CG of the whole system (pasteboard plus clay). This suggests that the CG has to do not only with the weight of the body, but also with the distribution of the matter of the body and its distance to the point of support or to the axis of rotation.

We will arrive here at a mathematical expression with which we can calculate the CG of any distribution of matter. To this end we need first to quantify the intuitive concept of weight. That is, to find a clear and objective way of measuring the weight of a body. This is the subject of the next few Sections.

Chapter 7

Balances and the Measurement of Weight

7.1 Building a Balance

The more basic or fundamental quantitative concepts we have in physics are those of the size of a body (or the distance between bodies), time between physical events, and weight of a body.

In order to measure the size of a body, or the distance between two bodies, we utilize essentially a rigid standard of length. By definition we say that two bodies have the same size when their extremities coincide. For example, we say that person A has the same height as person B if, when they are placed back to back, the heels and heads coincide with one another. By definition we say that body A is N times the size of body B if it is possible to superimpose in linear sequence N times body A between the extremities of body B. The simplest example of this is a 1 meter ruler divided into centimeters. We see that the ruler has 100 units of 1 cm between its ends, with these units stamped along the ruler. Utilizing a graduated ruler we can also measure the length of a body, or the distance between small bodies.

Time is the concept created by man in order to measure the changes which happen in nature. Any standard that repeats itself periodically can be utilized as a measure of time. Historically the most important and precise clock utilized in astronomy was the rotation of the Earth in relation to the background of stars seen with the naked eye. These stars are usually known as fixed stars, because they do not change noticeably their relative positions between one another while the Earth rotates relative to them. This leads to the definition of the unit of a sidereal day. Other astronomical clocks are given by the rotation of the Earth relative to the Sun, yielding the unit of a solar day, the phases of the Moon, or the variation of the position of the sunset in relation to the mountains and other terrestrial bodies, yielding the unit of a solar year. There are clocks with different degrees of precision. The simplest clock distinguish between darkness and light; others, the phases of the Moon, or the shadows of a gnomon. A gnomon is a vertical stick fixed in the ground which measures the height of the Sun in the sky through the orientation and size of its shadow. It is the basis of the construction of the sundials. Several other periodic phenomena have been utilized through the centuries to measure time: water clocks, mechanical clocks (based on a pendulum or a spring), electromagnetic clocks, atomic clocks, etc.

But the main concept we want to analyze in more detail here is the weight of a body. In this book we will deal only with the concept of weight, we will not deal with the concept of mass nor with the concept of quantity of matter. The concept of mass is related to dynamics and to the acceleration of bodies related to an inertial frame of reference. Classical mechanics was developed by Isaac Newton (1642-1727) in his book *Mathematical Principles of Natural Philosophy*, usually known by its first Latin name, *Principia*.[154] His other major book was the Opticks.[155]

We all have an intuitive notion of the weight of a body as a quantitative measure of the gravitational force. We say that body A is heavier than body B if it is more difficult to keep body A in our hands at a certain height from the surface of the Earth than to keep body B at the same height. This difficulty can be indicated by our sweat, or by the fatigue we feel in our outstretched arm. We also say that body A is heavier than body B when we need to make a larger physical effort to raise body A to a certain height h than to raise body B to the same height h. This sensorial and subjective notion can also be indicated by certain phenomena affecting other material bodies. For instance, the deformation caused by body A upon a material support holding it at rest relative to the ground. Let us suppose that this support is a spring. We can say that body A is heavier than body B if the same spring is more compressed supporting A than supporting B. In this case we would utilize a flexible and deformable body such as a spring

[154] See [New34] and [New99]. Portuguese translation in [New90], [New08] and [New10].
[155] See [New79] with Portuguese translation in [New96].

as a weight indicator. It is better to use an objective phenomenon like the deformation of a spring in order to quantify the notion of weight than to use a subjective phenomenon like our sensation of fatigue.

But historically the oldest and most important instrument utilized to quantify the notion of weight was the balance with equal arms. Balance is the name given to any instrument which determines quantitatively the weight of a body. The balance with equal arms has been known since ancient Egypt, if not longer. Figure 7.1 shows paintings from the time of the Pharaohs depicting balances in use around 1500 BC. It is interesting that three paintings show people holding a plumb bob which helps to indicate when the beam of the balance is horizontal.

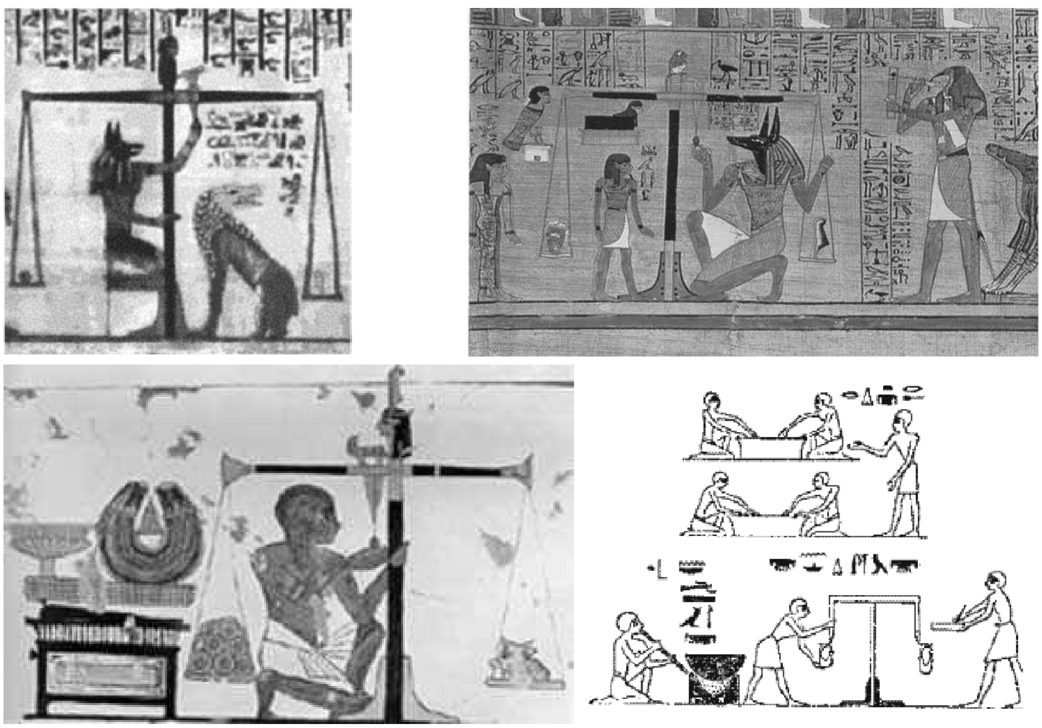


Figure 7.1: Balances in ancient Egypt.

According to Steve Hutcheon (private communication, which he obtained from Thompson[156]), the earliest record of the balance in astronomy is from circa 1350 BC when the Akkadians of Mesopotamia called a star group Ziban-

[156] [Thosd].

itum (the scales). These stars later became known as the Zodiac constellation Libra. In that period Zibanitum gave the location of the Sun at sunrise on the Autumnal Equinox when the lengths of day and night, and the seasons, were in balance.

The main components of a traditional weighing balance or scale with equal arms are:

> The *fulcrum* or fixed hinge of a balance is the horizontal axis about which the balance pivots.
>
> A homogeneous rigid beam (bar, rigid rod or crossbar) which is free to rotate around the fulcrum.
>
> A rigid support which keeps the fulcrum of the balance at rest relative to the ground.
>
> Two scale pans (plates or bowls), suspended at equal distances to the vertical plane passing through the fulcrum.

Normal conditions for a balance with equal arms:

> The fulcrum is orthogonal to the beam of the balance.
>
> The beam of a balance in equilibrium remains horizontal.
>
> A vertical plane passing through the fulcrum divides the beam of the balance into two equal parts.

An example of a balance is given in Figure 7.2.

The objects to be weighed are placed in these pans. In this case the fulcrum is part of the rigid vertical support, while the beam of the balance hangs from the needle, as in Figure 7.2.

Alternatively the fulcrum may be part of a rigid system connected to the beam of the balance, as in Figure 7.3. Here we have a cylindrical cork with its axis of symmetry along the letter E. The beam of the balance is a bamboo skewer passing through the cork and supporting two pans. It is parallel to the axis of the cork, but it passes through the cork below its axis. The needle A is perpendicular to the beam of the balance, passing through the cork above its axis. The needle is supported by two rectangular poles.

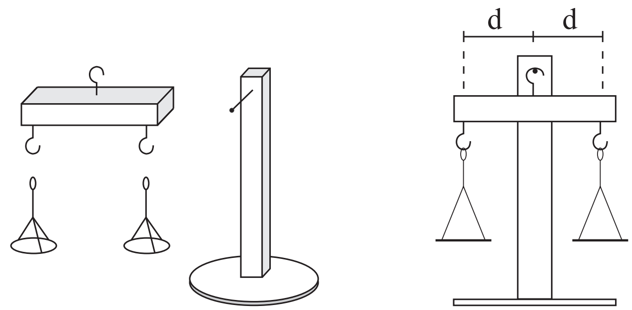


Figure 7.2: Components of a balance. The axis or fulcrum of this balance (the needle) is fixed to the vertical support.

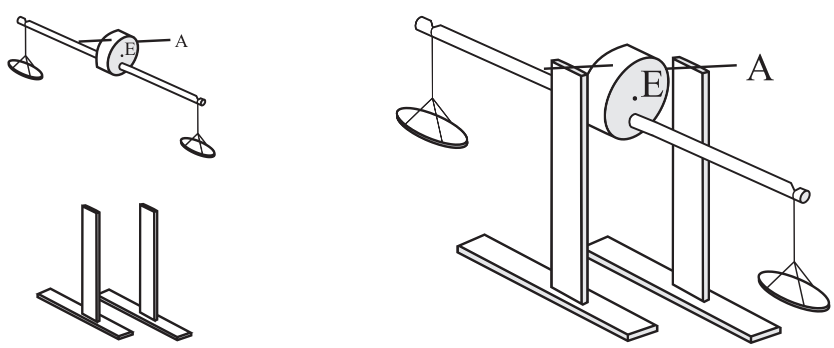


Figure 7.3: The axis or fulcrum of this balance (the needle A) is fixed to the cork, which is fixed to the horizontal beam of the balance (a bamboo skewer).

The system composed by the needle, cork and skewer behaves as a single rigid body. The system can turn around the needle A.

We call the *arm* of the balance, the horizontal distance, d, between the point of suspension of the pan from the beam and the vertical plane passing through the fulcrum of the balance. In some balances we will build, no scale pans will be employed, as the bodies to be weighed will be suspended directly from the beam of the balance.

When we mention the distance between a point Q and a straight line, this should be understood as the smallest distance between this point Q and any other point of the straight line. Consider, for instance, this straight line as being the x axis of a Cartesian coordinate system and the point Q localized at $(x,\ y,\ z) = (0,\ 0,\ d)$. Therefore, the distance between Q and the axis x is, by definition, equal to d. The same interpretation should be understood of the distance between a point Q and a plane. That is, this distance is, by definition, the smallest distance between Q and any other point of the plane. If this plane is the xy plane and Q is located at $(x,\ y,\ z) = (0,\ 0,\ d)$, than this distance between Q and the plane is defined as d.

Before weighing any body, the balance must be adjusted in such a way that its beam is horizontal without the scale pans. If necessary, this can be done by changing the location of the fulcrum on the beam, or the length of the arm on one side of the beam. In addition, the beam must remain horizontal with the scale pans added. If necessary, this can be adjusted by changing the exact location of the nails where the scale pans are hanging. If the beam does not remain horizontal after these adjustments, we can sometimes succeed by placing a small counterweight at some point on one side of the beam. This counterweight can be a piece of thread, wire, or clay.

We have seen that by definition the direction followed by a falling body is called vertical, which coincides with the direction of a plumb line in equilibrium. Horizontal is any straight line or plane perpendicular to the vertical. We also defined the equilibrium of a body as the situation in which the body and all its parts remain at rest relative to the Earth.

Definition of a balance in equilibrium: We define the meaning of the expression "balance in equilibrium" as the situation in which its arms remain at rest horizontally. This is the meaning given by most people to the equilibrium of balances, and we adopt it here. That is, even when the beam and scale pans are at rest relative to the ground, we will not say that the balance is in equilibrium if the beam is not horizontal.

Before using a balance scale to measure weights, it is necessary to construct it and ensure it is balanced horizontally without any additional objects to be weighed, only with its arms and pans. It is also important to verify that the wires attaching the pans to the arms of the balance scale are positioned at the same distance from the vertical plane passing through the fulcrum. For the balance scale to be accurate, it is essential that it rotates freely around the fulcrum, without being impeded by friction or being too tightly fixed at this point.

We have seen before that a rigid body suspended by a point is in stable equilibrium when the point of suspension PS is vertically above the CG of the body. If the CG is above the auxiliary point of support PA, the equilibrium tends to be unstable, unless the PA is not a point but an area of support.

For the time being we will deal only with balances suspended by a fulcrum located vertically above the CG of the empty balance. In some figures we will represent this fulcrum by the letters PS. One of the most important aspects to take notice in the construction of a balance is that the fulcrum must be vertically above the CG of the beam (without the scale pans and weights to be measured). This will guarantee the stable equilibrium of the beam. That is, it will return to the horizontal position after being released at rest, no matter what the initial inclination of the beam relative to the horizontal.

The beam of the balance can be made of any rigid material like wood, plastic, metal, or even pasteboard. It can be cylindrical (a bamboo barbecue skewer or a broomstick), rectangular (a ruler or a rectangular pasteboard), or like a parallelepiped (a wooden board). Close to the extremities of the beam, at equal distances to the vertical plane passing through the fulcrum, we can fix two equal nails, two needles or two hooks, which will support the scale pans by the threads. Alternatively, we can also make two equal holes at the extremities of the beam, where we will hang the pans (utilizing hooks). The pans of the balance may be two small bottle lids, two small plastic cups, or any other adequate supports for the bodies. We should make three holes symmetrically located around the edges of the lids or cups, where the threads will be tied. The threads holding the pans on each side of the beam should have the same length and should be made of the same material. Instead of lids or cups, we could also use small plastic or cloth bags suspended from the beam. Inside the bags we place the bodies to be weighted.

Here we present several types of balances with equal arms that are very sensitive and precise, even though they are made with cheap and easy to find materials. They illustrate different possibilities, and can also be adapted to make levers.

There are many ways in which the balance can have freedom of rotation around the fulcrum. One possibility is to have a hook on the upper part of the beam, above its midpoint. Another possibility is to make a horizontal hole halfway between its extremities, with the hole above the CG of the beam. In these two cases the balance hangs by a nail, needle, or bamboo skewer fixed horizontally in the rigid support, passing through the hook or hole made in the beam, as in Figure 7.2.

One of the simplest balances found in every home is the coat-hanger. The cylindrical horizontal bar holding the coat-hanger is the fulcrum of the balance, and we can hang objects to be measured on the beam of the coat-hanger, Figure 7.4.

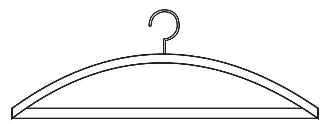


Figure 7.4: A coat-hanger can be utilized as a balance.

A simple and instructive balance model which we will utilize for some activities is made with a pasteboard in the form of the letter T. It has several holes along the body, as well as holes symmetrically located along the arms of the T, as in Figure 7.5. In this case it has 8 horizontal holes H_1 to H_8 and 10 vertical holes V_1 to V_{10}, with H_5 coinciding with V_1. This holes can be made with a single-hole punch-pliers, Figure 4.28.

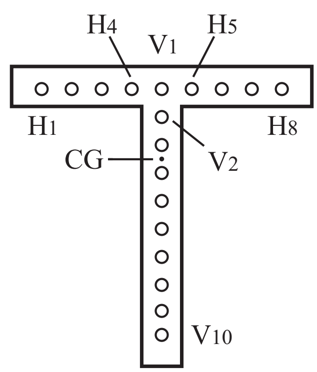


Figure 7.5: A pasteboard balance.

By the method of the plumb line described in Chapter 4 we can easily find the CG of the pierced figure, as indicated in Figure 7.5.

The T balance should be suspended by a fulcrum passing through a hole located above its CG. The scale pans can be suspended by any two holes along the arms, provided they are at equal distances from the vertical plane passing through the axis of symmetry of the T.

Another way to reduce friction from the wobbling of the balance is to use a horizontal stick fixed to the beam, orthogonal to it, supported on both sides by rigid, smooth stands of the same height. One example is made of a short cork (that is, a cork cut in half on a plane parallel to its faces), a needle (or toothpick), and a bamboo barbecue skewer, as in Figure 7.6.

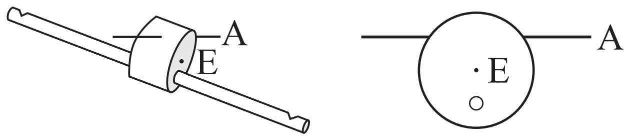


Figure 7.6: The beam of a balance made with a bamboo barbecue skewer, a cork and a needle.

Initially we pass a nail longitudinally through the cork, parallel to its axis of symmetry, but outside it. We represent the axis of symmetry of the cork by the letter E. We remove the nail and pass the bamboo barbecue skewer through this hole. We then remove the tip of the bamboo skewer, so that it becomes symmetrical. We then pass a needle or toothpick through the cork. The needle should be perpendicular to the bamboo skewer and parallel to the faces of the cork. We represent the needle by the letter A. The center of the cork must be located between the center of the bamboo skewer and the center of the needle. The bamboo skewer and the axis of the cork should be parallel, with the needle orthogonal to the plane formed by the axis and the bamboo skewer.

We support both sides of the needle on the back of two chairs, above two cans or over another appropriate stand. We adjust the center of the bamboo skewer in relation to the center of the cork in such a way that the bamboo skewer remains horizontal. We then make two cuts on the upper part of the bamboo skewer, perpendicular to its axis, symmetrically located relative to the needle. The scale pans should be hung from these cuts. The friction of the needle rotating over the smooth stands is very small and this balance allows good precision. The needle works as the fulcrum of the balance, that is, the horizontal axis around which it can turn, as in Figure 7.7.

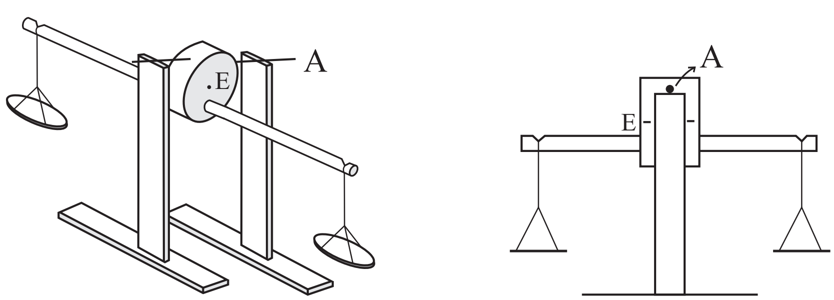


Figure 7.7: A complete balance on its support.

Another way to reduce friction during the oscillation of the balance is to fix vertical nails or needles in the beam, which will be supported over smooth stands. Figure 7.8 illustrates a balance of this kind made with cork, bamboo barbecue skewers, and pins or needles.

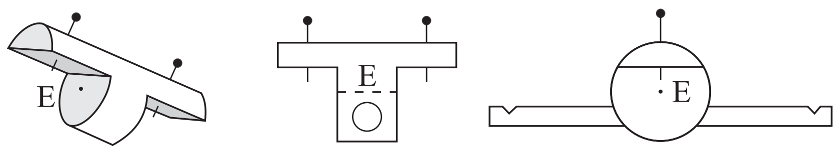


Figure 7.8: Another kind of balance with very small friction.

Initially we use a knife to cut equal pieces of the cork from both ends, each one 1/3 of the length of the cork, in such a way as to remove 3/4 of the circular part of the cork. Then we pass a bamboo barbecue skewer through the lower part of the cork, orthogonal to its axis of symmetry E. The bamboo skewer should be in a plane parallel to the plane of the longitudinal cuts of the cork, parallel to its axis of symmetry, but below the axis. We then remove the tip of the skewer in order to make it symmetrical. Before passing the bamboo skewer through the cork, we can pass a nail of the same thickness through the cork, in order to facilitate the later insertion of the bamboo skewer. We attach two vertical pins or needles in such a way that their tips are above the original axis of symmetry E of the cork. The bamboo barbecue skewer is set to horizontal, with the tips of the pins supported over appropriate stands of the same height. In order to stabilize the balance, it is crucial that the tips of the pins should be located above the CG of the system composed of cork, pins and bamboo skewer. We can make small cuts close to the extremities of the bamboo skewer, perpendicular to the beam and on its upper side, in order to attach the threads for the scale pans.

There are several other possibilities, but what we have shown here should give a good idea of how to build sensitive balances.

7.2 Measurement of Weight

7.2.1 Definitions of Equal Weights, Greater Weight, and Lesser Weight

We now show how to utilize a balance to weigh bodies. We will suppose that we have already built our balance with equal arms and that it is completely free to turn around the fulcrum. Moreover, we will assume that it is in equilibrium; that is, the beam is horizontal without other bodies, only with the scale pans suspended at equal distances to the vertical plane passing through the fulcrum.

Experiment 7.1

We place body A (for example, a large paper clip) in the left pan of the balance and a sequence of N other bodies B (for example, a small paper clip, a large paper clip, a coin, a piece of modeling clay, etc.) in the right pan of the balance. In each trial we place only one of the N bodies B in the right pan, always releasing the balance at rest with its beam in a horizontal position. We observe that in some cases A goes up while B goes down, in other cases both bodies remain at rest with the beam horizontal, and in other cases A goes down while B goes up.

We now present the fundamental practical definitions utilized in this work.

> We say that bodies A and B have the same weight P if, when A is placed on one pan of this balance and B is placed on the other pan, and the beam is released at rest horizontally, it remains at rest, as in Figure 7.9.
>
> We place two bodies A and B on different pans of a balance with equal arms, with the beam initially horizontal, releasing the system at rest. Let us suppose that the balance does not remain in equilibrium, but inclines towards one of the sides. We say that the body which moves toward the ground is the heavier than the other body (has a greater weight), while the body which moves away from the ground is the lighter than the first body (has a lesser weight).

In order to obtain better precision with the balance, it is important to swap the position of the bodies on the scale pans. If the balance remains in

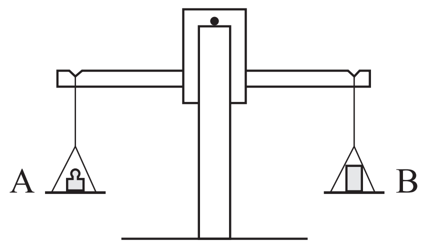


Figure 7.9: A balance in equilibrium with equal weights.

equilibrium before and after this swap of the position of the bodies, we can say that the two bodies really have the same weight.

There is one main reason for this precaution. It may happen that one of the arms (let us call it arm 1) is shorter than arm 2, this difference in lengths being difficult to detect with the naked eye. That is, the distance between the thread of the pan on arm 1 and the vertical plane passing through the fulcrum may be smaller than the distance between the thread of the pan on arm 2 and the fulcrum. We further suppose that body A placed on pan 1 balances body B placed on pan 2. If the arms have different lengths, then body A placed on pan 2 will not balance body B placed on pan 1.

The scale will only balance in both cases (A on pan 1 with B on pan 2; and A on pan 2 with B on pan 1) if the two arms are at the same distance from the vertical plane passing through the fulcrum. Switching the objects between scale pans is also necessary in the other cases where we use a balance with equal arms. We will not mention this again, and simply suppose it is implied in other definitions and procedures.

We said before that body A (or B) has weight P, as if the weight belonged to it or were a property of the body A. However, as a matter of fact, the weight arises from an interaction of A with the Earth (or from an interaction of B with the Earth). We call this interaction gravity. The tendency of gravity is to unite the bodies with the Earth. Therefore, it would be more appropriate to say that when the balance remains in equilibrium, the interaction of A with the Earth has the same value P as the interaction of B with the Earth. In any case, we will keep the previous definition of the weight P of A and B, as this is the usual way of expressing it. But it should not be forgotten that weight is really an interaction of each body with the Earth.

The previous definition is an operational procedure to find two bodies of the same weight. But it is not an experimental law. We are merely utilizing an empirical observation (the equilibrium of the balance supporting two bodies A and B) in order to arrive at a conceptual (or operational) definition.

This could only be an experimental law if we had some other way of knowing when two bodies have the same weight. If this were the case, then we could say that experiment teaches us that two bodies of the same weight keep a balance with equal arms in equilibrium. But historically we first found an objective procedure for quantifying the notion of weight utilizing the balance with equal arms. Therefore, the equality of weight of two bodies by this first procedure must be established by definition. Once we know one procedure for defining the equality of weight of two bodies, we can apply it to obtain other experimental laws. For instance, suppose we utilize the previous experimental procedure with a balance with equal arms to find two bodies A and B of equal weight. Then we can raise to an experimental law the empirical result that these two bodies compress a spring by the same amount when each one of them remains at rest on a vertical spring supported from below under the gravitational influence of the Earth.

The previous definition is the main operational procedure for quantifying the equality of weight of two bodies. We might think of an alternative procedure such as: we define two bodies made of the same material and having the same size and shape as having the same weight. But this alternative procedure has problems and limitations, for two principal reasons. The first limitation is that it is difficult to know in practice if the two bodies are really made of the same material. After all, microscopic differences may arise during the manufacturing process (impurities, internal bubbles, etc.) which are difficult to detect. Even disregarding this prospect, there is a second, even more serious problem. There is not the slightest possibility of comparing the weights of two bodies made of different materials, such as iron and wood, or corn and water, by this alternative definition. That is, when we have bodies of different chemical composition, we cannot compare their weights by this alternative definition.

Let us illustrate this point with a specific example, as this is a relevant issue that is rarely discussed in textbooks. When we buy a box of paper clips we observe visually that they have the same shape and size. As they are made of the same material, it is reasonable to suppose that they have the same weight. Despite this fact, there are always some microscopic variations between two clips which are difficult to detect macroscopically. In any event, even forgetting this fact, there is not the slightest possibility of visually comparing the weight of one of these clips with a certain amount of clay. After all, the clip and the clay have different shapes, sizes, volumes, textures, colors, etc. But the main difference is that they are made of different chemical substances. The only way of knowing if they have the same weight or not, is to utilize a measurable effect arising from the gravitational interaction. The first quantitative instrument that was devised historically to determine the weight of bodies was the balance with equal arms. Therefore we can say, by definition, that a paper clip and a certain volume of clay have the same weight if, when released at rest on the opposite pans of a balance with equal arms, the beam remains horizontally at rest.

The best procedure is to define the equality of weight between two bodies A and B through a gravitational effect produced by these two bodies. This effect can be the equilibrium of a balance with equal arms, as described previously. This effect could also be, for instance, the fact that they produce the same compression of a spring when placed above it. It could also be another gravitational effect. Historically the springs did arise thousands of years after the balances. Therefore, we will adopt the previous convention of equality of weight utilizing a balance with equal arms.

In principle the previous definition is only strictly valid when the balance is placed in a high vacuum. The reason is that if bodies A and B are immersed in a fluid like air, an upward force will be exerted upon them by the air. And this force is equal to the weight of the displaced air, as discovered by Archimedes himself. Therefore, the body with the larger volume will receive a larger upward force from the air. This force of the air will distort the comparison of weights of A and B. In our previous definition we are neglecting the effect of this upward force, considering only the downward forces upon A and B due to their interactions with the Earth.

7.2.2 Definition of Multiples of a Certain Weight

Now that we have defined the equality and inequality of weight between two bodies, we can continue to quantify the notion of weight with another definition. We begin with a simple experiment.

Experiment 7.2

We take 5 paper clips of the same weight. We place 2 of these clips together at a distance of 6 cm from the vertical plane passing through the fulcrum. We place the other 3 paper clips together at a distance of 6 cm from the other side of the vertical plane passing through the fulcrum, releasing the balance at rest horizontally. We observe that it turns around the fulcrum, with the 3 paper clips falling and the 2 paper clips rising, as in Figure 7.10.

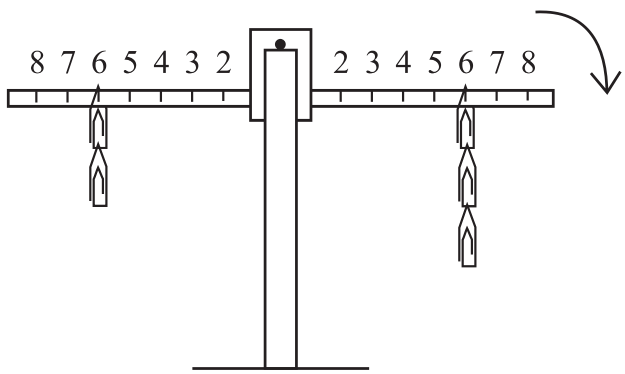


Figure 7.10: Three bodies of the same weight are heavier than two bodies of the same weight.

This experiment can be generalized to other cases. That is, suppose we have N bodies of equal weight P at a distance d from one side of the vertical plane passing through the fulcrum of a balance, and M other bodies of equal weight P at the same distance d on the other side of the fulcrum, with $M > N$. If we release the balance at rest horizontally, it turns around the fulcrum, with the set of M bodies falling and the set of N bodies rising. By the definitions we introduced in Subsection 7.2.1, we say that the set of M bodies is heavier than the set of N bodies.

The result of this experiment allows the following definition:

Definition: We say that N bodies of the same weight placed together on a scale pan have N times the weight of one of these bodies.

For example, suppose that with a balance with equal arms we discover that the bodies A, B, C and D all have the same weight P, that is, $P_A = P_B = P_C = P_D \equiv P$. Suppose we place these four bodies over one of the pans of a scale and verify that they balance another body E placed on the other pan. Then we say, by definition, that the weight of E is four times the weight of A, or, $P_E \equiv 4P_A$.

This may seem a trivial definition. But this is not the case. In order to see this, let us compare it with the temperature of a body. We define two bodies as having the same temperature T if, when they are placed in contact, they remain in thermal equilibrium. That is, their macroscopic variables, like the pressure or volume in the case of gases, do not change with the passage of time. But if we place N bodies together at the same temperature T, the system as a whole will also have the same temperature T, and not a temperature N times higher than T. The same holds for density. That is, when we place N cubic solid homogeneous bodies of the same density ρ together, the system as a whole will have the same density ρ. The system will not have N times this density.

Based on this definition, we can prepare a set of standard weights. We choose a specific object, such as a small paper clip, as our standard, and define its weight as 1. With a balance we can find many other objects (e.g., pieces of clay) which have the same weight. We then put five of these equal weights on one side of a balance and on the other side we place an appropriate amount of clay to balance these 5 objects. This clay will have, by definition, weight 5. We can mark this number in the clay. We can find other standards of weight: 10, 50 and 100, for instance. Now suppose we want to weigh an apple. We put it on one side of the balance and find how many units we need to place on the other side to balance it. If it is 327 units, we will say that the weight of the apple is the same as the weight of 327 paper clips, or simply 327 units.

7.2.3 The Weight does Not Depend upon the Height of the Body

Experiment 7.3

We utilize a balance with equal arms having threads of equal length holding the two pans. We find two bodies A and B which keep the balance in equilibrium, as in Figure 7.11 (a).

We then shorten one of the threads, placing the excess thread on the pan to which it belonged, and again release the beam at rest with bodies A and B on the two pans. We observe that the balance remains in equilibrium as in Figure 7.11 (b). In other words, experimentally the weight of a body does not depend on its height above the ground.

Experiment 7.4

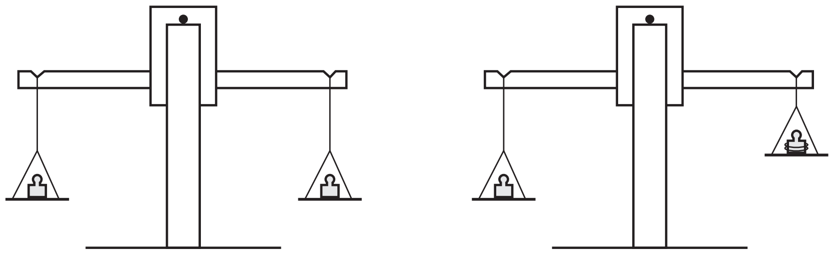


Figure 7.11: The weight does not depend upon the height of the body.

This fact can also be observed with another procedure. We utilize a balance with equal arms having threads of equal length holding four pans. We find two bodies A and B which keep the balance in equilibrium, as in Figure 7.12 (a).

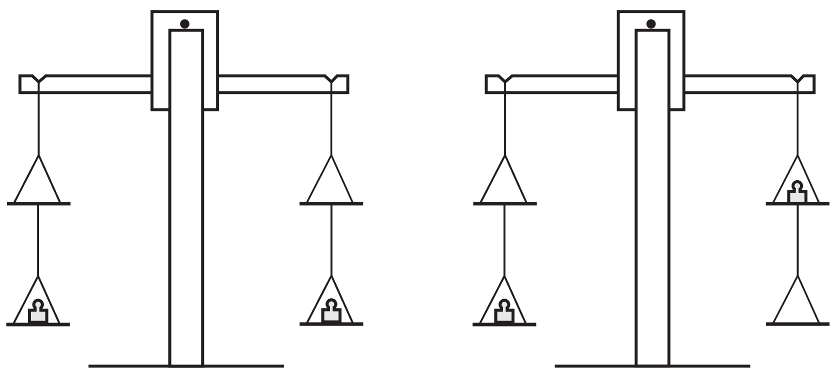


Figure 7.12: Another experimental procedure to show that the weight does not depend upon the height of the body.

We then change the place of one of these bodies, from the lower to the higher pan, Figure 7.12 (b). We observe that the balance remains in equilibrium after being released at rest. In other words, experimentally the weight of a body does not depend on its height above the ground.

