Equivalence between the formulas for inductance calculation

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Abstract: We demonstrate the equivalence for the self-inductance of closed circuits, with the formulas of Neumann, Weber, Maxwell, and Graneau.

Résumé : Nous démontrons que les formules de Neuman, Weber, Maxwell et Graneau donnent la même self-inductance pour des circuits fermés.
[Traduit par la rédaction]

1. Introduction

The concept of inductance arises naturally when studying the interaction energy between current-carrying circuits. This interaction energy has a factor that depends only on the geometry of the circuits. When we analyze the self-energy of a single circuit, this factor is called **self-inductance**; when we analyze the interaction energy of two distinct circuits, it is called **mutual inductance**.

With the theoretical development of electrodynamics three main formulas appeared to calculate inductance: the expressions of Neumann, Weber, and Maxwell. Recently, a new formula can be deduced from Graneau's work.

Our goal is to demonstrate the equivalence between these formulas for the self-inductance of a closed circuit. This equivalence is a known fact for the mutual inductance of two separate closed circuits [1], but there is no demonstration for a single closed circuit.

The demonstration we shall present here is a generalization of the equivalence we have recently shown in some specific configurations [2].

2. Inductance formulas

2.1. Neumann's formula

To explain Faraday's law of inductance with Ampère's force [3], Neumann introduced the concepts of vector potential and mutual inductance. Consider two closed circuits Γ_1 and Γ_2 carrying currents I_1 and I_2, respectively, Fig. 1. A current element of the circuit Γ_1 is $I_1 \mathrm{d}\boldsymbol{r}_i$, and a current element of the circuit Γ_2 is $I_2 \mathrm{d}\boldsymbol{r}_j$. They are located, respectively, at $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$.

The magnetic interaction energy U_{12}^{N}, between the circuits Γ_1 and Γ_2, derived by Neumann, is given by

$$U_{12}^{\mathrm{N}} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j}{r_{ij}} \tag{1}$$

where $\mu_0 \equiv 4\pi \times 10^{-7}$ kg m C^{-2} and $r_{ij} \equiv |\boldsymbol{r}_i - \boldsymbol{r}_j|$.

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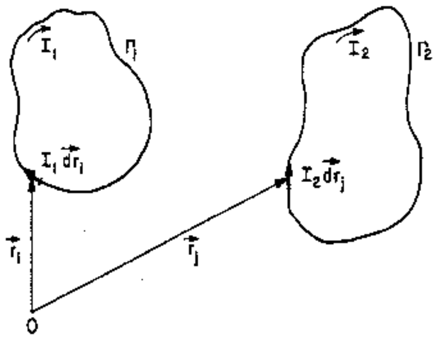


Fig. 1. Two closed circuits Γ_1 and Γ_2 with currents I_1 and I_2. The current element $I_1 \mathrm{d}\boldsymbol{r}_i$ is located at $\boldsymbol{r}_i$ while $I_2 \mathrm{d}\boldsymbol{r}_j$ is located at $\boldsymbol{r}_j$.

We can write the energy U_{12}^{N} as $I_1 I_2 M_{12}^{\mathrm{N}}$, where M_{12}^{N} is the geometric coefficient called the mutual inductance. Therefore, it follows from (1) that

$$M_{12}^{\mathrm{N}} \equiv \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j}{r_{ij}} \tag{2}$$

2.2. Weber's formula

Through Weber's force we can derive an interaction energy U_{12}^{W}, for the circuits Γ_1 and Γ_2, following the same reasoning shown in Sect. 2.1 [4, 5]. If we write U_{12}^{W} as $I_1 I_2 M_{12}^{\mathrm{W}}$, we obtain the coefficient of mutual inductance M_{12}^{W}, in Weber's electrodynamics:

$$M_{12}^{\mathrm{W}} \equiv \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i)(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_j)}{r_{ij}} \tag{3}$$

2.3. Maxwell's formula

In classical electrodynamics we utilize Darwin's energy [6] to obtain the interaction energy $U_{12}^{\mathrm{M}} \equiv I_1 I_2 M_{12}^{\mathrm{M}}$. The formula for the coefficient of mutual inductance M_{12}^{M} is [7, 8]

$$M_{12}^{\mathrm{M}} \equiv \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \left[\frac{(\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j) + (\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i)(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_j)}{2 r_{ij}} \right] \tag{4}$$

