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Calculation of the centre of gravity of the cone utilizing the method of Archimedes

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Abstract
Archimedes calculated the centre of gravity of the cone but the proof of this theorem is not extant in his works. Knorr made a reconstruction of this proof utilizing geometrical arguments. This paper proves this theorem by means of a physical demonstration utilizing the law of the lever, and by adapting from Archimedes the method of mechanical theorems that he described in his letter to Eratosthenes.

1. Introduction

Archimedes (287–212 BC) was born in Syracuse, where he spent most of his life. He is considered one of the greatest scientists of all time, the greatest mathematician of antiquity and is one of the founders of statics and hydrostatics. Although the works that have come down to us are related to mathematics and theoretical physics, his fame in antiquity was due to his work as an engineer and builder of war machines (catapults, burning mirrors, etc), connected with the second Punic war between Rome and Carthage, when he participated actively in the defense of Syracuse, allied with Carthage.

His extant works include studies of the centre of gravity of geometrical figures, the law of the lever, comparison of the area and volume of a sphere with those of the circumscribed cylinder and the properties of spiral lines, conoids and spheroids. The fundamental principle of hydrostatics is known today as Archimedes' principle in recognition of his ground-breaking proof.

The titles of his main surviving works are as follows: *On Spirals*, *On the Equilibrium of Planes*, *Quadrature of the Parabola*, *Measurement of a Circle*, *On the Sphere and Cylinder*, *On Conoids and Spheroids*, *On Floating Bodies* and *The Sand-Reckoner*.

In these works, Archimedes employed the theory of proportions and the application of areas as his main mathematical techniques. For the purpose of making Archimedes' work accessible to readers of this journal, his mathematics is transposed into algebraic notation in the discussion below.

Until 100 years ago, the oldest and most important manuscripts containing works of Archimedes in Greek were mainly of the 15th and 16th centuries, housed in libraries located in Europe. They had been copied from two other 9th and 10th century Greek manuscripts. One of these manuscripts belonged to the humanist George Valla, who taught at Venice between 1489 and 1499. This manuscript disappeared between 1544 and 1564, and it is not known if it still exists. The last record of the second manuscript was in the Vatican library in the years 1295 and 1311.

2. The method of Archimedes and the centre of gravity of the cone

In 1906, the Danish classical scholar L Heiberg discovered a medieval palimpsest containing many works of Archimedes in mid-10th century AD Greek, thus significantly predating the manuscripts referenced above. This collection includes the only source to survive into the modern era of *The Method of Mechanical Theorems*, dedicated to Eratosthenes. The *method* provides a core of our argument below. Heiberg produced transcriptions of the palimpsest, on which the standard translations are based. For most of the century following Heiberg's discovery, the whereabouts of this palimpsest were unknown. Now on loan at the Walters Museum in Baltimore, MD, it has undergone a decade of conservation and multispectral imaging, yielding new finds, as displayed in a recent exhibition there and other records[1].

There are 15 propositions in this work in which Archimedes presented a method showing how to calculate areas, volumes and centres of gravity of geometric figures. He premised the propositions with several lemmas. The eighth lemma reads as follows [1]:

> The center of gravity of any cone is (the point which divides its axis so that) the portion (adjacent to the vertex is) triple (of the portion adjacent to the base).

There is no demonstration of this lemma in his extant works. Knorr outlined a proof of this result following the geometrical reasoning of Archimedes [2]. A different approach is taken below, namely a physical demonstration of the centre of gravity of the cone utilizing the method of Archimedes.

3. The centre of gravity and the law of the lever

Although the centre of gravity is mentioned and utilized in many works of Archimedes, it is not defined in any extant memoir. Heath [1, 3], Duhem [4], Stein [5], Dijksterhuis [6], Assis [7] and many others have studied how Archimedes implicitly utilized this concept to calculate the centre of gravity of many figures. They also studied other authors such as Heron, Eutocius and Pappus, who had access to other works of Archimedes no longer extant. From these studies, the centre of gravity might be defined as follows:

> The center of gravity of any rigid body is a point such that, if the body be conceived to be suspended from that point, being released from rest and free to rotate in all directions around this point, the body so suspended will remain at rest and preserve its original position, no matter what the initial orientation of the body relative to the ground.