Since Newton's theory of universal gravitation, we have known that the result of Experiments 7.3 and 7.4 is only an approximation. The reason is that the gravitational force between two spherical bodies falls as the inverse square of the distance between their centers. But due to the huge radius of the Earth, compared with the difference in length between the two threads in Experiment 7.3, or compared with the difference in height between the two bodies in Figure 7.12 (b), the change of weight is negligible. Thus, it cannot be detected in this kind of experiment. We can therefore assume as an experimental result that the weight of a body upon the surface of the Earth, as detected by a balance with equal arms, does not depend upon its height above the ground.

7.3 Improving Balance Sensitivity

7.3.1 Definitions of Equal Sensitivities, Higher Sensitivity, and Lower Sensitivity

We now present a series of experiments that show how to improve the sensitivity of balances. The sensitivity of a balance is its ability to detect small differences in weight, which is often measured by the smallest weight that causes a perceptible change in its reading. A more sensitive balance provides more precise and reliable measurements.

Suppose we have two balances, 1 and 2, of equal arms which remain horizontal in equilibrium when they support equal weights P in their arms, Figure 7.13.

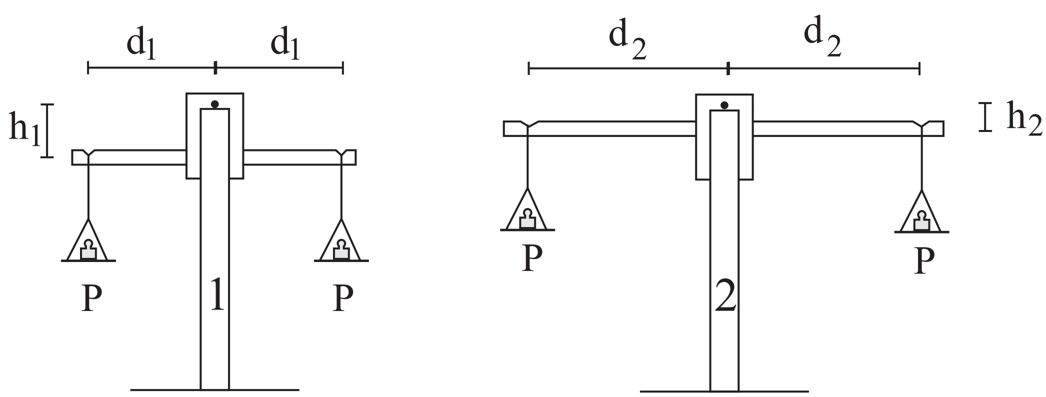


Figure 7.13: Balance 1 with equal arms remains in equilibrium when supporting two equal weights P in its two equal arms. The same happens with balance 2 with equal arms.

Suppose that now we hang two bodies A and B of different weights on opposite arms of balance 1. Its beam will not remain horizontal, but will be inclined by an angle θ_1 in relation to the horizon, Figure 7.14 (a). When we place these two bodies A and B on the opposite arms of balance 2 it will also be inclined to the horizon by and angle θ_2, Figure 7.14 (b).

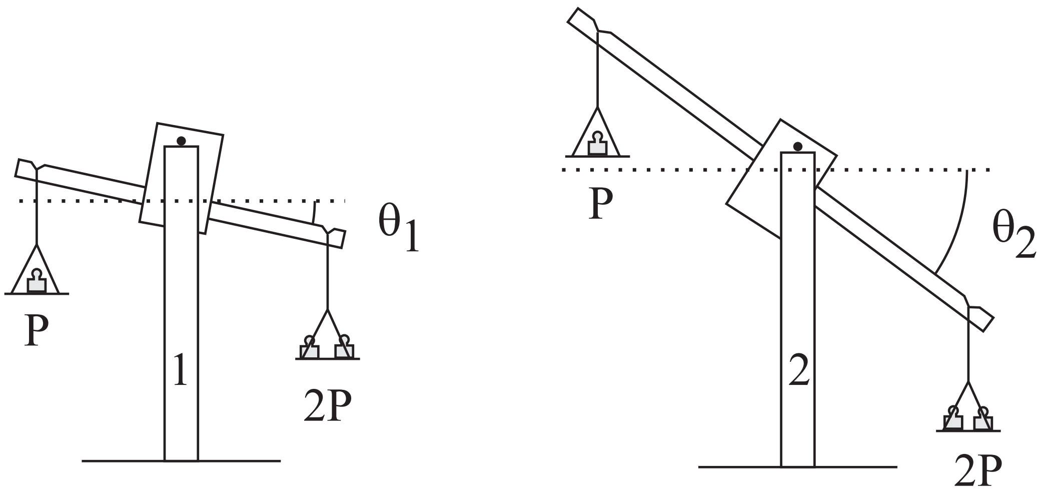


Figure 7.14: The beam of balance 1 remains inclined to the horizon by an angle θ_1 when supporting different weights A and B on its two arms. The beam of balance 2 remains inclined to the horizon by an angle θ_2 when supporting the same weights A and B on its two arms.

We say that balance 1 has a higher sensitivity than balance 2 if we can more easily distinguish the difference of weight in balance 1 than in balance 2. The sensitivity of a balance can be established quantitatively by the angle θ its arms make with the horizontal when it holds bodies A and B at equal distances to its fulcrum. The greater the value of θ, the greater the sensitivity of the balance. This allows the following definition.

Definition: We say that balance 1 has the same sensitivity of balance 2 if $\theta_1 = \theta_2$, a higher sensitivity if $\theta_1 > \theta_2$ and a lower sensitivity if $\theta_1 < \theta_2$.

7.3.2 Factors Affecting the Sensitivity of a Balance

We now perform some experiments which will indicate the main factors affecting the sensitivity of a balance.

We perform four experiments. Their results will show how to build balances with greater sensitivity.[157] All of these experiments utilize pasteboard figures in the shape of the letter T, as in Figure 7.15.

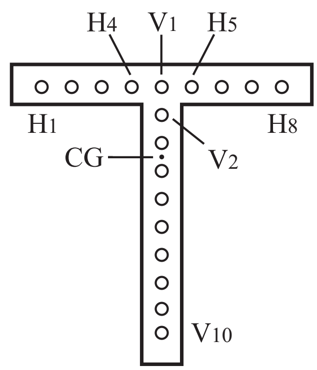


Figure 7.15: A balance made of a pierced pasteboard T.

This pierced pasteboard T will function as a balance. Its arms of equal length will be the beam of the balance. We will suppose that when we hang the T by the hole located at the intersection of the arms with its body, supporting it on a horizontal pin fixed in a vertical stand, the arms of the balance remain horizontal after the T stops swinging. We then find two bodies that keep the T in equilibrium when they are placed on opposite sides at equal distances from the axis of symmetry of the T. The balance is being utilized here to determine the equality of weight of these two bodies. But we can also use a balance to determine whether two bodies A and B have different weights. How should we build a balance capable of distinguishing, for instance, a difference in weight of 1% between A and B? We are interested here in finding the main features that increase the sensitivity of a balance, so that it can easily show that two bodies A and B have different weights. This is the goal of the next experiments.

In order to unbalance the beam we will use a paper clip placed on one of its arms. We want to know what makes the disequilibrium more visible, that is, what increases the angle θ indicated by the T balance.

The dimensions of the T do not need to be exactly as indicated. In the model used here, the length between the end of one arm and the end of the other arm is 15 cm. The height of the T is 16.5 cm. The width of the arms and body of the T is 3 cm. Holes separated by 1.5 cm are made along the

[157][Fer06].

axis of the arms and along the axis of symmetry of the T. Let us call the 10 holes along the axis of symmetry V_1 to V_{10}, with V_1 at the intersection of the arms with the body, and V_{10} at the bottom end of the body. The holes along the arms are called H_1 to H_8, with H_1 at the left of Figure 7.15 and H_8 at the right.

After making these holes we locate the CG of the T utilizing the experimental procedure described in Chapter 4. The simplest way to locate the CG is to hang the T by a pin passing through H_1, drawing the vertical with the help of a plumb line after the system has reached equilibrium. This procedure is repeated with the T hanging by H_8. The intersection of the two verticals is the CG of the T. With the previous dimensions it is located between V_3 and V_4, as indicated in Figure 7.15.

Experiment 7.5

Initially we have a balance in equilibrium, with its arms horizontal, suspended by hole V_1. We now disturb this equilibrium by placing a small piece of paper or clay, or a paper clip, at the end of one of the arms. The system turns around V_1, oscillates a few times, then stops with the extra weight lower than the opposite arm. Let us call θ_1 the smaller angle between the horizontal and the arm with the extra weight when the T is at rest, as in Figure 7.16 (a).

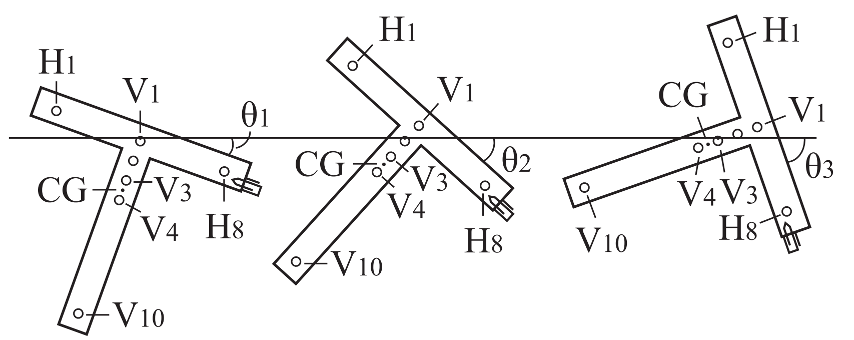


Figure 7.16: The greater the distance between the point of suspension PS and the CG of the balance, the smaller the sensitivity of the balance.

We repeat the experiment, but now with the T suspended by V_2. Initially the system is in equilibrium with the arms horizontal. We then disturb this equilibrium by placing the same extra weight in the same place as before.

After the system has come to rest, we measure the angle between the horizontal and the arm with the weight, calling it θ_2, Figure 7.16 (b). We repeat the procedure with the T suspended by V_3. In this case, the angle when the system is at rest is called θ_3, Figure 7.16 (c).

Experimentally it is found that the smaller the distance between the point of suspension (in this case, the pin) and the CG of the T, the greater the final angle when the system is at rest. That is, experiment shows that $\theta_1 < \theta_2 < \theta_3$, as in Figure 7.16.

If we try to keep the T in its normal position (with the arms above the body) by suspending it by holes which are below the CG, we do not succeed. In other words, if we try to suspend it by V_4, V_5, ..., V_{10}, the system turns upside down and remains at rest only with the horizontal arms below the body of the T, as we saw in Experiment 4.33. But even in these cases we can break the equilibrium as before, and obtain the same experimental results. That is, if we suspend the T by V_{10} and place an extra weight at the end of one of its arms, the system will reach a new position at rest with the arm inclined by an angle θ_{10} in relation to the horizon, as in Figure 7.17 (a). We now hang the T by V_9, ..., V_4, then put the same extra weight at the end of its arm, and wait until the system reaches equilibrium. In these cases the smaller angle between the horizontal and the arm with the extra weight is given by θ_9, ..., θ_4, respectively. Experimentally it is found that $\theta_{10} < \theta_9 < ... < \theta_4$, Figure 7.17.

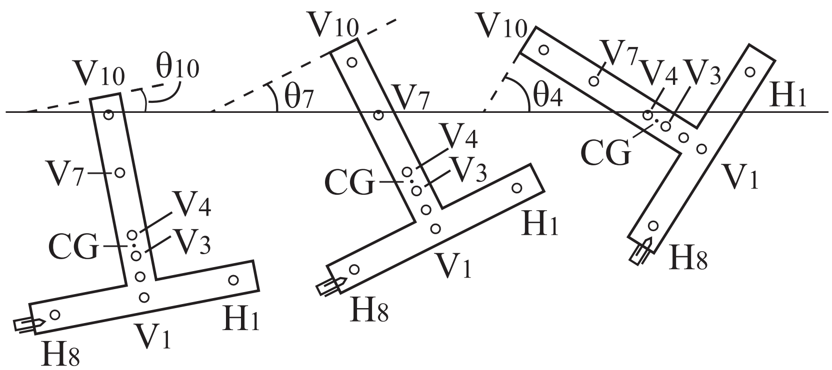


Figure 7.17: The same result as before with the T upside down.

In all these cases we placed the same weight acting at the same distance from the vertical plane passing through the fulcrum of the balance. And we discovered experimentally that the smaller the distance between the PS and the CG, the greater the angle of inclination of the beam with the horizontal after the system reached equilibrium. Therefore, the sensitivity of a balance increases with decreasing distance between the PS and the CG. As the distance between the PS and the CG gets smaller, it is easier to perceive that the beam is unbalanced, supporting different weights on both arms.

This experiment suggests that balances should be built to allow a variable distance between the PS and the CG, in order to control sensitivity. An example of a balance of this kind utilizes a cork, two bamboo barbecue skewers and two pins or needles, Figure 7.18.

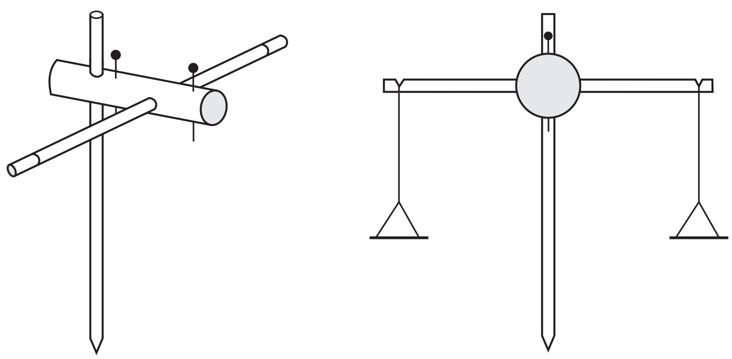


Figure 7.18: A balance with variable distance between the PS and the CG.

Initially we pass a bamboo skewer through the cork, orthogonally to its axis, at a distance of 1/3 of its length from one end. We remove the tip of the bamboo skewer and make two cuts on the upper face of the bamboo skewer, at the same distance from its center, in order to support the threads fixed to the scale pans. We then pass another bamboo skewer at a distance of 1/3 of its length from the other end, in such a way that it remains orthogonal to the axis of the cork and the first bamboo skewer. This second bamboo skewer will work as the pointer of the balance. We place a pin parallel to this second bamboo skewer, passing close to the center of the cork, to serve as the fulcrum of the balance. In order to prevent the beam from falling toward the ground when we place the threads and scale pans, raising the pointer, we place another pin parallel to the first one, this time on the front part of the cork, after the horizontal bamboo skewer. We then have along the length of the cork, from back to front: a vertical pointer, a vertical pin, the horizontal beam and another vertical pin, as in Figure 7.18. We fix the two scale pans and adjust the arms so that the beam becomes horizontal when supported by the two pins. We then support the balance with the two pins on the lid of a can or on another convenient support. The balance is then complete. By raising or lowering the vertical bamboo skewer, we can change the height of the CG of the balance. In this way we can change its sensitivity, as desired. This vertical bamboo skewer works as well as the pointer of the balance. For example, when the balance is in equilibrium with its arms horizontal, we can make a small mark on the support parallel to the location of the pointer indicating the zero (0) of the balance.

Another extremely creative idea to connect two bamboo skewers or two plastic straws, without a cork, is to make a loop out of pieces of a plastic straw.[158] To do this, we cut three small pieces of straw, one 4 cm in length and two 5 cm in length. The larger pieces are folded in two and we introduce them into the smaller piece. The angle between the planes of the two loops should be 90°, as in Figure 7.19 (a).

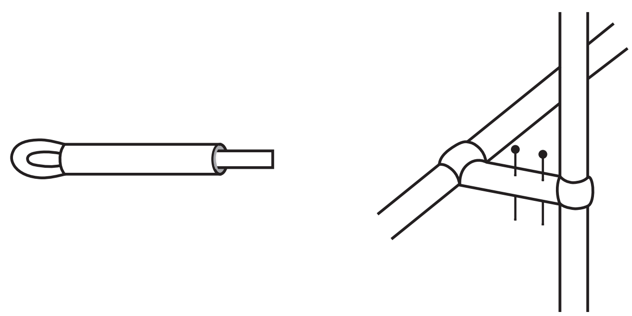


Figure 7.19: A balance with variable distance between the PS and the CG made of plastic straws.

We pass a whole straw or bamboo skewer through each loop and stick two pins or needles in the 4 cm long straw. The two bamboo skewers (or the two plastic straws) should be orthogonal to one another. The bamboo skewer parallel to the two pins or needles will be the pointer of the balance. In this way we can support the two pins over a rigid stand. The horizontal bamboo skewer will be the beam of the balance. The length of its two arms should be adjusted with the beam remaining horizontal at rest. After this procedure, we draw one mark on each arm at equal distances from the point of intersection. On these marks we hang the threads with scale pans. The vertical bamboo skewer (the pointer) can be adjusted at will, so that we

158 [Fer06].

can change the distance between the points of suspension (lower tip of the pins) and the CG of the system (composed of bamboo skewers, pieces of straw, pins and scale pans). In this way we can control the sensitivity of the balance. In order to prevent the balance from falling when we put objects on the scale pans, the objects should be very light, with a weight no larger than the weight of the system. If we wish to balance heavier bodies, then we will need to put extra weights over the pointer in order to prevent the whole balance from falling to the ground.

While increasing the balance's sensitivity is desirable, it also presents a drawback. If we remove the balance from its equilibrium position and release it, we will see that it oscillates for a while until it stops due to friction, returning to a stable equilibrium position. However, the smaller the distance between the point of suspension PS and the center of gravity CG, the longer the oscillation period. In other words, the oscillation will be slower, with the balance taking longer to complete each revolution. Therefore, when the PS is very close to the CG, a very long time must be waited until it stops oscillating. This creates problems because it takes a long time before a reading can be taken. This makes some measurements impractical, since small disturbances are frequent (air currents, room vibrations, disturbances when placing weights on the balance pans, etc.). To prevent this problem, one technique used in some balances is to place a damper (a pointer inside a container of oil, for example) that rapidly reduces the amplitude of the oscillations. This allows the PS to be brought closer to the CG, increasing the sensitivity of the balance, without significantly increasing the time for each reading of the balance after any disturbance.

In the next experiment we analyze another factor which increases the sensitivity of a balance.

Experiment 7.6

In this experiment we always hang the T by the same hole, such as V_1. Let us suppose that it remains at rest in this position with its arms horizontal. We now disturb this equilibrium by placing an extra weight (a piece of paper or clay, or a paper clip) over the hole H_5, releasing the system at rest. The T oscillates a few times, stopping with H_5 below H_1. Let θ_5 be the smaller angle between the horizontal and the beam in this final position, Figure 7.20 (a). We now remove the extra weight from H_5, and place it over H_6, releasing the beam at rest in a horizontal position. After a few oscillations the system stops with H_6 below H_1. Let θ_6 be the smaller angle between the beam and the horizontal in this final position. The procedure is repeated with the extra weight over H_7 and over H_8. These experiments show that $\theta_5 < \theta_6 < \theta_7 < \theta_8$, as in Figure 7.20.

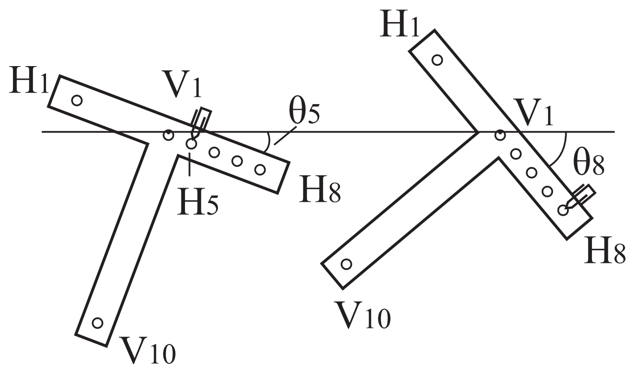


Figure 7.20: The longer the arms of a balance, the greater its sensitivity.

We can imagine that in these four situations we have the same balance, but with the scale pans hanging by equal arms of different lengths in each instance (by H_4 and H_5 in one situation, by H_3 and H_6 in another situation, by H_2 and H_7 in another situation, and by H_1 and H_8 in another situation). We conclude that the longer the arms of a balance, the greater its sensitivity. That is, by comparing two balances with the same distance between the PS and CG, the more sensitive balance is the one with the longer arms. After all, the longer the arm with the extra weight, the more visible will be the lack of equilibrium of this balance caused by objects A and B of different weights. This lack of equilibrium is indicated by the angle of inclination of the beam with the horizontal.

The results of these two experiments can be combined in a single expression. Let h be the vertical distance between the PS and the CG of the balance. Let d be the arm of the balance (horizontal distance between the point of suspension of the scale pans and the vertical plane passing through the fulcrum). The larger the ratio d/h, the greater the sensitivity of the balance. Or the larger the angle θ of inclination of the beam to the horizontal when there are different weights on the scale pans.

Experiment 7.7

A third effect that illustrates how to improve the sensitivity of a balance can also be easily seen with a pasteboard T. In this case we cut out three or four equal T figures, of the same size and shape. Two or three of them should be glued together, making a T of the same size as the original one, but now two or three times thicker than a single T. The two systems (the single T and the thick T) have holes in the same locations (V_1 to V_{10} and H_1 to H_8). We can determine the CG of both systems experimentally. They coincide with one another, being located between holes V_3 and V_4. We hang the single T by V_1 and wait until the system reaches equilibrium with its arms horizontal. We then suspend an extra weight, like a paper clip, at the extremity of one of its arms. We wait until the system stops its oscillations, with the arm containing the extra weight lower than the other arm. We measure the angle θ_S between the horizontal and this arm, Figure 7.21 (a). We remove the T from the support and hang the thick T by V_1. We suspend the same extra weight at the end of one of its arms. We wait for the system to stop moving and measure the angle θ_E between the horizontal and this arm, Figure 7.21 (b). Experimentally we observe that $\theta_S > \theta_E$, Figure 7.21.

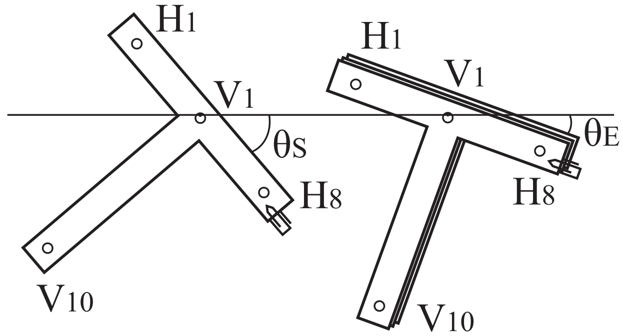


Figure 7.21: The lighter a balance is, the greater its sensitivity.

That is, the heavier the beam of a balance in comparison with the extra weight, the less sensitive it will be. In this experiment the distance between the PS and the CG of the balances was the same, and the extra weight always hung at the same distance from the vertical passing through the fulcrum. The different sensitivity of the two balances can only be due to the difference in their weights. We conclude that the lighter a balance is; the more sensitive it will be, as illustrated in Figure 7.21. That is, it will be easier to distinguish the same difference of weight between two bodies A and B with a lighter balance than with a heavy one.

Experiment 7.8

It is also easy to observe experimentally that the greater the extra weight placed upon one of the arms of a balance, the more inclined the beam will be in relation to the horizon. For example, we hang an extra weight upon one of the arms of a balance and wait until the system stops its oscillations. Let θ_L be the angle between the horizontal and this arm, Figure 7.22 (a). We now place two extra weights upon the same arm, at the same distance from the fulcrum. Once again, we release the balance at rest, with its beam horizontal, waiting until it stops its oscillations. Let θ_P be the new angle between the horizontal and this arm, Figure 7.22 (b). Experimentally it is found that $\theta_L < \theta_P$, as in Figure 7.22. This means that the greater the difference in weight between the bodies on the two equal arms of the balance, the more easily we will notice it. In other words, the greater the difference of weight between the two bodies, the greater the final angle of inclination between the beam and the horizontal.

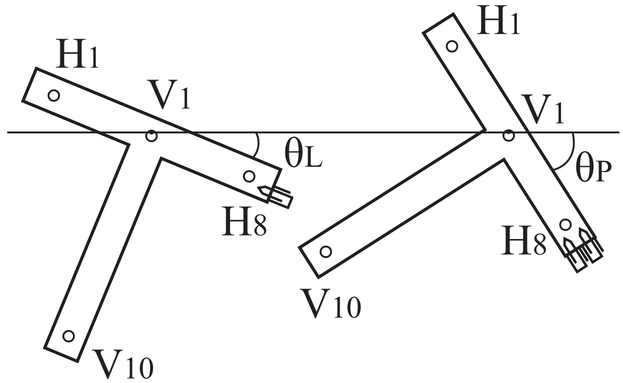


Figure 7.22: The greater the difference in weight between the two sides of a balance, the more easily will we notice it.

Once more we can combine the results of these last two experiments in a single expression. Let a body A of weight P_A be placed on one side of a balance, while body B of weight P_B is placed on the other side of this balance. Let $\triangle P \equiv |P_A - P_B|$ be the magnitude of the difference of weight between A and B. Let us represent the weight of the beam by P_{Beam}. Therefore, the greater the value of $\triangle P / P_{Beam}$, the greater will be the sensitivity of the balance. In other words, in this case the greater will be the angle θ of inclination of the beam in relation to the horizon when $\triangle P$ is different from zero. If $\triangle P$ is the same for two different balances, the balance with a lighter beam will be more sensitive.

After learning experimentally the factors which increase the sensitivity of a balance, we can ask similar questions for other measuring instruments. For instance, what is more sensitive, a thermometer with a larger internal diameter glass bulb or with a smaller one? Filled with mercury or alcohol? What is more sensitive, a galvanometer with a larger internal resistance or with a smaller one? We can also begin to think on how to test these factors in any equipment or measuring device.

7.4 Some Special Situations

7.4.1 Conditions of Equilibrium of a Suspended Body

Before studying levers it is worth making another experimental observation. Let us consider the balance with bamboo skewer, needle (A) and cork, where the axes of symmetry of these three bodies are in the horizontal position, as in Figure 7.6.

Experiment 7.9

The balance is in stable equilibrium when the needle is above the center of the cork and above the center of the bamboo skewer, with or without the scale pans, Figure 7.6. That is, when we lower one of the sides of the bamboo skewer and release it at rest, the balance oscillates a few times, stopping with horizontal arms (supposing there are equal weights on its scale pans suspended at equal distances from the fulcrum). It is easy to understand this fact by observing that in the orientation of stable equilibrium the CG of the system is in its lowest possible position, below the needle, along the vertical line passing through the center of the needle. Any perturbation raises the CG. Therefore, if the system is free to rotate after being released, it will return to the position of stable equilibrium.

Experiment 7.10

We now consider the opposite case in which the center of the needle is below the center of the cork and below the center of the bamboo skewer,

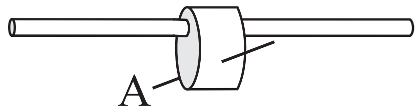


Figure 7.23: A beam with its CG above the fulcrum (needle A).

Figure 7.23. Let us suppose initially that there are no scale pans on the beam.

In this case the equilibrium is unstable with the horizontal needle. In other words, we cannot keep the balance at rest in this position after being released. That is, the balance tends to turn in the clockwise or in the anticlockwise direction after being released at rest. If the balance can make a complete turn, it will end up in the previous position of stable equilibrium of Experiment 7.9, Figure 7.6. It is also easy to understand the phenomenon by observing that in the orientation of unstable equilibrium the CG of the system is in its highest possible position, above the needle, along the vertical line passing through the center of the needle. Any perturbation in the system tends to lower its CG. Therefore, if the balance begins to turn in the clockwise direction after being released at rest, it will continue to turn in this direction, as the tendency of the CG is to fall toward the ground.

Experiment 7.11

The most curious situation is when the center of the needle is in the previous position, that of Figure 7.23. That is, when the center of the needle is below the center of the cork and below the center of the bamboo skewer, but now with equal weights M and N placed on arms of equal length. Let us suppose that the balance has the bamboo skewer (the beam) initially horizontal. Moreover, let us suppose that the weight of the set of threads and scale pans, together with objects M and N placed on these pans (CG of this first set located at P) is larger than the weight of the set of cork, needle and bamboo skewer (CG of this second set located at T), in such a way that the CG of both systems together is located at C, below the needle A, as in Figure 7.24 (a). Even in this case the system is in unstable equilibrium in this initial configuration. That is, if released at rest it tends to turn in the clockwise or in anticlockwise direction. The beam of the balance does not remain in this initial position if there is any perturbation in the system.

Let us try to understand what is happening here. We first analyze the situation for which the beam has rotated by an angle θ with respect to

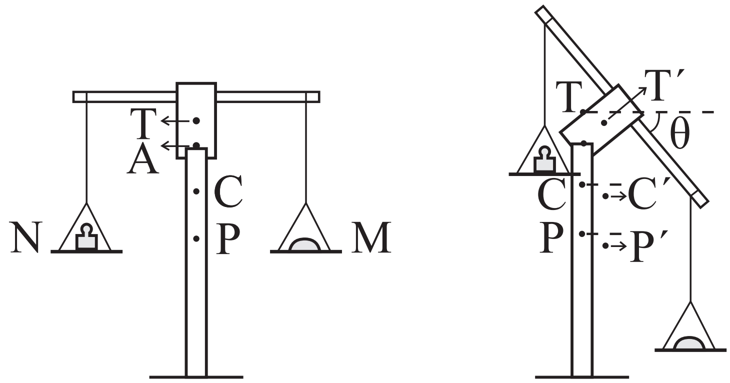


Figure 7.24: A balance in unstable equilibrium.

the horizontal, in such a way that body M moves downward and body N upward, as in Figure 7.24 (b). Body M, together with its pan and thread, fell a distance $H(\theta)$ relative to its original height above the ground. At the same time body N rose, together with its pan and thread, a distance $h(\theta)$ relative to its original height above the ground. As the center of the cork also fell below its original height, we have $H(\theta) > h(\theta)$. This means that the CG of the first set (bodies M and N, together with their pans and threads) fall relative to its original height above the ground, from P to P'. The same happened with the CG of the second set (cork, needle and bamboo skewer), moving from T to T', and with the CG of the whole system, which moved from C to C'. This means that the tendency of the system will be to increase the angle θ even more, as this will lower the CG of the whole system.

If the system had rotated by an angle θ with respect to the horizontal in such a way that N went downwards and M upwards, the CG of the whole system would have again moved downward relative to its original position. And the system would tend to increase angle θ even more. And this explains the unstable equilibrium in this case.

We call attention to this case because it brings something new. When we were considering the equilibrium of rigid bodies, we could only obtain unstable equilibria with the CG above the auxiliary point of support PA. This was the case, for instance, of Figure 4.38. In this situation we had a body with elliptic profile, with the body rotating around the point of support placed below its larger axis. This happened when any perturbation in the position of the body lowered its CG. On the other hand, we had seen stable equilibrium with the CG above the PA. This was the case of Figure 4.37, when the body with an elliptical profile was rotating around a point of support placed below its smaller axis. This was also the case of a rocking chair oscillating on a flat surface. We had also seen stable equilibrium with the CG below the point of suspension PS. This was the case of plane figures suspended by a needle passing through one of their holes, Figure 4.29. In these latter two cases the stable equilibria arose when any perturbation in the position of the body raised its CG.

In the present case we no longer have a rigid body. When the beam turns by an angle θ relative to the horizon, the angle between the beam and the threads supporting the scale pans is modified (it is no longer a right angle). Moreover, the distance between the center of each pan and the center of the beam has also been changed. We are now seeing a new kind of unstable equilibrium, a case where the CG of the whole system is below the PS. And we again conclude, but now in a more general situation not restricted to rigid bodies, that there will be stable (unstable) equilibrium whenever the CG of the whole system rises (falls) when there is any perturbation in the configuration of the system. There will be neutral equilibrium when the CG of the system remains at the same height for any perturbation of the system.

The key to obtaining stable equilibrium of a balance which is free to turn around a horizontal axis is that the point of suspension PS should be located vertically above the CG of the beam. We mentioned this earlier, but it is important to emphasize it here once more. For example, if the beam is a rectangular block of wood or a cylindrical rod, the fulcrum should not be placed at the center of the block or cylinder. In order to obtain stable equilibrium, the fulcrum or PS of the balance should be located above the center of the beam. This will guarantee the stability of the balance when it is placed with its beam horizontal. If the fulcrum is placed exactly at the center of the beam, then a procedure that will produce stable equilibrium is to fix an extra weight on the beam, located vertically below the fulcrum. This will lower the CG of the beam, in such a way that the new CG will be lower than the fulcrum (or PS).

7.4.2 Balances with the Center of Gravity Above the Fulcrum

Before moving on, we briefly mention balances which have the CG of the beam above the fulcrum. As there is unstable equilibrium in this case, the only way to build a working balance is to support it on a surface, not on a point or single horizontal line without thickness. An example of a balance of this kind is a horizontal ruler supported by a domino piece placed below its center, as in Figure 7.25. The ruler can only remain at rest if the width of the domino touching the ruler is not too small in comparison with the thickness of the ruler. For example, it is extremely difficult to balance a horizontal ruler on the edge of a vertical razor blade. In this case the ruler falls to one side or another even before we put the weights on it.

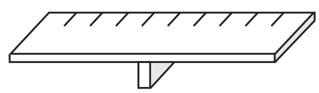


Figure 7.25: A balance with its CG above the fulcrum.

This setup limits the precision or sensitivity of the balance. After all, the surface on which the beam is supported does not allow a single distance between the weights above the pans and the vertical plane passing through the fulcrum. The distance of each arm to the vertical plane passing through the fulcrum will be between a minimum and a maximum value. This will allow not only bodies of the same weight to be balanced on this scale, but also some bodies of different weights (as established by previous precise scales, in which the fulcrum was above the center of gravity of the beam).

Another problem with these balances is that the supports for the weights to be measured (small cups, bottle caps, etc.) are normally attached to the beam. Therefore, the weights are not supported by a single point, as they are spread over a small region. This is another reason why it is difficult to find a single distance between each arm (or each weight) and the vertical passing through the fulcrum.

