Foundations of Physics, Vol. 30, No. 7, 2000

On the Propagation of Electromagnetic Signals in Wires and Coaxial Cables According to Weber's Electrodynamics

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Received October 30, 1999; revised February 3, 2000

We derive the equation describing the flow of a variable current in straight wires and in coaxial cables from Newton's second law of motion plus Weber's electrodynamics. We show that in both cases the signal propagates at light velocity.

1. INTRODUCTION

Our goal is to derive the equation describing the flow of a variable current in conducting wires like those of Figs. 1 to 4 from Weber's electrodynamics.[(1)] In Fig. 1 we have a cylindrical wire of radius a and length $\ell \gg a$ carrying a current $I(z, t)$ over its cross section πa^2. In Fig. 2 the symmetrical surface current $\vec{K} = K(z, t)\,\hat{z}$ flows only along the periphery $2\pi a$ of the cylindrical shell of radius a, $I(z, t) = 2\pi a K(z, t)$. In Figs. 3 and 4 we have coaxial cables with inner conductors like those of Fig. 1 and 2, respectively, but with an outer cylindrical shell of radius $b > a$ carrying the return current. As we are supposing symmetrical currents which do not depend on the polar angle φ, the source of the signal (closing a switch, an *ac* power supply,...) must have this property. The wires and cables are always supposed at rest in the laboratory.

In order to arrive at the desired equation we first calculate the force on a test charge q_1 located at $\vec{r}_1$, moving with velocity $\vec{v}_1$ and acceleration $\vec{a}_1$,

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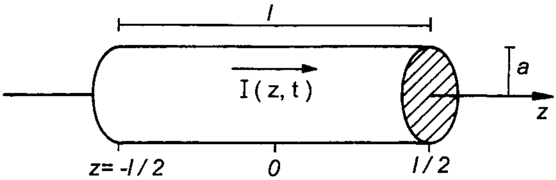

Fig. 1. Conducting cylindrical wire of radius a and length $\ell \gg a$ carrying a current $I(z, t)$ over its cross section.

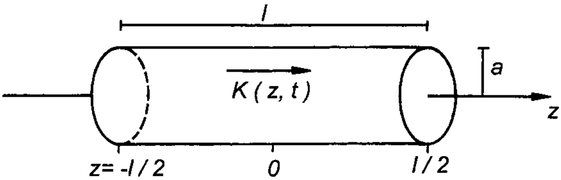

Fig. 2. Conducting cylindrical shell of radius a and length $\ell \gg a$ carrying a surface current density $\vec{K} = K(z, t)\,\hat{z}$ over its periphery: $I(z, t) = 2\pi a K(z, t)$.

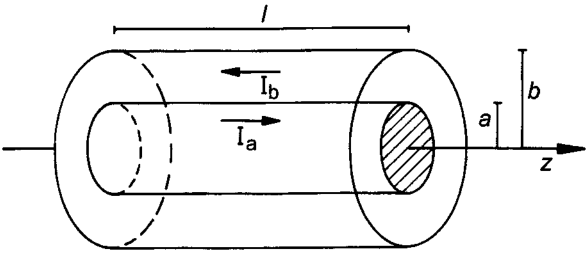

Fig. 3. Coaxial cable with inner wire of radius a carrying a current $I_a(z, t)$ over its cross section of area πa^2, with an outer cylindrical shell of radius b and length $\ell \gg b > a$ carrying the return current $I_b(z, t) = -I_a(z, t)$ over its periphery of length $2\pi b$.

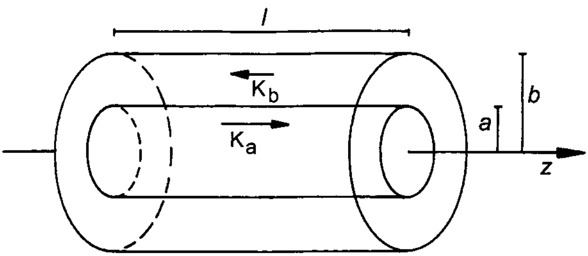

Fig. 4. Coaxial cable with inner cylindrical shell of radius a carrying a current $I_a(z, t)$ over its periphery $2\pi a$, with an outer cylindrical shell of radius b and length $\ell \gg b > a$ carrying the return current $I_b(z, t)$ over its periphery $2\pi b$.

relative to the center of the wire (origin of the coordinate system, with the z axis along the axis of the wire). There are three kinds of source charges in the wire exerting electromagnetic forces on the test one: (A) Free charges over the surface of the wire, (B) the stationary positive lattice making the body of the wire, and (C) the moving conduction electrons along the body of the wire which constitute the current. We consider separetely each one of them.

