DERIVING AMPÈRE'S LAW FROM WEBER'S LAW

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Abstract

We correct some misconceptions in the literature regarding Weber's law and show: (A) with it Ampère's law of force between current elements can be derived with the modern current element, (B) what are the correct expressions for the relative velocity and acceleration between point charges, (C) why the acceleration terms in Weber's law are essential and so we have arguments to show that Ritz's law is untenable, and (D) how to develop the energy of interaction between two modern current elements.

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In the last 10 years there has been a revival of interest in Ampère's law of force between current elements [1-6] and in Weber's law of force between point charges [7-9]. The main reason for this fact is that some recent experiments with a single circuit can only be explained by Ampère's law and not by Grassmann's law of force (sometimes known as Biot-Savart's law) [1-9]. Ampère's law states that the force which a usual current element $I_2 d\boldsymbol{l}_2$ exerts in another current element $I_1 d\boldsymbol{l}_1$ is given by [10]

$$d\boldsymbol{F} = \frac{-\mu_0}{4\pi} I_1 I_2 \frac{\hat{r}}{r^2}[2d\boldsymbol{l}_1 \cdot d\boldsymbol{l}_2 - 3(\hat{r} \cdot d\boldsymbol{l}_1)(\hat{r} \cdot d\boldsymbol{l}_2)], \tag{1}$$

where $\hat{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)/|\boldsymbol{r}_1 - \boldsymbol{r}_2|$, $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$.

Historically Weber's law appeared twenty years after this law. Weber's goal was to derive this law based in a law of force between point charges like Coulomb's, but modified when the charges have a relative velocity and acceleration . In this respect he was following the suggestion that Gauss gave in a letter to him [11]. Following also Fechner's hypothesis on the nature of the electric current, according to which it consists of a current of positive electricity in one direction combined with an exactly equal current of negative electricity in the opposite direction (equal as respects the quantity of electricity in motion and the velocity with which it is moving), Weber arrived at the formula [12]:

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2}\left[1 - \frac{\dot{r}^2}{2c^2} + \frac{r\,\ddot{r}}{c^2}\right]. \tag{2}$$

In this formula $\dot{r} = dr/dt$, $\ddot{r} = d^2r/dt^2$, $\mathbf{F}$ is the force that q_2 exerts on q_1, and c is a constant (the ratio between electromagnetic and electrostatic units of charge) with the same magnitude as the velocity of light, as Weber determined experimentally. With this equation and Fechner hypothesis Weber derived Eq. (1).

Nowadays one knows that Fechner's assumption regarding the nature of electric currents is wrong. In fact we have positive ions fixed in the lattice of a metal and only electrons are responsible for the current. Our aim is to show that even with this present model for the current we can derive Eq. (1) with Weber's force, Eq. (2). In this way we want to clear up some misunderstandings regarding Weber's law as this one expressed by P. Graneau in an excellent review article [13]: "Weber argued the forces on the charges were passed on to the metal but failed to explain how, at the same time, the charges could move freely through the conductor. This inconsistency and the subsequent discovery of the immobility of the lattice ion made Weber's current element untenable". As we will see, what is untenable is Fechner's current element, but not Weber's law.

In order to do so we first of all write each neutral current element in the form

$$\left.\begin{aligned} I_1 d\boldsymbol{l}_1 &= q_{1+}\boldsymbol{v}_{1+} + q_{1-}\boldsymbol{v}_{1-} = q_{1+}\left(\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}\right) , \\ I_2 d\boldsymbol{l}_2 &= q_{2+}\boldsymbol{v}_{2+} + q_{2-}\boldsymbol{v}_{2-} = q_{2+}\left(\boldsymbol{v}_{2+} - \boldsymbol{v}_{2-}\right) \cdot \end{aligned}\right\} \tag{3}$$

In these expressions we assumed $q_{i-} = -q_{i+}$ because we are considering only neutral current elements. Since $\boldsymbol{v}_{1+}$ and $\boldsymbol{v}_{2+}$ are arbitrary, Fechner hypothesis corresponds to a special case ($\boldsymbol{v}_{i+} = -\boldsymbol{v}_{i-}$), and now it can also be shown that Ampère's law can be applied to plasma physics where usually one has a neutral fluid with electrons and ions moving relative to the laboratory. Anyway, if we wish to particularize to metallic currents, we only need to put $\boldsymbol{v}_{1+} = \boldsymbol{v}_{2+} = 0$.