7.4.3 Other Types of Balance

Apart from the balance with equal arms, there are other types which utilize other measurable effects due to the action of gravity. A very common household balance is the spring scale. It utilizes the compression of a spring by a body resting on it, stationary relative to the Earth, as an indication of weight. Some high-precision piezoelectric balances utilize a phenomenon observed in anisotropic crystals as a weight indicator. Some crystals, when mechanically compressed, become electrically polarized in certain directions. This can be measured and calibrated to indicate the weight compressing the crystal. Some electronic balances transform mechanical deformations arising from the weight of a body into electrical voltage, which is measured electronically. There are several other kinds of balance, but we will not consider them here.

7.5 Using Weight as a Standard of Force

It is possible to keep the beam of an equal-arm balance horizontal by holding a body of weight P on one side, while on the other side, at the same distance from the vertical plane passing through the fulcrum, a force of another origin balances the weight. In order to simplify the analysis, we will suppose that the balance has no scale pans, in such a way that the weight P is suspended directly by the beam. The force on the other side which counterbalances the weight P can be, for instance, the finger of a person exerting a downward force. It can also be a stretched spring fixed at the ground below the balance, or a taught thread fixed to the ground, as in Figure 7.26. Forces of various other origins can act on the other side of the scale to balance the body's weight P (forces that depend, for example, on electrical or magnetic actions). This fact leads to an important definition.

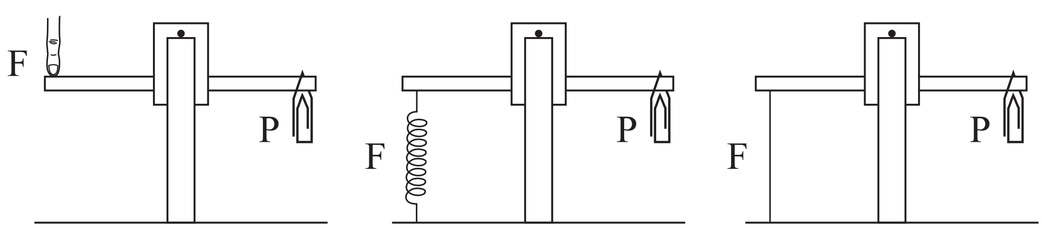


Figure 7.26: Utilizing the weight as a force standard.

Definition: Suppose that body A of weight P acting at a distance d from the fulcrum of a scale, is balanced by a second body B acting on the other side of the scale, at the same distance d from the fulcrum. We define that this second body B exerts a force of magnitude F equal to the weight P of the body A, regardless of the nature of this force (this other force can be elastic, electric, magnetic, etc.). That is, $F \equiv P$.

In other words, in these cases we define that the finger (or spring, or string, or magnet, or charged body, or ...) is exerting a force of intensity F equal to the weight P of the body. This is how we can initially calibrate or measure forces of another nature, not necessarily gravitational, by comparing them quantitatively with the weight force. These forces of another nature can be, for example, the contact force exerted by the finger, the elastic force resulting from the deformation of a spring, the magnetic force between magnets, the electric force between electrified bodies, etc.

This concept doesn't need to be limited to a balance with equal arms. We have seen that when we release a body at rest above the surface of the Earth, it falls to the ground. But this can be prevented by different means, for instance, by placing a rigid support or spring under the body, or suspending it by a thread or spring, etc. Figure 7.27 illustrates a few possibilities.

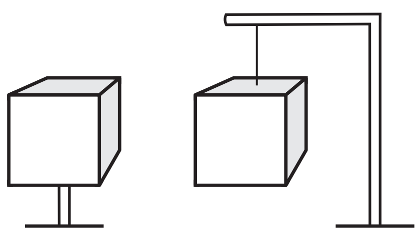


Figure 7.27: Different ways of equilibrating a weight.

Let us consider a spring at rest vertically, fixed at its upper end, with a total length L_0 in this vertical position, as in Figure 7.28 (a).

When a body of weight P is suspended and kept stationary at the lower end of this spring, the spring acquires a length $L_1 > L_0$, Figure 7.28 (b). Another way to keep this body at rest relative to the ground is to support it on the upper end of a vertical spring, which has its bottom end fixed on the ground. In this case, the spring is compressed to a length $L_2 < L_0$, Figure 7.28 (c).

By definition, we say that in these cases the stretched or compressed spring exerts an upward force F upon the body of weight P given by $F \equiv P$.

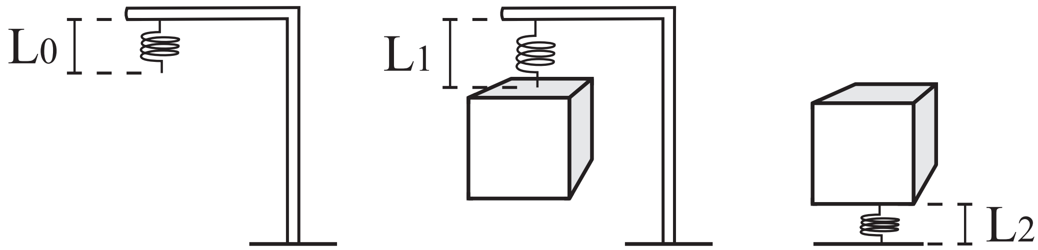


Figure 7.28: A stretched or compressed spring balancing a weight.

The same definition can also be applied to forces of another origin. If, instead of the spring, the body is suspended by a thread, supported on a stick, if the body gets stuck in a person's hands, etc., we can always say that the weight P of the body is balanced by this other force F of the same magnitude as the weight, that is, $F = P$, no matter the origin of this force F.

We have seen that if an object A is released at rest, it falls to the ground. In the previous experiments we saw that we can prevent this fall by connecting this body to a balance with equal arms and placing another body B on the other side of the balance. We define that these two bodies have the same weight if the balance remains in equilibrium. But body A is not connected directly to body B, as it is in contact only with the pan of the balance. We can then see that the downward weight acting upon A, due to the Earth's gravity and acting as if it were concentrated at the CG of A, is balanced by a normal upward force of magnitude N exerted by the pan of the balance acting upon A at the region of contact. That is, $N \equiv P$, as in Figure 7.29.

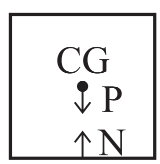


Figure 7.29: The downward weight P of the body balanced by the upward normal force N exerted by the pan.

This normal force N has its origin in the downward weight of body B, being transmitted by the curved pan and taught thread holding B, by the curved rigid beam, and then by the taught thread and curved pan holding A. The threads holding the scale pans are taught (that is, under tension) due to the gravity acting upon A and B. The scale pans are also under stress or tension, with the threads forcing them upward, while A and B force them downward.

We can then say that a first condition of equilibrium in order for a body to remain at rest relative to the ground is that the downward weight P must be counterbalanced by an upward force N of the same magnitude as the weight.

We can also investigate weight and forces in general by considering algebraic magnitudes, that is, positive and negative. We deal here with forces along the vertical direction and choose the downward direction as positive. In other words, forces exerted toward the Earth, such as the weight, are considered positive, while upward acting forces are considered negative. We can also choose, for instance, the right and forward directions as positive, while the left and backward directions will be negative. We then postulate that a body is in equilibrium when the sum of all forces acting upon it, in all directions, goes to zero. If this sum is different from zero, we postulate that the body will move toward the direction of the net force.

Chapter 8

The Law of the Lever

8.1 Building and Calibrating Levers

The lever is one of the simple machines studied in ancient Greece. The other simple machines were the windlass, the pulley, the wedge and the screw. The lever consists of a rigid body, normally linear, the beam, capable of turning around a fixed axis horizontal to the ground. This axis is called the fulcrum or point of suspension, PS, of the lever. This axis is orthogonal to the beam. The lever is like a balance, but now with the possibility of placing weights at different distances from the fulcrum. The models which we will consider here are analogous to the balances built earlier. We will consider only levers in stable equilibrium for which the fulcrum is vertically above the CG of the beam when it is at rest horizontally. We will suppose that the lever is symmetrical about the vertical plane passing through the fulcrum, with the beam horizontal and orthogonal to this vertical plane when there are no bodies supported by the lever.

As we did with the balance, we will define the expression "lever in equilibrium" when its beam remains at rest horizontally relative to the ground. We call the arm of the lever the horizontal distance d between the point of suspension of a body upon the beam and the vertical plane passing through the fulcrum. For brevity we sometimes say, simply, "distance between the body and the fulcrum;" but in general this should be understood as meaning the horizontal distance between the point of suspension of the body upon the beam and the vertical plane passing through the fulcrum. When we talk about the two arms of a lever, these should be understood as the opposite sides in relation to the vertical plane passing through the fulcrum.

In order to arrive at the oldest law of mechanics in a precise and quantitative way we need a sensitive lever. The conditions to obtain a good sensitivity are the same as for the balance:

> Freedom of rotation around the fulcrum.
>
> A high ratio $\triangle P/P_L$. Here $\triangle P$ is the difference of weight between the bodies suspended on the two sides of the lever, and P_L is the weight of the lever.
>
> A high ratio d/h. Here h is the vertical distance between the PS and the CG of the beam, while d is the smaller arm of the lever.

We also need to mark precisely upon the two arms several points at equal distances from the vertical plane passing through the fulcrum. There are two procedures for making these markings on both arms.

(A) The first procedure is to establish the fulcrum of the lever. To this end we can made a hole on the beam or attach a hook on the beam from which the lever will hang; alternatively, we can pass a needle through the beam, in such a way that the needle is attached to the beam, so that the needle can be supported over a stand, etc. After this procedure, we adjust the beam so that it lies horizontal without additional weights. Then markings are made on both sides of the crossbar, at equal distances from the vertical plane passing through the fulcrum.

(B) The second procedure is to initially make markings on the crossbar. This can easily be done, for instance, using a graduated ruler as the beam; or gluing graph paper onto a piece of wood used as the beam; or marking equally spaced points on a piece of wood, broom handle, or barbecue skewer with a ruler, then attaching nails or hooks to these markings, etc.

After the scale is established on the crossbar, the fulcrum is placed on the plane of symmetry that divides the horizontal crossbar into two equal parts. As we saw before, the fulcrum should not be at the center of the beam. The best place for the fulcrum is along the vertical plane of symmetry which divides the beam in two equal parts. The fulcrum should be above the center of the beam, generally offset vertically from the center, so that the point of suspension PS is above the CG of the crossbar, in order to produce stable equilibrium. It must then be verified whether the crossbar remains horizontal when the lever is free to rotate around the fulcrum. If this does not occur, some extra weight (a wire, piece of string, or modeling clay) can be placed somewhere on one of the arms to make the lever horizontal.

In Figure 8.1 we present several kinds of lever, analogous to the balances already built.

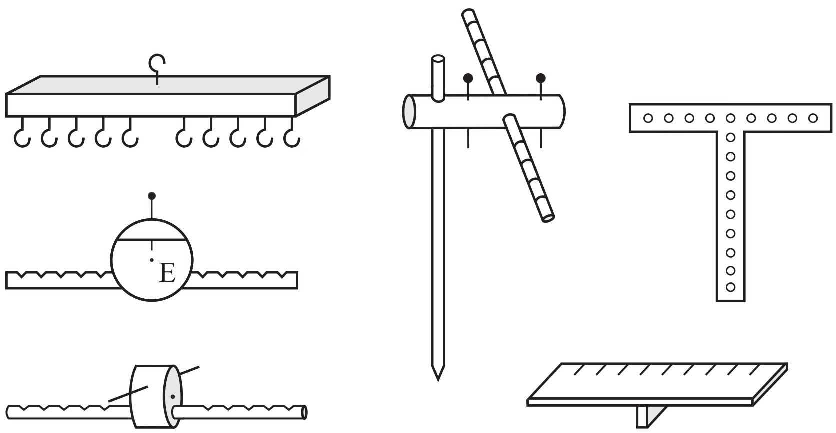


Figure 8.1: Examples of levers.

Before experimenting with the lever, we must test it in order to see if the lever is calibrated. Let us suppose that it remains horizontal after being released without any bodies upon it. We then suspend two equal weights ($P_A = W_B = P$) over two equal arms of the lever ($d_A = d_B = d$). The lever must remain in equilibrium when released at rest horizontally. After this, as we did with the balance, the positions of bodies A and B must be swapped and the lever must remain in equilibrium after being released at rest. Moreover, equilibrium must be maintained for all marks on the lever, that is, for all values of d. From now on we will assume that we are working with calibrated levers.

8.2 Experiments with Levers and the Oldest Law of Mechanics

We now begin experimenting with levers.

Experiment 8.1

We place a paper clip at the distance of 4 cm from the vertical plane passing through the fulcrum of the lever and another clip of the same weight at a distance of 6 cm from the fulcrum, on the other side of the lever. After the lever is released at rest horizontally, the clip at the larger distance from the fulcrum is observed to fall, while the other rises, as in Figure 8.2 (a).

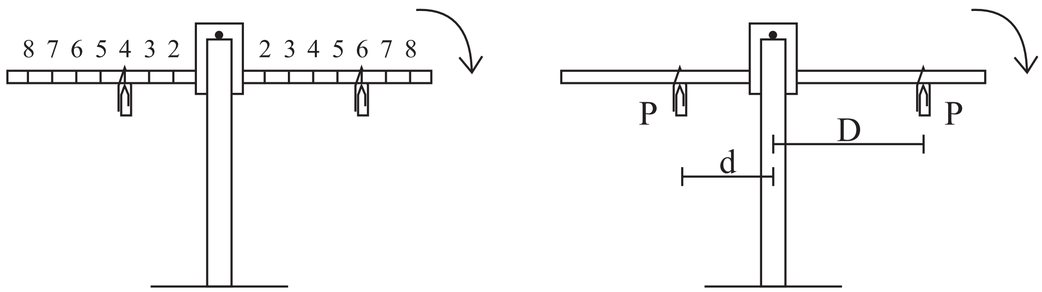


Figure 8.2: A weight at a greater distance from the fulcrum has a larger power to turn the lever than an equal weight at a smaller distance from the fulcrum.

The same phenomenon happens for other distances. That is, we place equal weights on arms of different lengths of the lever, $D > d$, releasing the lever at rest horizontally. We again observe that the weight at the larger distance, D, falls, while the other weight rises, as in Figure 8.2 (b).

This experiment shows that in order to obtain equilibrium, it is not enough to have equal weights on both sides of the fulcrum of a lever. The experiment shows that another relevant factor is the horizontal distance of the weights from the vertical plane passing through the fulcrum. Only experience tells us this; this fact does not come from theory. That is, experimentally we learn that for the equilibrium of two bodies on a lever, the relevant factors are their weights and distances from the fulcrum. On the other hand, other factors do not affect the equilibrium of the lever. Experience teaches that these other irrelevant factors are the colour, shape, texture or chemical composition of the bodies. Their volumes are also irrelevant, provided we are experimenting in a high vacuum.

This is one of the simplest and most intriguing experiments in mechanics. After all, there are equal weights on both sides of the lever. In spite of this, we observe that the weight at a larger distance from the fulcrum has a greater tendency or power to rotate the lever than the weight at a smaller distance. Although this fact is observed in everyday life, it is still extremely curious.

Experiment 8.2

We place 4 paper clips of the same weight at a distance of 6 cm from the fulcrum, equilibrating 4 other identical clips placed at 6 cm from the fulcrum on the other side of the lever. The lever remains at rest horizontally, Figure 8.3 (a). Experience shows that this equilibrium is not disturbed if on one of the sides we place 2 of the paper clips acting at a distance of 4 cm from the fulcrum, and the other 2 paper clips acting at a distance of 8 cm from the fulcrum, Figure 8.3 (b). Moreover, the equilibrium will continue by placing one of the clips at a distance of 3 cm from the fulcrum, another at the distance of 5 cm from the fulcrum, while the other 2 paper clips remain at a distance of 8 cm from the fulcrum, as in Figure 8.3 (c).

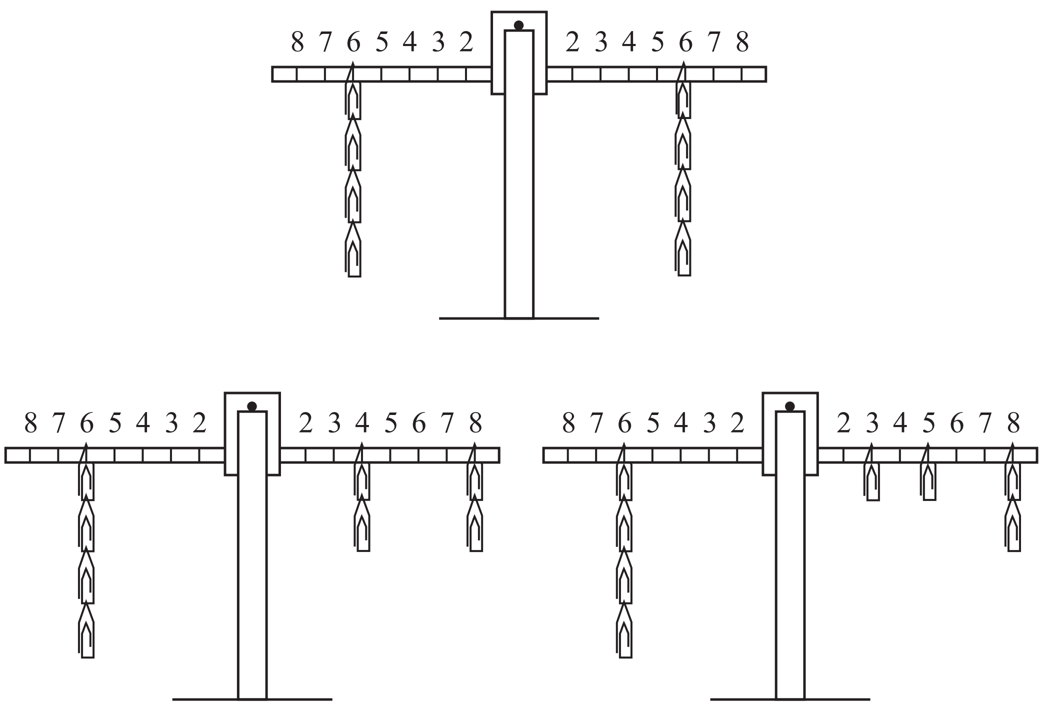


Figure 8.3: The equilibrium of the lever is not disturbed when we move simultaneously, by the same distance, a weight to the right and an equal weight to the left.

We can generalize this result as follows. We place N bodies of the same weight at a distance d from the fulcrum, and N other identical bodies at the same distance d from the other side of the fulcrum. The lever remains in equilibrium. Experimentally it is shown that it remains in equilibrium when we divide one of these groups into two or three parts, with M bodies at the distance d from the fulcrum (M can be equal to zero in the special case), $(N-M)/2$ bodies at a distance $d-x$ from the fulcrum and $(N-M)/2$ of these bodies at a distance $d+x$ from the fulcrum. On the other hand, equilibrium will not occur if we place $(N-M)/2$ bodies at a distance $d-x_1$ from the fulcrum and $(N-M)/2$ of these bodies at a distance $d+x_2$ from the fulcrum, if x_1 is different from x_2. Equilibrium will remain in the first case if we can divide one or more of the first groups of $(N-M)/2$ bodies into two or three sub-groups, by placing Q of them at the distance $d-x$ from the fulcrum, while $((N-M)/2-Q)/2$ are placed at a distance $(d-x)-y$ from the fulcrum and $((N-M)/2-Q)/2$ are placed at a distance $(d-x)+y$ from the fulcrum. And so on. In the previous example we had $N=4$, $M=Q=0$, $d=6$ cm, $x=2$ cm and $y=1$ cm.

This experiment is not trivial. It shows that a weight P placed at a distance d from the fulcrum is equivalent to a weight $P/2$ placed at a distance $d-x$ from the fulcrum, together with another weight $P/2$ at a distance $d+x$ from the fulcrum. That is, these two weights $P/2$ on one side of the fulcrum, at distances $d+x$ and $d-x$ from the fulcrum, balance a weight P on the other side of the fulcrum at a distance d from it. This experiment indicates that, as regards the rotation of the lever, the weights act independently of one another, following the principle of superposition, with a linear influence of their distances from the fulcrum.

If the influences of their distances to the fulcrum were not linear but followed another law (quadratic, cubic, inverse of the distance, inverse square, sinusoidal, logarithmic, etc.), then the equivalence already observed would no longer hold.[159] Once more, this conclusion comes from experiment; no logical argument obliges nature to behave like this.

We now analyze the equilibrium of a lever with different weights on its two arms.

Experiment 8.3

We take 5 paper clips of the same weight. We place 2 of these clips at a distance of 6 cm from the vertical plane passing through the fulcrum. We place the other 3 paper clips at a distance of 6 cm from the other side of the vertical plane passing through the fulcrum, releasing the lever at rest

[159] [AR08] and [AR09].

horizontally. We observe that it turns around the fulcrum, with the 3 paper clips falling and the 2 paper clips rising, as in Figure 8.4 (a).

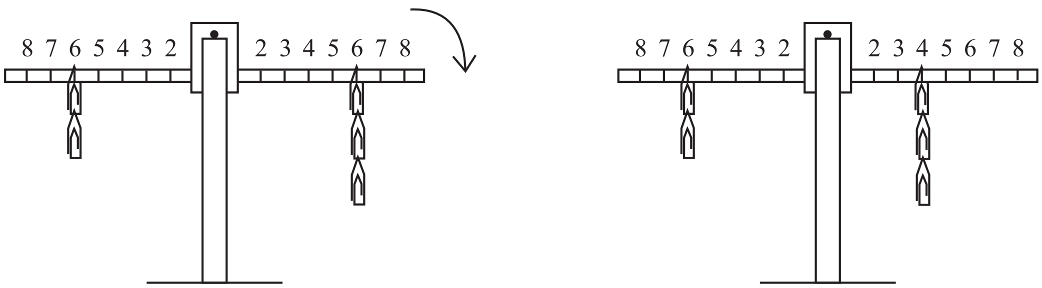


Figure 8.4: Imbalance of different weights placed at the same distance from the fulcrum and balance of different weights placed at different distances from the fulcrum.

This experiment can be generalized to other cases. That is, suppose we have N bodies of equal weight P at a distance d from one side of the vertical plane passing through the fulcrum of a lever, and M other bodies of equal weight P at the same distance d on the other side of the fulcrum, with $M > N$. If we release the lever at rest horizontally, it turns around the fulcrum, with the set of M bodies falling and the set of N bodies rising. By the definitions we introduced in Subsection 7.2.1, we say that the set of M bodies is heavier than the set of N bodies.

Now comes one of the most important experiments of all.

Experiment 8.4

We consider 5 paper clips of the same weight. We place 2 of these clips at the same distance of 6 cm from the vertical plane passing through the fulcrum. We want to find the distance from the other side of the fulcrum at which we should place the 3 other clips together in order to maintain the lever in equilibrium (that is, balanced at rest horizontally after being released). Experiment shows that this only happens when they are at a distance of 4 cm from the vertical plane passing through the fulcrum, as in Figure 8.4 (b).

When we place 2 paper clips at the same distance d_A from the vertical plane passing through the fulcrum, it is observed that the lever only remains in equilibrium with 3 other paper clips acting together at the same distance

d_B from the other side of the fulcrum when these two distances are related according to the following Table:

d_A (cm)	2	3	4	5	6	7	8
d_B (cm)	4/3	2	8/3	10/3	4	14/3	16/3

Experiment 8.5

We now consider 6 paper clips of the same weight. We place 1 of these paper clips at the distance of 5 cm from the vertical plane passing through the fulcrum. We want to find the distance from the other side of the fulcrum at which we should place the 5 other paper clips together in order to maintain the lever in equilibrium (that is, balanced at rest horizontally after being released). Experiment shows that this only happens when they are at a distance of 1 cm from the vertical plane passing through the fulcrum.

When we place 1 paper clip at the distance d_A from the vertical plane passing through the fulcrum, it is observed that the lever only remains in equilibrium with 5 other paper clips acting together at the same distance d_B from the other side of the fulcrum when these two distances are related according to the following Table:

d_A (cm)	5	6	7	8	9	10
d_B (cm)	1	6/5	7/5	8/5	9/5	2

8.2.1 First Part of the Law of the Lever

The result of Experiments 8.4 and 8.5 is also verified in other cases. We place N_A bodies of the same weight P acting together at a distance d_A from the fulcrum on one arm of a lever. Their total weight is given by $P_A \equiv N_A P$. We place N_B other bodies of the same weight P acting together on the other side of the lever at a distance d_B from the fulcrum. Their total weight is $P_B \equiv N_B P$. The lever is then released at rest horizontally. Experiment shows that it only remains in equilibrium if

$$\frac{d_B}{d_A} = \frac{P_A}{P_B} = \frac{N_A}{N_B} . \tag{8.1}$$

This is the initial part of the law of the lever. Sometimes it is called "the first law of mechanics."[160] The word "first" should be understood in a historical context, the law of the lever being the oldest law of mechanics in Western science.

Archimedes obtained the law of the lever in Proposition 6 of the first part of his work *On the Equilibrium of Plane Figures*:[161]

> *Commensurable magnitudes are in equilibrium at distances reciprocally proportional to the weights.*

By "magnitudes" we understand that Archimedes was referring to physical bodies. The idea behind commensurable magnitudes is measurement by comparison. That is, to measure two or more magnitudes with the same unit or standard of measure. Definition 1 of Book X of *The Elements* of Euclid stated the following:[162]

> Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

Moreover, according to Proposition 5 of Book X, we have the following:[163]

> Commensurable magnitudes have to one another the ratio which a number has to a number.

A number here should be understood as a natural number, namely, positive integers: 1, 2, 3, ...

If the weight of a body A is 5 times the weight of a body C, and the weight of a body B is 3 times the weight of C, we say that A and B are commensurable. In this example we can then say that the weight of A is to the weight of B as 5 is to 3. The weight of body C in this example would be the unit or standard of measure with which we can measure not only the weight of A, but also the weight of B.

On the other hand, if there is no body C such that the weight of A is a multiple of C, and the weight of B is another multiple of C, then we say that the weights of A and B are incommensurable.

[160] [BRS03, p. 14].
[161] [Dij87, p. 289].
[162] [Euc56, Vol. 3, p. 10].
[163] [Euc56, Vol. 3, p. 24].

The most famous example of incommensurable magnitudes is related to straight segments. The diagonal of a square, for instance, is incommensurable with the side of this square. That is, it is not possible to find a third segment such that the diagonal of the square is a multiple of this third segment, while the side of the square is another multiple of this third segment.

In Proposition 7 of his work *On the Equilibrium of Plane Figures* Archimedes generalized the law of the lever for incommensurable magnitudes:[164]

> *However, even if the magnitudes are incommensurable, they will be in equilibrium at distances reciprocally proportional to the magnitudes.*

In his English translation of Archimedes' work, Heath combined these two propositions in a single sentence, namely:[165]

> **Propositions 6, 7.** *Two magnitudes, whether commensurable* [Prop. 6] *or incommensurable* [Prop. 7], *balance at distances reciprocally proportional to the magnitudes.*

The law of the lever specifies the necessary condition in order to obtain equilibrium. It must be supplemented with the information of what happens when this condition is not satisfied.

Suppose that P_A/P_B is different from d_B/d_A, with the weight P_A at a distance d_A from one side of the fulcrum, while the weight P_B is at a distance d_B from the other side of the fulcrum. In this case there will be no equilibrium when the lever is released at rest. One of the bodies will move downward, and the other, upward.

The experimental results already presented, for the cases of equilibrium and for the cases in which there is lack of equilibrium, can be summarized as follows:

If $(P_A/P_B)(d_A/d_B) = 1$, the lever will remain in equilibrium.

If $(P_A/P_B)(d_A/d_B) > 1$, then A will fall and B will rise.

If $(P_A/P_B)(d_A/d_B) < 1$, then A will rise and B will fall.

[164] [Dij87, p. 305].

[165] [Arc02b, p. 192].

8.2.2 Experimental Mistakes which Prevent the Verification of the Law of the Lever

As the law of the lever is one of the most important laws of classical mechanics, it is worth while calling attention to some experimental errors which prevent the verification of this result.

Let us suppose that the lever remains initially at rest horizontally without the scale pans and also without bodies A and B. Let us suppose that both scale pans and their threads have the same weight. The most frequent mistake is to place weight P_A on one of the scale pans at a distance d_A from the fulcrum and the weight P_B on the other scale pan at a distance d_B from the fulcrum. In this case, even if $d_B/d_A = P_A/P_B$, the lever does not remain in equilibrium after being released. What happens is that the larger arm moves downward (assuming a lever with high sensitivity, with negligible friction, totally free to rotate around the fulcrum), as in Figure 8.5 (a).

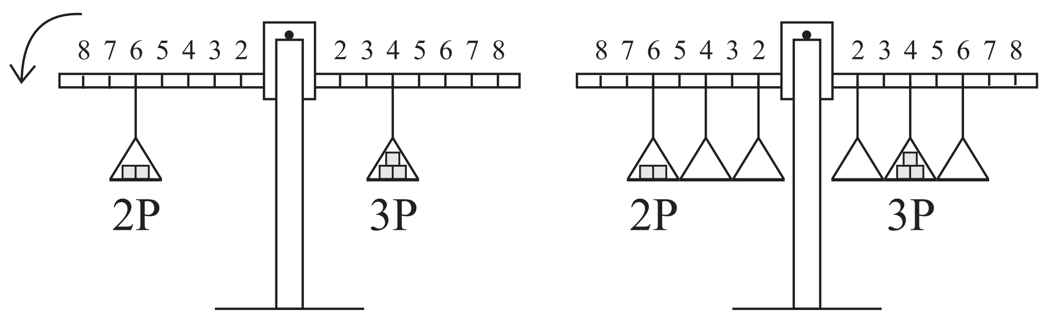


Figure 8.5: (a) Common mistake which prevents the verification of the law of the lever. (b) How to observe the law of the lever correctly.

The explanation of this phenomenon is related to the law of the lever itself. Although bodies A and B balance one another when placed at distances inversely proportional to their weights, the same does not hold for the two equal scale pans. Here we have two scale pans of the same weight placed at different distances from the fulcrum. By the previous experiments we know that they do not balance one another. Instead of this, the pan placed at the larger arm moves downward. In order to prevent this common mistake, we did not employ any scale pans in the experiments with levers performed thus far. Instead, we suspended the bodies directly from the beam. But it is possible to utilize scale pans in a lever, provided they are equal in number on both sides, with each pair of equal scale pans placed at the same distance from the fulcrum. For example, we can have 6 equal scale pans, three of them placed at distances of 2 cm, 4 cm and 6 cm on one side of the fulcrum, and the other three pans placed at the same distances on the other side of the fulcrum. In this case the lever remains in equilibrium even after A and B are placed on the scale pans, provided $d_B/d_A = P_A/P_B$, as in Figure 8.5 (b).

Another common mistake that is made even without scale pans is as follows. Suppose that a lever remains in equilibrium when the vertical plane passing through the fulcrum divides the homogeneous beam into two equal parts. We now place two bodies of different weights at the extremities of the beam and change the location of the fulcrum in such a way that $d_B/d_A = P_A/P_B$. The lever does not remain in equilibrium in this case after being released at rest. Instead of remaining in equilibrium, the side with longer arm falls to the ground, as in Figure 8.6.

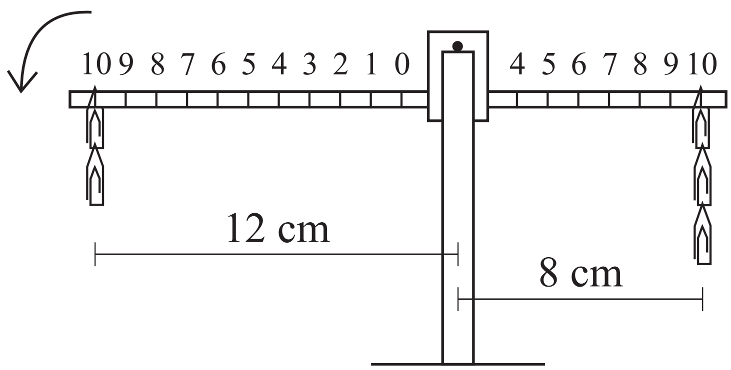


Figure 8.6: This is not the correct way to observe the law of the lever.

Once more, the explanation for this behavior is related to the law of the lever itself. Let us suppose that there are no scale pans and that the bodies A and B are suspended directly from the beam following the previous relation. Therefore they balance the fulcrum in its new position because they satisfy the relation $d_B/d_A = P_A/P_B$. But the beam itself is not in equilibrium for the new position of the fulcrum. When we changed the position of the fulcrum in relation to the center of the beam, the beam became unbalanced, regardless of the positions of bodies A and B. The longer arm of the homogeneous beam tends to fall to the ground, as it is heavier than the other side, which moves upward. Even placing bodies A and B on the beam satisfying the previous relation $d_B/d_A = P_A/P_B$ does not balance the beam. In order to avoid this mistake, the correct procedure is to balance the beam without bodies A and B, adjusting the fulcrum over the CG of the beam in such a way that it remains horizontally at rest relative to the ground. After this, without changing the position of the fulcrum in relation to the beam, we can place bodies A and B. In this case it will be seen that they will keep the beam in balance provided that $d_B/d_A = P_A/P_B$.

These two mistakes are related to the fact that the scale pans and the beam itself are material bodies with weight. Therefore, they may also influence the equilibrium of the lever. This aspect cannot be neglected when we work with sensitive levers and wish to identify precisely which quantitative factors determine the equilibrium of bodies.

8.2.3 Second Part of the Law of the Lever

In the first part of the law of the lever we consider a single body in each side of the fulcrum. For the second part of the law of the lever we consider the situation in which there is more than one body in at least one side of the lever, maybe also in both sides. The following experiments illustrate what happens in these cases.

Experiment 8.6

We take 16 paper clips of the same weight. On one side of the lever we place 1 paper clip at a distance of 10 cm from the fulcrum, 2 paper clips at 8 cm from the fulcrum, and 3 paper clips at 4 cm from the fulcrum. On the other side of the lever we put 1 paper clip at 2 cm from the fulcrum and 9 clips at 4 cm from the fulcrum. It is observed that the lever remains in equilibrium, as in Figure 8.7.

This experiment shows that as regards rotation of the lever, the weights act independently of one another, proportionately to their distances to the fulcrum. That is, the rotation effects due to the weights follow the law of addition. This is expressed in physics by saying that the law of the lever follows the principle of superposition.