Archimedes demonstrated the law of the lever in propositions 6 and 7 of his work *On the Equilibrium of Planes* or *Centers of Gravity of Planes* [6]. These propositions are as follows.

Proposition 6. *Commensurable magnitudes are in equilibrium at distances reciprocally proportional to the weights.*

[1] The Archimedes Palimpsest http://thewalters.org/exhibitions/archimedes/

Proposition 7. *However, even if the magnitudes are incommensurable, they will be in equilibrium at distances reciprocally proportional to the magnitudes.*

The word 'commensurable' here refers to rational fractions. Proposition 5 of the 10th book of *The Elements* by Euclid (flourished circa 300 BC) states the following [8]: 'Commensurable magnitudes have to one another the ratio which a number has to a number'. According to Heath, Euclid was alive between the first pupils of Plato and Archimedes [9, p 2].

Heath combined these two propositions in his paraphrase of Archimedes' work [1]: 'Propositions 6, 7. Two magnitudes, whether commensurable (Prop. 6) or incommensurable (Prop. 7), balance at distances reciprocally proportional to the magnitudes'.

Suppose we have weights W_A and W_B on two sides of a lever supported by their centres of gravity located at distances d_A and d_B from the fulcrum F. According to the law of the lever, equilibrium will prevail if

$$\frac{W_A}{W_B} = \frac{d_B}{d_A}. \tag{1}$$

To demonstrate the law of the lever, Archimedes utilized the famous sixth postulate of his work *On the Equilibrium of Planes*, namely [6] 'If magnitudes at certain distances be in equilibrium, other (magnitudes) equal to them will also be in equilibrium at the same distances'.

Stein [5], Dijksterhuis [6] and Assis [7] understood Archimedes to interpret 'magnitudes equal to other magnitudes' as 'magnitudes of the same weight' and 'magnitudes at the same distances' as 'magnitudes the centers of gravity of which lie at the same distances from the fulcrum'. This interpretation conferred a reasonable meaning to this sixth postulate and legitimated his demonstration of the law of the lever [10].

4. Physical calculation of the centre of gravity of the cone

The proof of the centre of gravity of the cone outlined here is similar to the second proposition of the work Archimedes sent to Eratosthenes, *The Method of Mechanical Theorems*. In this second proposition, he proved that the volume of any sphere is four times that of the cone which has its base equal to the greatest circle of the sphere and its height equal to the radius of the sphere, and that the volume of the cylinder which has its base equal to the greatest circle of a sphere and its height equal to the diameter of the sphere is one-and-a-half times that of the sphere.

The same physical method is applied here to the calculation of the centre of gravity of the cone. Figure 1 employed here is a simplified version of the figure of the second proposition of the work Archimedes sent to Eratosthenes.

In figure 1, let $\alpha\beta\gamma\delta$ be the greatest circle of the sphere, and $\alpha\gamma$ and $\beta\delta$ be its two diameters at right angles to each other intersecting at η, the centre of the sphere. Consider the cone with vertex α, whose base is the greatest circle in the plane through $\beta\delta$ at right angles to $\alpha\gamma$. The extended surface of this cone intersects the plane through γ at right angles to $\alpha\gamma$ in a circle on $\zeta\varepsilon$ as the diameter. A variable plane $\lambda\kappa$ at right angles to $\alpha\gamma$ intersects it at the point π. It also intersects the cone and sphere in circles whose diameters are successively $o\rho$ and $\lambda\kappa$. The segment $\alpha\gamma$ is extended to the left passing from ν and θ up to μ; it is also extended to the right up to ι, such that $\mu\theta = \theta\nu = \nu\alpha = \alpha\eta = \eta\gamma = \gamma\iota$. That is, all these segments are equal to the radius of the sphere.

