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RESISTIVE PLATES CARRYING A STEADY CURRENT: ELECTRIC POTENTIAL AND SURFACE CHARGES CLOSE TO THE BATTERY

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We treat the problem of two resistive plates carrying a steady current in the same direction. We consider a linear battery orthogonal to the direction of the current in the middle of the plates. We study the behavior of the surface charges close to the battery. We calculate the potential and electric field in the space outside the plates. We also consider the case of a single resistive plate carrying a steady current.

Key words: Electric potential, potential close to the battery, surface charges.

1. INTRODUCTION

The electric field outside resistive conductors carrying steady currents has been studied in many cases of interest, [1], [2, pp. 177–185], [3, 4, 5, 6, 7]. The surface charges that drive the steady current for these cases of straight conductors carrying steady longitudinal currents behave linearly with the longitudinal coordinate. These geometries have the problem of infinite length in the longitudinal coordinate, so that we cannot analyze the behavior of the potential, electric field and surface charges close to the battery.

To overcome this deficiency we consider in this work the location of the battery. Our goal is to analyse the electric potential, electric field and surface charges close to the battery in a simple geometry. In this respect our work is somewhat similar to the treatment made by Jefimenko, Heald, Griffiths and Jackson in other geometries, [8, Prob. 9.33 and Fig. 14.7], [9, p. 318], [10], [11, p. 279] and [12].

2. DESCRIPTION OF THE PROBLEM

We consider two parallel planes, separated by a distance $2a$, located at $y = a$ and $y = -a$, carrying a steady current in the same direction along the x axis, Fig. 1. The plates have dimensions $2\ell_x$ and $2\ell_z$ in the x and z axis, respectively, with $\ell_z \gg a$ and $\ell_x \gg a$. We suppose air or vacuum between the plates and also outside them. There are two identical linear batteries located at $x = 0$ on both plates, supplying an electromotive force of $2\phi_0$. The plates have potentials $\phi(x = -\ell_x) = \phi_L - \phi_0$ and $\phi(x = \ell_x) = \phi_R + \phi_0$ in the left and right extremities, respectively. The potential along the plates is described by, see Fig. 2:

$$\phi(x, y = \pm a, z) = \begin{cases} (\phi_R - \phi_L)x/(2\ell_x) + (\phi_R + \phi_L)/2 - \phi_0, & x < 0, \\ (\phi_R - \phi_L)x/(2\ell_x) + (\phi_R + \phi_L)/2 + \phi_0, & x > 0. \end{cases} \tag{1}$$

There is a discontinuity in $x = 0$ due to the presence of the battery.

Later on we consider the case of a single plate. This is analogous to our present case with the distance between the plates going to zero ($a \to 0$). We considered initially the double-plate case because it is more general than the single-plate and the mathematical difficulty is essentially the same in both cases.

This problem can clearly be separated in two parts: (a) the electrostatic problem of plates held at constant potentials ($-\phi_0$ for the region $x < 0$ and ϕ_0 for $x > 0$); and (b) the problem of a steady current without the discontinuity (with potentials $\phi(x = -\ell_x) = \phi_L$ and $\phi(x = \ell_x) = \phi_R$ in the extremities and $\phi(x = 0) = (\phi_L + \phi_R)/2$, see Fig. 3. That is, $\phi = (\phi_R - \phi_L)x/(2\ell_x) + (\phi_R + \phi_L)/2$). Both problems

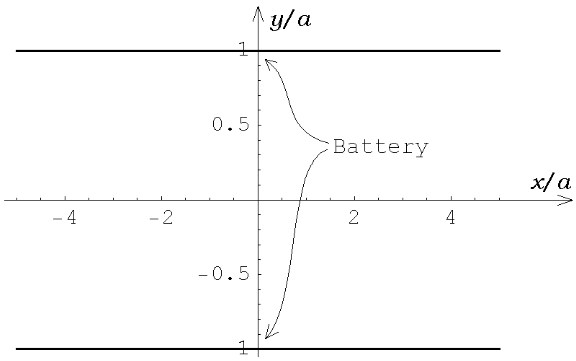

Fig. 1. Two parallel large resistive plates located in the planes $y = a$ and $y = -a$, carrying a steady current in the same direction along the x axis. There is a linear battery in each plate in $x = 0$, supplying an electromotive force $2\phi_0$.