According to Weber the net force of $I_2 d\boldsymbol{l}_2$ on $I_1 d\boldsymbol{l}_1$ will be given by a sum of four terms: the force of q_{2-} on q_{1+} and q_{1-}, plus the force of q_{2+} on q_{1+} and q_{1-}. Since

$$\begin{aligned} \dot{r} &\equiv \frac{dr}{dt} = \frac{d}{dt}\sqrt{(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)} \\ &= \frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \hat{r} \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2), \end{aligned} \tag{4}$$

$$\ddot{r} \equiv \frac{d^2 r}{dt^2} = \frac{1}{r}[(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2) - \dot{r}^2], \tag{5}$$

we have that the force of one of the charges q_2 on one of the charges q_1 (according to Eq. (2)), will be given by,

$$\begin{aligned} \boldsymbol{F} &= -\frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\Big\{1 - \frac{1}{2c^2}[(\hat{r}\cdot\boldsymbol{v}_1)^2 - 2(\hat{r}\cdot\boldsymbol{v}_1)(\hat{r}\cdot\boldsymbol{v}_2) + \\ &+ (\hat{r}\cdot\boldsymbol{v}_2)^2] + \frac{1}{c^2}[\boldsymbol{v}_1\cdot\boldsymbol{v}_1 - 2\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + \boldsymbol{v}_2\cdot\boldsymbol{v}_2 + (\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot\boldsymbol{a}_1 - \end{aligned}$$

$$- \quad (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{a}_2 - (\widehat{r} \cdot \boldsymbol{v}_1)^2 + (\widehat{r} \cdot \boldsymbol{v}_1)(\widehat{r} \cdot \boldsymbol{v}_2) - (\widehat{r} \cdot \boldsymbol{v}_2)^2] \Big\}. \tag{6}$$

When we add $\boldsymbol{F}$ of q_{2+} on q_{1+} with $\boldsymbol{F}$ of q_{2+} on q_{1-} we get:

$$\begin{aligned} \boldsymbol{F}_+ &= \frac{q_{1+}q_{2+}}{4\pi\varepsilon_0} \frac{\widehat{r}}{r^2} \Big\{ \frac{-3}{2c^2} [(\widehat{r} \cdot \boldsymbol{v}_{1+})^2 - (\widehat{r} \cdot \boldsymbol{v}_{1-})^2 - \\ &- 2(\widehat{r} \cdot (\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}))(\widehat{r} \cdot \boldsymbol{v}_{2+})] + \frac{1}{c^2} [\boldsymbol{v}_{1+} \cdot \boldsymbol{v}_{1+} - \boldsymbol{v}_{1-} \cdot \boldsymbol{v}_{1-} - \\ &- 2(\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}) \cdot \boldsymbol{v}_{2+} + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{a}_{1+} - \boldsymbol{a}_{1-})] \Big\}. \end{aligned} \tag{7}$$

Adding $\boldsymbol{F}$ of q_{2-} on q_{1+} with $\boldsymbol{F}$ of q_{2-} on q_{1-} yields:

$$\begin{aligned} \boldsymbol{F}_- &= \frac{q_{1+}q_{2-}}{4\pi\varepsilon_0} \frac{\widehat{r}}{r^2} \Big\{ \frac{-3}{2c^2} [(\widehat{r} \cdot \boldsymbol{v}_{1+})^2 - (\widehat{r} \cdot \boldsymbol{v}_{1-})^2 - \\ &- 2(\widehat{r} \cdot (\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}))(\widehat{r} \cdot \boldsymbol{v}_{2-})] + \frac{1}{c^2} [\boldsymbol{v}_{1+} \cdot \boldsymbol{v}_{1+} - \boldsymbol{v}_{1-} \cdot \boldsymbol{v}_{1-} - \\ &- 2(\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}) \cdot \boldsymbol{v}_{2-} + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{a}_{1+} - \boldsymbol{a}_{1-})] \Big\}. \end{aligned} \tag{8}$$