The method begins obtaining a simple mathematical relation from figure 1. By the similarity of the triangles $\alpha\lambda\pi$ and $\alpha\lambda\gamma$,

$$\frac{\alpha\gamma}{\alpha\lambda} = \frac{\alpha\lambda}{\alpha\pi}, \tag{2}$$

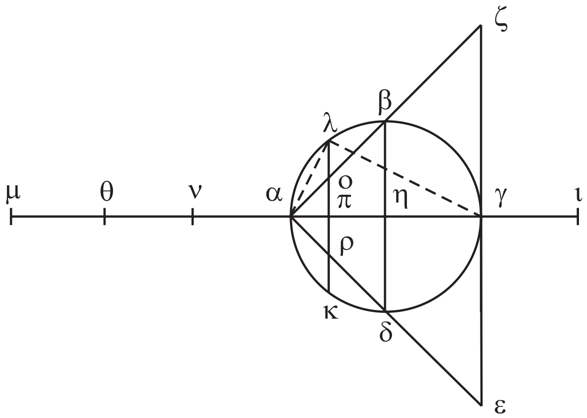

Figure 1. A sphere and a cone seen face on.

or

$$\alpha\gamma = \frac{\alpha\lambda \cdot \alpha\lambda}{\alpha\pi}. \tag{3}$$

This last equation is also equivalent to

$$\frac{\alpha\gamma}{\alpha\pi} = \frac{\alpha\lambda \cdot \alpha\lambda}{\alpha\pi \cdot \alpha\pi}. \tag{4}$$

A square with side $\alpha\lambda$ will be represented by $\alpha\lambda \cdot \alpha\lambda$. Analogously, squares with sides $\alpha\pi$ and $\lambda\pi$ will be represented by $\alpha\pi \cdot \alpha\pi$ and $\lambda\pi \cdot \lambda\pi$, respectively. By the theorem of Pythagoras applied to the triangle $\alpha\lambda\pi$ of figure 1,

$$\alpha\lambda \cdot \alpha\lambda = \alpha\pi \cdot \alpha\pi + \lambda\pi \cdot \lambda\pi. \tag{5}$$

Substituting equation (5) into equation (4) yields

$$\frac{\alpha\gamma}{\alpha\pi} = \frac{\alpha\pi \cdot \alpha\pi + \lambda\pi \cdot \lambda\pi}{\alpha\pi \cdot \alpha\pi}. \tag{6}$$

By the construction of figure 1, $\theta\alpha = \alpha\gamma$. The triangle $\alpha o\pi$ of figure 1 is isosceles. Therefore, $\alpha\pi = o\pi$. Utilizing these two relations at the left and right sides of equation (6), respectively, yields

$$\frac{\theta\alpha}{\alpha\pi} = \frac{o\pi \cdot o\pi + \lambda\pi \cdot \lambda\pi}{o\pi \cdot o\pi}. \tag{7}$$

The area of a circle is proportional to the square of its radius, or to the square of its diameter. Therefore, this equation can also be expressed as

$$\frac{\theta\alpha}{\alpha\pi} = \frac{\text{area of the circle with diameter } o\rho + \text{area of the circle with diameter } \lambda\kappa}{\text{area of the circle with diameter } o\rho}. \tag{8}$$

This is the basic mathematical relation necessary for the application of the physical method. Consider $\mu\iota$ as a lever with fulcrum α. Suppose geometric figures with weights uniformly distributed, that is, with weights proportional to the areas. A lever in equilibrium follows equation (1), which is similar to equation (8). This means that the circles $\lambda\kappa$ and $o\rho$, remaining where they are, with their centres acting at π, balance the circle $o\rho$ with its centre placed at θ. This situation of equilibrium is represented in figure 2(a). Figure 2(b) presents the same situation with circles suspended by weightless strings with their centres of gravity vertically below the points of suspension.