The result of this specific experiment is also true in other cases. It can be generalized as follows.

We place N weights P_1, P_2, ..., P_N on one side of the lever, acting at distances d_1, d_2, ..., d_N, respectively, from the vertical plane passing through the fulcrum. We place M other weights P_{N+1}, P_{N+2}, ..., P_{N+M} on the other

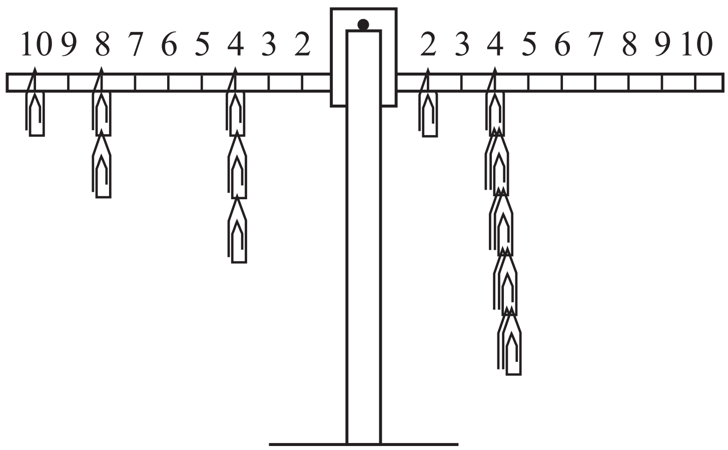


Figure 8.7: At the left side there is 1 clip at 10 cm, 2 clips at 8 cm and 3 clips at 4 cm from the fulcrum. At the right side there is 1 clip at 2 cm and 9 clips at 4 cm from the fulcrum. The lever remains in equilibrium, illustrating the principle of superposition.

side of the fulcrum, acting at distances d_{N+1}, d_{N+2}, ..., d_{N+M}, respectively, from the vertical plane passing through the fulcrum. The total weights on the left and right sides are defined by, respectively:

$$P_L \equiv \sum_{i=1}^{N} P_i \ , \tag{8.2}$$

and

$$P_R \equiv \sum_{i=N+1}^{N+M} P_i \ . \tag{8.3}$$

As will be discussed in Section 9.1, the centers of gravity of the weights on the left and right sides, d_L and d_R, are defined by the following expressions:

$$d_L \equiv \sum_{i=1}^{N} \frac{P_i}{P_L} d_i \ , \tag{8.4}$$

and

$$d_R \equiv \sum_{i=N+1}^{N+M} \frac{P_i}{P_R} d_i \ . \tag{8.5}$$

It is observed experimentally that the system only remains in equilibrium, after being released at rest horizontally, when the following equation is satisfied, namely:

$$\frac{d_L}{d_R} = \frac{P_R}{P_L} \ . \tag{8.6}$$

This equation might be expressed in words as follows:

> Two sets of weights are in equilibrium on both sides of a lever at distances of their centers of gravity reciprocally proportional to the total weights.

This is a generalization of Archimedes' statement of the law of the lever presented in Subsection 8.2.1. This is the final part of the oldest law of mechanics, that is, the law of the lever combined with the principle of superposition.

In our specific example we had $P_L = 6P_{clip}$ and $P_R = 10P_{clip}$, where P_{clip} is the weight of a single paper clip. The centers of gravity of the left and right sides will be located at, respectively:

$$d_L = \sum_{i=1}^{6} \frac{P_i}{6P_{clip}} d_i = \frac{3 \times 4 + 2 \times 8 + 1 \times 10}{6} = \frac{19}{3} \ , \tag{8.7}$$

and

$$d_R = \sum_{i=7}^{16} \frac{P_i}{10P_{clip}} d_i = \frac{1 \times 2 + 9 \times 4}{10} = \frac{19}{5} \ . \tag{8.8}$$

We then have:

$$\frac{d_L}{d_R} = \frac{19/3}{19/5} = \frac{5}{3} = \frac{10}{6} = \frac{10P_{clip}}{6P_{clip}} = \frac{P_R}{P_L} \ . \tag{8.9}$$

As Equation (8.6) is satisfied, the system will remain in equilibrium after being released at rest.

The general experimental results for the cases of equilibrium and for the cases in which there is lack of equilibrium, can be summarized as follows:

If $(P_L/P_R)(d_L/d_R) = 1$, the lever will remain in equilibrium.

If $(P_L/P_R)(d_L/d_R) > 1$, then the left side will fall and the right side will rise.

If $(P_L/P_R)(d_L/d_R) < 1$, then the left will rise and the right side will fall.

Experiment 8.7

We suspend a lever by the fulcrum on one of the sides of a balance with equal arms, in such a way that the lever remains horizontal without extra weights. On the other side of the balance we suspend a weight P_{Tr} equal to the weight of the lever, such that the balance remains in equilibrium horizontally, as in Figure 8.8 (a).

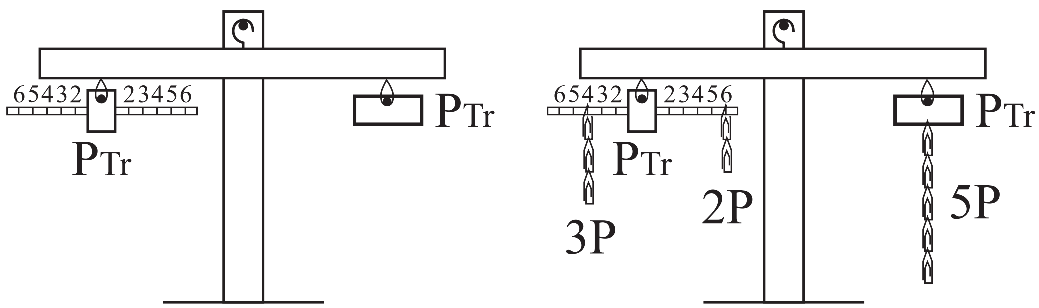


Figure 8.8: Levers in equilibrium.

We then take 10 paper clips of the same weight. We place 3 of them on one arm of the lever at a distance of 4 cm from the fulcrum, and 2 of them on the other side of the lever at a distance of 6 cm from the fulcrum. We then try to find how many paper clips of the same weight we need to suspend on the other side of the balance in order to keep it in equilibrium. Experimentally it is found that this only happens by hanging 5 paper clips, as in Figure 8.8 (b).

This and other analogous experiments show that the fulcrum of a balance in equilibrium with weights P_A and P_B at the distances d_A and d_B, respectively, on opposite sides of the fulcrum, in such a way that $P_A/P_B = d_B/d_A$, supports a total weight of $P_{Tr} + P_A + P_B$. Here P_{Tr} is the weight of the lever. We can then see that there are four forces acting upon the beam of a lever in equilibrium: (A) the downward weight of the beam acting as if it were concentrated at the CG of the beam; (B) the downward weight of body A

acting at a distance d_A from the fulcrum; (C) the downward weight of body B acting at a distance d_B from the other side of the fulcrum; and (D) the normal upward force N acting along the fulcrum, as in Figure 8.9.

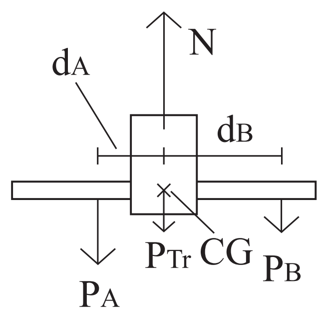


Figure 8.9: Conditions of equilibrium for a lever.

The weights of the beam and of bodies A and B are due to their gravitational interactions with the Earth. The normal upward force is exerted by the support upon the beam and arises due to the tension or compression of the support. The support will be stretched or under mechanical tension when it is a hook (or a thread, or a spring) attached to a rigid support at its upper end, holding the fulcrum of the lever at its lower end, as in the previous experiment. The support will be compressed when it is a rigid stand or a spring placed below the fulcrum, as in the majority of the situations considered up to now. We then see that there are two necessary conditions in order to have a balanced lever, namely:

$$N = P_{Tr} + P_A + P_B \ , \tag{8.10}$$

and

$$\frac{d_B}{d_A} = \frac{P_A}{P_B} \ . \tag{8.11}$$

The latter relation needs to be generalized if the fulcrum is not along the same vertical plane passing through the CG of the beam. Let us suppose that the CG of the lever is along the same side of the vertical plane passing through the fulcrum as the body B, at a distance d_{Tr} from this plane, as in Figure 8.10.

In this case the conditions for balancing the lever are given by:

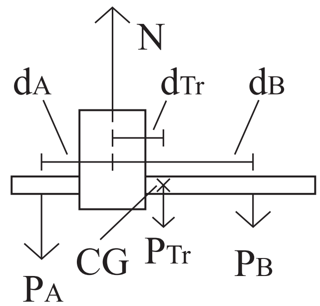


Figure 8.10: Conditions of equilibrium for an asymmetrical lever.

$$N = P_{Tr} + P_A + P_B \ , \tag{8.12}$$

and

$$\frac{d_L}{d_R} = \frac{P_R}{P_L} \ , \tag{8.13}$$

where $P_L = P_A$, $P_R = P_{Tr} + P_B$. Moreover, d_L and d_R in Equation (8.13) are given by, respectively:

$$d_L = \frac{P_A}{P_L} d_A \ , \tag{8.14}$$

$$d_R = \frac{P_{Tr}}{P_R} d_{Tr} + \frac{P_B}{P_R} d_B \ . \tag{8.15}$$

If $d_{Tr} = 0$, or if we can neglect the weight of the lever in comparison with the weights of bodies A and B, we return to the previous case.

When we have several bodies acting simultaneously on the lever, we can utilize the principle of superposition given earlier in order to establish the equilibrium conditions.

Complement to the law of the lever: The downward force exerted by the fulcrum upon the support when the lever is in equilibrium is given by the sum of the weights of the suspended bodies, plus the weight of the lever (that is, of its beam, threads, and scale pans).

8.3 Types of levers

We have seen in Section 7.5 how to use an equal-arm balance to quantify forces of any nature (contact, elastic, electric, magnetic, etc.) by comparing these other forces to the force due to the weight of a body. That is, how to compare, for instance, a magnetic force with the gravitational force between a body and the Earth. In other words, a force F of any origin acting on one side of a balance and equilibrating a weight P on the other side is defined as equal to this weight, $F = P$. This operational definition of force, together with the law of the lever, is related to the utilization of the lever as a simple machine. The law of the lever shows that a small weight can balance a large weight provided the small weight is at a greater distance from the fulcrum than the large weight. A simple machine is a device that can multiply the intensity of a force in order to do work.

In this Section we will neglect the weight of the lever as compared with the other forces acting upon it.

The law of the lever states that a weight P_A located at distance d_A from the vertical plane passing through the fulcrum balances another weight P_B at distance d_B from the vertical plane passing through the fulcrum if $P_A/P_B = d_B/d_A$. When we utilize a lever as a simple machine, it is more convenient to talk of forces than of weights, as the forces acting upon the lever do not need to be gravitational in origin. Let F_A be the applied force exerted upon the machine by the operator (a man, an animal, or a mechanical device) and F_R be the resistive force. That is, F_R is the force exerted by the machine upon the load (weight to be raised or pushed, body to be compressed or stretched, figure to be cut out, etc.). To simplify the analysis we will suppose that the points of application of F_A and F_R are aligned with the fulcrum of the lever, and that these two forces act orthogonally to this straight line. The arms of the lever (that is, the distances between the points of application of these forces and the fulcrum) will be represented by d_A and d_R, respectively. The equilibrium of the lever is then given by $F_A/F_R = d_R/d_A$.

We can see that there are three main elements to a lever working as a simple machine: the applied force (the effort), the resistive force (the load) and the fulcrum, which remains at rest relative to the Earth. Depending upon the position of the fulcrum in relation to the applied and resistive forces, there will be three basic kinds of lever.[166]

[166] [Netsd].

(A) Lever of the first class, with the fulcrum between the load and the effort, Figure 8.11 (a).

(B) Lever of the second class, with the load between the effort and the fulcrum, Figure 8.11 (b).

(C) Lever of the third class, with the effort between the fulcrum and the load, Figure 8.11 (c).

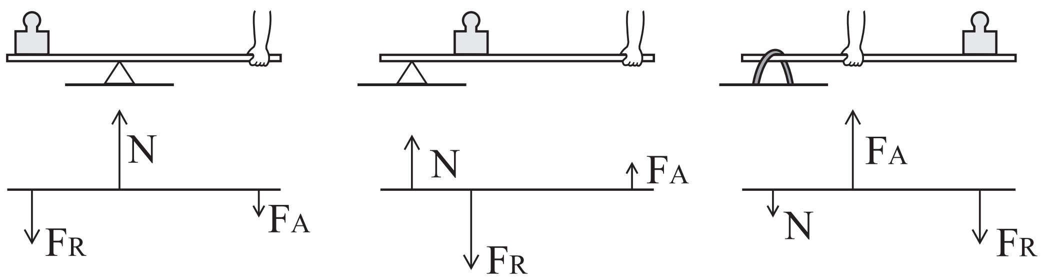


Figure 8.11: Types of lever.

Up to now we have worked only with levers of the first kind. Examples: the balance with equal arms, a Roman balance, a seesaw, a crowbar, the salad tongs, a pair of pliers, a pair of scissors, the handle of a water pump, a claw hammer used to pull a nail out of wood, etc.

Examples of levers of the second class: a wheelbarrow, a bottle opener, a wrench, a door, a pair of nutcrackers, a pair of bellows, a punching machine, a paper cutter, the brake pedal of a car, etc.

Examples of levers of the third class: a pair of tweezers, a hoe, a hammer, a broom, a pair of barbecue tongs, the forceps, the jaws, the human forearm, a fishing rod, a stapler, etc.

8.4 Limitations of the Law of the Lever

Archimedes' law of the lever reads as follows:[167]

> **Propositions 6, 7.** *Two magnitudes, whether commensurable* [Prop. 6] *or incommensurable* [Prop. 7], *balance at distances reciprocally proportional to the magnitudes.*

[167] [Arc02b, p. 192].

This law is an idealization, just like all laws of physics. Like any theoretical law, it has limitations or intervals of validity. It is impossible to know if it applies to enormous weights like the weight of a continent, the Earth itself, the Moon, or a planet. Nor is it possible to know if it applies to extremely small weights like the weight of an atom, an electron, or even smaller magnitudes. The same applies to distances. It is impossible to know if it applies to a distance like the diameter of the Earth or the distance between the Earth and the Sun. Similarly, it is impossible to know if it applies to a distance like the size of an atom or a subatomic particle. It is impossible to build scales and levers so large or so small that they can support such enormous or infinitesimal weights.

Furthermore, there are always unavoidable frictions that, in practice, lead to equilibrium situations that violate the law of the lever. Although these frictions may be small in very precise scales, there is always a minimum degree of friction still present even in the best scales and levers. Another relevant factor is that we cannot measure weights and distances with infinite precision. There is always uncertainty in these measurements. Although these uncertainties can be minimized, they cannot be completely eliminated in practice.

In any case, the main idea of a physical law is that it reasonably represents the behavior of nature within certain limits. A law that stated, for example, the following, would certainly be incorrect experimentally:

> *Commensurable magnitudes are in equilibrium at distances reciprocally proportional to the square of the weights.*

It would also be incorrect experimentally to state the following:

> There is equilibrium when the weights are inversely proportional to the squares of the distances to the fulcrum.

These two wrong laws could be expressed mathematically as follows: There is equilibrium between weights P_A and P_B at distances d_A and d_B from the fulcrum of a lever when these magnitudes behave as follows:

$$\left(\frac{P_A}{P_B}\right)^2 = \frac{d_B}{d_A} \ , \qquad (8.16)$$

or

$$\frac{P_A}{P_B} = \left(\frac{d_B}{d_A} \right)^2 . \tag{8.17}$$

In reality any physical law is an idealization and has limits of validity. In any event, inside its limits of validity, there are certainly other mathematical expressions which clearly do not correspond to reality. This is the essence of the concept that there are mathematical expressions which correspond to the behavior of bodies under certain specified conditions. These mathematical expressions which represent the known behavior of bodies are called physical laws of nature.

Chapter 9

Mathematical Definition of the Center of Gravity

9.1 Algebraic Expression of the CG in Cartesian Coordinates

The law of the lever and the principle of superposition allow a mathematical definition of the center of gravity of a body or of a system of bodies. We saw earlier that the condition of equilibrium of any body suspended by a horizontal axis (the fulcrum or PS) is that this axis and the CG of the body should be along a vertical. Equilibrium will be stable if any disturbance to the body raises the CG compared to its original height relative to the ground. Equilibrium will be unstable if any disturbance to the body lowers the CG compared to its original height relative to the ground. Equilibrium will be indifferent if any disturbance to the body does not alter the height of the CG compared to its original height relative to the ground.

We now consider a lever in stable equilibrium with its beam resting horizontally, without other bodies suspended on it. We imagine a homogeneous beam in such a way that the vertical plane passing through the fulcrum divides it into two equal halves. The CG of the beam is vertically below the fulcrum. We have seen that this equilibrium is not disturbed if two bodies A and B of weights P_A and P_B, respectively, are suspended on opposite sides of the fulcrum, provided that $d_B/d_A = P_A/P_B$. Here d_A and d_B are the horizontal distances between the points of suspension of A and B, respectively, and the vertical plane passing through the fulcrum. This means that the CG of these two bodies is also along the vertical plane passing through the fulcrum. If the ratio d_B/d_A is different from P_A/P_B, the beam does not remain in equilibrium after being released.

In order to find an algebraic expression yielding the location of the CG of the bodies A and B, we can imagine a horizontal axis x along the beam. The origin $x = 0$ can be chosen at any point, arbitrarily. Let us suppose that the ends of the beam of length L are located at x_E and $x_D = x_E + L$. Let x_A and x_B be the points of suspension of bodies A and B along the x axis, respectively. Moreover, let us assume that the lever continues in equilibrium after being released at rest horizontally with A and B acting upon these points, as in Figure 9.1.

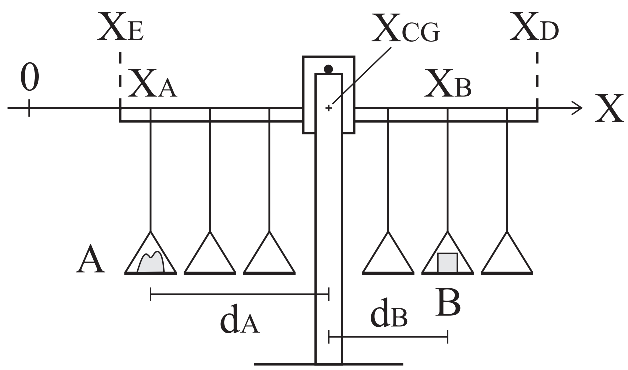


Figure 9.1: Finding an algebraic expression for the center of gravity.

The CG of this system must be along the vertical plane passing through the fulcrum, in such a way that $d_B/d_A = P_A/P_B$. Let x_{CG} be the location of the CG of bodies A and B along the x axis. From Figure 9.1 we have $d_A = x_{CG} - x_A$ and $d_B = x_B - x_{CG}$. From the law of the lever we can then define, mathematically, the position x_{CG} of the center of gravity of this system of two bodies along the x axis as given by

$$\frac{x_B - x_{CG}}{x_{CG} - x_A} \equiv \frac{P_A}{P_B} \ . \tag{9.1}$$

That is,

$$x_{CG} \equiv \frac{P_A}{P_T} x_A + \frac{P_B}{P_T} x_B \ , \tag{9.2}$$

where $P_T \equiv P_A + P_B$ is the total weight of the two bodies.

This theoretical definition of x_{CG} is made in such a way that it coincides with the previous experimental results on the CG of rigid bodies. In other words, in equilibrium the CG of the system of two bodies stays along the vertical plane passing through the fulcrum of the lever. If $P_A = P_B$, we can see from this expression that x_{CG} will be located at the midpoint between x_A and x_B. On the other hand, the larger the value of P_A/P_B, the closer x_{CG} will be from body A. Analogously, the smaller the value of P_A/P_B, the farther x_{CG} will be from body A.

From now on we will use the approximation of particles or point bodies. We consider bodies A and B as particles when the greatest dimensions of either (their diameters, or the greatest distance between any material points belonging to each one of these bodies) are much smaller than the distance between A and B. In this case we can treat the bodies as being concentrated in small regions as compared to the distance between them, as if they were concentrated into mathematical points.

Let us now imagine a rigid system of orthogonal axes xyz with origin O at $x = y = z = 0$. This system of axes is supposed at rest relative to the ground, with a fixed orientation relative to the Earth. The spatial location of body A will be represented by $(x_A,\ y_A,\ z_A)$, Figure 9.2, and the location of body B will be represented by $(x_B,\ y_B,\ z_B)$.

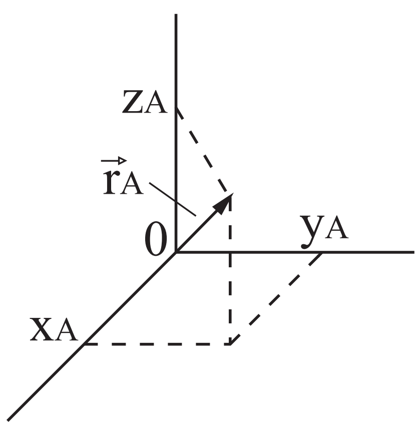


Figure 9.2: Finding the CG with vector notation.

In this way we can generalize for the y and z axes the previous mathematical definition for the CG of the system given by Equation (9.2). We then define the y and z coordinates of the CG, y_{CG} and z_{CG}, respectively, by the relations

$$y_{CG} \equiv \frac{P_A}{P_T} y_A + \frac{P_B}{P_T} y_B \ , \tag{9.3}$$

and

$$z_{CG} \equiv \frac{P_A}{P_T} z_A + \frac{P_B}{P_T} z_B \ . \tag{9.4}$$

In this way we can also utilize vector notation. We call $\vec{r}_A = (x_A,\ y_A,\ z_A)$ the position vector of body A, as in Figure 9.2, and $\vec{r}_B = (x_B,\ y_B,\ z_B)$ the position vector of body B.

The position vector of the CG, $\vec{r}_{CG}$, is defined by:

$$\vec{r}_{CG} \equiv \frac{P_A}{P_T} \vec{r}_A + \frac{P_B}{P_T} \vec{r}_B \ . \tag{9.5}$$

By the principle of superposition these relations can be extended to a set of N particles. Let P_i be the weight of body i located at the position vector $\vec{r}_i = (x_i,\ y_i,\ z_i)$, with $i = 1,\ 2,\ ...,\ N$. Let

1be the total weight of this system of particles. The x component of the CG of this system of particles is defined by:

$$x_{CG} \equiv \sum_{i=1}^{N} \frac{P_i}{P_T} x_i \ . \tag{9.6}$$

Analogous expressions are defined for the y and z components of the CG.

The position vector of the CG of this system of point particles is defined by:

$$\vec{r}_{CG} \equiv \sum_{i=1}^{N} \frac{P_i}{P_T} \vec{r}_i \ . \tag{9.7}$$

This is the modern mathematical definition of the CG of a system of particles. It makes a theoretical calculation of the CG possible, if the locations of particles and their weights are known.

If we have continuous distributions of matter, as in the case of one-, two- and three-dimensional bodies, the procedure is the same. In the first place we replace the summation by line, surface or volume integrals. And instead of the weight P_i of particle i we utilize an infinitesimal element of weight, dP, located at the position vector $\vec{r} = (x,\ y,\ z)$. This element of weight dP represents the weight contained in an infinitesimal element of length, $d\ell$, in an infinitesimal element of area da, or in an infinitesimal element of volume dV. The total weight is given by

$$P_T \equiv \int\int\int dP \ . \tag{9.8}$$

In this case the position vector of the CG can be defined by:

$$\vec{r}_{CG} \equiv \int\int\int \frac{dP}{P_T}\vec{r} \ . \tag{9.9}$$

These volume integrals should be performed over the whole space occupied by the body. If we have matter distributed continuously along a line or surface, we replace these volume integrals by line or surface integrals, respectively.

If we have combinations of discrete and continuous distributions of matter, we only need to add the corresponding expressions in order to obtain the CG of the system as a whole, because the CG follows the principle of superposition.

We will not go into mathematical details here, nor will we calculate the location of the CG for any distribution of matter, as this is not the goal of this book.

In the next Section we summarize the modern mathematical definition of the CG.

9.2 Mathematical Definition $CG9$

Consider a particle i of weight P_i located at the position vector $\vec{r}_i = (x_i,\ y_i,\ z_i)$ in relation to a Cartesian coordinate system fixed in the ground. The total weight P_T of a system of N particles is defined by:

$$P_T \equiv \sum_{i=1}^{N} P_i \ . \tag{9.10}$$

The center of gravity of this system of particles is defined by

$$\vec{r}_{CG} \equiv \sum_{i=1}^{N} \frac{P_i}{P_T}\vec{r}_i \ . \tag{9.11}$$

When we have a continuous distribution of matter we utilize analogous definitions. Let dP be the weight of an element of volume dV located at the position vector $\vec{r} = (x,\ y,\ z)$. The total weight of this distribution is defined by:

$$P_T \equiv \int\int\int dP \ . \tag{9.12}$$

The center of gravity of this continuous distribution of matter is defined by:

$$\vec{r}_{CG} \equiv \int\int\int \frac{dP}{P_T}\vec{r} \ . \tag{9.13}$$

Equations (9.11) and (9.13) constitute the mathematical definitions $CG9$ of the center of gravity of discrete and continuous distributions of matter, respectively.

9.3 Theorems to Simplify the Calculation of the CG

Equations (9.11) and (9.13) are the theoretical definitions in current use to calculate the CG of discrete and continuous distributions of matter, when the weights and locations of the bodies are known.

An important theorem which simplifies the location of the center of gravity states the following, adapted from Symon:[168]

> If a body is composed of two or more parts whose centers of gravity are known, then the center of gravity of the composite body can be computed by regarding its component parts as single particles located at their respective centers of gravity.

A proof of this theorem, beginning with definition $CG9$, can be given as follows. Let a body be composed of N parts of weights P_1, ..., P_N. Let any part P_k be composed of N_k parts of weights P_{k1}, ..., P_{kN_k}, whose centers of gravity are located at the points $\vec{r}_{k1}$, ..., $\vec{r}_{kN_k}$. Then the center of gravity of part P_k, according to definition $CG9$, is located at the point

$$\vec{r}_k \equiv \sum_{\ell=1}^{N_k} \frac{P_{k\ell}}{P_k}\vec{r}_{k\ell} \ , \tag{9.14}$$

[168] [Sym71, p. 221].

where

$$P_k \equiv \sum_{\ell=1}^{N_k} P_{k\ell} \ . \tag{9.15}$$

The center of gravity of the entire body is located at the point

$$\vec{r} \equiv \sum_{k=1}^{N} \sum_{\ell=1}^{N_k} \frac{P_{k\ell}}{P_T} \vec{r}_{k\ell} \ , \tag{9.16}$$

where

$$P_T \equiv \sum_{k=1}^{N} \sum_{\ell=1}^{N_k} P_{k\ell} \ . \tag{9.17}$$

This means that the center of gravity of the entire body can be written as

$$\vec{r} \equiv \sum_{k=1}^{N} \sum_{\ell=1}^{N_k} \frac{P_{k\ell}}{P_T} \vec{r}_{k\ell} = \sum_{k=1}^{N} \frac{P_k}{P_T} \left(\sum_{\ell=1}^{N_k} \frac{P_{k\ell}}{P_k} \vec{r}_{k\ell} \right) = \sum_{k=1}^{N} \frac{P_k}{P_T} \vec{r}_k \ . \tag{9.18}$$

The total weight can also be written as

$$P_T \equiv \sum_{k=1}^{N} \sum_{\ell=1}^{N_k} P_{k\ell} = \sum_{k=1}^{N} P_k \ . \tag{9.19}$$

Equations (9.18) and (9.19) embody the mathematical statement of the theorem to be proved.

Archimedes knew a theorem analogous to this one that "if a body is composed of two or more parts whose centers of gravity are known, then the center of gravity of the composite body can be computed by regarding its component parts as single particles located at their respective centers of gravity." It appears with different words in Proposition 8 of his work *On the Equilibrium of Plane Figures*:[169]

> If from a magnitude another magnitude be taken away which does not have the same centre as the whole, when the straight line joining the centres of gravity of the whole magnitude and the magnitude taken away be produced towards the side where the centre of the

[169] [Dij87, p. 306].

whole magnitude is situated, and when from the produced part of the line joining the said centres a segment be cut off such that it has to the segment between the centres the same ratio as the weight of the magnitude taken away has to the remaining magnitude, the extremity of the segment cut off will be the centre of gravity of the remaining magnitude.

Heath expressed this Proposition as follows, see also Figure 9.3:[170]

> If AB be a magnitude whose centre of gravity is C, and AD a part of it whose centre of gravity is F, then the centre of gravity of the remaining part will be a point G on FC produced such that
> $$GC : CF = (AD) : (DE) .$$

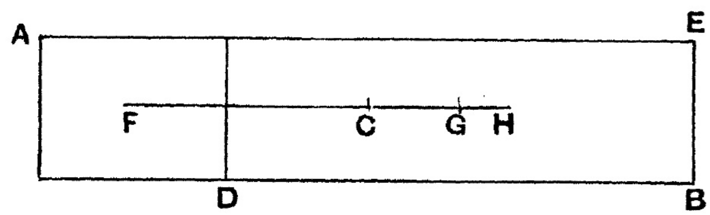


Figure 9.3: Proposition 8 of *On the Equilibrium of Plane Figures*.

Archimedes utilized this Proposition 8 in order to calculate the center of gravity of a trapezium (Proposition 15 of his work *On the Equilibrium of Plane Figures*). To this end he considered a large triangle and divided it into two parts by a straight segment parallel to the basis of the triangle. These two portions are a small triangle and a trapezium. Knowing the center of gravity of the large and small triangles, he then utilized this Proposition 8 in order to find the CG of the trapezium.

[170][Arc02b, p. 194].

Chapter 10

Deductions of the Law of the Lever

10.1 The Essence of Statics

What we have seen so far constitutes the most important aspects of statics. We can summarize the subject as follows:

Definitions:

> We say that a balance with equal arms is in equilibrium when its arms remain at rest horizontally, with the beam free to rotate around the fulcrum. The same definition is applied to a lever in equilibrium.
>
> Two bodies A and B have the same weight P if they keep this balance in equilibrium after being placed on its separate scale pans and released at rest.
>
> The body which balances N other bodies of the same weight P on an equal arm scale has N times the weight P.
>
> The center of gravity of a rigid body is a point such that, if the body is conceived to be suspended by this point, having freedom to rotate in all directions around this point, the body thus supported, when released at rest, will remain stationary and preserve its original position, whatever its initial orientation relative to the Earth. It is represented by the letters CG.

The CG of a set of N particles is defined by Equation (9.11), while for a continuous distribution of matter it is defined by Equation (9.13).

Experimental results:

Two bodies of weights P_A and P_B balance one another on opposite sides of a horizontal lever which has the CG of the beam along the vertical plane passing through the fulcrum, if $P_A/P_B = d_B/d_A$. Here d_A and d_B are the horizontal distances between the points of suspension bodies A and B, respectively, and the vertical plane passing through the fulcrum. This result is the mathematical law of the lever given by Equation (8.1).

If we have N bodies acting upon one side of the lever and M bodies acting on the other side, the equilibrium can be obtained by the principle of superposition, assuming that the weights act independently of one another in such a way that we can add their individual contributions. This last experimental result was expressed mathematically by Equations (8.2) to (8.6).

We can deduce an important result from this latter condition of equilibrium. A lever remains in equilibrium when two equal weights $2P$ are maintained at the same distance d on both sides of the fulcrum, Figure 10.1 (a).

Let us suppose that on one of the sides of the lever we place one weight $P_1 = P$ acting at a distance $d_1 = d - x$, while the other weight $P_2 = P$ is placed at a distance $d_2 = d + x$ from the fulcrum. The lever remains in equilibrium, as shown in Figure 10.1 (b). This fact can be deduced from Equations (8.2) to (8.6). At the right side we have $P_R = 2P_{clip}$ acting at their center of gravity $d_R = d$. The center of gravity of the left side, d_L, is given by:

$$d_L = \frac{P_1}{P_1 + P_2} d_1 + \frac{P_2}{P_1 + P_2} d_2 = \frac{d - x}{2} + \frac{d + x}{2} = d \; . \qquad (10.1)$$

That is, the center of gravity of these two equal weights is their middle point. In other words, these two weights are equivalent to a single weight $2P$ acting at a distance d from the fulcrum, as in Figure 10.1 (b). That is, Equation (8.6) is satisfied in situations (a) and (b) of Figure 10.1.

Equilibrium also remains with a weight $2P$ acting at a distance d at the left side, while at the right side we have a single weight P acting at a distance

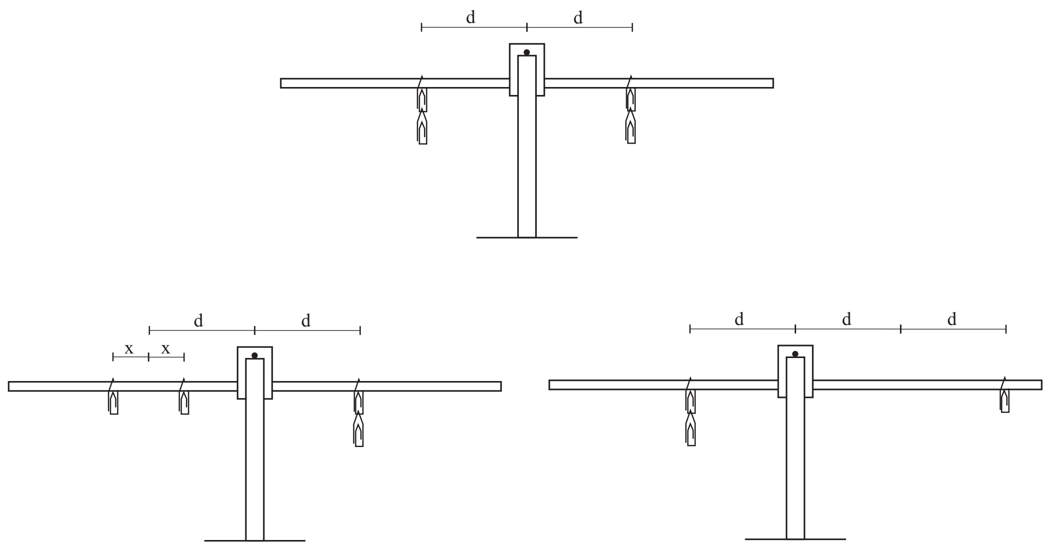


Figure 10.1: (a) A lever in equilibrium with the same weight $2P$ acting at both sides at the same distance d from the fulcrum. (b) It remains in equilibrium when on one side a weight P is placed at $x - d$, while the other is placed at $x + d$. (c) It also remains in equilibrium when we have on one side a weight $2P$ at a distance d, while in the other side we have a weight P at a distance $2d$ from the fulcrum.