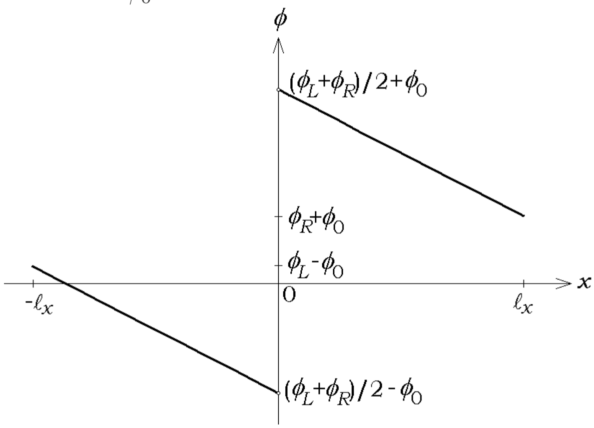

Fig. 2. Potential along the plates.

have already been solved separately in the literature. Part of problem (a) is treated in [13, pp. 309–313]. Problem (b) is treated by [14]. We discuss in more detail each solution and mainly the combination of both cases in the following sections.

3. ELECTROSTATIC SOLUTION OF PLATES HELD AT CONSTANT POTENTIALS

Suppose four semi-infinite parallel plates, located at $(x < 0, y = a)$, $(x > 0, y = a)$, $(x < 0, y = -a)$ and $(x > 0, y = -a)$, see Fig. 1. There is a thin insulating barrier at $(x, y) = (0, \pm a)$. The plates at

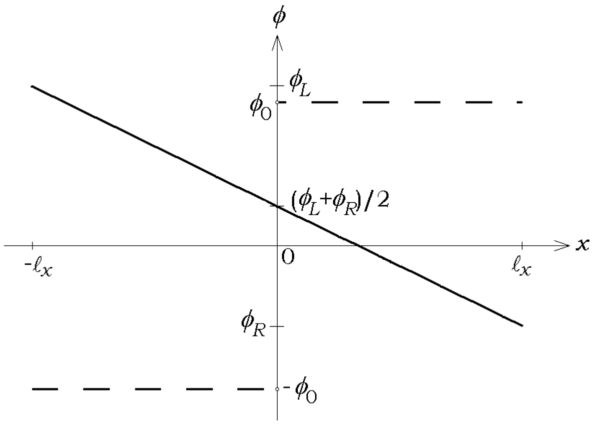

Fig. 3. The potential of Fig. 2 can be decomposed in two parts: the plates held at constant but discontinuous potentials (dashed lines), and the plates with a continuous potential varying linearly with the longitudinal coordinate (continuous line).

$x < 0$ are held at the constant potential $-\phi_0$, while the plates at $x > 0$ are held at ϕ_0, [13, pp. 309–313]. We can solve Laplace's equation, $\nabla^2\phi = 0$, in Cartesian coordinates using separation of variables in the form $\phi_k(x, y) = X_k(x)Y_k(y)$, where the functions X_k and Y_k obey the equations (k being an arbitrary constant):

$$\frac{d^2X_k}{dx^2} = -k^2X_k, \qquad \frac{d^2Y_k}{dy^2} = k^2Y_k. \tag{2}$$

The solutions of these equations are $X_k(x) = a_k\sin(kx)+b_k\cos(kx)$ and $Y_k(y) = c_k\mathrm{e}^{ky} + d_k\mathrm{e}^{-ky}$, with a_k, b_k, c_k and d_k being constants. The final solution $\phi(x, y)$ is a linear combination of all possible solutions $\phi_k(x, y)$.

As the boundary conditions are anti-symmetric around the x coordinate, $\phi(-x, \pm a) = -\phi(x, \pm a)$, the solution must also be anti-symmetric at all points: $\phi(-x, y) = -\phi(x, y)$ and $\phi(0, y) = 0$. The same reasoning applies to the y coordinate, but in this case the boundary conditions (and also the solution) are symmetric: $\phi(x, -y) = \phi(x, y)$. Additionally, we must have a limited solution in both x and y coordinates, $|\phi(|x| \to \infty, y)| \le \phi_0 < \infty$ and $|\phi(x, |y| \to \infty)| \to 0$. This means that our previous solutions are reduced to $X_k(x) = a_k\sin(kx)$ and $Y_k(y) = c_k(\mathrm{e}^{ky} + \mathrm{e}^{-ky}) \equiv e_k\cosh(ky)$ in the region between the plates, with $e_k = 2c_k$. For $y > a$ ($y < -a$) we must have $c_k = 0$ ($d_k = 0$). As $\phi(x, -y) = \phi(x, y)$ this implies that in the region outside the plates our solutions are reduced to $X_k(x) = a_k\sin(kx)$ and

