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Perplex numbers and quaternions

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We present some properties of mathematical and physical interest in generalized algebras of two, three and four dimensions. We give a new formulation for these algebras, stress some important applications related to the wave equation, and emphasize a novel didactic approach to this subject.

1. Introduction

In 1986 Fjelstad [1] presented an application of perplex numbers to interpret superluminal phenomena. In this way he extended special relativity to the case $|v|>c$. Band [2] obtained the essential mathematical results of Fjelstad from two simple postulates. As was shown by Ronveaux [3], the mathematics based on perplex numbers is not new [4]. Despite this fact the underlying mathematics and the possible physical applications are not known by most mathematicians and physicists in general, contrary to complex numbers.

The aim of this paper is to extend the analysis of Fjelstad showing some different and new applications and also to present a mathematical structure which allows an extension of the algebra in any number of dimensions [5]. This has not been done in this way before. In particular the algebras in three and four dimensions will be analysed and it will be seen that the quaternions are only a particular case of this four dimensional case.

2. Two dimensions

A general algebra in two dimensions is based on the variable s defined by

$$s \equiv x\tilde{a}_1 + y\tilde{a}_2 \tag{1}$$

where the coefficients x and y are real variables and the basal elements $\tilde{a}_1$ and $\tilde{a}_2$ are defined by their product. The sum of s_1 and s_2 and the product of s_1 by a scalar α ($\alpha \in \mathbb{R}$) are given by the usual rules

$$\begin{aligned} s_1 + s_2 &= (x_1\tilde{a}_1 + y_1\tilde{a}_2) + (x_2\tilde{a}_1 + y_2\tilde{a}_2) \\ &\equiv (x_1 + x_2)\tilde{a}_1 + (y_1 + y_2)\tilde{a}_2 \end{aligned} \tag{2}$$

$$\alpha s_1 = \alpha(x_1\tilde{a}_1 + y_1\tilde{a}_2) \equiv (\alpha x_1)\tilde{a}_1 + (\alpha y_1)\tilde{a}_2, \tag{3}$$

The algebra of the variable s is obtained with the definition of the products $\tilde{a}_1\tilde{a}_1$, $\tilde{a}_1\tilde{a}_2$, $\tilde{a}_2\tilde{a}_1$ and $\tilde{a}_2\tilde{a}_2$. We assume that each of these products gives a number proportional to a single basal element and not to a linear combination of them. In this way we restrict our arbitrariness so that we can have $\tilde{a}_1\tilde{a}_2 = \alpha\tilde{a}_1$ or $\tilde{a}_1\tilde{a}_2 = \alpha\tilde{a}_2$ but we cannot have $\tilde{a}_1\tilde{a}_2 = \alpha\tilde{a}_1 + \beta\tilde{a}_2$. If we further assume that the four products will generate two numbers proportional to $\tilde{a}_1$ and two numbers proportional to $\tilde{a}_2$ then we get six possibilities, two of which are represented below.

$$\left\{\begin{array}{l}\tilde{a}_1\tilde{a}_1=\alpha\tilde{a}_1\\ \tilde{a}_1\tilde{a}_2=\beta\tilde{a}_1\\ \tilde{a}_2\tilde{a}_1=\gamma\tilde{a}_2\\ \tilde{a}_2\tilde{a}_2=\delta\tilde{a}_2\end{array}\right\} \quad\text{and}\quad \left\{\begin{array}{l}\tilde{a}_1\tilde{a}_1=\alpha\tilde{a}_2\\ \tilde{a}_1\tilde{a}_2=\beta\tilde{a}_1\\ \tilde{a}_2\tilde{a}_1=\gamma\tilde{a}_1\\ \tilde{a}_2\tilde{a}_2=\delta\tilde{a}_2\end{array}\right\} \tag{4}$$

where α, β, γ and δ are arbitrary real constants. We could normalize to remove one of the constants but we leave it this way for the sake of clarity. Restricting our analysis to the situations in which $\tilde{a}_1\tilde{a}_2$ is proportional to $\tilde{a}_2\tilde{a}_1$ and assuming that the multiplication of s_1, s_2 and s_3 is associative $(s_1(s_2s_3)=(s_1s_2)s_3)$ we are left with only two possibilities out of these six, namely