$2d$ from the fulcrum, Figure 10.1 (c). That is, Equation (8.6) is satisfied in situations (a), (b) and (c) of Figure 10.1.

With the previous mathematical law of the lever we can explain the experimental result that in equilibrium, the CG of a rigid body is along the vertical line passing through the PS. Since the mathematical expression of the CG, that is, definition $CG9$, was made in order to agree with the law of the lever, this result follows automatically.

The law of the lever can be postulated independently of any other result. This approach will be followed in Section 10.2.

Conversely, if we assume some other result to be known, the law of the lever can be deduced from that other result. This alternative approach will be followed in Sections 10.3 to 10.8.

We can try to deduce the law of the lever experimentally or theoretically. In order to follow this approach, we need to begin with other experimental results, or we need to create other concepts and theoretical postulates. One motivation for following this route is that we want to find a simpler way to arrive at the law of the lever. An opposite motivation might be to begin with something more complex or more abstract than the law of the lever itself, in order to arrive not only at this law but also at other relevant results. For instance, it may be possible to utilize these new concepts and postulates to also arrive at results which are independent of the law of the lever, like the law of the inclined plane. Another reason to follow this new procedure is that we can utilize these new concepts and postulates in order to arrive at other laws and physical results which are valid not only in conditions of equilibrium but also, for instance, when the bodies are in motion in relation to the Earth. This might be the case, for instance, if we were studying the more general laws that govern the rotation and acceleration of rigid bodies moving relative to the ground.

Whenever we follow this alternative procedure, it should be kept in mind that we cannot explain everything. We can postulate the law of the lever (L) without explaining it and deduce from it consequences (C_1), (C_2), etc. This will be the approach followed in Section 10.2. Or, alternatively, we can postulate some other law (P) without explaining it and deduce from it results (L), (C_1), etc. The crux is that in all procedures we always need to begin with some axiom or postulate (which has no explanation) in order to deduce other things from it. The only justification of the basic axioms or postulates may be that they agree with experimental data or that they lead to verifiable experimental data. In Sections 10.3 to 10.8 we will see different approaches to deduce the law of the lever from other theoretical postulates.

There are still other ways to derive the law of the lever. In particular, there is a work from the Aristotelic line of reasoning, *Mechanica*, dealing with the law of the lever.[171] Duhem made a profound study of this work.[172] There is also an old Chinese work dating from about 300 BC dealing with the law of the lever. A discussion of this Chinese treatise has been given by Boltz, Renn and Schemme.[173] These other approaches to deduce the law of the lever will not be considered in this book.

[171] [Ari13].

[172] [Duh05, Chapitre I; Aristote et Archimede] and [Duh91, Chapter 1: Aristotle and Archimedes].

[173] [BRS03].

10.2 Postulating the Law of the Lever

It is possible to utilize the law of the lever to deduce more complex situations. That is, we do not need to explain the law of the lever; we can simply accept it as an empirical fact of nature, that is:

We postulate the law of the lever given by Equation (8.1) for the case of two bodies. When we have two sets of bodies on both sides of a lever, we then postulate Equations (8.2) to (8.6). That is, we postulate that a lever remains in equilibrium after being released at rest when these equations are satisfied.

The justification for this postulate is that it is compatible with experiments data.

We then utilize this law in order to explain the mechanism behind many types of toys and simple machines (such as the equilibrist and toys we saw earlier, or levers of the first, second and third classes). This is the simplest procedure, and there are no problems in assuming this point of view.

The only aspect to take into account with this procedure is that we are not discovering the law of the lever from experimental results.[174] What we are doing is to postulate the law of the lever. We then state that it agrees with experimental data.

10.3 Deducing the Law of the Lever from the Concept of Torque

In Section 7.5 we saw the first condition of equilibrium for a body to remain at rest relative to the Earth, in the presence of gravity. This condition is that the downward weight P acting upon the body must be counterbalanced by another upward force N, of the same magnitude as the weight. This prevents the motion of the body as a whole relative to the ground, when it is released at rest. In the case of the balance or lever, we have a horizontal axis fixed relative to the ground, its fulcrum. Therefore, the weight of the bodies placed upon the beam, together with the weight of the beam itself, must be counterbalanced by an upward normal force N acting at the fulcrum, exerted by the support of the lever. Nevertheless, the balance or the lever can turn around the fulcrum.

[174] [SO02].

We have seen that the concept of weight is not sufficient for the equilibrium of the lever. After all, two bodies of the same weight but placed on opposite sides of the fulcrum, at different distances from it, disturb the equilibrium of the lever. In this case the body acting at a larger distance from the fulcrum will fall toward the ground, with the other weight moving away from it, even though the fulcrum remains at rest relative to the ground. This shows that equal weights acting at different distances from the fulcrum tend to turn the lever.

Another very important example was the equilibrium of a triangle, Experiment 4.5. A triangle was balanced horizontally by placing it on the edge of a ruler in the vertical plane. The edge of the ruler should be parallel to the base of the triangle, passing through its barycenter. The extended vertical plane passing through the ruler divides the triangle into two different areas, that is, into two different weights. Nevertheless, the triangle remains in equilibrium when supported by this ruler, as in Figure 4.9, see also Figure 4.7.

Due to this fact we conclude that we need another concept, beyond the net force acting upon a rigid body, in order to establish the conditions of equilibrium of this body. This rigid body could be, for instance, the beam of a lever. We can utilize the lever in order to define this new concept related to the rotation of a rigid body around a horizontal axis which is fixed relative to the ground. Let us suppose the simplest case in which the fulcrum of the lever (that is, the horizontal axis around which it can turn) is vertically above the CG of the lever. We then suppose two new forces F_A and F_B acting in the same sense, vertically downwards, at horizontal distances d_A and d_B, respectively, from the vertical plane passing through the fulcrum. The experimental law of the lever informs us that if this lever is released at rest horizontally, being free to rotate around the fulcrum, it will remain at rest under the action of these two forces only if $F_A/F_B = d_B/d_A$.

We then define what causes the rotation of a rigid body around a horizontal axis which is fixed in relation to the ground as the "*torque*" or "*moment of a force*." We will represent this torque or moment by the letter T. The experimental law of the lever allows us to define the quantitative ratio T_A/T_B between the magnitudes of the torques exerted by the two forces F_A and F_B already mentioned as:

$$\frac{T_A}{T_B} \equiv \frac{F_A}{F_B}\frac{d_A}{d_B} \ . \tag{10.2}$$

This definition was suggested by an experimental result. But now that we have this definition, we can reverse the argument. The usual procedure is to postulate that the lever will remain in equilibrium if $T_A = T_B$. This postulate and the previous definition of the ratio of the magnitudes of two torques leads to the law of the lever, namely:

$$\frac{F_A}{F_B}\frac{d_A}{d_B} = 1 \ . \tag{10.3}$$

If $T_A/T_B > 1$ and the lever is released at rest horizontally, we postulate that body A will move towards the ground and body B will move away from it. If $T_A/T_B < 1$ and the lever is released at rest horizontally, we postulate that body A will move away from the ground and body B will move towards it.

It may seem that we do not gain anything with this theoretical deduction. After all, we are defining the ratio of torques according to the law of the lever. And in the end we are arriving at the law of the lever itself, by postulating that in equilibrium the torques acting on both sides of the lever have the same magnitude. But as already mentioned, this procedure may have some advantages if we utilize this torque concept not only for the case of a lever in equilibrium, but also as a basis for the study of more complex phenomena like the rotational motion of rigid bodies, etc.

With this torque concept we can also derive the empirical result that in equilibrium the CG of a rigid body must be along the vertical line passing through the point of suspension. To this end we need to postulate that the weight of any body behaves as if it were concentrated at its CG, acting downwards. As the force exerted upon the fulcrum or PS does not exert any torque upon the lever (because it acts at zero distance to the support and, therefore, has an arm of zero length), there remains only the torque exerted by the body. And this torque only goes to zero when the PS and the CG are along a vertical line.

We can also deal with the torque algebraically. In this case we define a tendency to rotate in one direction (for instance, the rotation of the lever in the vertical plane lowering body A and raising body B at the other side of the lever) as due to a positive torque. We also define a tendency to rotation in the opposite direction as due to a negative torque. In the case of the Figure 10.2, for instance, the weight of A would exert a positive torque upon the lever, while the weight of B would exert a negative torque.

In this case the fundamental postulate might be expressed as follows:

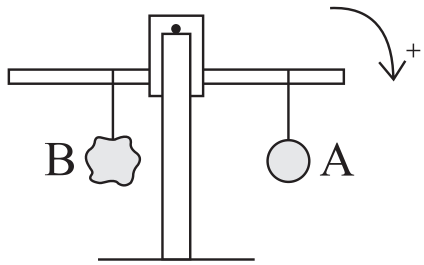


Figure 10.2: Algebraic torque.

> The algebraic sum of all torques acting upon a rigid body must be null in order for the body to remain in equilibrium after being released at rest, without rotation around a fixed axis.

If we have N bodies on one side of the lever and M bodies on the other side, the basic postulate can be generalized by the principle of superposition. That is, we postulate that the lever will remain in equilibrium if Equation (8.6) is valid with P_L, P_R, d_L and d_R defined by Equations (8.2) to (8.5), respectively.

If Equation (8.6) is not valid, we postulate that the side with the greater value of the magnitude $\sum_{i=1}^{N}(P_i/P_{total})d_i$ will move toward the Earth if the lever is released at rest, with the other side moving away from it.

Although this theoretical deduction of the law of the lever, beginning with the previous definitions and postulates, is correct, it should be emphasized that the concept of torque of a force was suggested historically from the empirical knowledge of the law of the lever. That is, it was the experimental fact that two bodies balance one another upon a lever when the ratio of their distances to the fulcrum is inversely proportional to the ratio of their weights which suggested the creation of the torque concept.

Suppose, for example, as a counterexample, that nature behaved in such a way that the experimental law of the lever were given by the following relation:[175]

$$\frac{P_A}{P_B} = \left(\frac{d_B}{d_A}\right)^{\alpha} , \tag{10.4}$$

with $\alpha = 2$ or another value different from 1. In this case it would be natural to define another magnitude proportional to $P_i(d_i)^{\alpha}$, instead of the usual

[175] [AR08] and [AR09].

torque proportional to $P_i d_i$. We could then postulate that the net algebraic value of this new magnitude must go to zero in order to have equilibrium. In this case we could derive the new law of the lever theoretically.

What we want to emphasize is that the traditional definitions of torque and center of gravity (as proportional to the distance between the fulcrum and the point of application of the force), together with the postulate that the algebraic sum of all torques acting upon a body in equilibrium must be zero, are only justifiable because they lead to the correct law observed in nature. These definitions and postulates were suggested by the experimental law. When we discover the limits of validity of any specific law, the relevant concepts and postulates must be modified or generalized in order to adapt to the new experimental knowledge.

10.4 Law of the Lever Deduced from the Experimental Result that a Weight $2P$ Acting at a Distance d from the Fulcrum is Equivalent to a Weight P Acting at a Distance $d - x$, Together with Another Weight P Acting at a Distance $d + x$ from the Fulcrum

A very simple way to arrive at the law of the lever utilizes two basic ingredients:

(I) Equal weights on opposite sides of the lever balance one another when they act at equal distances from the fulcrum.

(II) A weight $2P$ acting at a horizontal distance d to the vertical plane passing through the fulcrum is equivalent to a weight P acting at a distance $d - x$ from the fulcrum, together with another weight P acting at a distance $d + x$ from the fulcrum, as in Figure 10.3.

Here we are utilizing a coat-hanger as a lever. In this case the fulcrum or PS is the horizontal axis passing through the hook of the hanger. We

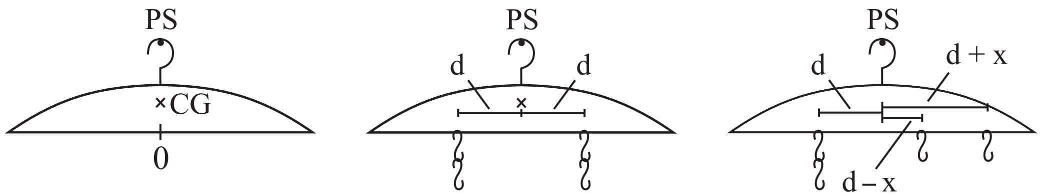


Figure 10.3: Experimental condition of equilibrium for a lever.

assume that in equilibrium this axis is vertically above the CG of the hanger and above the center 0 of the horizontal section of the hanger.

The "equivalence" mentioned in ingredient (II) refers to the tendency of the lever to rotate around the fulcrum. Ingredient (I) may be considered a definition of equality of weights, while ingredient (II) may be considered an experimental result, or a theoretical postulate. For the moment, we will utilize it as an experimental result. We will treat it as a primitive experimental fact, without trying to explain it.

Ingredient (I), a definition of equality of weights, is represented in the middle of Figure 10.3.

The experimental condition (II) is represented by Figure 10.3 (c). That is, if the situation of Figure 10.3 (b) is a configuration of equilibrium, then experience teaches us that the situation of Figure 10.3 is also a configuration of equilibrium.

Assuming condition (II), it is easy to arrive at the law of the lever without imposing any limit upon the possible values of x. To see how to arrive at the law of the lever with this procedure, we begin with two equal weights P acting at the same distance d on one side of the fulcrum, balanced by two other equal weights P acting at the same distance d on the other side, Figure 10.4 (a). By moving one of the weights on the right hand side to the position $d - x$ and the other weight on the right hand side to the position $d + x$, with $x = 2d$, we end up with the situation shown in Figure 10.4 (b). That is, a lever in equilibrium with a weight $3P$ at the distance d from the fulcrum, together with a weight P at the distance $3d$ in the other side of the fulcrum. This is a particular case of the law of the lever, since we have $P_A/P_B = d_B/d_A = 3$.

If we had made $x = d$ we would arrive at the equilibrium situation shown in Figure 10.5 (a). As one of the weights is along the vertical plane passing through the fulcrum and CG of the lever, it can be removed without affecting

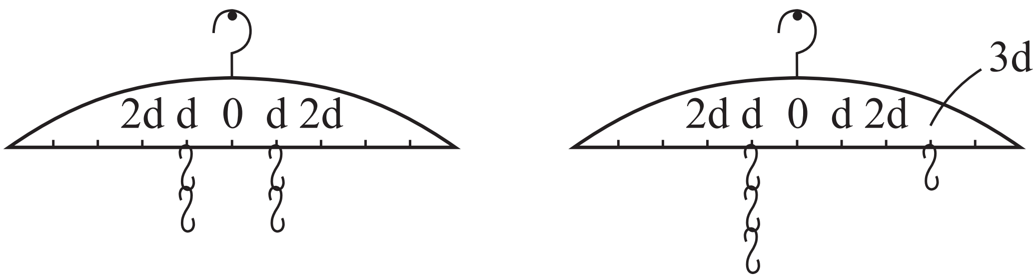


Figure 10.4: A particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 3$.

the equilibrium. In this case we end up in the situation of equilibrium shown in Figure 10.5 (b). That is, a lever in equilibrium with a weight $2P$ at the distance d from the fulcrum, together with another weight P at the distance $2d$ on the other side of the fulcrum. And this is another particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 2$.

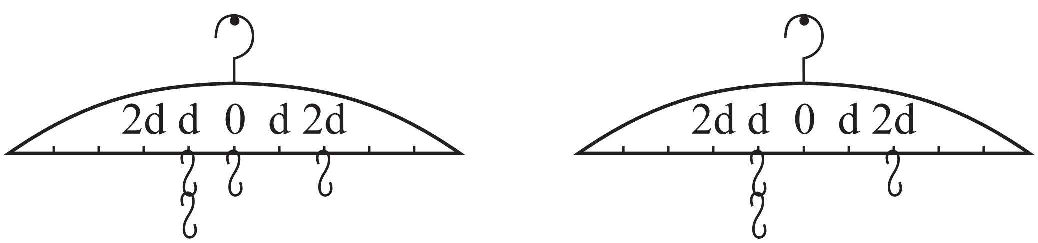


Figure 10.5: A particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 2$.

We now begin with three bodies of equal weight P acting at the same distance d on one side of the fulcrum, balanced by three other bodies of equal weight P acting at the same distance d on the other side of the fulcrum, Figure 10.6 (a). We do not touch the bodies on the left side and consider only the bodies on the right side. We can preserve equilibrium by moving one of these bodies to the right, away from the fulcrum by a distance $x = 2d$, provided that we move simultaneously one of these bodies to the left by the same distance $x = 2d$, while the third body remains fixed in its present position. We end up in the intermediate case shown in Figure 10.6 (b), that is, a weight $4P$ at the distance d on one side of the fulcrum, a weight P at the distance d on the other side of the fulcrum, and a weight P at a distance $3d$ on the same side of the fulcrum. We can preserve equilibrium by joining the latter two bodies at their midpoint, as in the situation of Figure 10.6 (c). We then end up with another special case of the law of the lever for which $P_A/P_B = d_B/d_A = 2$. This is the same value obtained before, although this time we did not need to remove a body from the lever.

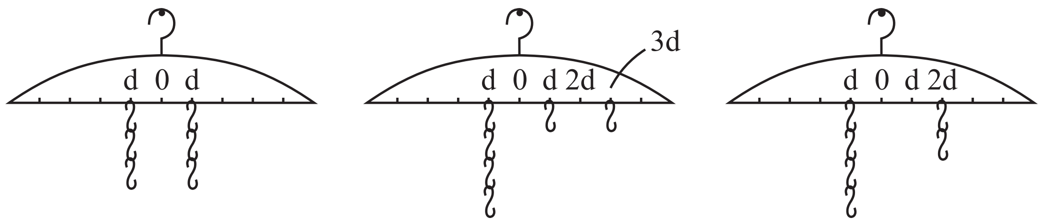


Figure 10.6: A particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 2$.

We now begin once more with three bodies of equal weight P on either side of the lever, acting at a distance d from the fulcrum, Figure 10.7 (a). By moving one of the bodies on the right hand side to the distance $d - x = 0$ from the fulcrum and another one to the distance $d + x = 2d$ from the fulcrum ($x = d$), we end up in the equilibrium situation shown in Figure 10.7 (b). As the body which is along the vertical plane passing through the fulcrum and CG of the lever does not disturb the equilibrium, we can remove it from the system. By joining the two weights on the right hand side at their midpoint, we end up with the third case of equilibrium shown in Figure 10.7 (c), that is, a weight $3P$ acting at a distance d from the fulcrum and another weight $2P$ acting at a distance $1.5d$ on the other side of the fulcrum. This is another special case of the law of the lever for which $P_A/P_B = d_B/d_A = 3/2 = 1.5$.

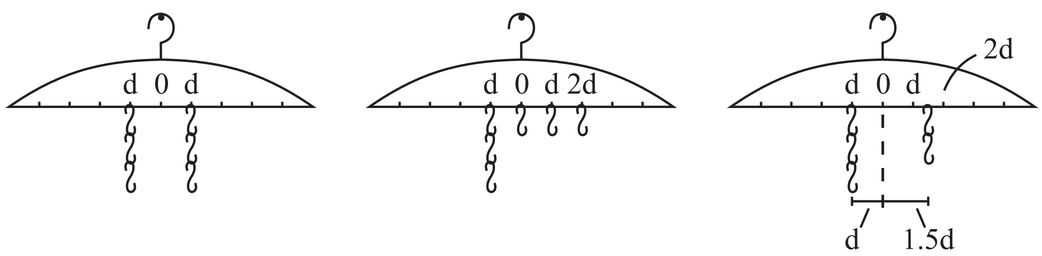


Figure 10.7: A particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 1.5$.

If we had begun with 5 equal paper clips in either side of the lever, acting at the same distance from the fulcrum, we could have arrived at the same relation without removing a body from the lever.

It is easy to extend this analysis to other cases. This shows how to deduce the law of the lever starting from the experimental result that a weight $2P$ acting at a horizontal distance d to the vertical plane passing through the fulcrum is equivalent to a weight P acting at a distance $d - x$ from the fulcrum, together with another weight P acting at a distance $d + x$ from the fulcrum.

10.5 Law of the Lever as Deduced by Duhem Utilizing a Modification of a Work Attributed to Euclid

The previous procedure seems to be at the origin of one of the oldest theoretical proofs of the law of the lever known to us. This information is taken from Duhem and Clagett.[176]

The main idea of Duhem to be discussed here is to consider the experimental condition (II) introduced in Section 10.4 as a theoretical postulate.

Here we present the main elements of a work of mechanics attributed to Euclid, the famous author of the geometry book *The Elements*, who lived in Alexandria around 300 BC. Although no works on mechanics were attributed to Euclid in Antiquity, many Arabic authors mention works by Euclid on this subject. Three fragments which survived are attributed to him. The titles given to these works are: *Book on the Balance*; *Book on the Heavy and Light*; and *Book on Weights According to the Circumference Described by the Extremities*. What interests us here is the first of these books, which was translated into French in 1851 from its Arabic version (there is no known version of this book in Greek or in Latin). An English translation of this work has been made by Woepcke.[177]

The book begins with a definition and two axioms.[178] Text between square brackets is Clagett's:

1. [DEFINITION] Weight is the measure of heaviness and lightness of one thing compared to another by means of a balance.

[176] [Duh05, Chapter V], [Duh91, Chapter V] and [Cla79].

[177] [Euc79].

[178] [Euc79, p. 24].

> 2. [Axiom I] When there is a straight beam of uniform thickness, and there are suspended on its extremities two equal weights, and the beam is suspended on an axis at the middle point between the two weights, then the beam will be parallel to the plane of the horizon.
>
> 3. [Axiom II] When two weights—either equal or unequal—are placed on the extremities of a beam, and the beam is suspended by an axis on some position of it such that the two weights keep the beam on the plane of the horizon, then if one of the two weights is left in its position on the extremity of the beam and from the other extremity of the beam a straight line is drawn at a right angle to the beam in any direction at all, and the other weight is suspended on any point at all of this line, then the beam will be parallel to the plane of the horizon as before.
>
> This is the reason that the weight is not changed when the cord of one of the two sides of the balance is shortened and that of the other is lengthened.
>
> [Propositions] [...]

The author of this work demonstrates four propositions. The last one contains the law of the lever. In Section 10.6 we will discuss this procedure.

For the time being we will follow a modification of this argument which was proposed by Pierre Duhem when he analyzed this work.[179] Duhem postulated two extra axioms, namely (text between square brackets is ours):

> Axiom III. If the weights are maintaining the beam of a balance parallel to the horizon and if one suspends an additional weight to the beam's point of suspension, the beam remains parallel to the horizon.
>
> Axiom IV. If any number of weights maintain the beam of a balance parallel to the horizon, and if Z and D are two of these weights [equal to one another] suspended from the same arm of the beam and if one moves weight Z by a given length away from the point of suspension of the balance and if one moves weight D by the same length towards the point of suspension, then the beam will remain parallel to the horizon.

[179] [Duh05, pp. 65-66] and [Duh91, pp. 47-51].

These axioms lead to an elegant demonstration of the law of the lever. It can be summarized as follows. Let BD be the beam of a lever with C being its fulcrum or point of support, and $BC = CD$, Figure 10.8.

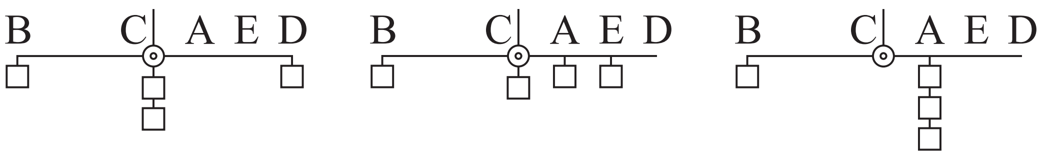


Figure 10.8: Duhem's proof of the law of the lever.

Suppose four equal bodies of weight P, one suspended at B, another at D, and two at C, as in Figure 10.8 (a). By Axioms I, II and III, the lever will remain in equilibrium, with its beam at rest horizontally. We divide CD into three equal parts by the points A and E, such that $CA = AE = ED = CD/3$. We now move one of the bodies which was at C to the point A, while simultaneously moving the body which was at D to the point E, as in Figure 10.8 (b). By Axiom IV the lever will remain in equilibrium horizontally. By Axiom IV it will remain in equilibrium if we move the body that remained in C to the point A, provided we simultaneously move the body that was at E to the point A, as in Figure 10.8 (c). We then see that the lever in its final configuration of equilibrium will have a weight P at a distance d from the fulcrum and another weight $3P$ at a distance $d/3$ on the other side of the fulcrum. In other words, we arrive at a particular case of the law of the lever. It is easy to generalize this demonstration.

This demonstration of the law of the lever depends not only on the equilibrium condition given by equal weights acting at equal distances from the fulcrum, but also on axiom IV. This is not an obvious axiom. It is only justified by its agreement with experimental results.

We can give a counter example. Consider a body A of weight P_A at a distance d_A from the fulcrum and another body B of weight P_B at a distance d_B on the other side of the fulcrum. Suppose that nature behaved in such a way that the experimental law of the lever was, for example, that there would be equilibrium when:[180]

$$\left(\frac{d_B}{d_A}\right)^\alpha = \frac{P_A}{P_B} \ , \tag{10.5}$$

[180] [AR08] and [AR09].

with $\alpha \neq 1$.

In this case Axiom IV would no longer hold.

As we will see in the next Section, the original procedure attributed to Euclid contains only the first two Axioms. Euclid derived an analogous to Duhem's Axiom IV as Proposition 2 of his work, based on the first two Axioms.

10.6 Proof of the Law of the Lever Utilizing an Experimental Procedure Suggested by a Work Attributed to Euclid

We present here some experiments which illustrate how to deduce the law of the lever in a very interesting way. These experiments were suggested by the *Book on the Balance*, attributed to Euclid.

Up to now we have been dealing with levers composed of horizontal beams which can turn in a vertical plane around a horizontal axis which is orthogonal to the beam. The procedure we will assume here is a different one. We now use a homogeneous rigid rectangle (or square) which remains in equilibrium in a horizontal plane, supported on a vertical stick placed under the center of the rectangle. We place three equal bodies upon this horizontal plane, studying the conditions in which the plane remains in equilibrium. We should attach a sheet of graph paper to the rectangle. This simplifies the analysis as we have now a Cartesian plane above it. We place two orthogonal axes, x and y, parallel to the sides of the rectangle and to the lines of the sheet of paper. We choose the origin of the coordinate system, $(0, 0)$, to be located at the center of the rectangle.

Materials: The rectangle can be made of pasteboard and the lines drawn upon it. An alternative procedure is to attach a sheet of graph paper to the pasteboard. The three bodies to be placed upon the rectangle should have the same size and weight. For instance, they could be three equal screw-nuts. During the experiments these nuts will slide over the plane and fall to the ground many times. To prevent this hindrance we should place some glue under the nuts, or attach a thin layer of modeling clay below them, so that they can be attached to any point of the rectangle. Another very interesting alternative is to utilize a metal rectangle (like the picture frame). In this case the three bodies can be small magnets like the ones used to attach the pictures to these frames. The size of the rectangle could be, for instance, 10 cm $\times$ 15 cm. The separation between the lines of the graph paper can be 0.5 cm or 1 cm. The vertical stick to be placed below the center of the rectangle can be a bamboo barbecue skewer with its tip stuck in a piece of clay. Any other appropriate support can be used.

It is important to note that the upper plane surface of the support (that is, the cross sectional area of the bamboo stick, bottle cap, etc.) should not be too small or too large. If this area is too small, the equilibrium becomes unstable and it may be difficult to balance the rectangle on it. If this area is too large, it will be very easy to balance the rectangle on it, but it will be difficult to establish the precise conditions which yield the equilibrium of the three equal bodies. As a convenient measure we can use a support such that, when the rectangle remains in equilibrium with the three pieces in adequate positions above it, this equilibrium will be disturbed when a single body moves one or two units of length along the x or y axes. That is, in such a way that the rectangle falls to the ground with this change of configuration, so that the lack of equilibrium can be easily perceived.

Let us then suppose that we have our graph paper rectangle. The first thing to do is to balance it horizontally, supporting it upon the stick placed under the origin $(0,\ 0)$ of the rectangle. Next, we balance the rectangle with the three pieces of equal weight placed above it. Let us call them P_1, P_2 and P_3. Initially we put them at $(x,\ y) = (-5,\ 0)$, $(0,\ 0)$ and $(5,\ 0)$, respectively, as in Figure 10.9.

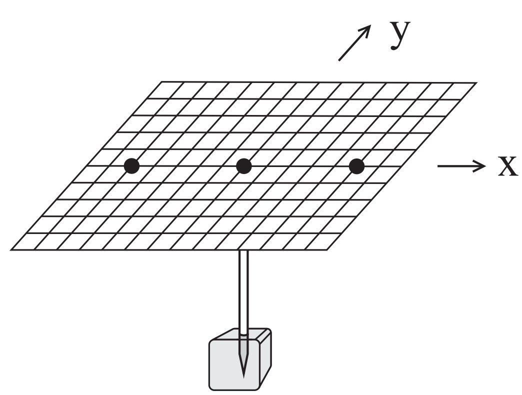


Figure 10.9: Another procedure to obtain the law of the lever.

Because this is a symmetrical configuration around the origin, equilibrium must be established. If this does not happen, we must first find the reason before we proceed. It may be due to the fact that the three pieces do not have exactly the same weight; or the stick may not be placed exactly below the center of the rectangle; or the skewer is not vertical; or its upper surface is not horizontal, etc.

When the equilibrium has been established, we are then ready to begin the main experiments.

Experiment 10.1

We move piece P_2 to the location $(x,\ y) = (0,\ 2)$. It should be observed that the system falls to the ground, with this piece moving toward the Earth. On the other hand, when we move P_2 to $(x,\ y) = (0,\ 2)$ and P_1 to $(x,\ y) = (-5,\ -2)$, leaving P_3 at $(x,\ y) = (5,\ 0)$, the system remains in equilibrium horizontally after being released at rest, as in Figure 10.10.

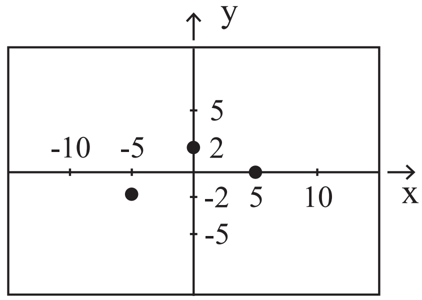


Figure 10.10: Experimental condition of equilibrium when this rectangle is supported by a vertical stick below the origin.

The result of this experiment can be generalized to other cases. Suppose that we have a set of bodies in equilibrium above a horizontal plane supported by a vertical stick placed below one of its points. We consider the position of this stick as the origin of an orthogonal system of coordinates $(x,\ y)$. If we move one of the bodies from position $(x_1,\ y_1)$ to position $(x_1 + d,\ y_1)$ and, simultaneously, move another body of equal weight from position $(x_2,\ y_2)$ to position $(x_2 - d,\ y_2)$, the system will remain in equilibrium. The same equilibrium will be maintained for equal and opposite displacements made simultaneously by two pieces of the same weight along the y axis, or for equal and opposite displacements perpendicular to any other line passing through the origin of the coordinate system and inclined at an arbitrary angle θ with respect to the x axis.

Experiment 10.2

Let us now invert the order of the movements. We again begin with the three bodies P_1, P_2 and P_3 located at $(x,\ y) = (-5,\ 0)$, $(0,\ 0)$ and $(5,\ 0)$, respectively. We now move only piece P_1 to $(x,\ y) = (-5,\ -2)$, holding the rectangle in place with our hands. We now look carefully at the rectangle. When we release the system slowly at rest, what is observed is that the whole side with $y < 0$ tends to fall to the ground, while the side $y > 0$ moves away from the Earth. On the other hand, we perceive no difference between the sides with $x > 0$ and $x < 0$. In other words, none of these sides tends to fall, as indicated by Figure 10.11. And this is somewhat surprising because body P_1 is not located symmetrically in relation to the origin of the axis x, as it is offset from the vertical extended upwards through the stick.

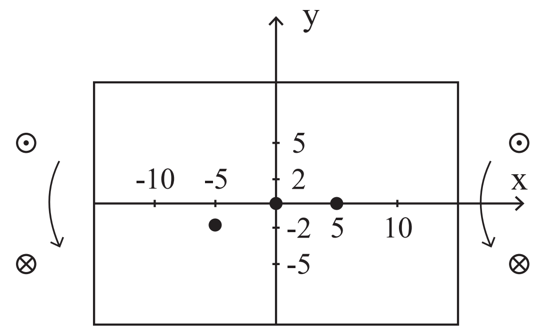


Figure 10.11: Direction of rotation of the plane.

We can express this finding as follows. Suppose we have a rigid system in equilibrium in a horizontal plane, capable of turning in any direction around a vertical stick, with several bodies on the horizontal plane. If only one of these bodies is moved in a certain direction on the horizontal plane, the system looses equilibrium only in this direction, tending to move closer to the ground, without disturbing the equilibrium in any direction orthogonal to this displacement. For instance, in the previous example the body P_1 moved along the negative y direction. The side of the rectangle characterized by $y < 0$ tended to fall to the ground, while the side $y > 0$ tended to move upwards. This experiment gives empirical support to the second Axiom of Euclid presented before.