Adding Eqs. (7) and (8) yields

$$\begin{aligned} dF &= \frac{q_{1+}q_{2+}}{4\pi\varepsilon_0} \frac{\widehat{r}}{r^2} \Big\{ \frac{3}{c^2} [\widehat{r} \cdot (\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-})] [(\widehat{r} \cdot (\boldsymbol{v}_{2+} - \boldsymbol{v}_{2-})] - \\ &- \frac{2}{c^2} (\boldsymbol{v}_{1+} - \boldsymbol{v}_{1-}) \cdot (\boldsymbol{v}_{2+} - \boldsymbol{v}_{2-}) \Big\}. \end{aligned} \tag{9}$$

Using (3) we can see straight away that Eq.(9) reduces to Eq. (1). As we let $\boldsymbol{v}_{1+}$ and $\boldsymbol{v}_{2+}$ completely arbitrary this completes the proof that Weber's law yields Ampère's law even without Fechner assumptions.

Some remarks must be made at this point. When deducing Weber's law from Ampère's law (the opposite path we followed in this paper) Maxwell supposed constant in magnitude velocities (that is $|v_i|$ = constant), [14]. As we showed here, this restriction is not necessary.

It should be remembered here that when Weber derived Eq. (1) from Eq. (2), he assumed in the beginning Fechner hypothesis, namely, $\boldsymbol{v}_{1-} = -\boldsymbol{v}_{1+}$ and $\boldsymbol{v}_{2-} = -\boldsymbol{v}_{2+}$. The same procedure was utilized by Maxwell, [14], and Whittaker, [15], p. 201. Then, when in their works we make $\boldsymbol{v}_{2+} = 0$, we will also have $\boldsymbol{v}_{2-} = 0$ because they imposed Fechner's hypothesis in the beginning, and so there will be no force between the current elements in this case according to them. But this will not happen in Eq. (9) because we don't need to impose Fechner's hypothesis. If we make $\boldsymbol{v}_{1+} = \boldsymbol{v}_{2+} = 0$ in (9), keeping $\boldsymbol{v}_{1-}$ and $\boldsymbol{v}_{2-}$ arbitrary, we still obtain Eq. (1) where now $I_1 d\mathbf{l}_1 = q_{1-}\boldsymbol{v}_{1-} = -q_{1+}\boldsymbol{v}_{1-}$ and $I_2 d\mathbf{l}_2 = q_{2-}\boldsymbol{v}_{2-} = -q_{2+}\boldsymbol{v}_{2-}$. This shows how careful we must be when arriving at some conclusions based on old works due to the implicit hypothesis they used. Fechner's current element was the most simple and natural one to be introduced at that time (1845) when the internal nature of a current was unknown. Nowadays we know it is untenable. Even so from Weber's law we still get Ampère's law using (3) and without Fechner's hypothesis, as we've shown.

Another aspect worth to note is that in the final expression, Eq. (9), the accelerations of the charges don't appear. Although each charge can have arbitrary acceleration, at the end this won't matter, in so far as Ampère's law is concerned. This is the reason why Gauss' law [14] can also yields Ampère's law, as it only differs from Weber's law in the acceleration terms:

$$\boldsymbol{F}_{\text{Gauss}} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\left[1 + \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)}{c^2} - \frac{3}{2}\frac{\dot{r}^2}{c^2}\right]. \tag{10}$$

Using Eqs. (4) and (5) in Eq. (2) we can show that Weber's law is equivalent to Gauss' law with $(\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot(\boldsymbol{a}_1 - \boldsymbol{a}_2)/c^2$ inside the square brackets of Eq. (10). As is well known, [14], despite this success Gauss' law is untenable because it is not consistent with the principle of conservation of energy and also because we can't derive Faraday's law of induction with it. To derive this induction law and to conserve energy the acceleration terms of Weber's law are essential.

With this in mind we can understand why Ritz's law [16, 17] is also untenable. The main reason is that Ritz's law for the force that q_2 exerts on q_1 depends only on the acceleration of q_2, but not at all on the acceleration of q_1. As we showed in another paper, [8], the acceleration of q_1 or m_1 in Weber's law is essential in order to derive an equation of motion similar to Newton's second law and the proportionality between inertial and gravitational masses, so that we need it in order to implement Mach's principle.