(A) When a current flows in a resistive wire connected to a battery or other power supply, the electric field driving the conduction electrons against the resistive friction of the wire is due to free charges distributed along the surface of the wire. The battery creates and maintains this distribution of charges but does not generate the electric field along the circuit. This was first pointed out by Kirchhoff and further analysed by Sommerfeld, Jefimenko, Heald, Jackson and many others: Refs. 2–4 with English translation in Ref. 5, Ref. 6, pp. 125–130, Refs. 7–9. We illustrate these surface charges in Fig. 5. The surface density of these free charges is represented by $\sigma_f(z, t)$. We need to integrate the force exerted on a test charge q_1 by a charge element $dq_f = \sigma_f(z_2, t)\, a\, d\varphi_2\, dz_2$ over the surface of the wire of radius a. Here φ_2 is the poloidal angle varying from 0 to 2π, $a\, d\varphi_2$ is an element of arc and dz_2 is an element of length along the wire.

(B) The stationary positive lattice also exerts force on the test charge. A charge element dq_{2+} of the lattice located at $\vec{r}_2$ is represented in Fig. 6. Its volume charge density for the case of Fig. 1 is represented by ρ_+ and its surface charge density for the case of Fig. 2 is represented by σ_+. For the homogeneous wire considered here, ρ_+ and σ_+ are constant over the wire and do not depend on time. We need to integrate the force on q_1

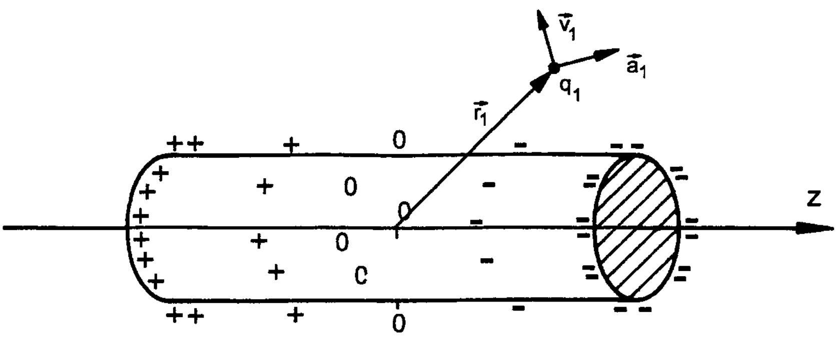

Fig. 5. Qualitative representation of the free surface charges with generate the electric field inside and outside the wire. They exert a force on a test charge q_1 located at $\vec{r}_1$ and moving with velocity $\vec{v}_1$ and acceleration $\vec{a}_1$ relative to the center of the wire.

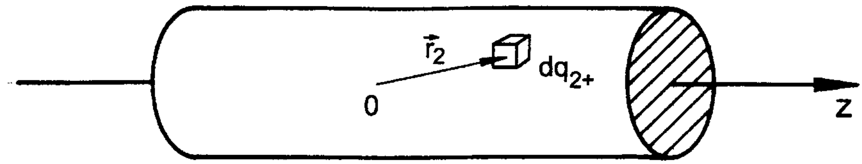

Fig. 6. Charge element of the lattice dq_{2+} located at $\vec{r}_2$.

due to $dq_{2+} = \rho_+ r_2\, d\varphi_2\, dr_2\, dz_2$ located at a distance r_2 from the axis of the wire over its volume in the case of Fig. 1 (or $dq_{2+} = \sigma_+ a\, d\varphi_2\, dz_2$ over the surface of the cylindrical shell of Fig. 2).

(C) The moving conduction electrons also exert force on the test charge. Their volume charge density for the case of Fig. 1 is represented by ρ_{c-} and their surface charge density for the case of Fig. 2 is represented by σ_{c-}. For the homogeneous wires considered here ρ_{c-} and σ_{c-} are constant over the wires and do not depend on time, as it happened with for ρ_+ and σ_+. The velocity and acceleration of the conduction electrons at a time t in a cross section located at z_2 are represented by $\vec{v}_{2-} = v_{2-}(z_2, t)\,\hat{z}$ and $\vec{a}_{2-} = a_{2-}(z_2, t)\,\hat{z}$, respectively, Fig. 7. We need to integrate the force on the test charge q_1 exerted by a charge element $dq_{c-} = \rho_{c-} r_2\, d\varphi_2\, dr_2\, dz_2$ located at a distance r_2 from the axis of the wire over its volume in the case of Fig. 1 (or $dq_{c-} = \sigma_{c-} a\, d\varphi_2\, dz_2$ over the surface of the cylindrical shell of Fig. 2).

As a first approximation we assume that the charge density of the conduction electrons is equal and opposite to the charge density of the positive lattice, namely,

$$\rho_{c-} = -\rho_+ \qquad \text{and} \qquad \rho_{c-} = -\sigma_+ \tag{1}$$

In order to integrate these three forces we utilize Weber's electrodynamics. Accordingly the force exerted by a charge element dq_2 located at $\vec{r}_2$, moving with velocity $\vec{v}_2$ and acceleration $\vec{a}_2$ on a point charge q_1

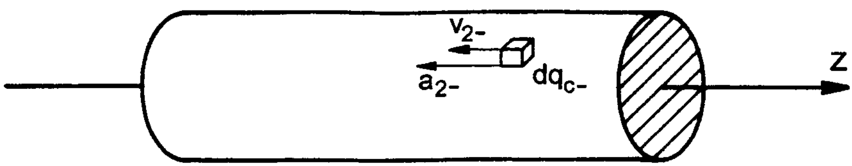

Fig. 7. Conduction charge element dq_{c-} located at $\vec{r}_2$, moving with velocity $\vec{v}_{2-} = v_{2-}(z_2, t)\,\hat{z}$ and acceleration $\vec{a}_{2-} = a_{2-}(z_2, t)\,\hat{z}$.

