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Consequences of a Quadratic Law of the Lever

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Abstract. We present the discussion which exists in the literature related to Archimedes's demonstration of the law of the lever. One important aspect of the argument concentrates on the meaning of his postulates. In order to clarify this whole subject, we analyze what consequences would arise if nature followed a different law of the lever. We concentrate, in particular, in the case of a torque proportional to the square of the distances of the bodies to the fulcrum. We consider not only a linear lever but also a horizontal triangle which can rotate around a horizontal axis parallel to one of its sides.

Keywords. Archimedes, Classical Mechanics, Law of the Lever.

1. Introduction

Archimedes (287-212 BCE) demonstrated the law of the lever in Propositions 6 and 7 of his work *On the Equilibrium of Planes*. In an earlier work, [1], we quoted all his words as taken from Dijksterhuis's book, [2]. In the present paper we quote all of them from Heath's translation, [3, p. 192]: "Propositions 6, 7. Two magnitudes, whether commensurable [Prop. 6] or incommensurable [Prop. 7], balance at distances reciprocally proportional to the magnitudes."

To demonstrate these results he utilized seven postulates, [3, p. 189-190]: "I postulate the following: 1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance. 2. If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards that weight to which the addition was made. 3. Similarly, if anything be taken away from one of the weights, they are not in equilibrium but incline towards the weight from which nothing was taken. 4. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide. 5. In figures which are unequal but similar the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides. 6. If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances. 7. In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure."

Although the concept of the centre of gravity appears in postulate 4, it is not defined in any extant work of Archimedes. Heath, Duhem, Stein, Dijksterhuis, Assis and many others have studied how Archimedes implicitly utilized this concept to calculate the centre of gravity of many figures. For references see [4] and [5]. From these studies it seems that Archimedes understood the centre of gravity to be a point such that if the body were suspended from that point, released from rest and free to rotate in all directions around that point, the body would remain at rest and would preserve its original position no matter what the initial orientation of the body relative to the ground.

Archimedes's demonstration of the law of the lever was criticized by Mach, [6]. He thought Archimedes's demonstration was a fallacy due to the fact that, according to Mach, Archimedes had utilized the law of the lever in his demonstration. Dijksterhuis and others objected to Mach's criticism, [2, p. 289-304], [4, p. 177-185]. They pointed out the relevance of Archimedes's sixth postulate. They understood Archimedes to interpret "magnitudes equal to other magnitudes" as "magnitudes of the same weight" and "magnitudes at the same distances" as "magnitudes the centres of gravity of which lie at the same distances from the fulcrum." This interpretation conferred a reasonable meaning to the sixth postulate and removed Mach's objection to Archimedes's demonstration of the law of the lever.

We agree with Dijksterhuis's points of view. To illustrate the crucial role played by postulate 6 in Archimedes's demonstration of the law of the lever, we consider what would be the consequences if nature behaved in such a way that the law of the lever were quadratic in the distances of the bodies.

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2. A generalized law of the lever

Suppose a horizontal beam acts as a lever that can rotate around another horizontal axis orthogonal to the beam of the lever and passing through its fulcrum. We consider N bodies on one side of the fulcrum and M bodies on the other side. A generic body i has weight W_i, with its centre of gravity being suspended by the beam of the lever at a distance d_i from the fulcrum. We define a generic "alpha" torque τ exerted by these bodies as $\sum_{i=1}^{N} W_i (d_i)^\alpha$ and $\sum_{i=N+1}^{M} W_i (d_i)^\alpha$. The exponent α characterizes the behaviour of the lever as a function of the distance to the fulcrum. In real life $\alpha = 1$. In this work we wish to compare this normal condition with hypothetical situations for which $\alpha \neq 1$. To this end we postulate what we call a generalized law of the lever. That is, we postulate the following behaviour for the lever released at rest horizontally, being free to rotate around the fulcrum: If $\tau_N = \tau_M$, the lever remains in equilibrium. If $\tau_N > \tau_M$, the set of N bodies inclines towards the ground. If $\tau_N < \tau_M$, the set of M bodies inclines towards the ground.

We now consider simple symmetrical situations of equilibrium. First we have two equal weights W suspended at points B and D from a lever which can rotate around a fulcrum located at C between B and D. If $BC = CD$, the lever will remain in equilibrium for all values of α. This is our configuration (I). The lever will also remain in equilibrium for any value of α when the two weights W are suspended together at C. This is our configuration (II). That is, in this case we can replace the two equal weights at B and D of configuration (I) by a single body of twice the weight at the midpoint C without disturbing the equilibrium of the lever for any value of α. The centre of gravity of the two equal weights W_B and W_D can be considered their midpoint. Archimedes proved this fact in Proposition 4 of his work, [3, p. 191]: "If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity."