2.4. Graneau's formula

In his book, ref. 9, p. 212, Graneau defined an electrodynamic energy $\mathrm{d}^2 V_{ij}^{\mathrm{G}}$ between current elements. Integrating for closed circuits we obtain the interaction energy $U_{12}^{\mathrm{G}} \equiv I_1 I_2 M_{12}^{\mathrm{G}}$. This results in

$$M_{12}^{\mathrm{G}} \equiv \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \left[\frac{3(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i)(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_j) - 2(\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j)}{r_{ij}} \right] \tag{5}$$

Analogous to Helmholtz's procedure [3, 7, 8, 10], these expressions for the mutual energy between two closed circuits can be written as $U_{12} = I_1 I_2 M_{12}$, with

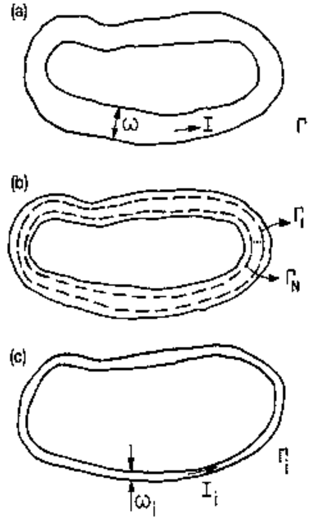


Fig. 2. (*a*) Circuit Γ of thickness ω, length ℓ and carrying an uniform and constant current I. (*b*) N circuits Γ_i dividing Γ. (*c*) Circuit Γ_i of thickness ω_i and carrying an uniform current $I_i \equiv I\omega_i/\omega$.

$$\begin{aligned} M_{12} &\equiv \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \left[\left(\frac{1+k}{2}\right) \frac{(\mathrm{d}\boldsymbol{r}_i \cdot \mathrm{d}\boldsymbol{r}_j)}{r_{ij}} + \left(\frac{1-k}{2}\right) \frac{(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_i)(\hat{r}_{ij} \cdot \mathrm{d}\boldsymbol{r}_j)}{r_{ij}} \right] \\ &= \left(\frac{1+k}{2}\right) M_{12}^{\mathrm{N}} + \left(\frac{1-k}{2}\right) M_{12}^{\mathrm{W}} \end{aligned} \tag{6}$$

where $M_{12} = M_{12}^{\mathrm{N}}$ for $k = 1$, $M_{12} = M_{12}^{\mathrm{W}}$ for $k = -1$, and $M_{12} = M_{12}^{\mathrm{M}}$ for $k = 0$, $M_{12} = M_{12}^{\mathrm{G}}$ for $k = -5$.

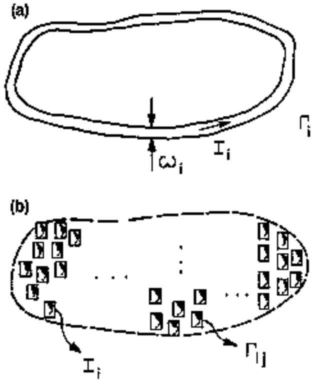
Changing the variables of integration in (6) we obtain $M_{21} = M_{12}$, for all formulas.

3. Generic proof of equivalence

As we want to prove the equivalence of the four formulas presented above when calculating the self-inductance of a single closed circuit, we cannot use the model of a linear current element. Expression (6) is not well defined when Γ_1 coincides with Γ_2. To overcome this difficulty we have to change the linear-current element to a surface- or volume-current element.

We now demonstrate the equivalence between the formulas of self-inductance given by Neumann, Weber, Maxwell, and Graneau. First, consider the circuit Γ described in Fig. 2*a*. We suppose this circuit to be composed of surface-current elements. The thickness of the circuit is ω. We divide Γ into N circuits Γ_i with thicknesses ω_i and carrying currents I_i, in such a way that $\omega = \sum_{i=1}^{N} \omega_i$, $I_i = I\omega_i/\omega$ (Figs. 2*b* and 2*c*). We choose a large N to make $\omega_i \ll \omega$ and $\omega_i \ll \ell$ (ℓ is the length of Γ).

Fig. 3. (*a*) Circuit Γ_i with thickness ω_i and carrying current I_i. *b* Circuit Γ_i replaced by M rectangular circuits Γ_{ij} ($j = 1, ..., M$), each carrying a current I_i in the same direction as Γ_i.