located at $\vec{r}_1$, moving with velocity $\vec{v}_1$ and acceleration $\vec{a}_1$ is given by [Ref. 1, Chapter 3]:

$$d\vec{F} = \frac{q_1\, dq_2}{4\pi\varepsilon_o} \frac{\hat{r}_{12}}{r_{12}^2} \left[1 + \frac{1}{c^2} \left(\vec{v}_{12} \cdot \vec{v}_{12} - \frac{3}{2} (\hat{r}_{12} \cdot \vec{v}_{12})^2 + \vec{r}_{12} \cdot \vec{a}_{12} \right) \right] \tag{2}$$

where $\varepsilon_o = 8.85 \times 10^{-12}\ \mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-2}$ is the permittivity of vacuum, $c = 3 \times 10^8\ \mathrm{m} \cdot \mathrm{s}^{-1}$, $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$, $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$, $\vec{a}_{12} = \vec{a}_1 - \vec{a}_2$, $r_{12} = |\vec{r}_{12}|$ and $\hat{r}_{12} = \vec{r}_{12}/r_{12}$ is the unit vector pointing from 2 to 1.

We begin with the cases of Figs. 1 and 2 and later on consider a coaxial cable. We follow essentially Kirchhoff's approach,[1] Sec. 3.1. After calculating the force on a generic test charge we consider it to be a conduction electron. With Newton's second law of motion $F = ma$ we get one equation with two unknowns, the current and the surface density of free electricity. The other equation connecting these two unknows is that for the conservation of charges. With these two relations we then obtain the equation describing the propagation of electromagnetic signals in wires and coaxial cables according to Weber's electrodynamics.

2. STRAIGHT WIRE

The first situation to be considered here is that of a straight wire of radius a and length $\ell \gg a$, Fig. 1. We suppose a symmetrical current density $\vec{J} = J(z, t)\, \hat{z}$, where the z axis has been chosen along the axis of the wire with $z = 0$ at its center. We consider cylindrical coordinates (r, φ, z), with r being the distance of the charge to the axis of the wire and not to the origin of the coordinate system (we do not employ the usual notation ρ to avoid confusion with the charge density). As pointed out above, our procedure will be to integrate Eq. (2) for the force acting on the test charge q_1 due to the coulombian, velocity and acceleration terms. The charges exerting the force will be the surface charges with density σ_f, the positive lattice with density ρ_+ and the conduction electrons with density ρ_{c-}. We begin with the force exerted by the free surface charges on the test charge.

As we are considering only the symmetrical situation in which the surface current does not depend on φ, the same will happen with the free surface charge density: $\sigma_f = \sigma_f(z, t)$. Accordingly the force on the test charge q_1 cannot depend on its poloidal angle φ_1. To simplify the calculations without any loss of generality we consider it located at $\varphi_1 = 0$, so that $\vec{r}_1 = r_1 \hat{x} + z_1 \hat{z}$, with velocity $\vec{v}_1 = \dot{x}_1 \hat{x} + \dot{y}_1 \hat{y} + \dot{z}_1 \hat{z}$ and acceleration $\vec{a}_1 = \ddot{x}_1 \hat{x} + \ddot{y}_1 \hat{y} + \ddot{z}_1 \hat{z}$.

Instead of integrating directly the coulombian force it is easier to integrate the electric potential and then obtain the force by taking the gradient of this potential. This was the approach employed by Kirchhoff and we follow it here.

The coulombian potential $\phi(r_1, z_1, t)$ where this test charge is located, due to the free surface charges in the current carrying wire, is then given by (with $dq_2 = \sigma_f a \, d\varphi_2 \, dz_2$ located at $\vec{r}_2 = a \cos \varphi_2 \hat{x} + a \sin \varphi_2 \hat{y} + z_2 \hat{z}$):

$$\phi(r_1, z_1, t)$$

$$= \frac{1}{4\pi\varepsilon_o} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} \frac{\sigma_f(z_2, t) \, a \, d\varphi_2 \, dz_2}{\sqrt{r_1^2 + a^2 - 2r_1 a \cos \varphi_2 + (z_2 - z_1)^2}}$$

$$= \frac{1}{4\pi\varepsilon_o} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} \frac{\sigma_f(z_2, t) \, d\varphi_2 \, dz_2}{\sqrt{(1 - 2(r_1/a) \cos \varphi_2 + (r_1^2/a^2)) + ((z_2 - z_1)/a)^2}} \tag{3}$$

Defining $s^2 \equiv 1 - 2(r_1/a) \cos \varphi_2 + (r_1^2/a^2) \geqslant 0$ and $u \equiv (z_2 - z_1)/a$ this can be written as

$$\phi(r_1, z_1, t) = \frac{a}{4\pi\varepsilon_o} \int_{\varphi_2=0}^{2\pi} \int_{u=-(\ell+2z_1)/2a}^{(\ell-2z_1)/2a} \frac{\sigma_f(au + z_1, t) \, d\varphi_2 \, du}{\sqrt{s^2 + u^2}}$$