$$\left\{\begin{array}{l}\tilde{a}_1\tilde{a}_1=\alpha\tilde{a}_1\\ \tilde{a}_1\tilde{a}_2=\alpha\tilde{a}_2\\ \tilde{a}_2\tilde{a}_1=\alpha\tilde{a}_2\\ \tilde{a}_2\tilde{a}_2=\beta\tilde{a}_1\end{array}\right\} \quad\text{and}\quad \left\{\begin{array}{l}\tilde{a}_1\tilde{a}_1=\alpha\tilde{a}_2\\ \tilde{a}_1\tilde{a}_2=\beta\tilde{a}_1\\ \tilde{a}_2\tilde{a}_1=\beta\tilde{a}_1\\ \tilde{a}_2\tilde{a}_2=\beta\tilde{a}_2\end{array}\right\} \tag{5}$$

By an index change and the permutation of α and β we can obtain the second set of equations (5) from the first one. For this reason we will analyse only the first set of equations (5) from now on.

A coherent definition of the quotient between these two basal numbers is given by

$$\frac{\tilde{a}_1}{\tilde{a}_1}=\frac{\tilde{a}_1}{\alpha} \qquad \frac{\tilde{a}_1}{\tilde{a}_2}=\frac{\tilde{a}_2}{\beta} \qquad \frac{\tilde{a}_2}{\tilde{a}_1}=\frac{\tilde{a}_2}{\beta} \qquad \frac{\tilde{a}_2}{\tilde{a}_2}=\frac{\tilde{a}_1}{\alpha} \tag{6}$$

where we are assuming $\alpha\neq 0$ and $\beta\neq 0$.

With equations (1) to (6) we have a complete commutative algebra. We did not begin with this last hypothesis and this was a consequence of the previous assumptions. As we will see later, this also happens in three dimensions but not necessarily in four dimensions. This was not noted by others before, and this will be an essential point in the following. It is easily seen that $\tilde{a}/\alpha$ works as a unit element in this algebra.

The algebra of complex numbers is a particular case of the first set of equations (5) with $\alpha=1$ and $\beta=-1$. Then the basal elements $\tilde{a}_1$ can be represented by 1 and $\tilde{a}_2$ can be represented by $i=(-1)^{1/2}$. The algebra of perplex numbers is obtained from the first set of equations (5) with $\alpha=1$ and $\beta=1$. Again $\tilde{a}_1$ can be represented by 1 and now $\tilde{a}_2$ represented by j such that $j^2=1$ and $j\neq 1$ (that is, $\tilde{a}_2\neq\tilde{a}_1$).

One of the main applications of perplex (or complex) numbers is through the equivalent Cauchy–Rieman equations and the wave equation (Laplace equation) that we obtain with it. It is a known fact that some students have great difficulty in accepting or in dealing with equations which have $c=1$, $\hbar=1$, etc. For this reason we will let α and β be arbitrary (but not zero) real constants and then we will get the wave velocity explicitly. We suggest this route when trying to present this subject to students, in order to avoid their resistance.

Any analytic function of s (for instance, s^3+5, or $\exp(s)$, or $\sin(s_1+s_2)$) can be written in the form

$$f(s)=u(x,y)\tilde{a}_1+v(x,y)\tilde{a}_2 \tag{7}$$

where u and v are real functions of x and y. We define in the usual way the derivative of $f(s)$.

$$f'(s)\equiv \lim_{\Delta s\to 0}\frac{f(s+\Delta s)-f(s)}{\Delta s} \tag{8}$$

at the points for which this limit exists. If we take two different limits (namely $\Delta s=\Delta x\tilde{a}_1$ and $\Delta s=\Delta y\tilde{a}_2$) then we obtain, equating these two derivatives, the generalized Cauchy–Riemann equations.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=\frac{\beta}{\alpha}\frac{\partial v}{\partial x} \tag{9}$$

This is a necessary condition for the analyticity of $f(s)$. In analogy with the complex functions we can show that this is also a necessary condition.

Assuming the existence and continuity of the second order derivative of u and v we obtain easily from equations (9)

$$\frac{\partial^2 u}{\partial x^2}=\frac{\alpha}{\beta}\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}=\frac{\alpha}{\beta}\frac{\partial^2 v}{\partial y^2} \tag{10}$$

If we have $\alpha=1=-\beta$ we get the usual Laplace equation. With perplex number, $\alpha=\beta=1$, we obtain the wave equation with unitary velocity.