Experiment 10.3

We draw two circles of the same radius on the rectangle in such a way that they touch one another at a single point. If the circles have a radius of 5 units, for instance, the centers of the circles can be located at $(x,\ y) = (-5,\ 0)$ and $(5,\ 0)$. In this case the point of contact is the origin $(0,\ 0)$. Let ACB be the straight line passing through the points $A = (8,\ -4)$, $C = (0,\ 0)$ and $B = (-8,\ 4)$. Let H and T be the edges of the circles along the x axis, that is, with $H = (-10,\ 0)$ and $T = (10,\ 0)$. We draw three parallel straight lines HB, CE, and AT, with $E = (2,\ 4)$. The projection of E on the x-axis and the projection of A on the x-axis are called Z and W, respectively, such that $Z = (2,\ 0)$ and $W = (8,\ 0)$, as in Figure 10.12. Experiment shows that this rectangle remains in equilibrium horizontally when supported on a stick placed under its origin. This can be understood by considerations of symmetry.

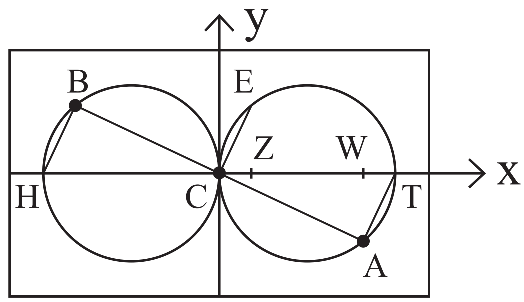


Figure 10.12: Euclid's procedure to derive the law of the lever.

Experiment 10.4

We place three bodies P_1, P_2, and P_3 of equal weight at positions B, C, and A, respectively. Experiment shows that this rectangle remains in equilibrium after being released at rest when supported under its origin $(0,\ 0)$, as in Figure 10.12. This can also be understood by considerations of symmetry.

We now move P_1 from B to H and, simultaneously, P_2 from C to E, while keeping P_3 at A. As these displacements were orthogonal to the straight line BCA, were of the same length, in opposite directions and, moreover, as P_1 and P_2 have the same weight, the system remains in equilibrium, based on what we discussed earlier, as in Figure 10.13.

Experiment 10.5

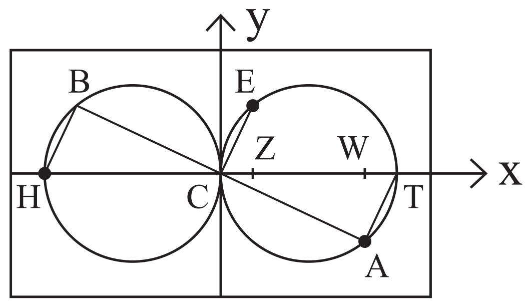


Figure 10.13: Second step to derive the law of the lever.

We now consider the straight line HCT. We begin with the previous equilibrium configuration with the three equal bodies at H, E, and A. We now move P_2 from E to Z, while simultaneously moving P_3 from A to W, keeping P_1 at H. Once more the system remains in equilibrium. After all we have displaced two equal weights by the same amount in opposite directions along the straight line HCT. We end up in the equilibrium configuration shown in Figure 10.14, in which the three equal weights are located at $H = (-10,\ 0)$, $Z = (2,\ 0)$ and $W = (8,\ 0)$.

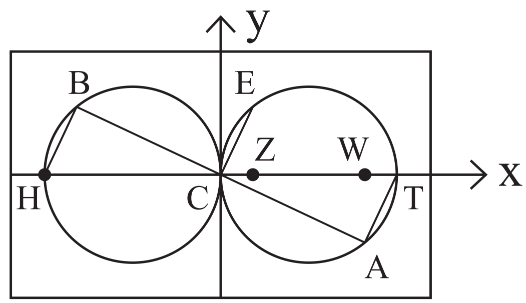


Figure 10.14: Third step to derive the law of the lever.

By changing the inclination of the straight line BCA to the x-axis and repeating the previous procedure, we will end up with the three equal bodies at the following positions: $P_1 = (-10,\ 0)$, $P_2 = (a,\ 0)$, $P_3 = (10 - a,\ 0)$, where magnitude a can have any value between 0 and 10. We conclude that a weight at a certain distance d from the origin is balanced by two other equal weights placed on the other side of the fulcrum at the following distances from the origin: a and $d - a$.

In particular, beginning with the straight line BCA at an inclination of

45° with the x-axis, we will end up with a weight P at the position $(-10,\ 0)$ and two other equal weights P at the position $(5,\ 0)$. Here we have $a = d/2$. This is a particular case of the law of the lever for which $P_A/P_B = d_B/d_A = 2$.

As we saw in Section 10.3, from this last result it is possible to derive the law of the lever experimentally.

The interesting aspect of this experimental procedure utilizing a rectangle in horizontal equilibrium is that we did not need to begin with this last result. Rather than start from it, we derived this result beginning from the fact that when we displace a body over a system which was originally in equilibrium supported on a vertical stick, the plane looses equilibrium only in this direction. That is, this displacement does not affect the equilibrium of the plane in directions orthogonal to the displacement.

10.7 Theoretical Proof of the Law of the Lever Attributed to Euclid

In the theoretical work attributed to Euclid, available in English,[181] the law of the lever is derived from the theoretical postulate of the previous experimental result. This is the essence of the second postulate presented earlier, namely:

> 2. [Axiom I] When there is a straight beam of uniform thickness, and there are suspended on its extremities two equal weights, and the beam is suspended on an axis at the middle point between the two weights, then the beam will be parallel to the plane of the horizon.
>
> 3. [Axiom II] When two weights—either equal or unequal—are placed on the extremities of a beam, and the beam is suspended by an axis on some position of it such that the two weights keep the beam on the plane of the horizon, then if one of the two weights is left in its position on the extremity of the beam and from the other extremity of the beam a straight line is drawn at a right angle to the beam in any direction at all, and the other weight is suspended on any point at all of this line, then the beam will be parallel to the plane of the horizon as before.

181 [Euc79].

> This is the reason that the weight is not changed when the cord of one of the two sides of the balance is shortened and that of the other is lengthened.
>
> [Propositions] (...)

The main aspect of this second Axiom is the postulate that the equilibrium of a horizontal beam is not disturbed when a body moves orthogonally to this beam "in any direction at all." Suppose the beam is along the x-axis in horizontal equilibrium, supported by a vertical stick placed under one of its points. In this case a body suspended on the beam can move a distance d in the vertical z direction, or along the horizontal y axis, or in any direction in the yz-plane, without disturbing the equilibrium of the beam along the x-axis. That is, displacement of the body along the yz-plane will not cause the side $x > 0$ of the beam to move upward or downward, the same being true for the side $x < 0$.

These two postulates are presented as follows in the English translation of Duhem's book:[182]

> Axiom I. When two equal weights are suspended from the two extremities of a straight beam of uniform thickness which, in turn, is suspended at the midpoint between the two weights, the beam remains parallel to the plane of the horizon.
>
> Axiom II. When two equal or unequal weights are attached to the two extremities of straight beam which at one of its points is suspended from a fulcrum so that the two weights maintain the beam parallel to the horizon, and if then, we leave one weight in its place at one extremity and draw a straight line from the other extremity of the beam which forms a right angle to the beam on either side of the beam and if one suspends the other weight from any point at all on this line, the beam will remain parallel to the plane of the horizon.
>
> This is the reason why the weight does not change if one shortens the strings of one of the two scale pans or lengthens the strings of the other.

The crucial words "in any direction at all" are replaced in this translation by "on either side of the beam."

[182] [Duh05, p. 65] and [Duh91, p. 50].

As these are axioms, we cannot derive them from other axioms. We simply postulate them. But we derive consequences from them.

This second theoretical axiom can be visualized by the experiments presented earlier, Section 10.6. From this axiom we can derive theoretically that a weight P at the position $x = -d$ on one side of the fulcrum, with the fulcrum located at $x = 0$, is balanced by two other equal weights P placed at positions $x = a$ and $x = d - a$. And from this result we can derive the law of the lever, as is done in the work attributed to Euclid.

10.8 Archimedes' Proof of the Law of the Lever and His Calculation of the Center of Gravity of a Triangle

10.8.1 Archimedes' Proof of the Law of the Lever

Archimedes presented a theoretical deduction of the law of the lever in his works *On the Equilibrium of Plane Figures*, or *The Centers of Gravity of Plane Figures*. The two parts of this work are available in English.[183]

What has reached us seems to have been only a part of a larger work. His proof of the law of the lever is based upon the concept of the center of gravity, which does not appear explicitly defined in any of his works now extant. But from what we have seen on Chapter 6 in the quotations of Heron, Pappus and Simplicius, who had access to works of Archimedes that are no longer extant, he seemed to have defined the CG along the lines of definition $CG8$ in Subsection 4.12.1, namely:

> The center of gravity of any rigid body is a point such that, if the body be conceived to be suspended from that point, being released at rest and free to rotate in all directions around this point, the body so suspended will remain at rest and preserve its original position, no matter what the initial orientation of the body relative to the ground.

In Proposition 6 of his work *Quadrature of the Parabola*, he mentioned the following:[184]

[183] [Arc79], [Dij87, pp. 286-313 and 346-360], [Arc01] and [Arc02b, pp. 189-220].
[184] [Duh91, p. 463], [Duh06, p. 307] and [Mug71a, p. 171].

> Every suspended body — no matter what its point of suspension — assumes an equilibrium state when the point of suspension and the center of gravity are on the same vertical line. This has been demonstrated.

This result is extremely important from the practical and theoretical points of view.

This proposition indicates a practical procedure to find the center of gravity of any rigid body, as was seen in definitions CG6 and CG7, Subsections 4.7.1 and 4.8.1. However, it must be noted that to Archimedes the intersection of the verticals was not a definition of the CG. To him this proposition was not a definition, but a theoretical result which he deduced elsewhere. Unfortunately his theoretical demonstration has not reached us. The proof of this proposition was probably included in his lost work *On Balances* or *On Levers*.

In order to prove the law of the lever in his work *On the Equilibrium of Plane Figures*, Archimedes begins with seven postulates, namely:[185]

> I postulate the following:
>
> 1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance.
>
> 2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards the weight to which the addition was made.
>
> 3. Similarly, if anything be taken away from one of the weights, they are not in equilibrium but incline towards the weight from which nothing was taken.
>
> 4. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide.
>
> 5. In figures which are unequal but similar the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

[185] [Arc02b, pp. 189-190].

6. If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances.

7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure.

The fundamental postulate which allows Archimedes not only to derive the law of the lever, but also to theoretically locate the CG of many two-dimensional (triangles, parallelograms, trapeziums, circles, semi-circles, parabolic segments) and three-dimensional figures (cones, hemispheres, semi-ellipsoids, paraboloids of revolution, hyperboloids of revolution), is his sixth postulate.

In Dijksterhuis' book, these seven postulates are translated as follows:[186]

> *Postulates.*
>
> I. *We postulate that equal weights at equal distances are in equilibrium, and that equal weights at unequal distances are not in equilibrium, but incline towards the weight which is at the greater distance.*
>
> II. *that if, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium, but incline towards that weight which something has been added.*
>
> III. *similarly that, if anything be taken away from one of the weights, they are not in equilibrium, but incline towards that weight from which nothing has been taken away.*
>
> IV. *when equal and similar figures are made to coincide, their centres of gravity likewise coincide.*
>
> V. *in figures which are unequal, but similar, the centres of gravity will be similarly situated.*[187]

[186] [Dij87, pp. 286-287].

[187] [Note by Dijksterhuis:]

> To this it is added: *we say that points are similarly situated in relation to similar figures if straight lines drawn from these points to the equal angles make equal angles with the homologous sides.*

VI. *if magnitudes at certain distances be in equilibrium, other [magnitudes] equal to them will also be in equilibrium at the same distances.*

VII. *in any figure whose perimeter is concave in the same direction the centre of gravity must be within the figure.*

The meaning of the crucial sixth postulate has been clarified by Vailati, Toeplitz, Stein and Dijksterhuis.[188] By "magnitudes equal to other magnitudes," Archimedes wished to say "magnitudes of the same weight." And by "magnitudes at the same distances," Archimedes understood "magnitudes the centers of gravity of which lie at the same distances from the fulcrum."

Suppose a system of bodies keeps a balance in equilibrium. According to this postulate, Archimedes can replace a certain body A suspended by the beam through its center of gravity located at a horizontal distance d from the vertical plane passing through the fulcrum, with another body B which has the same weight as A, without disturbing equilibrium, provided it is also suspended by the beam at its CG which is at the same horizontal distance d from the vertical plane passing through the fulcrum.

Instead of a body A, we can also think of a set of N bodies A_i, with $i = 1, ..., N$. Likewise, instead of a body B, we can think of a set of M bodies B_j, with $j = 1, ..., M$. Equilibrium will not be disturbed when we replace the N bodies A_i with the M bodies B_j, if the total weight of the two sets is the same and if the CG of the set of M bodies B_j acts at the same distance from the fulcrum as the CG of the N bodies A_i.

A particular example of this postulate is the replacement of a body of weight P located at the distance d from the vertical plane passing through the fulcrum of a lever in equilibrium by a set of two other bodies, namely: a weight $P/2$ located at the distance $d + x$ from the vertical plane passing through the fulcrum, and another weight $P/2$ located at the distance $d - x$ from the vertical plane passing through the fulcrum. In this case the two systems have the same total weight P. Moreover, the centers of gravity of the two systems are located at the same distance d from the vertical plane passing through the fulcrum. In the case of the second system composed of two weights $P/2$, this was proved by Archimedes in the fourth Proposition of this work:[189]

[188] [Ste30] and [Dij87, pp. 289-304 and 321-322].

[189] [Arc02b, p. 191].

> If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity.

The translation of this fourth Proposition in Dijksterhuis' work appears as follows:[190]

> If two equal magnitudes have not the same centre of gravity, the centre of gravity of the magnitude composed of the two magnitudes will be the middle point of the straight line joining the centres of gravity of the magnitudes.

From this particular example we can arrive at the law of the lever, as we saw earlier in the procedure attributed to Euclid. Archimedes presented a general proof of the law of the lever which is valid for commensurable magnitudes as well as incommensurable magnitudes.

The advantage of the postulate due to Archimedes as compared with the analogous postulate from Euclid is the generality implied by Archimedes' approach. Using this postulate, he derived both the law of the lever and the correct calculation of the CG of the one-, two-, and three-dimensional figures presented in Section 6.2.

We now present the main elements of his proof of the law of the lever. He considered three situations, namely:

> (A) A set of $2N_1$ magnitudes of the same weight P suspended by their centers of gravity along a rectilinear lever, with these magnitudes evenly spaced. We present a concrete example with $N_1 = 3$ and with the spacing between adjacent magnitudes given by the length w. The CG of this system of magnitudes is the point E, which is at the midpoint of these magnitudes,[191] as in the first situation in Figure 10.15 (a). The lever is free to turn around the fulcrum located at E.
>
> (B) The second situation considered by Archimedes is a system of $2N_2$ magnitudes of the same weight P suspended by their centers of gravity along a rectilinear lever, with these magnitudes evenly spaced. We present a concrete example with $N_2 = 2$ and with the spacing between

[190] [Dij87, p. 288]

[191] By Proposition 4 and Corollary 2 to be presented in Chapter 13.

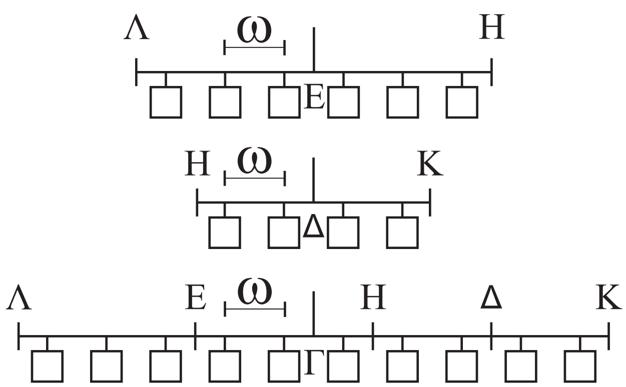


Figure 10.15: Archimedes' procedure for deducing the law of the lever.

adjacent magnitudes given by w. The CG of this system of magnitudes is the point $\triangle$, which is at the midpoint of these magnitudes, as in Figure 10.15 (b). The lever is free to turn around the fulcrum located at $\triangle$.

(C) The third situation considered by Archimedes is a system of $2N_1 + 2N_2$ magnitudes of the same weight P suspended by their centers of gravity along a rectilinear lever, with these magnitudes evenly spaced. We present a concrete example with $N_1 = 3$, $N_2 = 2$ and with the spacing between adjacent magnitudes given by w. The CG of this system of magnitudes is the point Γ, which is at the midpoint between these magnitudes, as in Figure 10.15 (c). The lever is free to turn around the fulcrum located at Γ.

By symmetry, situations (A), (B) and (C) of Figure 10.15 are configurations of equilibrium.

The CG of these three situations are located at the points E, $\triangle$ and Γ, respectively. This was proven by Archimedes in Corollary II of Proposition 5 of his work:[192]

> *If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity*

[192] [Arc02b, p. 191].

of the system is the middle point of the line joining the centres of gravity of the two middle ones.

This fact does not depend upon the linear aspect of the usual law of the lever. That is, this result can be proved by considerations of symmetry not only with the law of the lever given by Equation (10.5) with $\alpha = 1$, but also with $\alpha \neq 1$.

Suppose, for instance, that it were found experimentally that a lever only remained in equilibrium when Equation (10.5) was valid with a specific value of α. Even if this were the case, the CG of the three situations presented earlier would be located at the points E, $\triangle$ and Γ, regardless of the value of α. And by symmetry considerations these three levers would remain in equilibrium after being released at rest, no matter the value of α.

But now we must appeal to the crucial postulate 6. In the third situation already presented, Figure 10.15 (c), the set of $2N_1 = 6$ bodies on the left side can be replaced by a single body A of weight $P_A = 2N_1P = 6P$ acting at point E, Figure 10.16. That is, as the configuration of Figure 10.15 (c) was in equilibrium, then postulate 6 guarantees that the configuration of Figure 10.16 will also be an equilibrium situation with the lever free to turn around its fulcrum located at Γ.

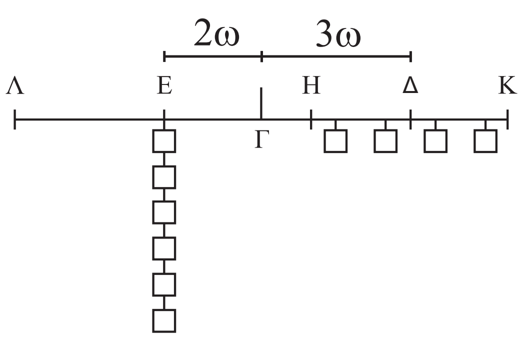


Figure 10.16: Second step in Archimedes' procedure for deducing the law of the lever.

By utilizing once more the sixth postulate, it is then possible to replace the set of $2N_2 = 4$ bodies on the right side with a single body B of weight $P_B = 2N_2P = 4P$ acting at point $\triangle$. This is shown in Figure 10.17. That is, postulate 6 guarantees that the configurations represented by Figures 10.16 and 10.17 will be equilibrium situations when the lever is free to turn around Γ, as was the case with the configuration of Figure 10.15 (c).

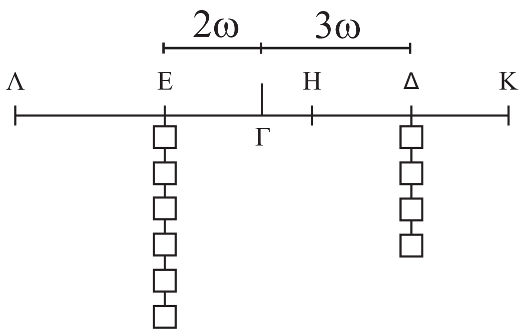


Figure 10.17: Third step in Archimedes' procedure for deducing the law of the lever.

Figure 10.17 represents the law of the lever, because the weight P_A is to the weight P_B as the distance $\triangle\Gamma$ is to the distance $E\Gamma$, namely:

$$\frac{P_A}{P_B} = \frac{6P}{4P} = \frac{3}{2} = \frac{3w}{2w} = \frac{\triangle\Gamma}{E\Gamma} . \tag{10.6}$$

That is, Figure 10.17, which can be expressed mathematically by Equation (10.6), represents the law of the lever given by Equation (8.1).

Suppose now that nature behaved in such a way that the law of the lever were given by Equation (10.5) with $\alpha \neq 1$. That is, suppose that a lever only remained in equilibrium when this equation was satisfied. In this case situations (A), (B) and (C) of Figure 10.15 would still be equilibrium configurations. But when we tried to go to situations (D) and (E), like Figures 10.16 and 10.17, equilibrium would be disturbed. These two configurations would not remain in equilibrium if the system were released at rest. This shows that in this hypothetical situation Archimedes' postulate 6 would no longer be valid.[193]

10.8.2 Archimedes' Calculation of the CG of a Triangle

We now analyze certain aspects of the calculation of the CG of a triangle given by Archimedes. This CG coincides with the intersection of the medians, which are the straight lines connecting the vertices to the midpoints of the opposite sides. The importance of this result is that it is only valid for a law of the lever which is linear with distance. On the other hand, the CG of a

[193] [AR08] and [AR09].

circle or rectangle would still be the geometric centers of these figures even if the law of the lever were quadratic or cubic in terms of distances, as can be seen by arguments of symmetry. This means that the calculation of the CG of a triangle is the first non trivial result for the CG of a two-dimensional figure.

Archimedes considered a generic scalene triangle $AB\Gamma$. In Proposition 13 he then showed that the CG must be along the straight line connecting any vertex to the midpoint of the opposite side:[194]

> *In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side.*

If $\triangle$ is the midpoint of the side $B\Gamma$ in the triangle of Figure 10.18, this means that the CG must be located at some point G along the straight line $A\triangle$.

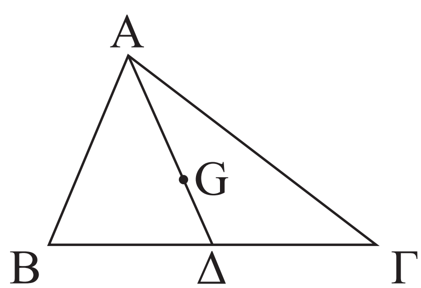


Figure 10.18: Center of gravity of a triangle.

In Proposition 14 he concludes that[195]

> *the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.*

Archimedes presented two proofs of this fact. The two proofs suppose that the CG is not along this line $A\triangle$, which leads to a logical contradiction. This means that the CG must be along the line $A\triangle$, which is what he wanted to prove.

[194] [Arc02b, p. 198].
[195] [Arc02b, p. 201].

Here we explore the opposite point of view. We begin supposing that the CG of the triangle is at some point G along the line $A\triangle$ connecting the vertex to the midpoint of the opposite side. We then show that we do not arrive at any logical contradiction with this reasoning. Moreover, we arrive at the ratio between AG and $G\triangle$. We hope this simplified analysis will aid in understanding Archimedes' proof. We explicitly present all the postulates being used in the proof.

From Postulate 7 we conclude that the CG must be inside the triangle $AB\Gamma$. We then suppose that it is at a point G along the straight line $A\triangle$, where $\triangle$ is the midpoint of the side $B\Gamma$. Let E be the midpoint of the side AB, and Z be the midpoint of the side $A\Gamma$, Figure 10.19.

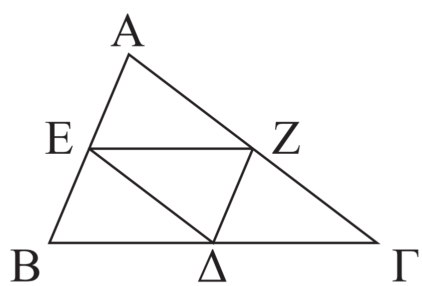


Figure 10.19: Intermediate step to find the CG of a triangle.

We join the segments $E\triangle$, $Z\triangle$ and EZ. The segment $E\triangle$ is parallel to the side $A\Gamma$; the segment EZ is parallel to the side $B\Gamma$; and the segment $Z\triangle$ is parallel to the side AB. This leads to the result that $B\triangle = \triangle\Gamma = EZ = B\Gamma/2$, $BE = EA = Z\triangle = BA/2$, and $AZ = Z\Gamma = E\triangle = A\Gamma/2$. We then obtain four equal triangles: $EB\triangle$, $Z\triangle\Gamma$, AEZ and $\triangle ZE$, as in Figure 10.19. These four triangles are similar to the original triangle $AB\Gamma$. The area and weight P of each one of them are equal to a quarter of the area and weight of the original triangle, namely: $P_{EB\triangle} = P_{Z\triangle\Gamma} = P_{AEZ} = P_{\triangle ZE} = P_{AB\Gamma}/4$.

Let M be the midpoint of the segment EZ, which is also the midpoint of the segment $A\triangle$. Let M_1 be the midpoint of the segment $B\triangle$ and M_2 the midpoint of the segment $\triangle\Gamma$. We join EM_1, ZM_2 and $A\triangle$. By postulate 5 the centers of gravity of the triangles $EB\triangle$, $Z\triangle\Gamma$, AEZ and $\triangle ZE$ will be located at the points G_1, G_2, G_3, and G_4 along the straight segments EM_1, ZM_2, AM, and $\triangle M$, respectively, situated in such a way that $EG_1 = ZG_2 = AG_3 = \triangle G_4 = AG/2$, as in Figures 10.18 and 10.20.

By postulates 1 and 6 we conclude that if the original triangle $AB\Gamma$ was

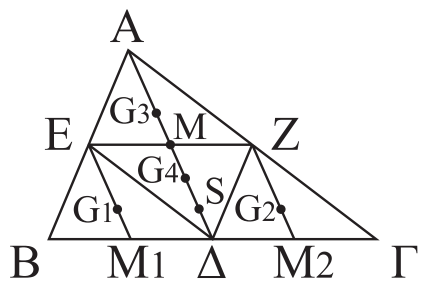


Figure 10.20: Final step to find the CG of a triangle.

in equilibrium when supported by point G indicated in Figure 10.18, then it will remain in equilibrium supported by G when we replace the two triangles $EB\triangle$ and $Z\triangle\Gamma$ by a single body of weight equal to the sum of the weight of these two smaller triangles, acting at the midpoint of the straight line G_1G_2. Let S be this midpoint, located along $A\triangle$. As a matter of fact, in Proposition 4 of this work Archimedes proved the following result:[196]

> *If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity.*

By the same token, we can replace the two triangles AEZ and $\triangle ZE$ with a single body of weight equal to the sum of the weights of these two smaller triangles, acting at the midpoint of the segment G_3G_4, that is, at the point M. This means that the system will remain in equilibrium supported by G after this replacement.

We then have only two equal weights acting at M and S. Once again we can replace these two weights with a single body having the total weight of the original triangle, acting at the midpoint of the segment MS. And this midpoint of MS must be the CG of the original triangle, the point G indicated by Figure 10.18. By postulate 5 we have $S\triangle = G_1M_1 = G_2M_2 = G_3M = G_4M = G\triangle/2$. As G is the midpoint of the segment MS, we have $G\triangle = (M\triangle + S\triangle)/2$. Combining the last two equalities, we obtain: $G\triangle = (M\triangle + G\triangle/2)/2$. As a result, $2G\triangle - G\triangle/2 = 3G\triangle/2 = M\triangle$. Since $M\triangle = A\triangle/2$, we obtain finally $A\triangle = 3G\triangle$. Since $A\triangle = AG + G\triangle$ we also find that $AG = 2G\triangle$.

[196] [Arc02b, p. 191].

We can then conclude that the supposition that the CG of the triangle is along the straight line joining each vertex to the midpoint of the opposite side is a coherent assumption. Moreover, this procedure shows that the CG given by the point G will divide this straight line $A\triangle$ in such a way that $AG = 2G\triangle$.

On the other hand, as $\triangle G_4 = AG/2$, we deduce from the last result that $\triangle G_4 = (2G\triangle)/2 = \triangle G$. In other words, the CG of the triangle $\triangle ZE$, which is located at the point G_4, coincides with the CG of the original triangle $AB\Gamma$, which is located at the point G.

We consider Archimedes' achievements to be some of the greatest scientific accomplishments humankind has produced.

Part IV

Commented Translation of Archimedes' Treatise *On the Equilibrium of Plane Figures* or *The Centers of Gravity of Plane Figures*, Book I

Chapter 11

General Comments on Archimedes' Treatise

Heath presented an English translation of Archimedes' treatise.[197] Heath's work is a paraphrase, that is, it conserves Archimedes' original ideas, but rephrases them in modern notation and omits parts of the text which he did not consider essential. Clagett presented a partial translation based on the original Greek text up to proposition 7.[198] Clagett has also a paraphrase of propositions 6 and 7 in another book.[199] Dijksterhuis has a detailed analysis of this memoir, although no full translation.[200] For this reason we decided to present here a full commented translation of Book I of this work based on Mugler's literal French translation of the Greek original text.[201]

The title of this work has been presented in various ways. Heath presented it as *On the Equilibrium of Planes* or as *The Centers of Gravity of Planes*.[202] Clagett presented it as *On the Equilibrium of Planes* or as *On the Centers of Gravity*.[203] Dijksterhuis presented it as *On the Equilibrium of Planes* or as *Centers of Gravity of Planes*.[204] Mugler presented it as *On the Equilibrium of Plane Figures* or as *On the Centers of Gravity of Plane Figures*.[205] Since

[197] [Arc02b, pp. 189-202].
[198] [Arc79].
[199] [Arc01].
[200] [Dij87, pp. 286-313].
[201] [Mug71a, pp. 75-100].
[202] [Arc02b, p. 189].
[203] [Arc79].
[204] [Dij87, p. 286].
[205] [Mug71a, p. 80].

we are translating here from Mugler, we will follow his title.

Before the translation, some comments. Archimedes' Postulate 7 states that[206]

> In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure.

In Book I of his work *On the Sphere and Cylinder*, Archimedes presented two definitions or axiomatic assumptions explaining what he understood by a line being concave in the same direction. In the first definition, he postulated the existence of a certain type of curve:[207]

> There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side.

By terminated lines he meant finite curves or curves that have two ends. Archimedes included in the expression "bent lines" not only curved lines with continuous curvature, but also lines made up of any number of other lines that can be curved or straight. Because of this fact, the curves he considered could also partially coincide with the straight line determined by their endpoints. Definition 2:[208]

> I apply the term concave in the same direction to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.

To illustrate this aspect, Dijksterhuis presented Figure 11.1.[209]

Of these figures with endpoints at A and B, curves a, b, and c are of the type described in Definition 2, that is, concave in the same direction. Curves d and e are not of this type. Let's analyze curve b, considering it closed by segment AB, Figure 11.2. All lines connecting two of its points either lie on the periphery (like segment AB), or inside the curve (like FG or HI). For this reason, it can be said that it is concave in the same direction, just like curves a and c of Figure 11.1.

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[206] [Arc02b, p. 190].
[207] [Arc02b, p. 2].
[208] [Arc02b, p. 2].
[209] [Dij87, p. 144].

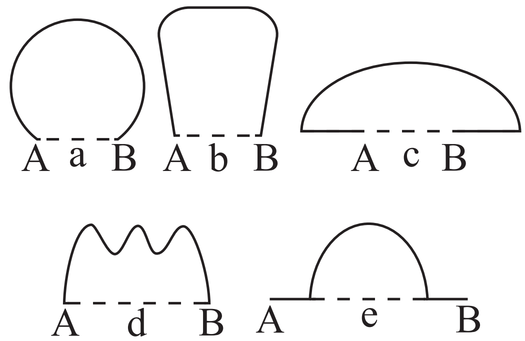

Figure 11.1: General curves.

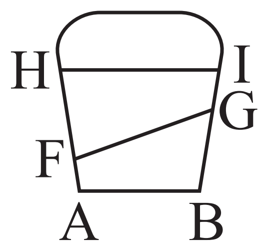

Figure 11.2: Concave curve in the same direction.

We now analyze curve d of Figure 11.1, considering it closed by the segment AB, Figure 11.3. Some lines connecting two of its points will lie on the periphery, such as segment AB. We can also connect two points of this curve such that this line segment lies entirely inside the curve, such as segment FG. On the other hand, we can find two other points such that the line segment connecting them lies entirely outside the closed curve, such as segment HI, Figure 11.3. For this reason, it is said that the curves d and e of Figure 11.1 are not concave in the same direction.

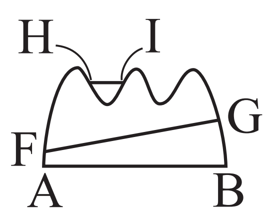

Figure 11.3: A curve that is not concave in the same direction.

In this sense, what Archimedes meant in Definition 2 about "all the straight lines connecting the points fall on the same side of the line," seems to be that all lines connecting any two points on the curve will be on the periphery of the curve or inside it. On the other hand, when he stated that "none fall on the other side," he seemed to be referring to the fact that none of these lines will be outside the closed curve.

We will present here the demonstration of the law of the lever given by Archimedes in Proposition 6, step by step, to clarify all the aspects he used. He called by the letters A, B, and Z the magnitudes, the location of their centers of gravity, and also, implicitly, their weights. Let's call the magnitudes and the location of their centers of gravity A, B, and Z. But let's represent the weights of these magnitudes, respectively, by P_A, P_B, and P_Z. He stated that the weights of A and B are commensurable in this proposition. Therefore, there is a common measure of the weights P_A and P_B. That is, there will be a weight P_Z such that P_A is a multiple of P_Z and P_B is another multiple of P_Z. We can then write $P_A/P_B \equiv i/j$, where i and j are two integers. Following Dijksterhuis,[210] we will present here a concrete example with $i = 3$ and $j = 2$. We will refer to the line segments always ordering the letters that represent the endpoints of the segments from left to right, according to Dijksterhuis's figure. Archimedes also chose a line segment $E\triangle$ divided at the point Γ such that $\Gamma\triangle/E\Gamma \equiv P_A/P_B$. Let's call $E\Gamma \equiv x$ and $\Gamma\triangle \equiv y$. Since P_A and P_B are commensurable, it follows from the two previous equalities that

$$\frac{\Gamma\triangle}{E\Gamma} \equiv \frac{y}{x} \equiv \frac{P_A}{P_B} \equiv \frac{i}{j} \ . \tag{11.1}$$

That is, it follows that $\Gamma\triangle$ and $E\Gamma$ are also commensurable. With our choice of i and j we then arrive at $y/x = 3/2$, see Figure 11.4.