Kirchhoff was able to solve these integrals utilizing the approximations

$$\ell \gg a, \qquad \ell \gg r_1, \qquad \text{and} \qquad \ell \gg |z_1| \tag{4}$$

We present here the main ideas behind his approach although not repeating his steps and derivation. For any given r_1 and φ_2 the maximum value of $1/\sqrt{s^2 + u^2}$ is at $u = 0$ (or $z_2 = z_1$). For z_2 far from z_1 the value of u will be of the order $\ell/a \gg 1$ due to the approximation (4). This means that $1/\sqrt{s^2 + u^2}$ will be close to zero if z_2 is far from z_1, as s is of the order of unity. Kirchhoff could then remove $\sigma_f(z_2, t)$ from the integrand taking its value at $z_2 = z_1$. We are then led to the approximate result

$$\phi(r_1, z_1, t) = \frac{a\sigma_f(z_1, t)}{4\pi\varepsilon_o} \int_{\varphi_2=0}^{2\pi} \int_{u=-(\ell+2z_1)/2a}^{(\ell-2z_1)/2a} \frac{d\varphi_2 \, du}{\sqrt{s^2 + u^2}} \tag{5}$$

These integrals can be solved with the approximations (4) utilized here, yielding (see Appendix):

$$\phi(r_1, z_1, t) = \frac{a\sigma_f(z_1, t)}{\varepsilon_o} \ln \frac{\ell}{a} \qquad \text{if} \quad r_1 \leqslant a \tag{6}$$

$$\phi(r_1, z_1, t) = \frac{a\sigma_f(z_1, t)}{\varepsilon_o} \ln \frac{\ell}{r_1} \qquad \text{if} \quad r_1 \geqslant a \tag{7}$$

The coulombian force is then given by (with $\vec{F} = -q_1 \nabla_1 \phi$):

$$\vec{F} = -\frac{q_1 a}{\varepsilon_o} \frac{\partial \sigma_f}{\partial z_1} \left(\ln \frac{\ell}{a} \right) \hat{z} \qquad \text{if} \quad r_1 < a \tag{8}$$

$$\vec{F} = \frac{q_1 a \sigma_f}{\varepsilon_o} \frac{\hat{r}_1}{r_1} - \frac{q_1 a}{\varepsilon_o} \frac{\partial \sigma_f}{\partial z_1} \left(\ln \frac{\ell}{r_1} \right) \hat{z} \qquad \text{if} \quad r_1 \geqslant a \tag{9}$$

Later on we consider the contribution to this force due to the motion of the test charge and of the free surface charges showing that they are negligible. These two last equations are then the expression for the force on the test charge due to the free surface charges.

We turn now to the force on the test charge due to the positive stationary lattice and to the conduction electrons. As the lattice is at rest we have $\vec{v}_{2+} = 0$ and $\vec{a}_{2+} = 0$. According to Eq. (2) the force of the lattice of Fig. 1 on the test charge q_1 is then given by:

$$\vec{F} = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} \frac{q_1\, dq_{2+}}{4\pi\varepsilon_o} \frac{\hat{r}_{12}}{r_{12}^2} \left[1 + \frac{1}{c^2} \left(\vec{v}_1 \cdot \vec{v}_1 - \frac{3}{2} (\hat{r}_{12} \cdot \vec{v}_1)^2 + \vec{r}_{12} \cdot \vec{a}_1 \right) \right] \tag{10}$$

Before integrating this force we consider the force on the test charge due to the moving conduction electrons (with velocity $\vec{v}_{2-}$ and acceleration $\vec{a}_{2-}$). From Eq. (2) this is given by:

$$\vec{F} = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} \frac{q_1\, dq_{c-}}{4\pi\varepsilon_o} \frac{\hat{r}_{12}}{r_{12}^2} \times \left[1 + \frac{1}{c^2} \left(\vec{v}_{12-} \cdot \vec{v}_{12-} - \frac{3}{2} (\hat{r}_{12} \cdot \vec{v}_{12-})^2 + \vec{r}_{12} \cdot \vec{a}_{12-} \right) \right] \tag{11}$$

We need to integrate these two forces over the volume of the wire of Fig. 1. To this end we replace dq_{2+} by $\rho_+ r_2\, d\varphi_2\, dz_2\, dr_2$ and dq_{2-} by