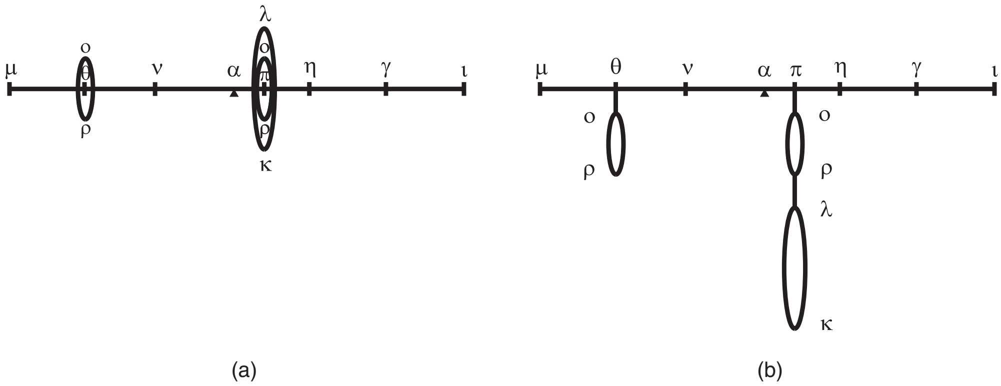

Figure 2. Lever in equilibrium. (a) Circles acting on the lever at their centres of gravity. (b) Circles suspended by weightless strings.

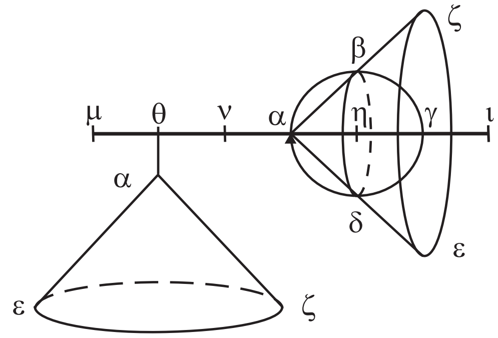

Figure 3. There is a cone suspended at θ, while the sphere and another cone are distributed over one arm of the lever.

This equilibrium is valid no matter the location of the variable plane $\lambda\kappa$ between α and γ. By considering all planes together, with the distance $\alpha\pi$ going from zero to $\alpha\gamma$, the circles $\lambda\kappa$ fill up the sphere $\alpha\beta\gamma\delta$, while the circles $o\rho$ fill up the cone $\alpha\zeta\varepsilon$. Consequently, by equation (8), there will also be equilibrium between the sphere $\alpha\beta\gamma\delta$ and cone $\alpha\zeta\varepsilon$ distributed over one arm of the lever, remaining where they are, together with another cone $\alpha\zeta\varepsilon$ acting only in θ. This is represented in figure 3 with the second cone $\alpha\zeta\varepsilon$ suspended by a weightless string at θ in such a way that its centre of gravity is vertically below θ.

By the sixth postulate of *On the Equilibrium of Planes*, quoted in section 3, equilibrium will remain when the sphere is suspended only by its centre of gravity, that is, by its centre η, figure 4.

In the second proposition of *The Method of Mechanical Theorems*, Archimedes proved that the volume of the sphere $\alpha\beta\gamma\delta$ of figure 1 is four times that of the cone $\alpha\beta\delta$:

$$\text{sphere } \alpha\beta\gamma\delta = 4(\text{cone } \alpha\beta\delta). \tag{9}$$

The volume of the cone $\alpha\zeta\varepsilon$ is eight times that of the cone $\alpha\beta\delta$ which has half its height, because the diameter $\zeta\varepsilon$ is twice the diameter $\beta\delta$:

$$\text{cone } \alpha\zeta\varepsilon = 8(\text{cone } \alpha\beta\delta). \tag{10}$$

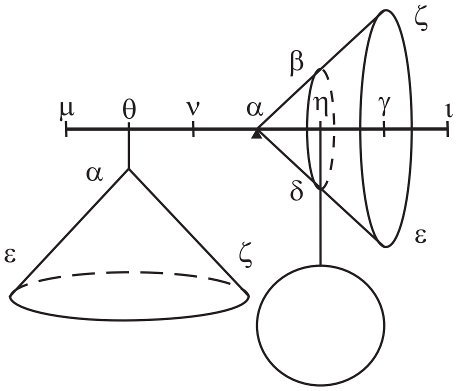

Figure 4. The sphere of figure 3 is now suspended at η by a weightless string.