It is important to emphasize once more that Eq. (1) is the correct expression for the force between two *neutral current elements.* This was the scope of Ampère's experiments (he worked only with neutral metallic currents). When we derived Eq. (1) from Weber's force, Eq. (2), we supposed $q_{1-} = -q_{1+}$ and $q_{2-} = -q_{2+}$ so that the net charge of each current element was zero. This shows that we cannot apply Eq. (1) as being the correct expression for the force between two charges in motion. An expression for the force in this case (electric charges in motion) is given by Weber's law, Eq. (2), and another expression is Lorentz's force law together with the Lienard - Wiechert retarded potentials. For instance, the force between two electrons in motion in the laboratory is given by (2) but not by (1), because each electron has a net charge so that they cannot be considered *neutral* current elements.

We should mention here another misconception regarding Weber's law, this time related to the relative velocity and acceleration between two charges. The correct expressions are those given by Weber, Eqs. (4) and (5). These are truly relational quantities as they have the same value for any observer, even for noninertial observers. On the other hand some authors, for instance see [18, 19], when discussing Weber's law and some modifications of it talk of the relative velocity between two charges as

$$u \equiv |\boldsymbol{v}_1 - \boldsymbol{v}_2| = \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)}. \tag{11}$$

But this is not the correct expression for the relative velocity in Weber's sense because the value of u depends on the observer. To see it, consider two charges at rest in the laboratory separated by a distance r, one of them being at the origin and the other on the x axis. To an observer at the origin spinning with a constant angular velocity $w\hat{z}$, the values he will find are: $\dot{r} = \hat{r}_{12}\cdot\boldsymbol{v}_{12} = 0$ as in the laboratory frame, but $u = |\boldsymbol{v}_{12}| = wr$ while in the laboratory frame $u = 0$. This simple example illustrates the relational character of $\dot{r}$ while it shows that $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$ has not always the same value for any observer. Concerning the relative acceleration, the correct expression is that given by Eq. (5), which, in general, is different from $\hat{r}\cdot(\boldsymbol{a}_1 - \boldsymbol{a}_2)$ and also from $|\boldsymbol{a}_1 - \boldsymbol{a}_2|$.

Another aspect to be touched upon refers to Ampère's law, Eq. (1). It is often claimed that Ampère's law has a weakness because it doesn't predict a torque between two current elements (as it is a central force). For instance, in Whittaker's classical book one reads [15], p. 86: *The weakness of Ampère's work evidently lies in the assumption that the force is directed along the line joining the two elements; for in the analogous case of the action between two magnetic molecules, we know that the force is* not *directed along the line joining the molecules"*. And in the next page (p. 87): *"Helmholtz assumes that the interaction between two current elements is derivable from a potential, and this entails the existence of a couple in addition to a force along the line joining the elements"*. In our opinion, this is not a fair statement relative to Ampère's law. Each current element Idl has, besides its location in space, a special direction, namely, that of the electric current. As such it has a vectorial character and is not a scalar quantity. So, even when a force between two current elements is directed along the line joining them we can have a torque between the current elements. This can be seen from Eq.(1) which involves the angle between the current elements and also the angle between each current element and the line joining them. This torque has received a special name by P. Graneau, namely: Alpha-torque forces, [20]. Its action has been seen in many experiments performed by Graneau. He states the origin of these torques in this way: "If the stored energy changes when one of the circuits is rotating with respect to the other, then there must exist a mutual torque between the circuits". Another way of understanding the origin of these torques is to remember that each current element cannot be a material point since it has a direction in space. So, we can imagine each one of them with a linear dimension dl (small but not negligible). In this view the torque arises because the force on the tip of the current element will be different from the force on the tail of it and so a torque can be produced.