The advantage of retaining α and β as arbitrary real numbers is that now we can obtain equations in which the wave velocity appears explicitly. For instance, the wave equation in a stretched string can be obtained from equations (9) and (10) with the substitutions, $u\to -F/\sigma$, $\beta/\alpha\to\tau/\sigma$, $y\to t$. $F(x,t)$ is the upward component of force exerted from left to right due to tension in a stretched string, $v(x,t)$ is the upward velocity of any point on the string, σ is the density of the string per unit length, τ is the tension in the string and t is time [6]. Similarly the equations of a plane electromagnetic wave travelling in the x-direction and linearly polarized in the y-direction can be obtained from equations (9) and (10) with substitutions $u\to -cE_y(x,t)$, $v\to B_z(x,t)$, $\beta/\alpha\to c^2$, $y\to t$ (Gaussian units) [6]. As an application in electrical circuits we can obtain the equations relating the voltage $E(x,t)$ across the transmission line and the current $I(x,t)$ with the substitutions $u\to -E/L$, $v\to I$, $\beta/\alpha\to(CL)^{-1}$, $y\to t$ (where C is the shunt capacitance per unit length and L is the series inductance per unit length) [6]. In this way we can see that with a single mathematical formulation we can construct solutions to many equations which are very important in physics, e.g. the $u(x,y)$ or $v(x,y)$ of any analytic function $f(s)$ like

$$s^3+5 \text{ or } \exp(s)\equiv\sum_{n=0}^{\infty}\frac{s^n}{n!}$$

When we develop these generalized algebras we need to take care with the 'divisors of zero', namely, numbers $s_1\neq 0$ and $s_2\neq 0$ such that $s_1s_2=0$ [7]. It is easy to see that $ss^*\equiv(x\tilde{a}_1+y\tilde{a}_2)(x\tilde{a}_1-y\tilde{a}_2)=(\alpha x^2-\beta y^2)\tilde{a}_1$, so that the points which satisfy $\alpha x^2-\beta y^2=0$ are divisors of zero. As we are taking x and y to be real numbers, if $\alpha\beta<0$ then the algebra will not have divisors of zero, as is the case with complex numbers.

On the other hand if $\alpha\beta > 0$ then the algebra will have divisors of zero, as is the case with perplex numbers. In particular the function $f(s) = (s)^{-1} = (x + jy)^{-1}$ with $j^2 = 1$, is not defined at the points which satisfy $x = \pm y$. So we need to take care when working in these generalized algebras.

The perplex algebra can be written in the form: $w = x + yj$, with $j^2 = 1$. When explaining this algebra to some people I observed that they had a resistance to accept it. They usually asked me: how can $j^2 = 1$ and $j \neq 1$. An analogy with complex numbers may help to understand this point. It is a known fact that the algebra of complex numbers can be expressed in a matrix form utilizing as our basal elements $\mathbb{I}$ and $\mathbb{J}$ defined by

$$\mathbb{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{J} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

Using the usual matrix operations we can show that $\mathbb{I} \cdot \mathbb{I} = \mathbb{I}$, $\mathbb{I} \cdot \mathbb{J} = \mathbb{J}$, $\mathbb{J} \cdot \mathbb{I} = \mathbb{J}$ and $\mathbb{J} \cdot \mathbb{J} = -\mathbb{I}$, so we can write the complex 'number' $\mathbb{Z}$ and the function $\mathbb{F}(\mathbb{Z})$ in the form

$$\mathbb{Z} = x\mathbb{I} + y\mathbb{J}$$

$$\mathbb{F}(\mathbb{Z}) = u(x, y)\mathbb{I} + v(x, y)\mathbb{J}$$

and then we have a representation of the complex algebra. A similar representation exists for the perplex numbers if we take as our basal elements the matrices $\mathbb{I}$ and $\mathbb{P}$ defined by

$$\mathbb{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{P} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

Observing that $\mathbb{I} \cdot \mathbb{I} = \mathbb{I}, \mathbb{I} \cdot \mathbb{P} = \mathbb{P}$, $\mathbb{P} \cdot \mathbb{I} = \mathbb{P}$ and $\mathbb{P} \cdot \mathbb{P} = \mathbb{I}$ we see that the perplex algebra can be obtained based on the perplex 'number' given by $\mathbb{Q} = x\mathbb{I} + y\mathbb{P}$. As we are utilizing the normal rules for matrix product and as we can see from matrices (12) that $\mathbb{I} \neq \mathbb{P}$, this can be a better form to introduce the perplex algebra to the students. As was pointed out by de Boer [8], this subring received many different names by different authors.