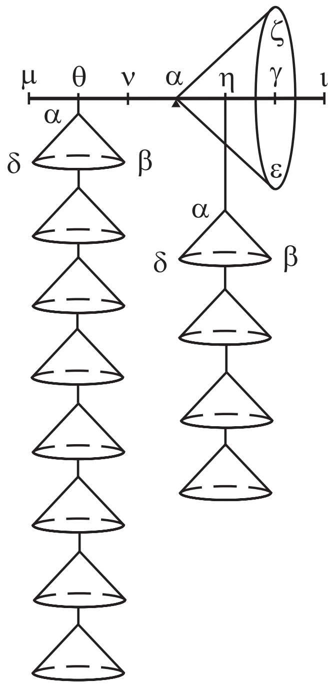

Figure 5. The sphere of figure 4 has been replaced by four cones $\alpha\beta\delta$, while the large cone acting at θ has been replaced by eight cones $\alpha\beta\delta$.

Accordingly, equilibrium will remain, replacing in figure 4 the sphere by four cones $\alpha\beta\delta$ and the large cone suspended at θ by eight cones $\alpha\beta\delta$, as in figure 5.

In the fourth proposition of *On the Equilibrium of Planes*, Archimedes proved that [1] 'If two equal weights have not the same center of gravity, the center of gravity of both taken together is at the middle point of the line joining their centers of gravity'. This means that we can replace in figure 5 the eight cones $\alpha\beta\delta$ acting at θ by four cones $\alpha\beta\delta$ acting at μ, together with four cones $\alpha\beta\delta$ acting at ν, without disturbing the equilibrium of the lever, figure 6.

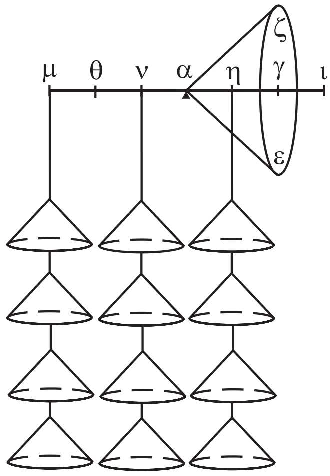

Figure 6. The eight cones acting at θ of figure 5 have been replaced by four cones acting at μ, together with other four cones acting at ν.

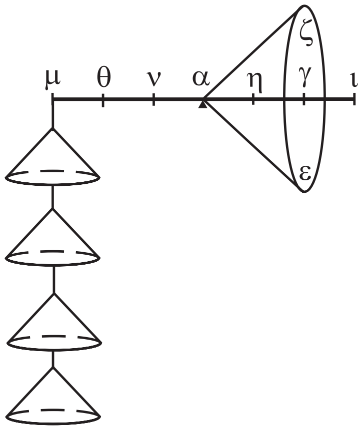

Figure 7. The equilibrium of the lever of figure 6 is not disturbed by simultaneously removing the four cones acting at ν and the four cones acting at η.

Equilibrium will remain by removing simultaneously the four cones $\alpha\beta\delta$ acting at ν and the four cones $\alpha\beta\delta$ acting at η, as $\nu\alpha = \alpha\eta$, figure 7.

By the sixth postulate of *On the Equilibrium of Planes*, the equilibrium of the lever of figure 7 will not be disturbed by replacing the cone $\alpha\zeta\varepsilon$ distributed over the arm of the lever by another cone $\alpha\zeta\varepsilon$ of the same weight, but acting only at its centre of gravity ξ. The goal is to find the distance $\alpha\xi$. This equilibrium is represented in figure 8, with the cone suspended at ξ by a weightless string.

According to the law of the lever, equation (1),

$$\frac{4\,(\text{cone } \alpha\beta\delta)}{\text{cone } \alpha\zeta\varepsilon} = \frac{\alpha\xi}{\mu\alpha}. \tag{11}$$

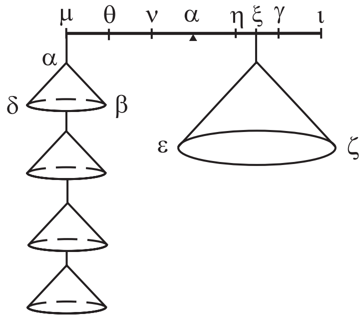

Figure 8. The equilibrium of the lever of figure 7 is not disturbed by replacing the cone distributed over one arm of the lever by the same cone acting only at its centre of gravity ξ.