$Y_k(y) = f_k e^{-k|y|}$, with f_k being constant. The particular solutions are then given by (with A_k and B_k being constants):

$$\phi_k(x, |y| \le a) = A_k \sin(kx)\cosh(ky), \tag{3}$$
$$\phi_k(x, |y| \ge a) = B_k \sin(kx) e^{-k|y|}. \tag{4}$$

The general solution which is a combination of all possible particular solutions has then the form

$$\phi(x, |y| \le a) = \int_0^\infty A_k \cosh(ky)\sin(kx)dk, \tag{5}$$
$$\phi(x, |y| \ge a) = \int_0^\infty B_k e^{-k|y|}\sin(kx)dk, \tag{6}$$

where the coefficients A_k and B_k must be determined by the boundary conditions. Eqs. (5) and (6) can be seen as sine Fourier transforms of a function Φ:

$$\Phi(k, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \phi(x, y)\sin(kx)dx = \begin{cases} \sqrt{\pi/2}A_k \cosh(ky), & |y| \le a, \\ \sqrt{\pi/2}B_k e^{-k|y|}, & |y| \ge a. \end{cases} \tag{7}$$

Calculating Eq. (7) in $y = a$ and applying the boundary conditions $\phi(x > 0, \pm a) = \phi_0$ and $\phi(x < 0, \pm a) = -\phi_0$ yields the coefficients A_k and B_k:

$$A_k = \frac{2\phi_0}{\pi k \cosh(ka)}, \qquad B_k = \frac{2\phi_0 e^{ka}}{\pi k}. \tag{8}$$

The final solution can be written as (see Appendix)

$$\phi(x, |y| \le a) = \frac{2\phi_0}{\pi}\int_0^\infty \frac{\cosh(ky)}{\cosh(ka)}\frac{\sin(kx)}{k}dk = \frac{2\phi_0}{\pi}\text{arccot}\left[\frac{\cos(\pi y/2a)}{\sinh(\pi x/2a)}\right], \tag{9}$$

$$\phi(x, |y| \ge a) = \frac{2\phi_0}{\pi}\int_0^\infty \frac{e^{-k(|y|-a)}\sin(kx)}{k}dk = \frac{2\phi_0}{\pi}\arctan\left[\frac{x}{|y| - a}\right]. \tag{10}$$

4. SOLUTION OF PLATES WITH STEADY CURRENT AND BATTERY

Equation (5) of [14] gives the solution of parallel plates separated by a distance $2a$, with dimensions $2\ell_z$ and $2\ell_x$, with $\ell_z \gg \ell_x \gg a$ and carrying a steady current:

$$\phi(x,y) = \frac{1}{\varepsilon_0}\left[(\alpha x+\beta)\left(\frac{2\ell_x}{\pi} - \frac{|y-a|+|y+a|}{2}\right) + \frac{2\beta\ell_x}{\pi}\ln\frac{2\ell_z}{\ell_x}\right], \tag{11}$$

where $\varepsilon_0 = 8.85\times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$ is the permittivity of free space and the constants α and β must be determined by the boundary conditions. In the present case,

$$\alpha = \frac{\varepsilon_0(\phi_R-\phi_L)}{2\ell_x(2\ell_x/\pi - a)}, \qquad \beta = \frac{\varepsilon_0(\phi_R+\phi_L)}{2(2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x) - a)}. \tag{12}$$

The solution of the problem with the battery is the sum of Eqs. (9) and (11), for the region between the plates, and Eqs. (10) and (11) outside the plates:

$$\phi(x,|y|\le a) = \frac{2\phi_0}{\pi}\operatorname{arccot}\left[\frac{\cos(\pi y/2a)}{\sinh(\pi x/2a)}\right] + \frac{\phi_R-\phi_L}{2\ell_x}x + \frac{\phi_R+\phi_L}{2}, \tag{13}$$

$$\phi(x,|y|\ge a) = \frac{2\phi_0}{\pi}\arctan\left(\frac{x}{|y|-a}\right) + \frac{2\ell_x/\pi - |y|}{2\ell_x/\pi - a}\frac{\phi_R-\phi_L}{2\ell_x}x + \frac{2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x) - |y|}{2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x) - a}\frac{\phi_R+\phi_L}{2}. \tag{14}$$

When $y = \pm a$ the potentials reduce to the boundary conditions given by Eq. (1) as expected.