$\rho_{c-} r_2 \, d\varphi_2 \, dz_2 \, dr_2$. Due to our approximation (1) that the wire is essentially neutral, except for the surface charges considered above, we can then add Eqs. (10) and (11). The coulombian term and the terms proportional to v_1^2, to $(\hat{r}_{12} \cdot \vec{v}_1)^2$ and to $\vec{r}_{12} \cdot \vec{a}_1$ cancel out. The remaining terms are (with $c^2 = 1/\mu_o \varepsilon_o$ and taking out of the integral the constant ρ_{c-}):

$$\vec{F} = \frac{\mu_o q_1 \rho_{c-}}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} \frac{\hat{r}_{12}}{r_{12}^2} \left[v_{2-}^2 - 2\vec{v}_1 \cdot \vec{v}_{2-} + 3(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_{2-}) - \frac{3}{2} (\hat{r}_{12} \cdot \vec{v}_{2-})^2 - \vec{r}_{12} \cdot \vec{a}_{2-} \right] r_2 \, d\varphi_2 \, dz_2 \, dr_2 \tag{12}$$

The terms proportional to v_{2-}^2 and to $(\hat{r}_{12} \cdot \vec{v}_{2-})^2$ are usually small compared to the coulombian forces (8) and (9). Moreover, they point towards the radial direction $\hat{r}_1$.[10] As we are interested only in the longitudinal propagation of the signal we will neglect these terms.

The terms depending on $\vec{v}_1 \cdot \vec{v}_{2-}$ and $(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_{2-})$ give rise to the magnetic force $q_1 \vec{v}_1 \times \vec{B}$,[10–12] [1, Secs. 6.6 and 7.4].[13] As the current is in the longitudinal direction $\hat{z}$, the magnetic field will be in the poloidal direction $\hat{\varphi}$. The test charge considered here will be a conduction electron moving in the $\hat{z}$ direction, so that $q_1 \vec{v}_1 \times \vec{B}$ will be in the radial direction $\hat{r}_1$. As we are interested only in the longitudinal propagation of the signal along the z direction, we will not consider these terms either.

We then need to take into account the acceleration term. As we are considering a straight wire with $\vec{v}_{2-} = v_{2-}(z_2, t) \, \hat{z}$ and $\vec{a}_{2-} = a_{2-}(z_2, t) \, \hat{z}$, this term will appear when there is alteration of the strength of the current (acceleration of the conduction electrons). With $\vec{r}_1 = r_1 \hat{x} + z_1 \hat{z}$ and $\vec{r}_2 = r_2 \cos \varphi_2 \hat{x} + r_2 \sin \varphi_2 \hat{y} + z_2 \hat{z}$ this term can be written as:

$$\vec{F} = \frac{\mu_o q_1 \rho_{c-}}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} r_2 \, d\varphi_2 \, dz_2 \, dr_2 \times \frac{(r_1 - r_2 \cos \varphi_2) \, \hat{x} - r_2 \sin \varphi_2 \hat{y} - (z_2 - z_1) \, \hat{z}}{[r_1^2 + r_2^2 - 2 r_1 r_2 \cos \varphi_2 + (z_2 - z_1)^2]^{3/2}} (z_2 - z_1) \, a_{2-}(z_2, t) \tag{13}$$

Integrating in φ_2 the y component goes to zero. Once more with Eq. (4) and Kirchhoff's great idea of approximation we remove $a_{2-}(z_2, t)$ from the integrand taking its value at $z_2 = z_1$, yielding

$$\vec{F} = \frac{\mu_o q_1 \rho_{c-} a_{2-}(z_1, t)}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-\ell/2}^{\ell/2} r_2 \, d\varphi_2 \, dz_2 \, dr_2 \times \frac{(r_1 - r_2 \cos \varphi_2) \, \hat{x} - (z_2 - z_1) \, \hat{z}}{[r_1^2 + r_2^2 - 2 r_1 r_2 \cos \varphi_2 + (z_2 - z_1)^2]^{3/2}} (z_2 - z_1) \tag{14}$$

Integrating in z_2 the x component goes to zero, as we are supposing $\ell \gg |z_1|$. Calling $z_2 - z_1 \equiv m$ and $r_1^2 + r_2^2 - 2r_1 r_2 \cos\varphi_2 \equiv n^2$ we are then led to:

$$\vec{F} = -\frac{\mu_o q_1 \rho_{c-} a_{2-}(z_1, t)}{4\pi} \hat{z} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{m=-\ell/2}^{\ell/2} r_2 \, d\varphi_2 \, dm \, dr_2 \frac{m^2}{(n^2+m^2)^{3/2}} \tag{15}$$

These integrals can be solved utilizing (4), see Appendix, yielding

$$\vec{F} = -\frac{q_1 \mu_o a^2 \rho_{c-} a_{2-}(z_1, t)}{2} \left(\ln \frac{\ell}{a} \right) \hat{z} \qquad \text{if} \quad r_1 \leqslant a \tag{16}$$

$$\vec{F} = -\frac{q_1 \mu_o a^2 \rho_{c-} a_{2-}(z_1, t)}{2} \left(\ln \frac{\ell}{r_1} \right) \hat{z} \qquad \text{if} \quad r_1 \geqslant a \tag{17}$$

These equations can be written as

$$\vec{F} = -q_1 \frac{\partial \vec{A}}{\partial t} \tag{18}$$

where

$$\vec{A}(r_1, z_1, t) = \frac{\mu_o}{2\pi} I(z_1, t) \left(\ln \frac{\ell}{a} \right) \hat{z} \qquad \text{if} \quad r_1 \leqslant a \tag{19}$$

$$\vec{A}(r_1, z_1, t) = \frac{\mu_o}{2\pi} I(z_1, t) \left(\ln \frac{\ell}{r_1} \right) \hat{z} \qquad \text{if} \quad r_1 \geqslant a \tag{20}$$

and

$$I(z_1, t) = \pi a^2 \rho_{c-} v_{2-}(z_1, t) \tag{21}$$

$$\frac{\partial I}{\partial t} = \pi a^2 \rho_{c-} a_{2-}(z_1, t) \tag{22}$$

Here $I(z_1, t)$ is the total current through the cross section πa^2 in $z = z_1$, at the time t.