Now let us see how Archimedes demonstrated the law of the lever considering a very simple case. Consider three equal weights suspended at points A, B, and D. The lever is free to rotate around the middle point B. If $AB = BD$, the lever will remain in equilibrium no matter the value of α. This is our configuration (III). Let us call C the midpoint of the segment BD. By postulate 6 we will not disturb the equilibrium of the lever by replacing bodies B and D by a single body of twice the weight acting at C. This new configuration (IV) is a special case of the law of the lever because $W_A/W_C = BC/AB = 1/2$, or $BC = AB/2$.

Let us now assume that $\alpha \neq 1$ and our generalized law of the lever. In this case the configuration (III) continues to be an equilibrium configuration, no matter the value of α. But configuration (IV) is no longer in equilibrium. If $\alpha < 1$, the weights at C will incline toward the ground. In contrast, if $\alpha > 1$, the weight A will incline toward the ground. The new equilibrium situation according to the generalized law of the lever and the definition of the "alpha" torque is the configuration with the equal weights W_B and W_D acting together at another point E such that $W_A/W_E = (BE/AB)^\alpha$, that is, $BE = (1/2)^{1/\alpha}$. If $\alpha = 2$, $BE = (\sqrt{2}/2)AB \approx 0.707AB$. If $\alpha = 0$, the solution diverges. If $\alpha = 1/2$, we have $BE = AB/4$.

We can go from configuration (I) to configuration (II) without disturbing the equilibrium of the lever for all values of α. On the other hand, we can go from configuration (III) to configuration (IV) without disturbing the equilibrium of the lever only if $\alpha = 1$. If $\alpha = 2$, we can maintain the equilibrium of the lever only by combining the weights W_B and W_D at another point E given by $BE = \sqrt{2}AB/2 \approx 0.707AB$. This last situation shows that Archimedes's postulate 6, as interpreted by Dijksterhuis, would not be valid if $\alpha = 2$. This conclusion lends support to his interpretation of this postulate and to the fact that this postulate was essential in order to allow Archimedes to demonstrate the law of the lever.

3. Equilibrium of a Triangle

Archimedes also demonstrated how to locate the centre of gravity of a triangle, [3, p. 198 and 201]: "Proposition 13. In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side." "Proposition 14. If follows at once from the last proposition that the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively."

We now consider a generic horizontal triangle ABC with height H and base BC. This triangle can rotate freely around the horizontal axis DE which is fixed relative to the ground and is parallel to BC. We want to find the distance R

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between this axis and the side BC that will let the triangle be in equilibrium for a given value of α, with $0 < R < H$.

Our generalized law of the lever implies that equilibrium will happen when the alpha torque exerted by one side of the axis, $\int r^{\alpha} dW$, is equal to the alpha torque exerted by the other side of the axis, $\int r'^{\alpha} dW'$. Here r and r' are the distances between the rotation axis and the strips of weight dW and dW' on either side of the axis.

After performing these integrals we obtain that equilibrium will happen when [1]:

$$k^{\alpha+2} - (\alpha+2)k - (\alpha-1) = 0 . \quad (1)$$

The constant k is defined by $k = (H - R)/R$.

For $\alpha = 1$ there are three solutions to this equation, namely, $k_1 = 2$, $k_2 = -1$ and $k_3 = -1$. Only the first solution is physically reasonable, implying $R = H/3 \approx 0.333H$. This is the usual solution of an axis passing through the centre of gravity of the triangle, which was Archimedes's solution. To demonstrate this result he also utilized implicitly postulate 6.

For $\alpha = 0$, there are two solutions to Eq. (1), namely, $k_1 = 1 + \sqrt{2} \approx 2.414$ and $k_2 = 1 - \sqrt{2} \approx -0.414$. Only the first solution is physically reasonable, leading to $R \approx H/3.414 \approx 0.293H$. This axis parallel to the side BC will not pass through the intersection of the medians. It will be closer to the base BC than the previous equilibrium axis for the case $\alpha = 1$.

For $\alpha = 2$, there are four solutions to Eq. (1), namely,

$$k_1 \approx -0.693,\ k_2 \approx -0.546 - 1.459i$$

$$k_3 \approx -0.546 + 1.459i \text{ and } k_4 \approx 1.784$$

Only the fourth solution is compatible with the condition $0 < R < H$. We are then led to $R \approx H/2.784 \approx 0.359H$. This axis parallel to the side BC will not pass through the intersection of the medians. It will be closer to the vertex A than the equilibrium axis for the case $\alpha = 1$.

This conclusion shows once more that postulate 6 is essential to demonstrate not only the usual law of the lever, but also to find the usual centre of gravity of a triangle. If nature behaved with a generalized power law with $\alpha \neq 1$, the results demonstrated by Archimedes would not remain valid.

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