The self-inductance L_Γ of the circuit Γ, in Figs. 2*a* and 2*b*, can be written as[3]

$$\begin{aligned} L_\Gamma &= \int\!\!\int_{S_\Gamma}\int\!\!\int_{S_\Gamma} \mathrm{d}^4 M_{ij} = \left(\sum_{n=1}^{N}\int\!\!\int_{S_{\Gamma_n}}\right)\left(\sum_{n=1}^{N}\int\!\!\int_{S_{\Gamma_n}}\right)\mathrm{d}^4 M_{ij} \\ &= \sum_{n=1}^{N} L_{\Gamma_n} + \sum_{\substack{m,n=1 \\ m\neq n}}^{N} M_{\Gamma_m\Gamma_n} \end{aligned} \tag{7}$$

where S_Γ is the surface of the circuit Γ. To arrive at (7) we defined $L_{\Gamma_n} \equiv \int\int_{S_{\Gamma_n}}\int\int_{S_{\Gamma_n}} \mathrm{d}^4M_{ij}$ and $M_{\Gamma_m\Gamma_n} \equiv \int\int_{S_{\Gamma_m}}\int\int_{S_{\Gamma_n}} \mathrm{d}^4M_{ij}$.

Now, we approximate the circuit Γ_i, in Fig. 2*c*, by M rectangular closed circuits Γ_{ij} with currents I_i, in the same direction as in Γ_i (Figs. 3*a* and 3*b*). This approximation can be improved to any desired degree by decreasing the rectangle's areas and increasing their number M accordingly . We can write

$$L_{\Gamma_i} = \sum_{j=1}^{M} L_{\Gamma_{ij}} + \sum_{\substack{j,k=1 \\ j\neq k}}^{M} M_{\Gamma_{ij}\Gamma_{ik}} \tag{8}$$

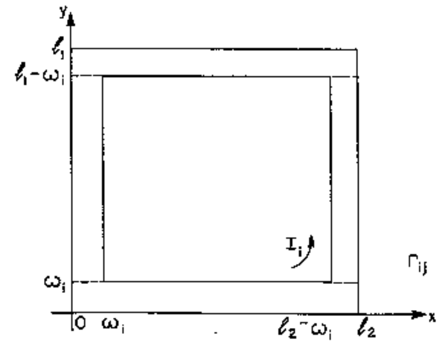
The self-inductance $L_{\Gamma_{ij}}$ of the rectangle Γ_{ij} can be calculated with the geometry of Fig. 4.

As we have surface-current elements in the rectangle of Fig. 4, we have to make use of the equivalence $I\mathrm{d}\boldsymbol{r} \leftrightarrow \boldsymbol{K}\mathrm{d}a$ in expressions (3) to (6), where $\boldsymbol{K}$ is the surface-current density ($|\boldsymbol{K}| = I/\omega_i$) and $\mathrm{d}a$ is the area element. Calculating the integrals [2], supposing $\omega_i \ll \ell_1$ and $\omega_i \ll \ell_2$, yields (neglecting terms of the orders $(\omega_i/\ell_1)^3$, $(\omega_i/\ell_2)^3$, and above)

$$\begin{aligned} L^{\mathrm{N}}_{\Gamma_{ij}} = L^{\mathrm{W}}_{\Gamma_{ij}} = L^{\mathrm{M}}_{\Gamma_{ij}} = L^{\mathrm{G}}_{\Gamma_{ij}} \approx \frac{\mu_0}{2\pi}\bigg[& 2\ell_2 \ln\left(\frac{2\ell_2}{\omega_i}\right) + 2\ell_1 \ln\left(\frac{2\ell_1}{\omega_i}\right) - 2\ell_2 \sinh^{-1}\left(\frac{\ell_2}{\ell_1}\right) \\ & -2\ell_1 \sinh^{-1}\left(\frac{\ell_1}{\ell_2}\right) + 4(\ell_1^2+\ell_2^2)^{1/2} - \ell_1 - \ell_2 \bigg] \end{aligned} \tag{9}$$

[3] With this definition, the self-energy of the circuit Γ is $I^2L_\Gamma/2$.

Fig. 4. Rectangular closed circuit Γ_{ij} with sides ℓ_1 and ℓ_2, thickness ω_i, and current I.



This is a very important result. It shows that for the closed circuit of Fig. 4, we have the same coefficient of self-inductance according to the expressions of Neumann, Weber, Maxwell, and Graneau.