Up to now we included the forces on a test charge due to the free electricity considered at rest and to the motion of the conduction electrons. We might think that the free electricity is moving together with the conduction electrons, so that we would need to calculate the force of this free electricity on a test charge taking into account the acceleration of σ_f. If this is done, we obtain essentially Eqs. (16) and (17) with σ_f replacing $a\rho_{c-}/2$. The density of conduction electrons in a typical metallic conductor is of the order of one electron per atom, yielding: $|\rho_{c-}| \approx 10^{10}$ C $\cdot$ m^{-3}. We can estimate σ_f observing that in linear conductors it is a linear function of the axial coordinate.[8] For instance, consider a coaxial cable of inner radius a and outer radius b, with conductivity g. Then the surface charge density of the inner conductor σ_f^a when it is flowing a current I is given by (see Ref. 6, pp. 125–130): $\sigma_f^a = -\varepsilon_o Iz/\pi g a^3 \ln(b/a)$. With a copper wire of inner radius 1 mm, outer radius 2 mm, carrying a current of 100 A the charge density at the large distance of 100 m is only $|\sigma_f^a| \approx 10^{-7}$ C/m^2. We then have $\sigma_f^a \approx 10^{-7}$ C/m$^2 \ll a\rho_{c-}/2 \approx 10^7$ C/m^2. This means that in these calculations it does not matter if this free electricity is moving or not with the conduction electrons. The effect of their motion is negligible compared with the effect of the moving conduction electrons. We can then say that all relevant electromagnetic effects have been taken into account here.

We now suppose the test charge to be a conduction electron: $q_1 = -e = -1.6 \times 10^{-19}$ C, $\vec{v}_1 = v_{2-}(z_1, t)\,\hat{z}$ and $\vec{a}_1 = a_{2-}(z_1, t)\,\hat{z}$. In this case we must also include the frictional force due to its collisions with the lattice. The average value of this force can be represented by $-b\vec{v}_1$, where the coefficient of friction b is given by $b = \rho_+ e/g = -e\rho_{c-}/g$, g being the conductivity of the wire,[14] Sec. 7.7. Writing the resistance R of the wire of radius a as $R = \ell/g\pi a^2$ this can also be written as $b = -e\rho_{c-}\pi a^2 R/\ell$.

We can now write down the z component of the equation of motion for a conduction electron applying Newton's second law of motion $F_z = ma_z$. Considering the frictional force plus Eqs. (8), (16) and dropping the subscript 1 yields:

$$\frac{ea}{\varepsilon_o}\left(\ln\frac{\ell}{a}\right)\frac{\partial\sigma_f(z,t)}{\partial z} + \frac{e\mu_o a^2\rho_{c-}}{2}\left(\ln\frac{\ell}{a}\right)a_{2-}(z,t) + \frac{e\pi a^2 R\rho_{c-}}{\ell}v_{2-}(z,t) = ma_{2-}(z,t) \tag{23}$$

Usually $|e\mu_o a^2\rho_{c-}(\ln(\ell/a))/2| \gg m$,[15] so that we can neglect the term ma_{2-} in this equation. For instance, for a one meter wire with one millimeter diameter we have, with $e = 1.6 \times 10^{-19}$ C and $\rho_{c-} \approx -10^{10}$ C $\cdot$ m^{-3}, $|e\mu_o a^2\rho_{c-}(\ln(\ell/a))/2| \approx 2 \times 10^{-21}$ kg, which is much greater than the electron mass $m = 9 \times 10^{-31}$ kg.

With Eqs. (21) and (22) this equation can then be written as (multiplying it by $\varepsilon_o/ea\ln(\ell/a)$ and utilizing $c^2 = 1/\mu_o\varepsilon_o$):

$$\frac{\partial\sigma_f}{\partial z} + \frac{1}{2\pi a}\frac{1}{c^2}\frac{\partial I}{\partial t} = -\frac{\varepsilon_o R}{a\ell\ln(\ell/a)}I \tag{24}$$

There are two unknowns in this equation, σ_f and I. In order to relate them we utilize the equation for the conservation of charges, $\nabla \cdot \vec{J} = -\partial\rho_f/\partial t$. For the case considered here of a current flowing in the z direction over the cross section πa^2 of the wire of radius a this is equivalent to:

$$\frac{\partial I}{\partial z} = -2\pi a \frac{\partial \sigma_f}{\partial t} \tag{25}$$

Applying $\partial/\partial t$ in Eq. (24), multiplying it by $-2\pi a$, and utilizing Eq. (25) yields:

$$\frac{\partial^2 I}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 I}{\partial t^2} = \frac{2\pi\varepsilon_o R}{\ell \ln(\ell/a)}\frac{\partial I}{\partial t} \tag{26}$$

This is the equation of telegraphy, which will also be satisfied by σ_f, by $\phi(a, z, t)$ and by the z component of $\vec{A}$ at $r = a$, $A_z(a, z, t)$.