The electric field $\mathbf{E}$ can be obtained from the potential utilizing the relation $\mathbf{E} = -\nabla\phi$. This yields

$$\mathbf{E}(x,|y|<a) = -\left[\frac{2\phi_0}{a}\frac{\cosh(\pi x/2a)\cos(\pi y/2a)}{\cosh(\pi x/a)+\cos(\pi y/a)} + \frac{\phi_R-\phi_L}{2\ell_x}\right]\hat{\mathbf{x}} - \frac{2\phi_0}{a}\frac{\sinh(\pi x/2a)\sin(\pi y/2a)}{\cosh(\pi x/a)+\cos(\pi y/a)}\hat{\mathbf{y}}, \tag{15}$$

$$\mathbf{E}(x,|y|>a) = -\left[\frac{2\phi_0}{\pi}\frac{|y|-a}{x^2+(|y|-a)^2} + \frac{2\ell_x/\pi - |y|}{2\ell_x/\pi - a}\frac{\phi_R-\phi_L}{2\ell_x}\right]\hat{\mathbf{x}} + \frac{y}{|y|}\left[\frac{2\phi_0}{\pi}\frac{x}{[x^2+(|y|-a)^2]} + \frac{\phi_R-\phi_L}{2\ell_x(2\ell_x/\pi - a)}x + \frac{\phi_R+\phi_L}{2(2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x) - a)}\right]\hat{\mathbf{y}}. \tag{16}$$

We can see that this electric field satisfies Maxwell's equations for steady currents because $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$ between the plates ($|y| < a$) and outside them ($|y| > a$), as expected.

The electric field lines can be obtained by a similar procedure to that given by Sommerfeld, [15, p. 128]. That is, we look for a function $\xi(x, y)$ such that $\nabla\xi \cdot \nabla\phi = 0$. This yields:

$$\xi(x, |y| < a) = \operatorname{arctanh}\left(\frac{\sin(\pi y/2a)}{\cosh(\pi x/2a)}\right) - \frac{y}{\ell_x}\frac{\pi}{2}\frac{\phi_L - \phi_R}{2\phi_0}, \tag{17}$$

$$\begin{aligned}\xi(x, |y| > a) = &\frac{1}{2}\ln\left(\frac{x^2 + (|y| - a)^2}{\ell_x^2}\right) - \frac{x^2 - y^2}{\ell_x^2}\frac{\pi}{4}\frac{\phi_L - \phi_R}{2\phi_0} \\ &- \frac{|y|}{\ell_x}\frac{\ell_x}{2\ell_x/\pi - a}\frac{\phi_L - \phi_R}{2\phi_0} \\ &+ \frac{x}{2\ell_x/\pi[1 + \ln(2\ell_z/\ell_x)] - a}\frac{\pi}{2}\frac{\phi_R + \phi_L}{2\phi_0}.\end{aligned} \tag{18}$$

Figure 4 shows the equipotentials and lines of electric field utilizing Eqs. (13), (14), (17) and (18) with $\phi_L = \phi_0$, $\phi_R = -\phi_0$ and $\ell_z/\ell_x = \ell_x/a = 10$.

The surface charge distribution $\sigma(x, y = \pm a)$ can be found by applying Gauss' law and choosing a gaussian volume surrounding a small piece of the conductor surface. In the limit of an infinitesimal surface we have the internal and external densities of surface charges of the upper plates as given by, respectively:

$$\sigma(x, a^-) = -\lim_{y\to a}\varepsilon_0 E_y(x, 0 < y < a) = \frac{\varepsilon_0\phi_0}{a\sinh(\pi x/2a)}, \tag{19}$$

$$\begin{aligned}\sigma(x, a^+) = &\lim_{y\to a}\varepsilon_0 E_y(x, y > a) = \frac{2\varepsilon_0\phi_0}{\pi x} + \frac{\varepsilon_0(\phi_R - \phi_L)}{2\ell_x(2\ell_x/\pi - a)}x \\ &+ \frac{\varepsilon_0(\phi_R + \phi_L)}{2[(2\ell_x/\pi)(1 + \ln(2\ell_z/\ell_x)) - a]}.\end{aligned} \tag{20}$$