Figure 11.4: Commensurable line segments.

Arquimedes extended the line segment $E\Gamma\triangle$ to the right by the segment $\triangle K = x$ and chose a point H between Γ and $\triangle$ such that $H\triangle = E\Gamma = \triangle K = x$. Since $E\Gamma = H\triangle = x$ and $\Gamma\triangle = y = \Gamma H + x$, it follows that

[210][Dij87, pp. 289-290].

$\Gamma H = y - x$. Therefore, $E\Gamma = x + \Gamma H = x + (y - x) = y$. He extended the line segment $E\triangle$ to the left by the segment $\Lambda E \equiv y$. We then have the result shown in Figure 11.5.

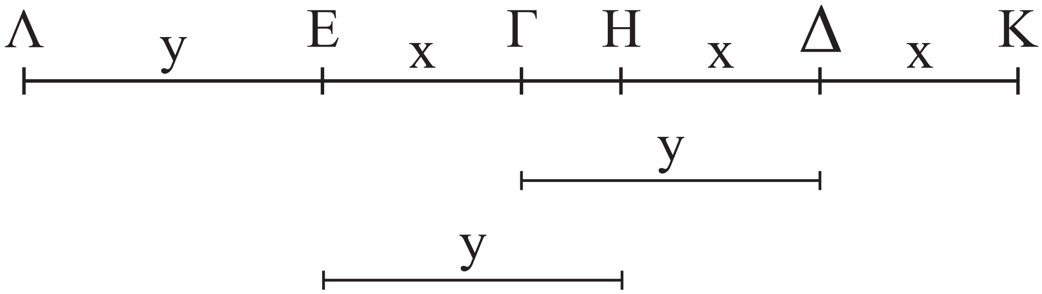

Figure 11.5: Archimedes' geometrical construction.

Archimedes chose N as a line segment that is a common measure of $\Gamma\triangle$ and $E\Gamma$. Let's call $N \equiv w$. He also chose a magnitude Z of weight P_Z which was a common measure of P_A and P_B, such that $P_A/P_Z \equiv \Lambda H/N = 2y/w$. With this choice and the previous relation, we obtain the following: $P_B/P_Z = (P_B/P_A)(P_A/P_Z) = (j/i)(2y/w) = (x/y)(2y/w) = 2x/w$. Since P_Z is a common measure of P_A and P_B, we have that P_A and P_B are integer multiples of P_Z. Since we have already chosen $P_A/P_B = 3/2$, let's choose a concrete example in which $P_A/P_Z = 6/1$ and $P_B/P_Z = 4/1$, Figure 11.6.

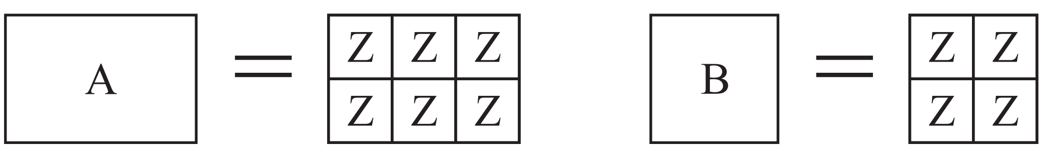

Figure 11.6: We choose the weights of A and B to be multiples of a weight Z.

From the previous relationships, it also follows that: $y/w = 3/1$ and $x/w = 2/1$, see Figure 11.7.

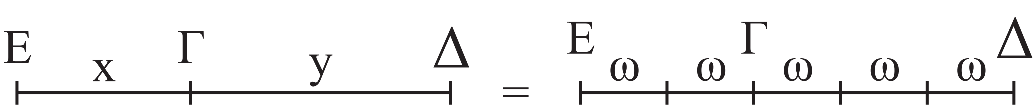

Figure 11.7: Segments x and y are also commensurable, having a common measure w.

Archimedes then divided the entire segment $\Lambda K = 2y + 2x$ into parts equal to $N = w$. In our particular example, we will have 10 pieces of equal length. Since $P_A/P_Z = 2y/w$, there is the same number of pieces equal to w in $\Lambda H = 2y$ as the number of magnitudes Z contained in magnitude A. Similarly, since $P_B/P_Z = 2x/w$, there is the same number of pieces equal to w in $HK = 2x$ as the number of magnitudes Z contained in magnitude B. He then placed the center of gravity of each magnitude Z on the midpoint of each segment w, as shown in Figure 11.8.

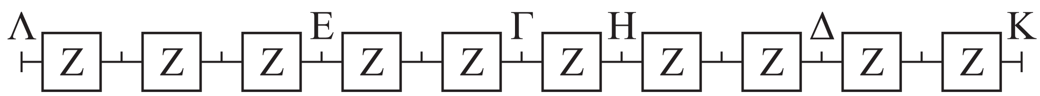

Figure 11.8: Segment ΛK divided into 10 equal pieces.

Another way to represent this situation is by suspending the magnitudes by their centers of gravity, but placing them below the crossbar of the lever, that is, using the method adopted by Archimedes in his work *Quadrature of the Parabola*. Let's disregard the weights of the threads attaching the bodies to the lever. This situation is represented in Figure 11.9.

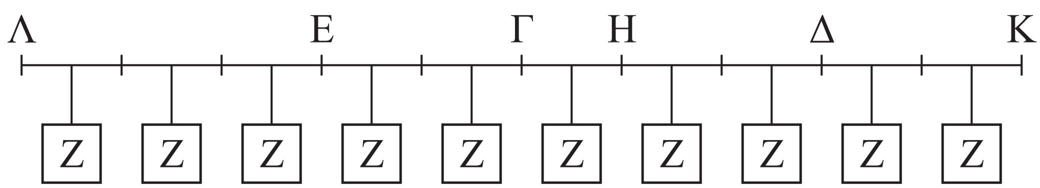

Figure 11.9: Ten magnitudes Z suspended by their centers of gravity at the centers of equal segments of length ΛK.

Archimedes now imagined the line segment ΛK as being the crossbar of a horizontal lever. This lever was free to rotate around a horizontal axis passing through Γ, the midpoint of the segment. This axis was orthogonal to the crossbar of the lever. From the second Corollary of Proposition 5, it follows that the center of gravity CG of this set with an even number of magnitudes of equal weight P_Z situated at equal distances from each other is the point Γ. That is, if the lever is supported by this point Γ and released at rest, it will remain in equilibrium, stationary horizontally.

Moreover, from Corollary 2 of Proposition 5, it also follows that the CG of the equal magnitudes P_Z contained in the segment ΛH is the point E, the midpoint of this segment. But the magnitude A has the same weight as the sum of the weights of the partial magnitudes Z contained in the segment ΛH. Archimedes then replaced the set of magnitudes contained in segment ΛH with the magnitude A acting on the CG of this set (that is, at point E). By Postulate 6, the equilibrium of the lever with the fulcrum located at Γ is not disturbed by this substitution. We then obtain the equilibrium situation represented by Figure 11.10.

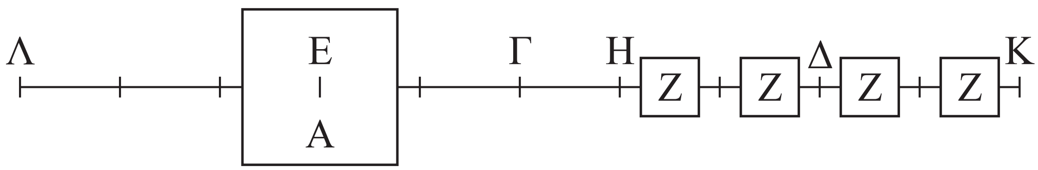

Figure 11.10: Figure 11.9 was in equilibrium. Postulate 6 guarantees that this new situation is also in equilibrium. That is, the lever will not rotate around Γ when released at rest.

Another way to represent this new equilibrium situation is by placing the suspended magnitudes below the crossbar, as in Figure 11.11.

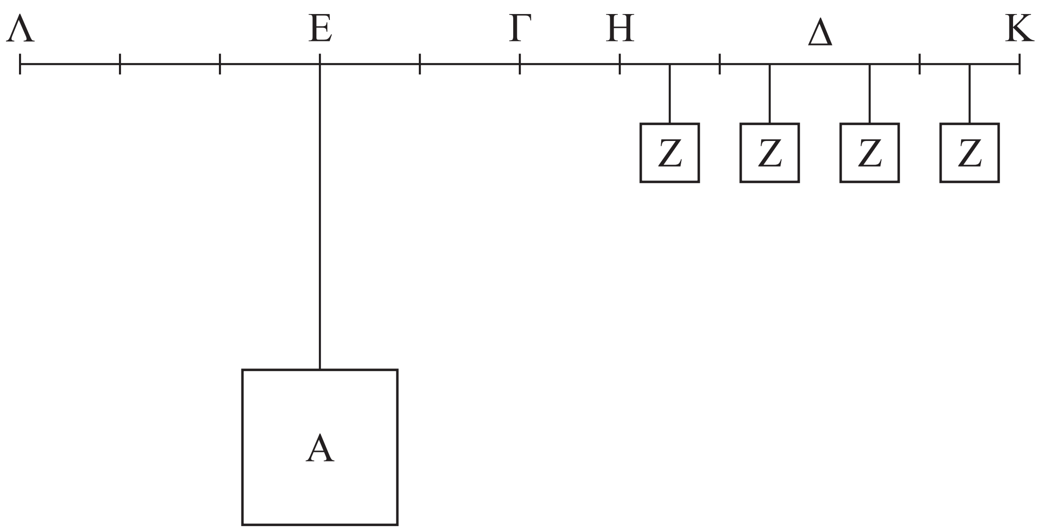

Figure 11.11: Another representation of the equilibrium situation shown in Figure 11.10.

Similarly, the CG of the equal weights P_Z contained in the segment HK is the point $\triangle$, the midpoint of this segment. But the magnitude B has the same weight as the sum of the weights of the partial magnitudes Z contained in the segment HK. Archimedes replaced the set of magnitudes contained in the segment HK with the magnitude B acting on the CG of this set (that is, at the point $\triangle$). It follows again from Postulate 6 that the equilibrium of the lever with the fulcrum located at Γ is not disturbed by this substitution. Archimedes then ends up in the equilibrium situation shown in Figure 11.12.

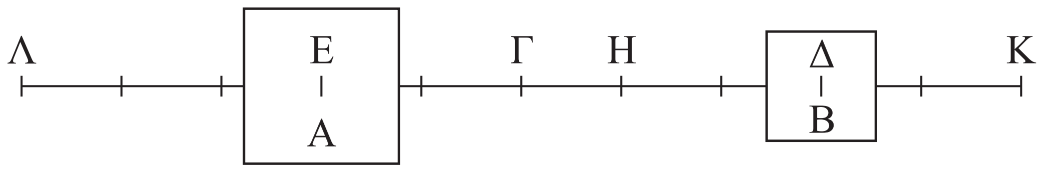

Figure 11.12: Final equilibrium situation that is equivalent to the law of the lever.

We can also represent this situation by placing the magnitudes A and B suspended by their centers of gravity using threads of negligible weight such that A and B are at a level below the crossbar of the lever, as in Figure 11.13.

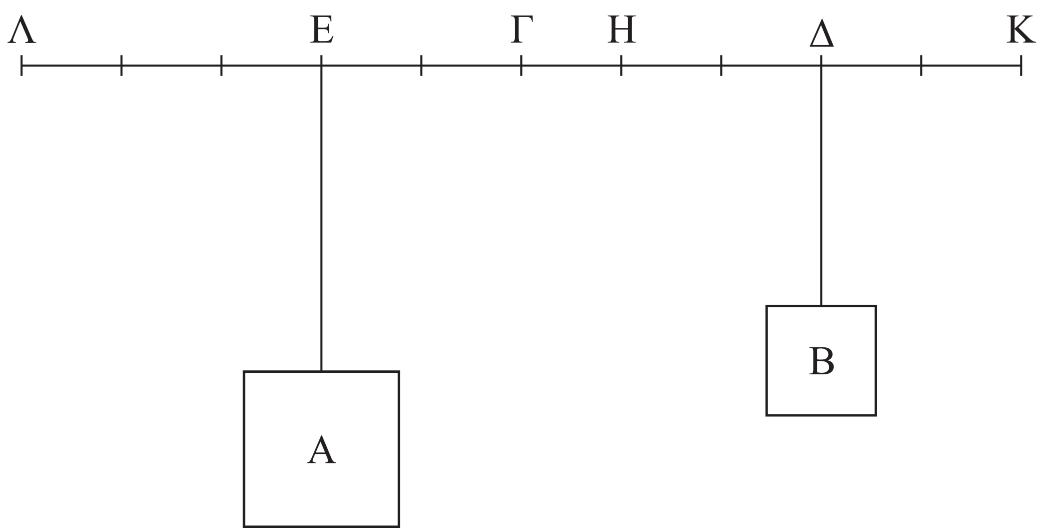

Figure 11.13: Another way to present the final equilibrium configuration, equivalent to the law of the lever.

The use of Postulate 6 guarantees that the center of gravity (CG) of this last situation is the same point as the starting point, that is, the CG is given by the point Γ. In other words, a lever supported at the point Γ is kept in equilibrium if the CG of a magnitude of weight P_A acts at the point E and the CG of a magnitude of weight P_B acts at the point $\triangle$, provided that $P_A/P_B = \Gamma\triangle/E\Gamma$, which is what Archimedes wanted to demonstrate.

Following these clarifications, we present the commented translation of Archimedes' treatise.

Chapter 12

Mugler's Introduction

I present here Mugler's introduction to his French translation of Archimedes' work *On the Equilibrium of Plane Figures*, namely:[211]

In this treatise, where mathematical analysis is applied to the statics of the lever, Archimedes lays the foundations of a new exact science, destined for a great future, rational mechanics.

The flat shapes that Archimedes seeks to determine the equilibrium and centers of gravity of are actually flattened solid bodies of uniform density, treated as flat shapes. The first book contains the fundamental propositions of statics: equilibrium of equal weights suspended from equal lever arms; equilibrium of unequal weights suspended from unequal lever arms; position of the center of gravity of the sum of two or more magnitudes of equal weight; inverse proportionality of two unequal magnitudes in equilibrium to the distances of their suspension points from the fulcrum; determination of the centers of gravity of the parallelogram, triangle, and trapezium by dividing these figures into parallel slices. The second book is entirely devoted to the determination of the center of gravity of the segment of a parabola by the "exact" inscription in this segment of a sequence of triangles, that is to say by the process which consists of constructing a first triangle having the same base and the same height as the segment of a parabola, to repeat this

[211][Mug71a, pp. 77-79].

construction in the partial segments cut out by the sides of the triangle, and so on.

The importance of the treatise *On the Equilibrium of Plane Figures* for the fundamental concepts of solid mechanics has earned it numerous studies and even criticism in modern times. Archimedes has been criticized for using the center of gravity of the figures examined in his demonstrations without providing a definition of it. This omission can be explained by the knowledge of the reality designated by this name that Archimedes was entitled to assume his readers possessed, following one of his earlier treatises, now lost, or by the dissemination of this concept by other authors dealing with questions of mechanics. A more serious reservation has been formulated by the historian of mechanics E. Mach,[212,213] who reproaches Archimedes for tacitly admitting as true the property that he intends to demonstrate in proposition 6 of this treatise, namely the law of inverse proportionality of weights and distances in the equilibrium between unequal weights. But Mach's criticism ignores the scope of the postulates placed by Archimedes at the head of the first book of this treatise, in particular postulate 6, to the significance of which O. Toeplitz[214,215] and W. Stein[216,217] were the first to draw attention. In his insightful analysis of Archimedes' reasoning, E. J. Dijksterhuis[218,219] highlights the analogy between Euclid's fifth postulate and the sixth postulate of the treatise *On the Equilibrium of Plane Figures*, showing that proposition 6 of this treatise is based on postulate 6 in the same way as proposition I, 27 of Euclid is based on the parallel postulate.[220] Today, thanks to these works,

[212][Note by Mugler:] E. Mach (1839-1916), *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt*, seventh edition, Leipzig 1912, p. 14 sq.

[213][Mac12] with Spanish translation in [Mac49] and English translation in [Mac60].

[214][Note by Mugler:] In his interviews and correspondence cited by E. J. Dijksterhuis; O. Toepliz, 20th-century theorist of mathematical knowledge, author of studies on Archimedes and his geometric and mechanical representations.

[215]See [Dij87].

[216][Note by Mugler:] W. Stein, *Der Begriff des Schwerpunktes bei Archimedes*. Quellen und Studien zur Gesch. der Math., Phys. und Astron., B, I, 1930, pp. 221-244.

[217][Ste30].

[218][Note by Mugler:] *Op. laud.*, p. 293 sq.

[219][Dij87, p. 293 and the following].

[220]Euclid's fifth postulate, [Euc56, Vol. 1, p. 155] and [Euc09, p. 98]:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right

this treatise by Archimedes on the foundations of the statics of the lever is considered one of the great texts in the history of applied mathematics.

The text of this treatise is established according to the manuscripts $\mathcal{L}$, D, E, G, H and the preserved, and readable, parts of C.[221]

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angles.

Proposition 27 of Book I of Euclid's Elements, [Euc56, Vol. 1, p. 307] and [Euc09, p. 119]:

> If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

[221] A discussion of these manuscripts can be found in [Arc02b, Chapter II] and [Dij87, Chapter II].

Chapter 13

Translation

On the Equilibrium of Plane Figures

or

The Centers of Gravity of Plane Figures

Book I

1. We postulate that equal weights balance at equal distances,[222] and that equal weights suspended at unequal distances do not balance, but that there is an inclination to the side of the weight suspended at the greater distance.

2. If weights suspended at certain distances are in equilibrium, and one of the weights is increased, the weights are no longer in equilibrium, but there is an inclination toward the side of the weight that was increased.

3. Similarly, if something is subtracted from one of the two weights, the weights no longer balance, but there is an inclination on the side of the weight from which nothing has been subtracted.

4. In equal and similar flat figures that can be superimposed on one another, the centers of gravity also superimpose on one another.

5. In unequal but similar plane figures, the centers of gravity will be

[222] [Note by Mugler:] This first part of postulate 1 is quoted by Proclus, *In Eucl.* (ed. Friedlein), p. 181, 18.

similarly located. We call similarly located in similar figures points such that the straight lines joining them to the vertices of equal angles make equal angles with the corresponding sides.

6. If magnitudes balance each other at certain distances, magnitudes equivalent to these magnitudes will in turn balance each other at the same distances.

7. In any figure, the perimeter of which turns its concavity to the same side,[223,224] the center of gravity must be inside the figure.

These principles being admitted (sc.[225] we will demonstrate the propositions)

1.

Weights that balance at equal distances are equal to each other.[226]

If, in fact, the weights were unequal, the excess of the larger having been subtracted, the remaining weights would no longer balance, since something would have been subtracted from one of the two weights in balance.[227] It follows[228] that weights that balance at equal distances are equal.

2.

Unequal weights suspended at equal distances do not balance each other, but there is an inclination toward the heavier side.

If we remove the excess, the weights will balance, since equal weights

[223] [Note by Mugler:] Compare *On the Sphere and Cylinder*, [Book] I, definition 2.

[224] Definition 2 of Book I of *On the Sphere and Cylinder* according to Heath, [Arc02b, p. 2]:

> I apply the term **concave in the same direction** to a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side.

[225] The abbreviation "sc." stands for *scilicet*, a Latin adverb meaning "namely," "that is to say," or "it is permitted to know." It is sometimes used like here to indicate that a word (or a group of words) is missing from a text.

[226] I have italicized all the propositions and corollaries, even though Mugler did not use italics in his French translation.

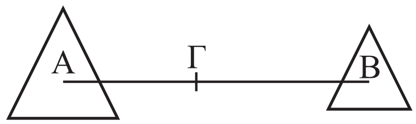
[227] [Note by Mugler:] Compare postulate 3.

[228] [Note by Mugler:] Namely, of the preceding absurdity.

balance at equal distances.[229] Therefore, if we add back what was removed, there will be an inclination toward the heavier side, since we have added to one of the two weights that balance each other.[230]

3.

Unequal weights will balance at unequal distances, with the greater weight at the shorter distance.



Let the unequal weights A and B be given, A being greater than B, and that they balance at the distances $A\Gamma$ and ΓB. It must be shown that $A\Gamma$ is less than ΓB.

Let $A\Gamma$ not be less than ΓB. Let us subtract the excess of A from B. Since we have subtracted something from one of the two weights, which balance each other,[231] there will be an inclination on the side (sc. of the other weight, namely) of B. But this inclination will not occur; for if ΓA is equal to ΓB, the weights will balance,[232] and if ΓA is greater than ΓB, there will be an inclination on the side of A, because equal weights at unequal distances do not balance, but there is an inclination on the side of the weight suspended at the greater distance. For this reason, $A\Gamma$ is less than ΓB.

It is also evident that weights balanced at unequal distances are unequal, and that the weight suspended at the smallest distance is the largest.

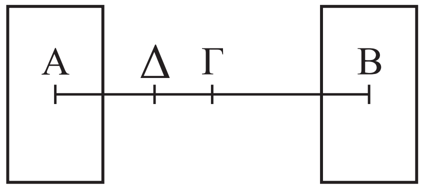
4.

If two equal magnitudes do not have the same center of gravity, the center of gravity of the magnitude composed of these magnitudes will be the midpoint of the line segment joining the centers of gravity of the magnitudes.

Let A be the center of gravity of magnitude A, B that of magnitude B,

[229] [Note by Mugler:] Compare postulate 1.
[230] [Note by Mugler:] Compare postulate 2.
[231] [Note by Mugler:] Compare postulate 3.
[232] [Note by Mugler:] Compare postulate 1.

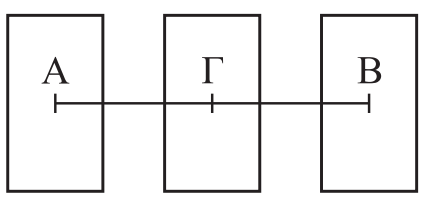
and Γ the midpoint of line segment AB; I say that Γ is the center (sc. of gravity) of the magnitude composed of the two magnitudes.



Indeed, if Γ is not (sc. this center of gravity), let the center of gravity of the magnitude composed of the magnitudes A and B be the point Δ, if this is possible; we have, in fact, shown previously that this center is located on the line AB.[233] From the moment therefore that Δ is the center of gravity of the magnitude composed of A and B, (sc. this magnitude) will be in equilibrium if the point Δ is fixed; consequently, the magnitudes A and B will be balanced at the distances $A\Delta$ and ΔB, which is impossible,[234] since equal magnitudes do not balance at unequal distances. It is therefore evident that the point Γ is the center of gravity of the magnitude composed of the magnitudes A and B.

5.

If the centers of gravity of three magnitudes are located on the same straight line, if these magnitudes have the same weight, and if the line segments between the centers (sc. of gravity) are equal, the center of gravity of the magnitude composed of the sum of the three magnitudes will be the point which is also the center of gravity of the magnitude located in the middle.



Let A, B, and Γ be three magnitudes whose centers of gravity are the

[233] Heath mentions that this allusion is no doubt to the lost treatise *On Levers* ($\pi\epsilon\rho\iota$ $\zeta\nu\gamma\omega\nu$), [Arc02b, p. 191].
[234] [Note by Mugler:] Compare postulate 1.

aligned points A, B, and Γ. Let the magnitudes A, B, and Γ be equal to each other, and let $A\Gamma$ be equal to ΓB. I say that the center of gravity of the magnitude composed of the sum of the (sc. three) magnitudes is the point Γ.

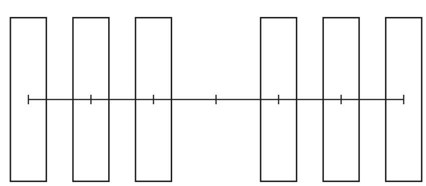
Since the magnitudes A and B have the same weight, the center of gravity will be the point Γ, since the line segments $A\Gamma$ and ΓB are equal.[235] But the point Γ is also the center of gravity of the magnitude Γ; it is therefore obvious that the center of gravity of the magnitude composed of the sum of the three magnitudes is the same point which is also the center of gravity of the middle magnitude.

Corollary I

It is clear from the above that for any odd number of magnitudes whose centers of gravity are aligned and whose distances from the middle are equal, and which are arranged in such a way that the line segments between the centers of gravity of the magnitudes are equal,[236] *the center of gravity of the magnitude composed of the sum of the magnitudes will be the point that is also the center of gravity of the middle magnitude.*

Corollary II

Even when the magnitudes are even in number, if their centers of gravity are aligned, if the (sc. two) middle magnitudes and those equidistant from the middle ones have the same weight, and if the straight line segments between the centers (sc. of gravity) are equal, the center of gravity of the magnitude composed of the sum of the magnitudes will be the midpoint of the straight line segment joining the centers of gravity of the magnitudes, as shown in the figure.



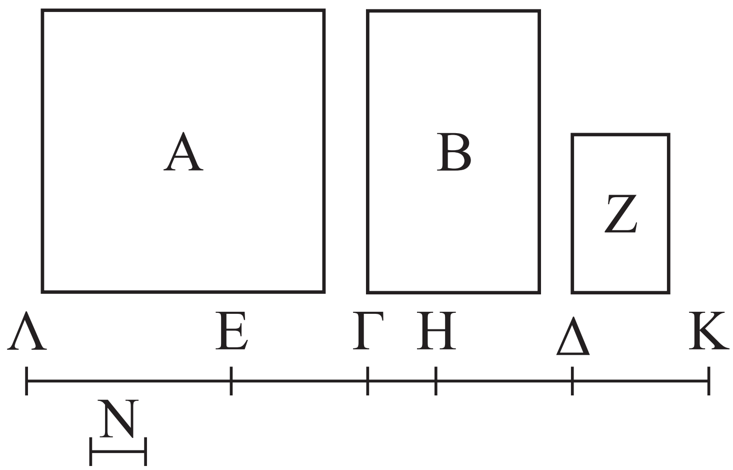
[235] [Note by Mugler:] Compare proposition 4.

[236] [Note by Mugler:] Sc. on either side of the middle magnitude.

6.

Commensurable magnitudes balance each other at distances inversely proportional to their weights.

Let A and B be commensurable magnitudes, A and B their centers (sc. of gravity); let $E\Delta$ be a length, and let the length $\Delta\Gamma$ be to the length ΓE as A is to B. It must be shown that the center of gravity of the magnitude composed of the two magnitudes A and B is the point Γ.



From the moment, in fact, that $\Delta\Gamma$ is to ΓE as A is to B, and that A and B are commensurable, it follows that the two line segments $\Gamma\Delta$ and ΓE are in turn commensurable. Let N be their common measure. Let us assume that the two line segments ΔH and ΔK are equal, each, to $E\Gamma$, and the line segment $E\Lambda$ equal to $\Delta\Gamma$. Since ΔH is equal to ΓE, $\Delta\Gamma$ is equal to EH, so that ΛE is also equal to EH. Consequently ΛH is double $\Delta\Gamma$, and HK double ΓE. Therefore N also measures each of the line segments ΛH and HK, since N measures their halves.[237,238] And since, on the one hand, $\Delta\Gamma$ is to ΓE as A is to B, and on the other hand, ΛH is to HK as $\Delta\Gamma$ is to ΓE, — each of the first segments is in fact double each of the second —, the ratio of

[237] [Note by Mugler:] Compare Euclid X, 12.

[238] Proposition 12 of Book X of Euclid's Elements, [Euc56, Vol. 3, p. 34] and [Euc09, p. 365]:

> Magnitudes commensurable with the same magnitude are commensurable with one another also.

ΛH to HK is also equal to the ratio of A to B. Let A contain the magnitude Z as many times as ΛH contains N. It follows[239,240] that ΛH is to N as A is to Z. But KH is also[241,242] to ΛH as B is to A. By identity,[243,244] therefore, KH is to N as B is to Z. Consequently, B is a multiple of Z as many times as KH is a multiple of N. But we have shown that A is also a multiple of Z, so that Z is a common measure of A and B. If now the segment ΛH is divided into parts equal to N, and the magnitude A into parts equal to Z, the segments equal to N contained in the segment ΛH will be equal in number to the partial magnitudes equal to Z contained in the magnitude A. Therefore, if we place on each of the segments of ΛH a magnitude equal to Z having its center of gravity in the middle of the segment, the sum of these magnitudes

[239][Note by Mugler:] Compare Euclid V, definition 5.

[240]Definition 5 of Book V of Euclid's Elements, [Euc56, Vol. 2, p. 114] and [Euc09, p. 205]:

> Magnitudes are said to **be in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

[241][Note by Mugler:] Compare Euclid V, 7, corollary.

[242]Corollary of Proposition 7 of Book V of Euclid's Elements, [Euc56, Vol. 2, p. 149] and [Euc09, p. 213]:

> From this it is manifest that, if any magnitudes are proportional, they will also be proportional inversely.

[243][Note by Mugler:] Compare Euclid V, 22.

[244]Proposition 22 of Book V of Euclid's Elements, [Euc56, Vol. 2, p. 179] and [Euc09, p. 227]:

> If there be any number of magnitudes whatsoever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

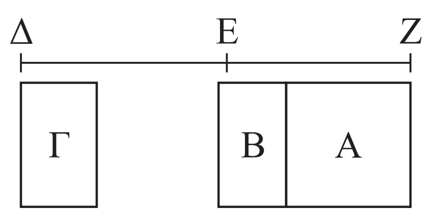
Definition 17 of Book V of Euclid's *Elements*, [Euc56, Vol. 2, p. 115] and [Euc09, p. 206]:

> A ratio ex aequali arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes;
>
> Or, in other words, it means taking the extreme terms by virtue of removal of the intermediate terms.

is equal to the magnitude A, and the center of gravity of the magnitude which is the sum of all these partial magnitudes will be the point E; all these magnitudes are, in fact, even in number, and there is the same number on either side of E, since the segment ΛE is equal to the segment HE. We will demonstrate in the same way that, even if on each of the partial segments of the segment KH we place a magnitude equal to Z, having its center of gravity in the middle of the segment, the sum of these partial magnitudes will be equal to B, and that the center of gravity of the magnitude which is the sum of all these partial magnitudes will be the point Δ.[245] The magnitude A will therefore be placed at point E, and magnitude B at point Δ. We will therefore have magnitudes that are equal to each other, whose centers of gravity are also equidistant from each other and which are placed in even numbers on a line segment. It is therefore obvious that the center of gravity of the magnitude that is the sum of all these magnitudes is the midpoint of the line segment that contains the centers of the middle magnitudes.[246] But since the line segment ΛE is equal to the segment $\Lambda\Delta$, and the segment $E\Gamma$ is equal to the segment ΔK, the entire segment $\Lambda\Gamma$ is equal to the segment ΓK. It follows that the center of gravity of the magnitude that is the sum of all the partial magnitudes is the point Γ. Therefore, the magnitude A, located at point E, and the magnitude B, located at point Δ, will balance each other at point Γ.

7.

In the same way, if the magnitudes are incommensurable, they will balance at distances inversely proportional to the magnitudes.



Let the incommensurable magnitudes be AB and Γ and the distances ΔE

[245][Note by Mugler:] Since Z measures the magnitude B; compare proposition 5, corollary 2.
[246][Note by Mugler:] According to proposition 5, corollary 2.

and EZ; let the ratio of AB to Γ be equal to the ratio of the distance $E\Delta$ to the distance EZ; I say that the center of gravity of the magnitude composed of the two magnitudes[247] AB and Γ is the point E.

For if AB, placed at point Z, does not balance Γ, placed at point Δ, the magnitude AB will either be too large compared to the magnitude Γ for there to be balance, or it will not be too large. Let it be too large, and let a magnitude less than this excess of AB over Γ which prevents balance be subtracted from AB, such that the remaining magnitude A is commensurable with Γ. From the moment, therefore, that the magnitudes A and Γ are commensurable, and that the ratio of A to Γ is less than the ratio of the line segment ΔE to the segment EZ, the magnitudes A and Γ will not balance[248] at the distances ΔE and EZ if the magnitude A is placed[249] at point Z and the magnitude Γ at point Δ. For the same reasons, there will be no equilibrium either, if the magnitude Γ is too large to balance the magnitude AB.

8.

If we subtract from a certain magnitude a magnitude which does not have the same center (sc. of gravity) as the whole, the center of gravity of the magnitude which remains is the end of the straight line segment cut from the extension, on the side of the center of the whole magnitude, of the straight line joining the centers of gravity of the whole magnitude and of the magnitude subtracted, and cut in such a way that the ratio of this segment to the straight line segment between the centers is equal to the ratio of the weight of the magnitude subtracted to the weight of the remaining magnitude.[250,251]

[247] [Note by Mugler:] With the implied assumption that AB is placed on Z, and Γ on Δ.

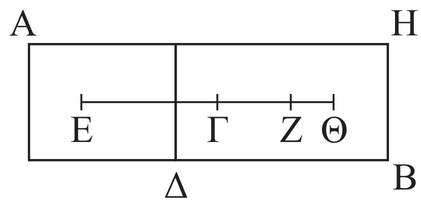
[248] [Note by Mugler:] Compare proposition 6.

[249] [Note by Mugler:] As long as $\frac{A}{\Gamma} < \frac{\Delta E}{EZ}$, there is an inclination on the side of Δ, which is impossible, since the subtracted magnitude of AB is too small for the remainder, that is, A, to balance Γ.

[250] [Note by Mugler:] Proposition cited in the treatise *On Floating Bodies*, II, 2.