3. Three and four dimensions

We can easily extend this formulation [5] to any number of dimensions through an extension of equation (1).

$$s = \sum_{j=1}^{n} x_j \tilde{a}_j \tag{13}$$

[5], where we restrict our coefficients to be real variables. We will consider here algebras of three and four dimensions. The general treatment which we present for these algebras is new in the literature. It allows a better understanding of this subject and is more appropriate for applications in mathematical physics.

In three dimensions we can have 1680 $(= 9!/(3!\,3!\,3!))$ different possibilities of defining the products $\tilde{a}_1\tilde{a}_2$, $\tilde{a}_3\tilde{a}_1$, etc., if we restrict the product of two basal elements to be a number proportional to a single basal element and the nine products $\tilde{a}_1\tilde{a}_2$, $\tilde{a}_1\tilde{a}_3$, etc. to give three numbers proportional $\tilde{a}_1$, three to $\tilde{a}_2$ and three to $\tilde{a}_3$. If moreover we assume $\tilde{a}_i\tilde{a}_j$ is proportional to $\tilde{a}_j\tilde{a}_i$ then we get only 36 possibilities.

Imposing also associativity in the multiplication we are left with only three possibilities

$$\left\{\begin{array}{l}\tilde{a}_1\tilde{a}_2=\tilde{a}_2\tilde{a}_1=\alpha\tilde{a}_2\\ \tilde{a}_1\tilde{a}_3=\tilde{a}_3\tilde{a}_1=\alpha\tilde{a}_3\\ \tilde{a}_2\tilde{a}_3=\tilde{a}_3\tilde{a}_2=\gamma\tilde{a}_1\\ \tilde{a}_1\tilde{a}_1=\alpha\tilde{a}_1\\ \tilde{a}_2\tilde{a}_2=\beta\tilde{a}_3\\ \tilde{a}_3\tilde{a}_3=\dfrac{\alpha\gamma}{\beta}\tilde{a}_2\end{array}\right\}\qquad\left\{\begin{array}{l}\tilde{a}_1\tilde{a}_2=\tilde{a}_2\tilde{a}_1=\alpha\tilde{a}_1\\ \tilde{a}_1\tilde{a}_3=\tilde{a}_3\tilde{a}_1=\gamma\tilde{a}_2\\ \tilde{a}_2\tilde{a}_3=\tilde{a}_3\tilde{a}_2=\alpha\tilde{a}_3\\ \tilde{a}_1\tilde{a}_1=\beta\tilde{a}_3\\ \tilde{a}_2\tilde{a}_2=\alpha\tilde{a}_2\\ \tilde{a}_3\tilde{a}_3=\dfrac{\alpha\gamma}{\beta}\tilde{a}_1\end{array}\right\}$$

$$\text{and}\qquad\left\{\begin{array}{l}\tilde{a}_1\tilde{a}_2=\tilde{a}_2\tilde{a}_1=\gamma\tilde{a}_3\\ \tilde{a}_1\tilde{a}_3=\tilde{a}_3\tilde{a}_1=\alpha\tilde{a}_1\\ \tilde{a}_2\tilde{a}_3=\tilde{a}_3\tilde{a}_2=\alpha\tilde{a}_2\\ \tilde{a}_1\tilde{a}_1=\beta\tilde{a}_2\\ \tilde{a}_2\tilde{a}_2=\dfrac{\alpha\gamma}{\beta}\tilde{a}_1\\ \tilde{a}_3\tilde{a}_3=\alpha\tilde{a}_3\end{array}\right\}\tag{14}$$