The circuits Γ_{ij} and Γ_{ik} of Fig. 3*b* are two distinct closed circuits, with $j \neq k$. Therefore, $M^{\mathrm{N}}_{\Gamma_{ij}\Gamma_{ik}} = M^{\mathrm{W}}_{\Gamma_{ij}\Gamma_{ik}} = M^{\mathrm{M}}_{\Gamma_{ij}\Gamma_{ik}} = M^{\mathrm{G}}_{\Gamma_{ij}\Gamma_{ik}}$ [1]. This fact, and the equivalence in (9) substituted in (8), shows that for the circuit Γ_i of Fig. 2*c*,

$$L^{\mathrm{N}}_{\Gamma_i} = L^{\mathrm{W}}_{\Gamma_i} = L^{\mathrm{M}}_{\Gamma_i} = L^{\mathrm{G}}_{\Gamma_i} \tag{10}$$

As $M^{\mathrm{N}}_{\Gamma_i\Gamma_j} = M^{\mathrm{W}}_{\Gamma_i\Gamma_j} = M^{\mathrm{M}}_{\Gamma_i\Gamma_j} = M^{\mathrm{G}}_{\Gamma_i\Gamma_j}$ (two distinct closed circuits of Fig. 2*b*) we finally obtain from (7), with the equivalence in (10),

$$L^{\mathrm{N}}_{\Gamma} = L^{\mathrm{W}}_{\Gamma} = L^{\mathrm{M}}_{\Gamma} = L^{\mathrm{G}}_{\Gamma} \tag{11}$$

This is the proof of equivalence between the expressions of Neumann, Weber, Maxwell, and Graneau obtained utilizing a generic circuit Γ with surface-current elements (Fig. 2*a*). Instead, we could have utilized a circuit Γ with volume-current elements. The demonstration of the equivalence in this case follows the same reasoning presented above. It is only necessary to calculate the self-inductance of the rectangular closed circuit with volume-current elements, analogous to the one presented in Fig. 4 with surface-current elements. With the same dimensions as in Fig. 4, but now with the square cross section of sides ω_i, and utilizing the equivalence $I\mathrm{d}\boldsymbol{r} \leftrightarrow \boldsymbol{J}\mathrm{d}V$ ($\boldsymbol{J}$ is the volume-current density, $|\boldsymbol{J}| = I/\omega_i^2$, and $\mathrm{d}V$ is the volume element) in (3) to (6), we obtain [2] (supposing $\omega_i \ll \ell_1$, $\omega_i \ll \ell_2$ and neglecting terms of the orders $(\omega_i/\ell_1)^3$, $(\omega_i/\ell_2)^3$ and above)

$$\begin{aligned} L^{\mathrm{N}}_{\Gamma_{ij}} = L^{\mathrm{W}}_{\Gamma_{ij}} = L^{\mathrm{M}}_{\Gamma_{ij}} = L^{\mathrm{N}}_{\Gamma_{ij}} \approx \frac{\mu_0}{2\pi}\bigg[& 2\ell_2 \ln\left(\frac{2\ell_2}{\omega_i}\right) + 2\ell_1 \ln\left(\frac{2\ell_1}{\omega_i}\right) - 2\ell_2 \sinh^{-1}\left(\frac{\ell_2}{\ell_1}\right) \\ & -2\ell_1 \sinh^{-1}\left(\frac{\ell_1}{\ell_2}\right) + 4(\ell_1^2+\ell_2^2)^{1/2} + (\ell_1+\ell_2)\left(\frac{1}{6} - \frac{2}{3}\ln 2 - \frac{2\pi}{3}\right) \bigg] \end{aligned} \tag{12}$$

As in expression (9), the same coefficient of self-induction is obtained according to the expressions of Neumann, Weber, Maxwell, and Graneau.

4. Conclusions

The fact that the mutual inductance of two separate closed circuits is the same according to the formulas of Neumann, (2), Weber, (3), Maxwell, (4), and Graneau, (5), has been known for a long time [1]. We conclude in this work that this result remains valid even when we are calculating the self-inductance of a single closed circuit.

Therefore, there is no distinction between these formulas when dealing with closed circuits.

In some situations it is easier to calculate the force between closed circuits (or between a closed circuit and a part of itself) by deriving it from the inductance (see ref. 9, p. 204, and refs. 11 and 12) not calculating it directly by means of Ampère's force or Grassmann's force. Thus, the fact that the self-inductance with Maxwell's formula and Weber's formula is the same implies that classical electrodynamics (Grassmann's force) and Weber's electrodynamics (Ampère's force) agree as regards the resultant force in closed circuits [13–16].

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