There is an important point which should be discussed here related with the conduction of current in metals. As has been correctly pointed out by Graneau more than once, Ampère's force, Eq. (1), acts on the fixed lattice of the metal (straining atomic bonds, etc). On the other hand when we derive Eq. (1) from Eq. (2) we need to take into account the force of the electrons on the positive ions of the lattice and vice-versa, as well as the force of the positive ions on one another and the force of the electrons on one another. In order to make these two approaches compatible it would be necessary a transfer of momentum from the electrons to the lattice, so that Ampère's force would act between conductor atoms and not between conduction electrons. It is not clear how this transfer of momentum can happen because in the modern theory of conduction the electrons in metals are said to drift freely in the "Fermi sea". We can only suspect that this happens through the friction which makes the electrons flow at a constant drift speed when there is an applied constant voltage in the metallic wire (voltage due to a battery, for instance). The imperfections in the lattice cause this friction which blocks the free motion of the electrons and these, in their turn, according to Newton's third law, make an opposite force on the lattice. This would be the origin of the ponderomotive force on the lattice. That is, this force is exerted indirectly through the currents. Anyway it should be emphasized here that this problem is not restricted to Weber's law, but is implicit in any theory which derives the force between current elements based in a generalization of Coulomb's law. For instance, if we begin with Lorentz's force law, together with the Lienard - Wiechert retarded potentials, and follow a procedure like the one presented here, adding the force of q_{2-} on q_{1+} and q_{1-}, plus the force of q_{2+} on q_{1+} and q_{1-}, we end with Grassmann's law (see [16], pp. 518 - 523). Puting aside the controversy surrounding Ampère's force law versus Grassmann's force law, the same problem of the mechanism responsible for the transfer of momentum between the mobile electrons and the fixed ions appears here in Lorentz's force law yielding Grassmann's law, as with Weber's law yielding Ampère's law. A correct understanding of this problem is greatly desirable but has not been supplied up to now.

It is also of interest here to generalize, without making resort to Fechner's assumptions, the energy of interation between two current elements according to Weber's law. The mutual energy of two moving charges is, according to Weber:

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}\left(1 - \frac{\dot{r}^2}{2c^2}\right). \tag{12}$$

Weber showed that his force law, Eq. (2), could be derived from this velocity - dependent potential energy and that it was consistent with the principle of conservation of energy. As in the derivation of Ampère's law we must add four terms to get the mutual energy between $I_1 d\boldsymbol{l}_1$ and $I_2 d\boldsymbol{l}_2$, namely $U_{2+,1+}$, $U_{2+,1-}$, $U_{2-,1+}$ and $U_{2-,1-}$. Neglecting the energy of formation of each current element (the self–energy) we get for this mutual energy:

$$dU = \frac{\mu_0}{4\pi}\,\frac{I_1 I_2}{r}\,(\hat{r}\cdot d\boldsymbol{l}_1)(\hat{r}\cdot d\boldsymbol{l}_2), \tag{13}$$

where $I_1 dl_1$ and $I_2 dl_2$ are given by (3). If we put $v_{1+} = v_{2+} = 0$ this result will remain valid. It must be emphasized that this is not the same energy as that given by F. Neumann [15], which is

$$dU_N = \frac{\mu_0}{4\pi} \frac{I_1 I_2}{r} (dl_1 \cdot dl_2). \tag{14}$$

But it has been proved elsewhere that these two expressions give the same result when integrated round either closed circuit [15], p. 233. The same considerations are valid for the vector potential which in Neumann's and Weber's theories are given by, respectively , [15]:

$$\left.\begin{aligned} \mathbf{A}_N &= \frac{\mu_0}{4\pi} I_2 \int \frac{dl_2}{r}, \\ \mathbf{A}_W &= \frac{\mu_0}{4\pi} I_2 \int \frac{(\hat{r} \cdot dl_2)}{r} \hat{r}, \end{aligned}\right\} \tag{15}$$

where $I_2 dl_2$ in Weber's case is given by (3).

This completes the revision of Weber's law and its correct interpretation and use.

In conclusion, we can say that even with the modern current element (fixed positive ions and free electrons being the responsible for the current), we can derive Ampère's law from Weber's law. Also the mutual energy between two current elements, Eq. (13), can be obtained with this modern current element, expression (3). To do so we only need to use the correct relative velocity and acceleration between two point charges, Eqs. (4) and (5).

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