By the construction of figure 1, we have

$$\frac{\mu\alpha}{\alpha\gamma} = \frac{3}{2}. \tag{12}$$

Combining equations (10)–(12) yields

$$\frac{1}{2} = \frac{\alpha\xi}{(3/2)\alpha\gamma}. \tag{13}$$

That is,

$$\alpha\xi = \tfrac{3}{4}\alpha\gamma. \tag{14}$$

This is the final result which Archimedes expressed in the following words [1]:

The center of gravity of any cone is (the point which divides its axis so that) the portion (adjacent to the vertex is) triple (of the portion adjacent to the base).

5. Insights into the essence of Archimedes method

From what has been seen in this work, it is possible to identify features in our proof that correlate with the ways Archimedes applied geometry in his extant works. These features provide us with insight regarding how he might haven proven the lemma that we discuss.

(1) By geometrical considerations, we can obtain a proportion stating the equality of two ratios. One ratio is that of two distances. The other ratio can be that of lengths belonging to certain figures, or that of areas belonging to certain figures, as in equation (8).
(2) Weight is assumed to be uniformly distributed in the geometric figures. In particular, the weight of each figure will be supposed proportional to its length, area or volume.
(3) These magnitudes are imagined to be suspended upon a lever in equilibrium, according to equation (1). The configuration of the lever in equilibrium is represented here by figure 2.
(4) Each plane figure is considered as being filled up by all straight segments contained in it parallel to a certain direction. Analogously, each solid figure is considered as being filled up by all planes contained in it orthogonal to a certain direction.
(5) This analysis produces a lever in equilibrium with one or more bodies suspended on one arm of the lever by their centres of gravity, while other bodies are distributed along the second arm of the lever. This configuration is represented here by figure 3.

(6) By the crucial sixth postulate of his work *On the Equilibrium of Planes*, quoted in section 3, Archimedes could then replace the bodies distributed along the second arm of the lever by bodies of the same weight suspended only by their centres of gravity. The sixth postulate guarantees that the lever will remain in equilibrium when these substitutions are made. These configurations of equilibrium are represented here by figures 4 and 8.

(7) The combination of the law of the lever, equation (1), yields then the area, volume or centre of gravity of a geometric figure when the area, volume or centre of gravity of another figure is known.

6. Pedagogical value of the analysis

The letter of Archimedes addressed to Eratosthenes, *The Method of Mechanical Theorems*, may be considered one of the earliest works on mathematical physics. It utilizes the physical law of the lever to calculate the area, volume or centre of gravity of bodies. This paper illustrates the law of the lever by using it to calculate the centre of gravity of the cone. While Archimedes correctly calculated the location of the centre of gravity of the cone, his demonstration is no longer extant. What we did was to apply the *method* of Archimedes as we understood it, to construct a proof in the spirit of the physical analysis that distinguishes his work, setting it apart from others which employ mathematics without a physical grounding for the argument. The analysis presented here, concentrating on the main aspects of the method, may help to illustrate the power of his reasoning. The figures included in this presentation aim to make the method more intuitive from the physical point of view.

In this demonstration, the physical lever is used to cancel and redistribute equal weight quantities in balance. By thinking about the lever construction in analogy with algebraic operations, students might further their awareness of how mathematical methods and demonstrations interpret physical behaviours. There is a great educational potential in this analogy which might be explored by undergraduate-level physics teachers. It is also possible to utilize physical demonstrations to accompany this analysis.

In fact, one of our undergraduate students took on the project of building balances and levers in equilibrium reproducing the steps leading to the proofs of theorems 1 of 2 of the Archimedes method (area of a parabola and volume of a sphere) [11]. He utilized metal rods of appropriate lengths, triangular and parabolic plane sheets made of hard rubber, together with spheres, cones and cylinders made of gypsum. All these heavy bodies had lengths, areas and volumes following the values presented by Archimedes. The student observed that when these bodies were suspended at the required distances from the fulcrum of the lever, the lever always remained in equilibrium. These physical demonstrations of levers in equilibrium could accompany the geometrical proofs of these theorems, lending support to Archimedes amazing reasoning.

The educational advantages of employing physical properties in the argument are evident in the figures utilized in this work, as the mathematics is kept to a minimum. The physical properties of the lever in equilibrium, together with the crucial sixth postulate of his work *On the Equilibrium of Planes*, are the key to solving a mathematical problem, as illustrated in this paper.

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