If we had performed the calculations with the surface current K flowing over the periphery $2\pi a$ of the cylindrical shell of radius a, then the longitudinal equation of motion, instead of Eq. (23), would be:

$$\frac{ea}{\varepsilon_o}\left(\ln\frac{\ell}{a}\right)\frac{\partial \sigma_f}{\partial t} + e\mu_o a\sigma_{c-}\left(\ln\frac{\ell}{a}\right)a_{2-} + \frac{2\pi a e R\sigma_{c-}}{\ell}v_{2-} = ma_{2-} \tag{27}$$

The equation for the conservation of charges in this case would read

$$\frac{\partial K}{\partial z} = -\frac{\partial \sigma_f}{\partial t} \tag{28}$$

The final equation for K, for $I = 2\pi a K$, for σ_f, for $\phi(a, z, t)$ and for $A_z(a, z, t)$ would also be given by (26).

If the resistance of the wire is negligible, Weber's electrodynamics plus Newton's second law of motion predicts a current flow obeying the wave equation. That is, with a signal propagating at light velocity.

3. COAXIAL CABLE

We now perform the same calculations as above but considering a coaxial cable composed of two cylindrical shells of radii a and $b > a$, with currents flowing longitudinally along the z axis, Fig. 4. The return conductor at $\rho = b$ is supposed to have zero resistivity. The length ℓ of the cable is supposed to be much larger than b and a. This case was not considered by Kirchhoff and has never been treated by Weber's electrodynamics.

Beyond the conditions above, we need only two further relations connecting the free surface charge densities (σ_f^a and σ_f^b) and surface current densities (K^a and K^b) in the inner and outer conductors. The most reasonable conditions are that (see Ref. 6, pp. 125–130):

$$dQ_a(z,t) = -dQ_b(z,t) \qquad \text{or} \qquad 2\pi a dz \sigma_f^a(z,t) = -2\pi b dz \sigma_f^b(z,t) \tag{29}$$

$$I_a(z,t) = -I_b(z,t) \qquad \text{or} \qquad 2\pi a K^a(z,t) = -2\pi b K^b(z,t) \tag{30}$$

The approximations utilized here are

$$\ell \gg b > a, \qquad \ell \gg r \qquad \text{and} \qquad \ell \gg |z| \tag{31}$$

With these conditions, integrating for both shells as in the previous section and utilizing Kirchhoff's approximation method yields:

$$\phi(z,t) = \begin{cases} a\sigma_f^a \ln(b/a)/\varepsilon_o, & \text{if} \quad r \leqslant a \\ a\sigma_f^a \ln(b/r)/\varepsilon_o, & \text{if} \quad a \leqslant r \leqslant b \\ 0, & \text{if} \quad \ell \gg r \geqslant b \end{cases} \tag{32}$$

$$\vec{A} = \begin{cases} \mu_o I_a \ln(b/a)\, \hat{z}/2\pi, & \text{if} \quad r \leqslant a \\ \mu_o I_a \ln(b/r)\, \hat{z}/2\pi, & \text{if} \quad a \leqslant r \leqslant b \\ 0, & \text{if} \quad \ell \gg r \geqslant b \end{cases} \tag{33}$$

$$\vec{B} = \begin{cases} 0, & \text{if} \quad r < a \\ \mu_o I_a \hat{\varphi}/2\pi r, & \text{if} \quad a \leqslant r \leqslant b \\ 0, & \text{if} \quad \ell \gg r > b \end{cases} \tag{34}$$

According to Weber's electrodynamics the force on a test charge is then given by (with $I_a(z,t) = 2\pi a K^a(z,t)$):

$$\vec{F} = -q\,\nabla\phi - q\frac{\partial \vec{A}}{\partial t} + q\vec{v}\times\vec{B} \tag{35}$$

If we have $r < a$, $a \leqslant r \leqslant b$ or $\ell \gg r > b$ this yields, respectively:

$$\vec{F} = -\frac{qa}{\varepsilon_o}\ln\frac{b}{a}\left(\frac{\partial\sigma_f^a}{\partial z} + \frac{1}{c^2}\frac{\partial K^a}{\partial t}\right)\hat{z} \tag{36}$$

$$\vec{F} = \frac{qa}{\varepsilon_o}\frac{\sigma_f^a}{r}\hat{r} - \frac{qa}{\varepsilon_o}\ln\frac{b}{r}\left(\frac{\partial\sigma_f^a}{\partial z} + \frac{1}{c^2}\frac{\partial K^a}{\partial t}\right)\hat{z} + q\vec{v}\times\frac{\mu_o I_a}{2\pi r}\hat{\varphi} \tag{37}$$