And similarly for the lower plates:

$$\sigma(x, -a^-) = \lim_{y\to -a}\varepsilon_0 E_y(x, -a < y < 0) = \sigma(x, a^-), \tag{21}$$

$$\sigma(x, -a^+) = \lim_{y\to -a}\varepsilon_0 E_y(x, y < -a) = \sigma(x, a^+). \tag{22}$$

When $|x|/a \gg 1$ the terms proportional to ϕ_0 can be neglected and we recover the linear behaviour found before, [14]. For $|x|/a \ll 1$ the terms proportional to ϕ_0 will be the dominant ones and the surface charge densities will be reduced to

$$\sigma(x, \pm a^\pm) \approx \frac{2\varepsilon_0\phi_0}{\pi x}. \tag{23}$$

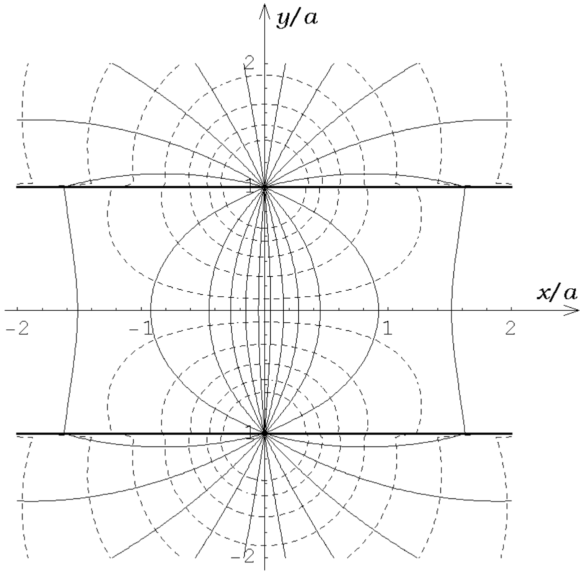

Fig. 4. Equipotentials (continuous lines) and electric field lines (dashed lines) of resistive parallel plates with a battery in $x = 0$. This is a plot of Eqs. (13), (14), (17) and (18) with $\phi_L = -\phi_R = \phi_0$, $\ell_z/\ell_x = \ell_x/a = 10$. There is a steady current flowing in the plates along the positive x direction.

Figure 5 shows the normalized surface charge densities given by Eqs. (19) to (22) as a function of x/a. Figure 6 shows Eqs. (19) and (20) normalized by (20).

We can see from these expressions that, for $|x| \ll \ell_x$, the electrostatic solution has the main contribution to the values of the potential, electric field and surface charges. It has been noted elsewhere [12] that the difference in the surface charges is small between the electrostatic situation and its equivalent with steady current. Even so, these physical situations are completely different.

When $a \to 0$ these solutions reduce to that of a single resistive plate in the $y = 0$ plane carrying a steady current along the positive x direction, namely:

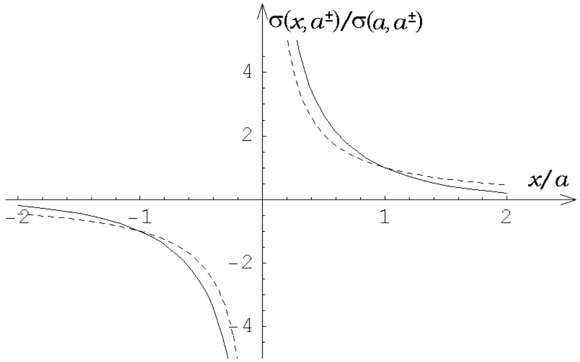

Fig. 5. Normalized surface charge distributions as a function of the x (longitudinal) coordinate for two parallel plates conducting a steady current. The internal surface charge distributions given by Eqs. (19) and (21) are represented by the continuous lines, while Eqs. (20) and (22) expressing the external surface charge distributions are represented by the dashed lines. The internal (external) surface charge density is normalized by the internal (external) surface charge density at $x = a$. We utilized $\phi_L = -\phi_R = \phi_0$ and $\ell_x/a = 10$.