Again we can see that any of these three remaining algebras are also commutative. A coherent definition of the quotient between two base elements is given by (first possibility in equations (14))

$$\left\{\begin{array}{c}\dfrac{\tilde{a}_1}{\tilde{a}_1}=\dfrac{\tilde{a}_2}{\tilde{a}_2}=\dfrac{\tilde{a}_3}{\tilde{a}_3}=\dfrac{\tilde{a}_1}{\alpha}\\ \dfrac{\tilde{a}_1}{\tilde{a}_2}=\dfrac{\tilde{a}_3}{\gamma},\quad \dfrac{\tilde{a}_2}{\tilde{a}_1}=\dfrac{\tilde{a}_2}{\alpha}\\ \dfrac{\tilde{a}_1}{\tilde{a}_3}=\dfrac{\tilde{a}_2}{\gamma},\quad \dfrac{\tilde{a}_3}{\tilde{a}_1}=\dfrac{\tilde{a}_3}{\alpha}\\ \dfrac{\tilde{a}_2}{\tilde{a}_3}=\dfrac{\beta}{\alpha\gamma}\tilde{a}_3,\quad \dfrac{\tilde{a}_3}{\tilde{a}_2}=\dfrac{\tilde{a}_2}{\beta}\end{array}\right\}\tag{15}$$

The unit element in this algebra is again given by $\tilde{a}_1/\alpha$. The quotients in equations (15) exist because a multiplicative group was singled out (the semi-explicit assumption that $\tilde{a}_i\tilde{a}_j=\tilde{a}_i\tilde{a}_k$ if and only if $\tilde{a}_j=\tilde{a}_k$ guarantees the existence of inverses). The simplest conjugate of $s_1=x_1\tilde{a}_1+y_1\tilde{a}_2+z_1\tilde{a}_3$ can be shown to be $s_1^*=(\alpha x_1^2-\gamma y_1z_1)\tilde{a}_1+\alpha/\beta(\gamma z_1^2-\beta x_1y_1)\tilde{a}_2+(\beta y_1^2-\alpha x_1z_1)\tilde{a}_3$, so that $s_1s_1^*=-1/\beta(3\alpha\beta\gamma x_1y_1z_1-\alpha^2/\beta x_1^3-\beta^2\gamma y_1^3-\gamma^2\alpha z_1^3)\tilde{a}_1$. All points other than $x_1=y_1=z_1=0$ which satisfy $s_1s_1^*=0$ are divisors of zero. As there are many such points the algebra in three dimensions is not a division algebra [9]. So the existence of this algebra in three dimensions does not violate the Frobenius theorem which asserts that the only *division algebras* over the real field $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and the algebra of quaternions [10, 11].

As in two dimensions, a function of s can be written in the form

$$f(s)=u(x,y,z)\tilde{a}_1+v(x,y,z)\tilde{a}_2+p(x,y,z)\tilde{a}_3\tag{16}$$

where u, v and p are real functions of x, y and z. Taking

$$f'(s)=\lim_{\Delta s\to 0}[f(s+\Delta s)-f(s)]/\Delta s$$

and Δs to be equal to, alternatively, $\Delta x\tilde{a}_1$ or $\Delta y\tilde{a}_2$ or $\Delta z\tilde{a}_3$ we obtain that u, v and p satisfy the partial differential equations

$$\left\{\begin{aligned}&\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial p}{\partial z}\\&\frac{\partial u}{\partial y}=\frac{\beta}{\alpha}\frac{\partial v}{\partial z}=\frac{\gamma}{\alpha}\frac{\partial p}{\partial x}\\&\frac{\partial u}{\partial z}=\frac{\gamma}{\alpha}\frac{\partial v}{\partial x}=\frac{\gamma}{\beta}\frac{\partial p}{\partial y}\end{aligned}\right\}\tag{17}$$

$$\left\{\begin{aligned}&\frac{\partial^2 u}{\partial x^2}=\frac{\beta}{\gamma}\frac{\partial^2 v}{\partial z^2}=\frac{\alpha}{\beta}\frac{\partial^2 p}{\partial y^2}\\&\frac{\partial^2 u}{\partial y^2}=\frac{\beta\gamma}{\alpha^2}\frac{\partial^2 v}{\partial x^2}=\frac{\beta}{\alpha}\frac{\partial^2 p}{\partial z^2}\\&\frac{\partial^2 u}{\partial z^2}=\frac{\gamma}{\beta}\frac{\partial^2 v}{\partial y^2}=\frac{\gamma^2}{\alpha\beta}\frac{\partial^2 p}{\partial x^2}\end{aligned}\right\}\tag{18}$$