$$\vec{F} = 0 \tag{38}$$

If we apply Newton's second law of motion to a conduction electron at $r = a$ we must include Eq. (35) and also the resistive force $-b\vec{v}_{2-}$ due to its collision with the lattice (at $\rho = b$ there is no resistive drag as we are supposing a superconducting return conductor). With the equation for the conservation of charges (28) and considering only the z component of the total force this yields, by a similar reasoning as in the previous section (considering now $|e\mu_o a\sigma_{c-} \ln(b/a)| \gg m$ which is the case for most coaxial cables):

$$\frac{\partial^2 \xi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = \frac{2\pi\varepsilon_o R_a}{\ell \ln(b/a)}\frac{\partial \xi}{\partial t} \tag{39}$$

where R_a is the resistance of the internal wire and ξ represents any one of the following quantities: σ_f^a, σ_f^b, K_a, K_b, I_a, I_b, $\phi(a)$ or $A_z(a)$.

These results are essentially the same as for a straight wire, but now replacing $\ln(\ell/a)$ with $\ln(b/a)$.

If the current were flowing all over the cross-section πa^2 of the inner conductor the result would be essentially the same.

In conclusion we may say that the equation describing the current flow in a conducting wire or in a coaxial cable according to Weber's electrodynamics plus Newton's second law of motion is given by Eqs. (26) or (39), respectively.

APPENDIX

We now show how to calculate the integrals of Eq. (5) for $|z_1| \ll \ell$:

$$I \equiv \int_{\varphi_2=0}^{2\pi} \int_{u=-\ell/2a}^{\ell/2a} \frac{d\varphi_2\, du}{\sqrt{s^2+u^2}} \tag{40}$$

where $s^2 \equiv 1 - 2(r_1/a)\cos\varphi_2 + (r_1^2/a^2)$.

Integration with respect to u yields

$$I = \int_0^{2\pi} d\varphi_2 \ln \frac{\sqrt{s^2+(\ell/2a)^2}+(\ell/2a)}{\sqrt{s^2+(\ell/2a)^2}-(\ell/2a)} \tag{41}$$

With approximation (4) this can be written as

$$\begin{aligned} I &= \int_0^{2\pi} d\varphi_2 \ln \frac{(\ell/a)^2}{s^2} \\ &= 4\pi \ln\frac{\ell}{a} - \int_0^{2\pi} \left[\ln\left(1 - 2\frac{r_1}{a}\cos\varphi_2 + \frac{r_1^2}{a^2}\right)\right] d\varphi_2 \end{aligned} \tag{42}$$

This last integral is equal to zero if $r_1 \leqslant a$. If $r_1 > a$ we can put r_1^2/a^2 in evidence and utilize once more this result to solve the last integral, namely:

$$\int_0^{2\pi} \left[\ln \left(1 - 2 \frac{r_1}{a} \cos \varphi_2 + \frac{r_1^2}{a^2} \right) \right] d\varphi_2 = 0 \qquad \text{if} \quad r_1 \leqslant a \tag{43}$$

$$\int_0^{2\pi} \left[\ln \left(1 - 2 \frac{r_1}{a} \cos \varphi_2 + \frac{r_1^2}{a^2} \right) \right] d\varphi_2 = 2\pi \ln \frac{r_1^2}{a^2} \qquad \text{if} \quad r_1 \geqslant a \tag{44}$$

This means that the final value of I is found to be

$$I = 4\pi \ln \frac{\ell}{a} \qquad \text{if} \quad r_1 \leqslant a \tag{45}$$

$$I = 4\pi \ln \frac{\ell}{r_1} \qquad \text{if} \quad r_1 \geqslant a \tag{46}$$

We now solve Eq. (15):

$$J \equiv \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{m=-\ell/2}^{l/2} \frac{m^2\, dm}{(n^2+m^2)^{3/2}}\, r_2\, d\varphi_2\, dr_2 \tag{47}$$

where $n^2 \equiv r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi_2$.

The indefinite integral in m yields

$$-\frac{m}{\sqrt{n^2+m^2}} + \ln(\sqrt{n^2+m^2} + m)$$

From approximation (4) and taking the two limits of the integral in m we are then led to

$$J = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \left(-2 + \ln \frac{\ell^2}{n^2} \right) r_2\, d\varphi_2\, dr_2$$

$$= 2\pi a^2 (\ln \ell - 1) - \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \ln(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi_2)\, r_2\, d\varphi_2\, dr_2 \tag{48}$$

From Eqs. (43) and (44) we can solve this last integral, yielding $2\pi a^2 \ln a + \pi(r_1^2 - a^2)$ if $r_1 \leqslant a$ or $2\pi a^2 \ln r_1$ if $r_1 \geqslant a$. Utilizing once more the approximation (4) we are then led to:

$$J = 2\pi a^2 \ln \frac{\ell}{a} \qquad \text{if} \quad r_1 \leqslant a \tag{49}$$

$$J = 2\pi a^2 \ln \frac{\ell}{r_1} \qquad \text{if} \quad r_1 \geqslant a \tag{50}$$

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