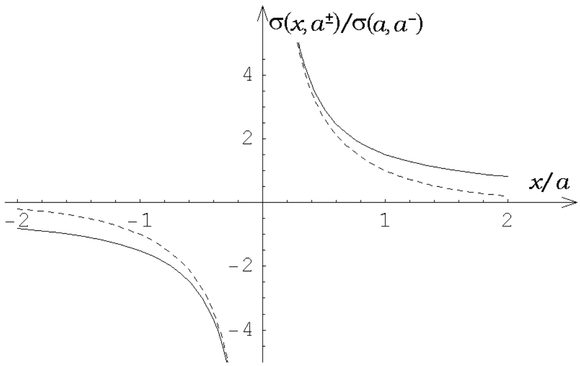

Fig. 6. External ($y \to a^+$, continuous lines) and internal ($y \to a^-$, dashed lines) surface charge distributions on parallel plates carrying a steady current in the x direction. There are line batteries at $x = 0$. Both distributions were normalized by the internal ($y \to a^-$) surface charge distribution at $x = a$. We utilized $\phi_L = -\phi_R = \phi_0$ and $\ell_x/a = 10$.

$$\phi(x,y) = \frac{2\phi_0}{\pi}\arctan\left(\frac{x}{|y|}\right) + \frac{2\ell_x/\pi - |y|}{2\ell_x/\pi}\frac{\phi_R - \phi_L}{2\ell_x}x$$
$$+ \frac{2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x) - |y|}{2\ell_x/\pi + (2\ell_x/\pi)\ln(2\ell_z/\ell_x)}\frac{\phi_R + \phi_L}{2}, \quad (24)$$
$$\mathbf{E}(x,y) = \left[\frac{2\phi_0|y|}{\pi(x^2+y^2)} - \frac{2\ell_x/\pi - |y|}{2\ell_x/\pi}\frac{\phi_R - \phi_L}{2\ell_x}\right]\hat{\mathbf{x}} + \frac{y}{|y|}\left[\frac{2\phi_0 x}{\pi(x^2+y^2)}\right.$$
$$\left. + \frac{\pi(\phi_R - \phi_L)}{4\ell_x^2}x + \frac{\pi(\phi_R + \phi_L)}{4\ell_x(1 + \ln(2\ell_z/\ell_x))}\right]\hat{\mathbf{y}}, \quad (25)$$
$$\sigma(x, 0^\pm) = \frac{2\varepsilon_0\phi_0}{\pi x} + \frac{\varepsilon_0\pi(\phi_R - \phi_L)}{4\ell_x^2}x + \frac{\varepsilon_0\pi(\phi_R + \phi_L)}{4\ell_x(1 + \ln(2\ell_z/\ell_x))}. \quad (26)$$

The electric field lines are given by

$$\xi(x,y) = \frac{1}{2}\ln\left(\frac{x^2+y^2}{\ell_x^2}\right) - \frac{x^2-y^2}{\ell_x^2}\frac{\pi^2}{8}\left(\frac{\phi_L - \phi_R}{2\phi_0}\right) - \frac{|y|}{\ell_x}\frac{\pi}{2}\left(\frac{\phi_L - \phi_R}{2\phi_0}\right)$$
$$+ \frac{x}{\ell_x}\frac{\pi^2}{4[1 + \ln(2\ell_z/\ell_x)]}\left(\frac{\phi_R + \phi_L}{2\phi_0}\right). \quad (27)$$

Figure 7 is a plot of Eqs. (24) and (27) with $\phi_L = -\phi_R = \phi_0$.

5. DISCUSSION

Jefimenko, [9, p. 509–511], and Heald, [10] studied the problem of an infinite resistive cylindrical shell of radius R, centered on the z axis and carrying an azimuthal steady current. They supposed the battery to be an infinite line, located at $\varphi = \pm\pi$ rad, in cylindrical coordinates. Supposing that the terminals of the battery were at potentials $\pm\phi_0$ they found the potential at all points in space to be given by (with $r = \sqrt{x^2+y^2}$):

$$\phi(x,y) = \frac{2\phi_0}{\pi}\arctan\left(\frac{y}{R+x}\right), \quad \text{for } r \le R, \quad (28)$$
$$\phi(x,y) = \frac{2\phi_0}{\pi}\arctan\left(\frac{Ry}{x^2+Rx+y^2}\right), \quad \text{for } r \ge R. \quad (29)$$

When we take $R \gg \sqrt{x^2+y^2}$ in Eqs. (28) and (29), we obtain the same result as Eqs. (13) and (14) with $a \to 0$ and $\ell_x \gg a, |x|, |y|$.