$$\left\{\begin{aligned}&\frac{\partial^3 u}{\partial x^3}=\frac{\alpha^2}{\beta\gamma}\frac{\partial^3 u}{\partial y^3}=\frac{\alpha\beta}{\gamma^2}\frac{\partial^3 u}{\partial z^3}\\&\frac{\partial^3 v}{\partial x^3}=\frac{\alpha^2}{\beta\gamma}\frac{\partial^3 v}{\partial y^3}=\frac{\alpha\beta}{\gamma^2}\frac{\partial^3 v}{\partial z^3}\\&\frac{\partial^3 p}{\partial x^3}=\frac{\alpha^2}{\beta\gamma}\frac{\partial^3 p}{\partial y^3}=\frac{\alpha\beta}{\gamma^2}\frac{\partial^3 p}{\partial z^3}\end{aligned}\right\}\tag{19}$$

where we assumed that the partial derivatives of second and third order u, v and p exist and are continuous in a region $\mathbb{R}$ around (x, y, z).

We now analyse the quaternions and we show that they are a particular case of the general case equation (13). The quaternion is a four dimensional vector $q=x_1+x_2i+x_3j+x_4k$, with real coefficients x_1, x_2, x_3, x_4. The product of two quaternions is obtained using the rules

$$\left\{\begin{aligned}&i^2=j^2=k^2=-1\\&ij=-ji=k\\&jk=-kj=i\\&ki=-ik=j\end{aligned}\right\}\tag{20}$$

(1 is an identity).

An algebra in four dimensions is given by equation (13) with $n=4$.

$$s=x_1\tilde{a}_1+x_2\tilde{a}_2+x_3\tilde{a}_3+x_4\tilde{a}_4\tag{21}$$

A particular choice of the product between two basal elements $\tilde{a}_i$ and $\tilde{a}_j$ which mimics the quaternion product is given by

$$\tilde{a}_m \tilde{a}_n = r_{mn} \tag{22}$$

where r_{mn} is an element of the matrix $\mathbb{R}$ given by

$$\mathbb{R} \equiv \begin{pmatrix} \alpha_{11}\tilde{a}_1 & \alpha_{21}\tilde{a}_2 & \alpha_{31}\tilde{a}_3 & \alpha_{41}\tilde{a}_4 \\ \alpha_{12}\tilde{a}_2 & \alpha_{22}\tilde{a}_1 & \alpha_{32}\tilde{a}_4 & \alpha_{42}\tilde{a}_3 \\ \alpha_{13}\tilde{a}_3 & \alpha_{23}\tilde{a}_4 & \alpha_{33}\tilde{a}_1 & \alpha_{43}\tilde{a}_2 \\ \alpha_{14}\tilde{a}_4 & \alpha_{24}\tilde{a}_3 & \alpha_{34}\tilde{a}_2 & \alpha_{44}\tilde{a}_1 \end{pmatrix} \tag{23}$$

where the α_{mn}s are real constants.

Imposing associativity in the multiplication $(\tilde{a}_m(\tilde{a}_n\tilde{a}_l) = (\tilde{a}_m\tilde{a}_n)\tilde{a}_l)$ we restrict our matrix $\mathbb{R}$ to the form

$$\mathbb{R} \equiv \begin{pmatrix} \alpha_{11}\tilde{a}_1 & \alpha_{11}\tilde{a}_2 & \alpha_{11}\tilde{a}_3 & \alpha_{11}\tilde{a}_4 \\ \alpha_{11}\tilde{a}_2 & \alpha_{22}\tilde{a}_1 & \alpha_{33}\tilde{a}_4 & \dfrac{\alpha_{11}\alpha_{22}}{\alpha_{33}}\tilde{a}_3 \\ \alpha_{11}\tilde{a}_3 & \pm\alpha_{33}\tilde{a}_4 & \alpha_{33}\tilde{a}_1 & \pm\alpha_{11}\tilde{a}_2 \\ \alpha_{11}\tilde{a}_4 & \pm\dfrac{\alpha_{11}\alpha_{22}}{\alpha_{33}}\tilde{a}_3 & \alpha_{11}\tilde{a}_2 & \pm\dfrac{\alpha_{11}\alpha_{22}}{\alpha_{33}}\tilde{a}_1 \end{pmatrix} \tag{24}$$

We then reduce our 16 α_{mn}s to only three arbitrary constants and a possible choice of sign. If we choose the upper (lower) sign in r_{32} the same sign should be utilized in r_{42}, r_{34} and r_{44}. We will get a commutative algebra if we choose the upper sign and an algebra equivalent to the quaternion algebra if we choose the lower sign. This is only evident due to this treatment of the matter. Perhaps this is the reason why these points were not observed by others previously.