[251] In the middle of the proof of Proposition 2 of Book II of Archimedes' treatise *On Floating Bodies* we have the following sentence, [Mug71b, pp. 25-26]:

> [...] because it has been demonstrated in the *Elements of Mechanics* that if a magnitude not having the same center of gravity is subtracted from an integer magnitude, the center of gravity of the magnitude which remains will be located on the line joining the centers of gravity of the integer magnitude and the magnitude subtracted, this line being extended on the side where the center



Let Γ be the center of gravity of a magnitude AB; let us subtract the magnitude $A\Delta$ from the magnitude AB, and let E be the center of gravity of $A\Delta$; draw the line $E\Gamma$ and subtract the segment ΓZ from its extension (sc. towards Γ) so that the ratio of ΓZ to ΓE is equal to the ratio of the magnitude $A\Delta$ to the magnitude ΔH; it must be shown that the center of gravity of the magnitude ΔH is the point Z.

Indeed, let this center not be Z, but the point Θ. From the moment therefore that the magnitude $A\Delta$ has as its center of gravity the point E, and the magnitude ΔH the point Θ, the center of gravity of the magnitude composed of the magnitudes $A\Delta$ and ΔH will be on the line segment $E\Theta$, cut in such a way that its partial segments have between them the inverse ratio of the magnitudes;[252] consequently[253] the point Γ will not be located at the place assigned to it by the division corresponding to that which we have just indicated. The point Γ is therefore not the center (sc. of gravity) of the magnitude composed of the magnitudes $A\Delta$, ΔH, that is to say of the magnitude AB. But it is by hypothesis; it follows that Θ is not the center of gravity of the magnitude ΔH.

9.

In any parallelogram, the center of gravity is located on the line connecting the midpoints of the opposite sides of the parallelogram.

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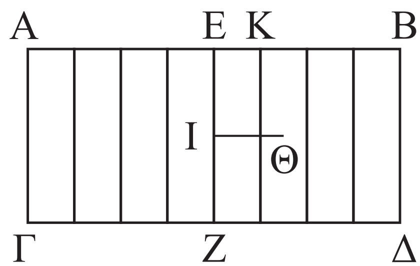
of gravity of the integer magnitude is located.

Mugler then mentioned in a footnote to compare Proposition 8, Book I of Archimedes' treatise *On the Equilibrium of Plane Figures*.

[252][Note by Mugler:] Compare propositions 6 and 7.

[253][Note by Mugler:] There are several missing links in the reasoning here, namely: "but, by hypothesis, the point Γ already marks in the segment EZ a division such that the partial segments have between them the inverse ratio of that of the magnitudes; it could therefore not mark the same division also in the segment $E\Theta$." To fill the gap, the origin of which is unknown, Heiberg proposes to insert, between *μεγεθεσιν* and *ωστε*, the sentence: *το δε* Γ *επι ταζ EZ εστι τμαθεισαζ, ωστε τα τμηματα αντιπεπονθεμεν τοιζ μεγεθεσιν*.



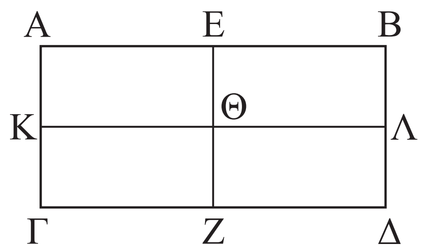
Let $AB\Gamma\Delta$ be the parallelogram, and EZ the line connecting the midpoints of sides AB and $\Gamma\Delta$. I say that the center of gravity of parallelogram $AB\Gamma\Delta$ will be located on line EZ.

Let it not be so, but let the center of gravity be, if possible, the point Θ. Let us draw the line ΘI parallel to AB. The line segment EB being continually divided into two equal parts, there will come a time when the remaining segment will be less than $I\Theta$. Let each of the two line segments AE and EB be divided into partial segments equal to the segment EK (sc. less than $I\Theta$). Let us draw the parallels to EZ through the points of division. The entire parallelogram will thus be divided into parallelograms equal and similar to the parallelogram KZ. These parallelograms, equal and similar to KZ, overlap each other, so their centers of gravity will coincide.[254] Therefore, there will be: certain magnitudes, parallelograms equal to KZ, in even numbers; their centers of gravity located on a straight line; the magnitudes in the middle equal. Furthermore, all the magnitudes on either side of the middle magnitudes are equal, and the line segments between the centers are equal. It follows that the center of gravity of the magnitude that is the sum of all these parallelograms will be located on the line joining the centers of gravity of the middle areas.[255] But this is not the case, since point Θ lies outside the middle parallelograms.[256] It is therefore clear that the center of gravity of parallelogram $AB\Gamma\Delta$ lies on line EZ.

10.

In any parallelogram the center of gravity is the point where the diagonals meet.

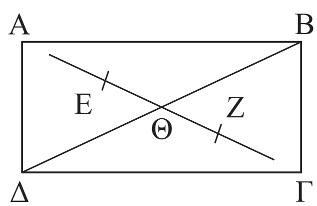
[254] [Note by Mugler:] Compare postulate 4.
[255] [Note by Mugler:] Compare proposition 5, corollary 2.
[256] [Note by Mugler:] Since EK is less than $I\Theta$, by hypothesis.



Consider the parallelogram $AB\Gamma\Delta$, and in this parallelogram the line EZ dividing the sides AB and $\Gamma\Delta$ into two equal parts, and the line $K\Lambda$ dividing the sides $A\Gamma$ and $B\Delta$.[257] The center of gravity of the parallelogram $AB\Gamma\Delta$ is therefore located on the line EZ, since this has been demonstrated.[258] But for the same reasons it is also located on the line $K\Lambda$. It follows that the point Θ is the center of gravity. Now it is at the point Θ that the diagonals of the parallelogram meet, so that the proposition is demonstrated.

Another demonstration

But it is possible to demonstrate the same proposition in yet another way.



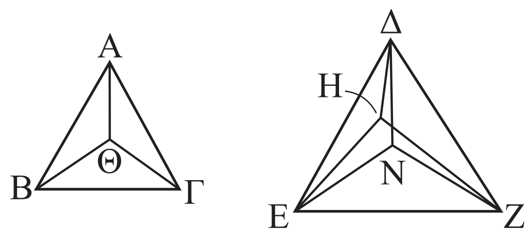
Let $AB\Gamma\Delta$ be a parallelogram, and ΔB one of its diagonals; the triangles $AB\Delta$ and $B\Delta\Gamma$ are therefore equal and similar to each other; if these triangles are superimposed on each other, their centers of gravity will coincide. Let E be the center of gravity of triangle $AB\Delta$; divide ΔB into two equal parts at point Θ, draw $E\Theta$ and take segment $Z\Theta$ equal to ΘE on the extension of $E\Theta$. If we now apply triangle $AB\Delta$ to triangle $B\Delta\Gamma$, with side AB placed on $\Delta\Gamma$ and side $A\Delta$ on $B\Gamma$, segment ΘE will in turn be applied to $Z\Theta$, and point E will coincide with point Z. But it will also coincide with the center of gravity of triangle $B\Delta\Gamma$.[259] Therefore, since E is the center of gravity of

[257] The line $K\Lambda$ also divides the sides $A\Gamma$ and $B\Delta$ into two equal parts.
[258] [Note by Mugler:] Compare proposition 9.
[259] [Note by Mugler:] Compare postulate 4. At this point, the conclusion is missing: "the

triangle $AB\Delta$, and Z is the center of gravity of triangle $\Delta B\Gamma$, it is clear that the center of gravity of the composite of these two triangles is the midpoint of the line segment EZ,[260] namely the point Θ.

11.

Given two similar triangles, and in these triangles points similarly situated with respect to the triangles, if one of these points is the center of gravity of the triangle in which it is situated, the other point is also the center of gravity of the triangle in which it is situated. We call similarly situated with respect to similar figures points such that the straight lines which join them to the vertices of equal angles make equal angles with the corresponding sides.[261]



Let the two triangles be $AB\Gamma$ and ΔEZ, and let $A\Gamma$ be to ΔZ as AB is to ΔE and as $B\Gamma$ is to EZ;[262,263] let Θ and N be two points similarly located with respect to the triangles $AB\Gamma$ and ΔEZ, and let Θ be the center of gravity of the triangle $AB\Gamma$; I say that the point N is the center of gravity of the triangle ΔEZ.

Let it not be so, and let the center of gravity of the triangle ΔEZ be the

point Z is thus the center of gravity of the triangle $B\Delta\Gamma$. Heiberg suggests completing the text with: $\tau o\ Z\ \alpha\rho\alpha\ \chi\epsilon\nu\tau\rho o\nu\ \tau o\upsilon\ \beta\alpha\rho\epsilon o\zeta\ \epsilon\sigma\tau\iota\ \tau o\upsilon\ B\Delta\Gamma\ \tau\rho\iota\gamma\omega\nu o\upsilon$.

[260][Note by Mugler:] Compare proposition 4.

[261][Note by Mugler:] This definition is rejected, as a copyist's addition, by Barrowius and by Heiberg.

[262][Note by Mugler:] Similarity conditions of triangles $AB\Gamma$ and ΔEZ; compare Euclid VI, 4.

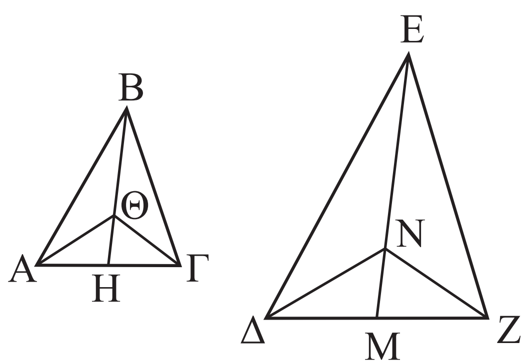
[263]Proposition 4 of Book VI of Euclid's Elements, [Euc56, Vol. 2, p. 200] and [Euc09, p. 235]:

> In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

point H. Let us draw the lines ΘA, ΘB, $\Theta\Gamma$, ΔN, EN, ZN, ΔH, EH, ZH. From the moment, then, that the triangle $AB\Gamma$ is similar to the triangle ΔEZ, and that the centers of gravity are the points Θ and H, and that, moreover, the centers of gravity of similar figures are similarly situated, so that the lines which join them at the vertices make equal angles with the corresponding sides, each with each,[264] the angle $H\Delta E$ is equal to the angle ΘAB.[265] But the angle ΘAB is equal to the angle $E\Delta N$ by virtue of the similar situation of the points Θ and N; it follows that the angle $E\Delta N$ is also equal to the angle $E\Delta H$, that is to say that a larger angle is equal to a smaller angle, which is impossible. The point N cannot therefore not be the center of gravity of the triangle ΔEZ; N is therefore the center of gravity.

12.

Given two similar triangles, if the center of gravity of one is located on the line drawn from one of the vertices to the middle of the base (sc. on the opposite side), the center of gravity of the other triangle will also be located on the line drawn in the same way.



Let the two triangles be $AB\Gamma$ and ΔEZ; let $A\Gamma$ be to ΔZ as AB is to ΔE and as $B\Gamma$ is to ZE;[266,267] let us divide the side $A\Gamma$ into two equal parts at the point H and draw the line BH; let the center of gravity of the triangle $AB\Gamma$, be Θ, located on BH; I say that, also for the triangle $E\Delta Z$, the center

[264] [Note by Mugler:] This sequence is considered an interpolation by Heiberg.
[265] [Note by Mugler:] Compare postulate 5.
[266] [Note by Mugler:] Compare Euclid VI, 4.
[267] See footnote 263.

of gravity is located on the line drawn similarly.

Let us divide ΔZ into two equal parts at point M, draw EM and (sc. take a point N on EM such that) BH is to $B\Theta$ as ME is to EN; draw the lines $A\Theta$, $\Theta\Gamma$, ΔN, NZ. Since AH is half of ΓA and ΔM is half of ΔZ, we also have equality between the ratio of BA to $E\Delta$ and the ratio of AH to ΔM. Sides with equal angles[268] are also proportional; it follows that angle AHB is equal[269,270] to angle ΔME and that AH is to ΔM as[271,272] BH is to EM. But we also have BH to $B\Theta$ as ME to EN; by identity, the ratio of AB to ΔE is therefore equal[273,274] to the ratio of $B\Theta$ to EN. The sides comprising equal angles are, moreover, proportional; under these conditions, the angle $BA\Theta$ is equal to the angle $E\Delta N$; consequently the angle $\Theta A\Gamma$ which remains[275,276] is equal to the angle $N\Delta Z$. But for the same reasons the angle $B\Gamma\Theta$ is equal to the angle EZN, and the angle $\Theta\Gamma H$ equal to the angle NZM. As we had shown that also the angle $AB\Theta$ is equal to the angle

[268] [Note by Mugler:] Sc. the angles BAH and $E\Delta M$, equal by virtue of the similarity of the triangles $AB\Gamma$ and ΔEZ.

[269] [Note by Mugler:] Compare Euclid VI, 6.

[270] Proposition 6 of Book VI of Euclid's Elements, [Euc56, Vol. 2, p. 237] and [Euc09, p. 227]:

> If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

[271] [Note by Mugler:] Compare Euclid VI, 4.

[272] See footnote 263.

[273] [Note by Mugler:] Compare Euclid V, 17 and 22.

[274] Propositions 17 and 22 of Book V of Euclid's Elements, [Euc56, Vol. 2, pp. 228 and 240] and [Euc09, pp. 222 and 227]:

> If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.
>
> If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and, if the rectilineal figures similar and similarly described upon them be proportional, the straight lines will themselves also be proportional.

[275] [Note by Mugler:] Sc. after subtracting the angle $BA\Theta$ from the angle BAH, and the angle $E\Delta N$ from the angle $E\Delta M$; compare Euclid I, Common Notion 3.

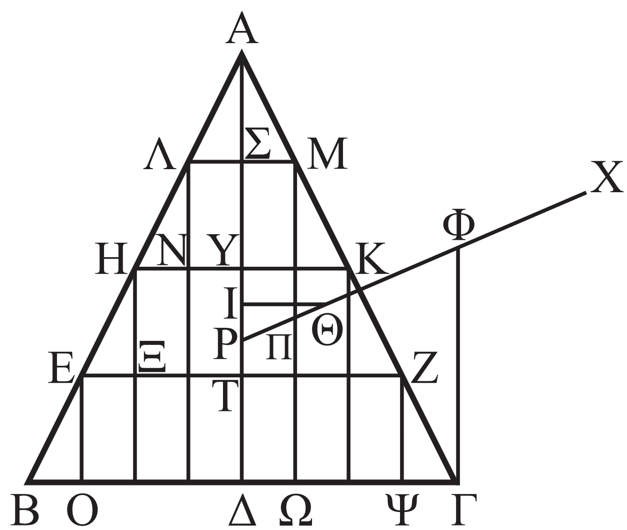
[276] Common Notion 3 of Book I of Euclid's Elements, [Euc56, Vol. 1, p. 155] and [Euc09, p. 99]:

> If equals be subtracted from equals, the remainders are equal.

ΔEM, the angle $\Theta B\Gamma$ which remains is equal[277] in turn to the angle NEZ. For all these reasons, the points Θ and N are similarly located, forming equal angles with the corresponding sides. Since, therefore, the points Θ and N are similarly located and Θ is the center of gravity of the triangle $AB\Gamma$, the point N is also the center of gravity of the triangle ΔEZ.

13.

In any triangle, the center of gravity is located on the line from a vertex to the midpoint of the opposite side.



Consider the triangle $AB\Gamma$, and let $A\Delta$ be the line segment drawn (sc. from vertex A) to the midpoint of side $B\Gamma$. We must prove that the center of gravity of triangle $AB\Gamma$ lies on line segment $A\Delta$.

Let it not be so, but let the center of gravity be the point Θ, if possible; let us draw through Θ the parallel ΘI to $B\Gamma$. If, from then on, the segment $\Delta\Gamma$ is continually divided into two equal parts, there will come a time when the remainder segment will be less than ΘI; let us divide each of the segments $B\Delta$ and $\Delta\Gamma$ into equal parts (sc. to this remainder), and through the points of division draw lines each parallel to $A\Delta$ and draw the lines EZ, HK and

[277][Note by Mugler:] Sc. after subtracting the sum of angles $AB\Theta$, $B\Gamma\Theta$, $\Theta\Gamma H$, $HA\Theta$, ΘAB from the sum of angles of the triangle $AB\Gamma$, and subtracting the sum of angles ΔEN, EZN, NZM, $M\Delta N$, $N\Delta E$ from the sum of angles of the triangle ΔEZ.

ΛM, which will then be parallel to $B\Gamma$. Therefore, the center of gravity is, in the parallelogram MN on the line $\Upsilon\Sigma$, in the parallelogram $K\Xi$ on $T\Upsilon$, in the parallelogram ZO on $T\Delta$;[278] it follows that in the magnitude composed of all these parallelograms the center of gravity is on the line[279] $\Sigma\Delta$. Let this center be point P; join P to Θ, extend $P\Theta$ and draw $\Gamma\Phi$ parallel to $A\Delta$. The ratio of triangle $A\Delta\Gamma$ to the sum of the triangles constructed on AM, MK, KZ, $Z\Gamma$, similar to triangle $A\Delta\Gamma$, is therefore equal[280,281] to the ratio of ΓA to AM, by virtue of the equality of segments AM, MK, $Z\Gamma$, KZ. But since, on the other hand, the ratio of triangle $A\Delta B$ to the sum of the similar triangles constructed on $A\Lambda$, ΛH, HE, EB is equal to the ratio of BA to $A\Lambda$, triangle $AB\Gamma$ is to the sum of the triangles indicated as ΓA is to AM. But the ratio of ΓA to AM is greater than the ratio of ΦP to $P\Theta$; for the ratio of ΓA to AM is equal to the ratio of ΦP to $P\Pi$ by virtue of the similarity of the triangles; thus the ratio of the triangle $AB\Gamma$ to the sum of the triangles indicated is greater than the ratio of ΦP to $P\Theta$; it follows, by dissociation, that also the ratio of the sum of the parallelograms MN, $K\Xi$, ZO to the remaining triangles is greater[282,283] than the ratio of $\Phi\Theta$ to

[278] [Note by Mugler:] Compare proposition 9.

[279] [Note by Mugler:] Compare proposition 4.

[280] [Note by Mugler:] Compare Euclid V, 16 and 18; VI, 2.

[281] Propositions 16 and 18 of Book V of Euclid's Elements, [Euc56, Vol. 2, pp. 164 and 169] and [Euc09, pp. 221 and 223]:

> If four magnitudes be proportional, they will also be proportional alternately.

> If magnitudes be proportional *separando*, they will also be proportional *componendo*.

Proposition 2 of Book VI of Euclid's Elements, [Euc56, Vol. 2, p. 194] and [Euc09, p. 233]:

> If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

[282] [Note by Mugler:] Compare *On the Sphere and Cylinder* II, 7 and Pappus VII, 45.

[283] Proposition 7 of Book II of Archimedes' treatise *On the Sphere and Cylinder*, [Arc02b, p. 84]:

> From a given square to cut off a segment by a plane so that the segment may have a given ratio to the cone which has the same base as the segment and equal height.

ΘP. Let us consider a ratio $X\Theta$ to ΘP equal to the ratio of the sum of the parallelograms to the sum of the triangles.[284,285] Since we then have a certain magnitude, the triangle $AB\Gamma$, whose center of gravity is the point Θ, from which we have subtracted the magnitude composed of the parallelograms MN, $K\Xi$, ZO, and since the center of gravity of the subtracted magnitude is the point P, the center of gravity of the remaining magnitude, composed of the remaining triangles, is located on the extension of the line $P\Theta$, on which we have cut a segment having to ΘP the ratio that the subtracted magnitude has to the remaining magnitude.[286] The point X is therefore the center of gravity of the magnitude composed of the remaining triangles, which is impossible, since all (sc. these triangles) are located on the same side[287] of the line drawn by the point X parallel to $A\Delta$. The proposition is therefore obvious.

Another demonstration of the same proposition

Let us consider triangle $AB\Gamma$; let us draw line $A\Delta$ joining A to the midpoint of side $B\Gamma$. I say that the center of gravity of triangle $AB\Gamma$ lies on $A\Delta$.

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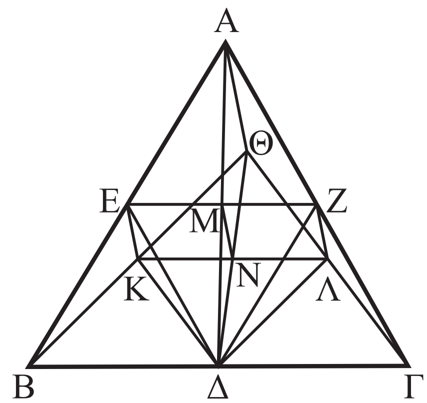
[284][Note by Mugler:] $X\Theta$ is indeed greater than $P\Theta$; compare Euclid V, 8.

[285]Proposition 8 of Book V of Euclid's Elements, [Euc56, Vol. 2, p. 149] and [Euc09, p. 214]:

> Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

[286][Note by Mugler:] Compare proposition 8.

[287][Note by Mugler:] The four words in the text $\tau o \upsilon \tau \varepsilon o \tau \iota \nu$ $\varepsilon \pi \iota$ $\theta \alpha \tau \varepsilon \rho o \nu$ $\mu \varepsilon \rho o \zeta$ are condemned by Heiberg as an interpolation from Eutocius' commentary; compare Heiberg, *Archimedis Opera* II, page 155.



Indeed, let it not be so, but let the center of gravity be, if possible, the point Θ. Let us draw the lines $A\Theta$, ΘB and $\Theta\Gamma$ and join the midpoints of the sides BA, $A\Gamma$ (sc. and $B\Gamma$) by the lines $E\Delta$ and ZE. Let us draw EK and $Z\Lambda$ parallel to $A\Theta$, and draw the lines $K\Lambda$, $\Lambda\Delta$, ΔK, $\Delta\Theta$ and MN. Since the triangle $AB\Gamma$ is similar to the triangle $\Theta Z\Gamma$, because BA is parallel[288,289] to $Z\Delta$, since, moreover, the center of gravity of the triangle $AB\Gamma$ is the point Θ (sc. by hypothesis), the center of gravity of the triangle $Z\Delta\Gamma$ is the point[290] Λ; because the points Θ and Λ are similarly located in each of the triangles.[291,292] For the same reasons, in triangle $EB\Delta$, the center of gravity is point K, so that the center of gravity of the sum of the two triangles $EB\Delta$ and $Z\Delta\Gamma$ is located in the middle[293] of the line segment $K\Lambda$. But the midpoint of the segment $K\Lambda$ is the point N, since[294,295] BK is to ΘK as BE is to EA, ΓA is to $\Lambda\Theta$ as ΓZ is to ZA, and, under these conditions,

[288] [Note by Mugler:] Because $\frac{A\Gamma}{Z\Gamma} = \frac{B\Gamma}{\Delta\Gamma} = 2$; compare Euclid VI, 2.

[289] See footnote 281.

[290] [Note by Mugler:] Λ is the homologous point of Θ according to Proposition 11.

[291] [Note by Mugler:] Compare Euclid I, 29.

[292] Proposition 29 of Book I of Euclid's Elements, [Euc56, Vol. 1, p. 311] and [Euc09, p. 120]:

> A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

[293] [Note by Mugler:] Compare proposition 4.

[294] [Note by Mugler:] Compare Euclid VI, 2.

[295] See footnote 281.

$B\Gamma$ is parallel[296,297] to $K\Lambda$. Furthermore, we have drawn the line $\Delta\Theta$; the segment KN is therefore to the segment $N\Lambda$ as $B\Delta$ is to $\Delta\Gamma$. It follows that the center of gravity of the area that is the sum of the two triangles shown is the point N. On the other hand, in the parallelogram $AE\Delta Z$, the center of gravity[298] is the point M, so that the center of gravity of the sum of all the magnitudes is located on the line MN. But the center of gravity of the triangle $AB\Gamma$ is also (sc. by hypothesis) the point Θ. Consequently, the extended line MN will pass through the point Θ, which is impossible.[299] The center of gravity of the triangle $AB\Gamma$ cannot therefore not be located on the line $A\Delta$. It is therefore located on this line.

14.

In any triangle the center of gravity is the point of intersection[300,301] *of*

[296][Note by Mugler:] Since $\frac{BE}{EA} = \frac{\Gamma Z}{AZ}$, we have $\frac{\Gamma\Lambda}{\Lambda\Theta} = \frac{BK}{K\Theta}$; compare Euclid VI, 2.

[297]See footnote 281.

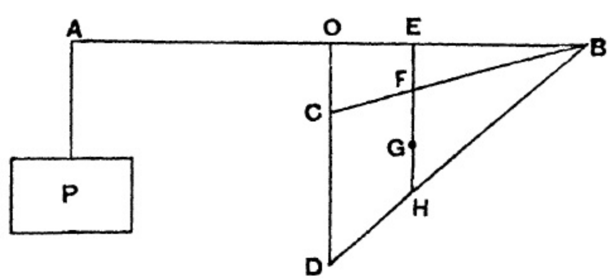
[298][Note by Mugler:] Compare proposition 10.

[299][Note by Mugler:] The lines MN, $Z\Lambda$ and $A\Theta$ are indeed parallel, since $EM = MZ$ and $KN = N\Lambda$.

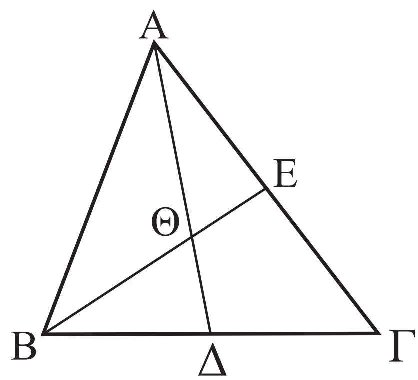
[300][Note by Mugler:] Compare *Quadrature of the Parabola*, proposition 6.

[301]Combined propositions 6 and 7 of Archimedes' treatise *Quadrature of the Parabola*, [Arc02b, p. 238]:

> Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD in which the angle C is right or obtuse be suspended from B and O, so that C is attached to O and CD is in the same vertical line with O. Then, if P be such an area as, when suspended from A, will keep the system in equilibrium,
>
> $$P = \frac{1}{3}\Delta BCD\ .$$



the lines joining the vertices of the triangle to the midpoints of the sides.



Consider the triangle $AB\Gamma$; draw the line $A\Delta$ joining A to the midpoint Δ of $B\Gamma$, and the line BE joining B to the midpoint E of $A\Gamma$. The center of gravity of the triangle $AB\Gamma$ will thus be located on each of the two lines $A\Delta$ and BE, as this has been demonstrated.[302] The point Θ is therefore the center of gravity.

15.

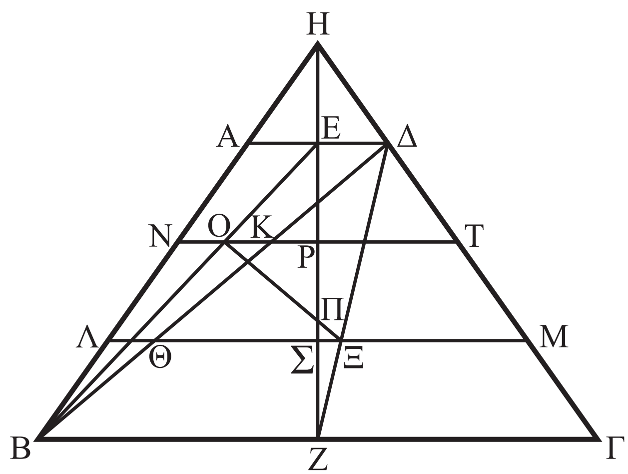
In any trapezium with two parallel sides, the center of gravity is located on the line segment joining the midpoints of the parallel sides at a point that divides this segment so that the partial segment ending at the midpoint of the smaller of the parallel sides is to the remaining segment as the sum of twice the larger parallel side and the smaller parallel side is to the sum of twice the smaller parallel side and the larger parallel side.[303,304]

[302] [Note by Mugler:] Compare proposition 13.

[303] [Note by Mugler:] Quoted proposition in *Quadrature of the Parabola* 10.

[304] Combined Propositions 10 and 11 of Archimedes' treatise *Quadrature of the Parabola*, [Arc02b, pp. 239-240]:

> Suppose a lever AOB placed horizontally and supported at O, its middle point. Let $CDEF$ be a trapezium which can be so placed that its parallel sides CD, FE are vertical, while C is vertically below O, and the other sides CF, DE meet in B. Let EF meet BO in H, and let the trapezium be suspended by attaching F to H and C to O. Further, suppose Q to be an area such that
>
> $$AO : OH = (trapezium\ CDEF) : Q\ .$$



Let $AB\Gamma\Delta$ be the trapezium with sides $A\Delta$ and $B\Gamma$ parallel, and the line segment EZ joining the midpoints of sides $A\Delta$ and $B\Gamma$. It is first obvious that the center of gravity of the trapezium is located on EZ. By extending the segments $\Gamma\Delta H$, ZEH and BAH, it is clear that they converge at the same point, that the center of gravity of triangle $HB\Gamma$ will be located on the line segment HZ, and that the center of gravity of triangle $AH\Delta$ will similarly be located[305] on the segment EH. It follows that the center of gravity of the remaining trapezium $AB\Gamma\Delta$ will be located[306] on EZ. Let us

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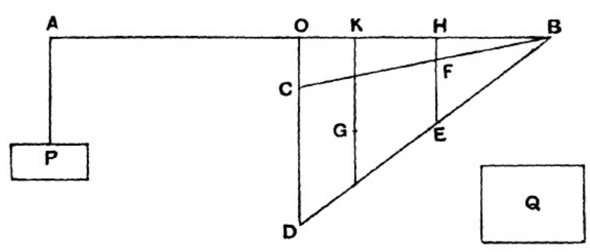
Then, if P be the area which, when suspended from A, keeps the system in equilibrium,

$$P < Q .$$

The same is true in the particular case where the angles at C, F are right, and consequently C, F coincide with O, H respectively.



[305] [Note by Mugler:] Compare proposition 13.
[306] [Note by Mugler:] Compare proposition 8.

join points B and Δ and divide $B\Delta$ into three equal parts by points K and Θ; let us draw parallels $\Lambda\Theta M$ and NKT to $B\Gamma$ through points K and Θ, and let us draw lines ΔZ, BE, and $O\Xi$; the center of gravity of triangle $\Delta B\Gamma$ will thus be located on ΘM, since ΘB is the third part[307] of $B\Delta$ and $M\Theta$ has been drawn through Θ parallel to the base. But the center of gravity of the triangle $\Delta B\Gamma$ is also located on ΔZ, so that the point Ξ is the center of gravity of the indicated triangle. Now for the same reasons the point O is the center of gravity of the triangle $AB\Delta$. It follows that for the magnitude which is the sum of the triangles $AB\Delta$ and $B\Delta\Gamma$, namely for the trapezium, the center of gravity is located on $O\Xi$. But the center of gravity of the indicated trapezium is also located on EZ, so that in the trapezium $AB\Gamma\Delta$ the center of gravity is the point Π. On the other hand, the ratio of triangle $B\Delta\Gamma$ to triangle $AB\Delta$ will be equal[308] to the ratio of $O\Pi$ to $\Pi\Xi$. But the ratio of triangle $B\Delta\Gamma$ to triangle $AB\Delta$ is equal[309,310] to the ratio of $B\Gamma$ to $A\Delta$, and the ratio of $O\Pi$ to $\Pi\Xi$ is equal[311,312] to the ratio of $P\Pi$ to $\Pi\Sigma$, so that $B\Gamma$ is to $A\Delta$ as $P\Pi$ is to $\Pi\Sigma$; therefore the ratio of the sum of twice $B\Gamma$ and $A\Delta$ to the sum of twice $A\Delta$ and $B\Gamma$ is equal to the ratio of the sum of twice $P\Pi$ and $\Pi\Sigma$ to the sum of twice $\Pi\Sigma$ and ΠP. But twice $P\Pi$ plus $\Pi\Sigma$ is equal to the sum of ΣP and $P\Sigma$, which is itself equal[313] to ΠE, and twice $\Pi\Sigma$ plus ΠP is equal to the sum of $P\Sigma$ and $\Sigma\Pi$, which is itself equal to ΠZ.[314] The proposition is therefore proven.

[307] [Note by Mugler:] Compare proposition 14.

[308] [Note by Mugler:] Compare propositions 6 and 7.

[309] [Note by Mugler:] Compare Euclid VI, 1.

[310] Proposition 1 of Book VI of Euclid's Elements, [Euc56, Vol. 2, p. 191] and [Euc09, p. 231]:

> Triangles and parallelograms which are under the same height are to one another as their bases.

[311] [Note by Mugler:] Because of the similarity of the triangles $OP\Pi$ and $\Sigma\Pi\Xi$; compare Euclid VI, 4.

[312] See footnote 263.

[313] [Note by Mugler:] We have in fact: $EP = P\Sigma = \Sigma Z$, since $AN = N\Lambda = BA$, and the lines NT, ΛM and $B\Gamma$ are parallel.

[314] [Note by Mugler:] The demonstration must be completed with the conclusion: $\frac{2B\Gamma + A\Delta}{2A\Delta + B\Gamma}$ is therefore equal to $\frac{\Pi E}{\Pi Z}$.

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Archimedes, the Center of Gravity and the Law of the Lever deals with the most fundamental aspects of physics. The book describes the main events in the life of Archimedes and the content of his works. It goes on to discuss a large number of experiments relating to the equilibrium of suspended bodies under the influence of Earth's gravitational force. All experiments are clearly described and performed with simple, inexpensive materials. These experiments lead to a clear conceptual definition of the center of gravity of material bodies and illustrate practical procedures for locating it precisely.

The conditions of stable, neutral, and unstable equilibrium are analyzed. Many equilibrium toys and games are described and explained. Historical aspects of the concept are presented, together with the theoretical values of center of gravity obtained by Archimedes.

The book also explains how to build and calibrate precise balances and levers. Several experiments are performed leading to a mathematical definition of the center of gravity. These experiments are compatible with the law of the lever, the oldest law of mechanics. Consequences of this law and different explanations of it are described at the end of the book. There is also an exhaustive analysis of the works of Euclid and Archimedes on equilibrium. A commented translation of Archimedes' work "*On the Equilibrium of Plane Figures*" is included at the end of this book.

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