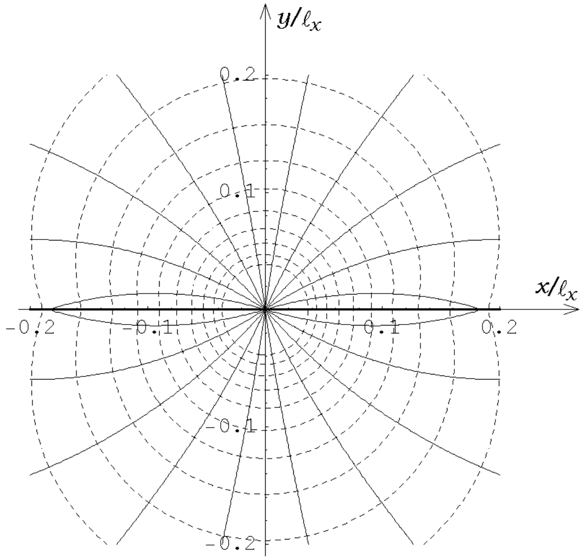

Fig. 7. Equipotentials (continuous lines) and electric field lines (dashed lines) of a single resistive plate carrying a steady current in the x direction. There is a line battery at $x = 0$. We utilized $\phi_L = -\phi_R = \phi_0$.

Something similar occurs with the surface charge distribution close to the battery, given by Heald as:

$$\sigma(\varphi) = \frac{\varepsilon_0 \phi_0}{\pi R} \tan\frac{\varphi}{2}. \tag{30}$$

For angles close to the battery, $\varphi = (\pi \pm \delta)$ rad with $0 < \delta \ll \pi$, we have $\sigma \approx \pm 2\varepsilon_0\phi_0/\pi R\delta$. The same happens with Eq. (23) close to the battery, observing that $R\delta$ is analogous to x in our linear case.

These results prove that the discontinuity in the potential due to the battery creates a nonlinear divergence of the densities of surface charge close to the battery. This happens even in very long straight conductors, as we have seen here.

Eqs. (9) and (10) are valid for semi-infinite plates. Eq. (11), on the other hand, is valid at points (x, y, z) such that $\sqrt{x^2 + y^2 + z^2} \ll \ell_x \ll \ell_z$, [14]. This means that the combination of both solutions given by Eqs. (13) and (14) is valid only close to the battery, but not close to the extremities $x = \pm\ell_x$.

For long and wide plates ($\ell_z \gg \ell_x \gg a$), Eqs. (13) to (22) show that close to the battery the contribution from the electrostatic solution is greater than the contribution from the terms that maintains the current flow. This means that there is little difference between the situation with and without current, and has already been noted elsewhere, [12]. Despite this fact, these two situations are completely different, namely: plates kept at constant potentials, and plates with a steady current.

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APPENDIX

Calculation of the integral of Eq. (9). As the integral is an even function of k it can be written as

$$\int_0^\infty \frac{\cosh(ky)}{\cosh(ka)}\frac{\sin(kx)}{k}dk = \frac{1}{2}\int_{-\infty}^\infty \frac{\cosh(ky)}{\cosh(ka)}\frac{\sin(kx)}{k}dk$$
$$= \frac{1}{2i}\int_{-\infty}^\infty \frac{\cosh(ky)}{\cosh(ka)}\frac{e^{ikx}}{k}dk. \tag{31}$$