With the upper sign the best choice for the conjugate is given by

$$s^* = \overline{x_1}\tilde{a}_1 + \overline{x_2}\tilde{a}_2 + \overline{x_3}\tilde{a}_3 + \overline{x_4}\tilde{a}_4 \tag{25}$$

where

$$\overline{x_1} = x_1 x_2\left(-\alpha_{11}^2 x_1^2 + \alpha_{11}\alpha_{22}x_2^2 \pm \alpha_{11}\alpha_{33}x_3^2 \pm \frac{\alpha_{11}^2\alpha_{22}}{\alpha_{33}}x_4^2\right) - 2\alpha_{11}\alpha_{22}x_2x_3^2x_4$$

$$\overline{x_2} = \pm\alpha_{33}x_2x_3\left(\alpha_{11}x_3^2 \mp \frac{\alpha_{11}\alpha_{22}}{\alpha_{33}}x_2^2\right) + \alpha_{11}x_2x_3\left(\pm\frac{\alpha_{11}\alpha_{22}}{\alpha_{33}}x_4^2 + \alpha_{11}x_1^2\right) - 2\alpha_{11}^2x_1x_3^2x_4$$

$$\overline{x_3} = x_3^2\left(\alpha_{11}^2x_1^2 + \alpha_{11}\alpha_{22}x_2^2 \mp \alpha_{11}\alpha_{33}x_3^2 \pm \frac{\alpha_{11}^2\alpha_{22}}{\alpha_{33}}x_4^2\right) \mp 2\frac{\alpha_{11}^2\alpha_{22}}{\alpha_{33}}x_1x_2x_3x_4$$

$$\overline{x_4} = x_3\left(\pm\alpha_{11}\alpha_{33}x_3^2x_4 \mp 2\alpha_{11}\alpha_{33}x_1x_2x_3 \mp \frac{\alpha_{11}^2\alpha_{22}}{\alpha_{33}}x_4^3 + \alpha_{11}\alpha_{22}x_4 + \alpha_{11}^2x_1^2x_4\right)$$

so that

$$ss^* = \left(\alpha_{11}x_1\overline{x_1} + \alpha_{22}x_2\overline{x_2} + \alpha_{33}x_3\overline{x_3} + \frac{\alpha_{11}\alpha_{22}}{\alpha_{33}}x_4\overline{x_4}\right)\tilde{a}_1 \tag{26}$$

It must be remarked that this is the first time in which this generalized conjugate appears in print. A more complete study including its application in solving physical problems will appear in another paper.

With the lower sign the conjugate is much simpler, namely

$$s^* = x_1\tilde{a}_1 - x_2\tilde{a}_2 - x_3\tilde{a}_3 - x_4\tilde{a}_4 \tag{27}$$

so that

$$ss^* = \left(\alpha_{11}x_1^2 - \alpha_{22}x_2^2 - \alpha_{33}x_3^2 + \frac{\alpha_{11}\alpha_{22}}{\alpha_{33}}x_4^2\right)\tilde{a}_1 \tag{28}$$

The algebra of quaternions is obtained with the lower sign and with the particular choice of the coefficients such that $\alpha_{11}=1$, $\alpha_{22}=-1$ and $\alpha_{33}=-1$.

The important point to note in this exercise is that the quaternion algebra is only a very particular one between many possible algebras of four dimensions. This should be explored and emphasized in the classroom. Interesting applications of quaternions in electromagnetism are given [12]. Applications of perplex numbers to problems in hydrodynamics are well described in an interesting and very readable book by Lavrentiev and Chabat [13]. We suggest this book, especially Chapter II, be used in the classroom.

In a later work we will concentrate on the application of these results in specific problems relevant to mathematical physics (electromagnetism, hydrodynamics and elasticity).

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