The integral above is part of a contour integral I on the complex variable z:

$$I = \oint_C \frac{\cosh(yz)}{\cosh(az)}\frac{e^{izx}}{z}dz, \tag{32}$$

with an appropriate contour C. The integrand has a simple pole in $z_0 = 0$ and infinite simple poles in z_n such that

$$\cosh(az_n) = 0 \quad \rightarrow \quad z_n = \frac{2n+1}{2a}\pi i, \tag{33}$$

for integer n. As we have the term $e^{izx} = e^{i\alpha x - \beta x}$, for $z = \alpha + i\beta$ (α and β being real numbers), the integral converges for $x\beta > 0$. We choose a contour of the type shown in Fig. 8 for $x > 0$. The integral I can thus be divided in three terms: along the real z axis, along the path C_r and along the path C_R. The integral along the path C_r is given by

$$\lim_{r\to 0}\int_{C_r} \frac{\cosh(zy)}{\cosh(za)}\frac{e^{izx}}{z}dz = \lim_{r\to 0}\int_\pi^0 \frac{\cosh(re^{i\theta}y)}{\cosh(re^{i\theta}a)}\frac{\exp(ire^{i\theta}x)}{re^{i\theta}}ire^{i\theta}d\theta = -i\pi. \tag{34}$$

The integral along the path C_R is limited and vanishes for $R \to \infty$:

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{\cosh(zy)}{\cosh(za)}\frac{e^{izx}}{z}dz\right| \leq \lim_{R\to\infty}\int_{C_R}\left|\frac{e^{izx}}{z}\right|dz = 0. \tag{35}$$

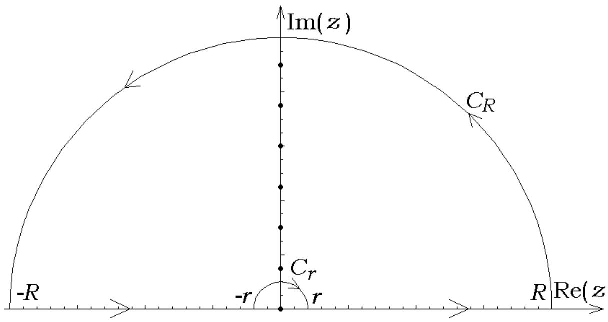

Fig. 8. Contour to calculate the integral of Eq. (9), for $x > 0$. For $x < 0$ we choose a symmetrical contour reflected at the horizontal (real z) axis.

Using Cauchy's theorem, the integral I is given by:

$$\lim_{r\to 0, R\to\infty} I = \int_{-\infty}^{\infty} \frac{\cosh(ky)}{\cosh(ka)} \frac{e^{ikx}}{k} dk - \pi i = 2\pi i \sum_{n=1}^{\infty} \text{Res}(z_n)$$

$$= 2\pi i \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{\pi(2n+1)} e^{-\pi x(2n+1)/2a} \cos\left[\frac{\pi y}{2a}(2n+1)\right], \tag{36}$$

where $\text{Res}(z_n)$ is the residue of the integrand in $z = z_n$. Therefore, we have the integral of Eq. (9) for $x > 0$ as:

$$\int_0^{\infty} \frac{\cosh(ky)}{\cosh(ka)} \frac{\sin(kx)}{k} dk = \frac{\pi}{2} + 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} e^{-\pi x(2n+1)/2a} \cos\left[\frac{\pi y}{2a}(2n+1)\right]. \tag{37}$$

Using the series

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}, \tag{38}$$

we can write the above integral as

$$\int_0^{\infty} \frac{\cosh(ky)}{\cosh(ka)} \frac{\sin(kx)}{k} dk = \frac{\pi}{2} + 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} e^{-\pi x(2n+1)/2a} \cos\left[\frac{\pi y}{2a}(2n+1)\right]$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-\pi(x-iy)(2n+1)/2a} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-\pi(x+iy)(2n+1)/2a}$$

$$= \frac{\pi}{2} - \arctan\left[e^{-\pi(x-iy)/2a}\right] - \arctan\left[e^{-\pi(x+iy)/2a}\right] = \frac{\pi}{2} - \arctan\left[\frac{\cos(\pi y/2a)}{\sinh(\pi x/2a)}\right]$$

$$= \operatorname{arccot}\left[\frac{\cos(\pi y/2a)}{\sinh(\pi x/2a)}\right]. \quad (39)$$

For $x < 0$, the integral of Eq. (9) is analogously given by

$$\int_0^\infty \frac{\cosh(ky)}{\cosh(ka)} \frac{\sin(kx)}{k} dk = -\frac{\pi}{2} - 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} e^{\pi x(2n+1)/2a} \cos\left[\frac{\pi y}{2a}(2n+1)\right]$$

$$= \operatorname{arccot}\left[\frac{\cos(\pi y/2a)}{\sinh(\pi x/2a)}\right]. \quad (40)$$

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