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Electric potential due to an infinite conducting cylinder with internal or external point charge

J.A. Hernandes*, A.K.T. Assis

Instituto de Física 'Gleb Wataghin', Universidade Estadual de Campinas—Unicamp, 13083-970 Campinas, São Paulo, Brasil

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Abstract

We utilize the Green's function method in order to calculate the electric potential due to an infinite conducting cylinder held at zero potential and a point charge inside and outside it. We calculate and plot the net force upon the point charge as a function of its distance to the axis of the cylinder. We show that this force goes to zero when the radius of the cylinder goes to zero, no matter the distance of the external point charge to the conducting line.
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1. Introduction

The goal of this work is to calculate the electrostatic force between an infinite conducting cylinder of radius a held at zero potential and an external point charge q. To our knowledge this has never been done before. To this end we consider the Green's function method, [1, Chapters 1–3]. We begin reviewing a known solution of the potential inside a grounded, closed, hollow and finite cylindrical box with a point charge inside it [1, p. 143]. We analyse the limit of an infinite cylinder and explore the

*Corresponding author.
E-mail addresses: julioher@ifi.unicamp.br (J.A. Hernandes), assis@ifi.unicamp.br (A.K.T. Assis).
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force exerted upon the point charge. We then perform a similar analysis for the case of an external point charge. We consider in detail the particular situation of a thin wire, that is, with the point charge many radii away from the axis of the cylinder.

2. Finite conducting cylinder with internal point charge: solution to Poisson's equation

Consider a finite conducting cylindrical box of radius a and length $L \gg a$, with z being its axis of symmetry, see Fig. 1. With cylindrical coordinates (ρ, ϕ, z) the center of the box is supposed to be at $(\rho, z) = (0, L/2)$. We consider also a point charge q located at $\vec{x}' = (\rho' < a, \phi', z')$ inside the box. We wish to calculate the electric potential of the system, the electric field, the surface charge distribution induced by q and the net force between the cylinder and q.

The electrostatic potential Φ obeys Poisson's equation:

$$\nabla_x^2 \Phi = -4\pi\xi\rho,$$

where $\xi = 1/4\pi\varepsilon_0$ in SI units (ε_0 is the electric permittivity of vacuum) or $\xi = 1$ in Gaussian units. In this work, we will suppose a vacuum inside and outside the cylinder. The derivation and results might also be useful with a homogeneous dielectric or insulating liquid inside and outside the cylinder, by utilizing the standard approach described in most textbooks dealing with electromagnetism.

By the standard Green's function method, the solution of Poisson's equation for this case with Dirichlet boundary conditions (potential specified on a closed surface) is given by

$$\Phi(\vec{x}) = \xi \int\int\int_V \rho(\vec{x}'') G(\vec{x}, \vec{x}'')\, \mathrm{d}\vec{x}'' - \frac{1}{4\pi} \oiint_S \Phi(\vec{x}'') \frac{\partial G}{\partial n''}\, \mathrm{d}a'', \tag{1}$$

where V is the volume of the cylindrical box, S its closed surface and $\partial/\partial n''$ is the normal derivative at the surface S of the box directed outwards. Here, $G(\vec{x}, \vec{x}'')$ is Green's function satisfying the equation

$$\nabla_{x''}^2 G(\vec{x}, \vec{x}'') = -4\pi\delta(\vec{x} - \vec{x}''). \tag{2}$$

As the surface of the cylinder in electrostatic equilibrium is at a constant potential Φ_0 we imposed that $G(\vec{x}, \vec{x}'') = 0$ at this surface.

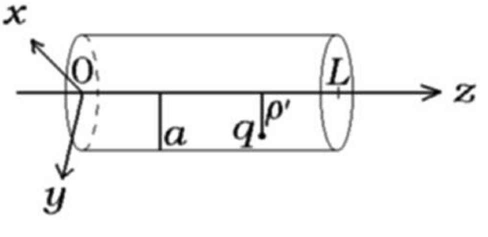

Fig. 1. Finite conducting cylinder of length L and radius a centered at $(\rho, z) = (0, L/2)$, with z being its axis of symmetry. There is a point charge q located at $(\rho, \phi, z) = (\rho' < a, \phi', 0 < z' < L)$.

We can expand the Dirac delta function in cylindrical coordinates as given by

$$\delta(\vec{x}-\vec{x}'')=\delta(\rho-\rho'')\frac{\delta(\phi-\phi'')}{\rho}\,\delta(z-z''). \tag{3}$$

The delta functions for ϕ and z can be written in terms of orthonormal functions:

$$\delta(z-z'')=\frac{2}{L}\left[\sum_{n=1}^{\infty}\sin\frac{n\pi z}{L}\,\sin\frac{n\pi z''}{L}\right], \tag{4}$$

$$\delta(\phi-\phi'')=\frac{1}{2\pi}\left[\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi'')}\right]. \tag{5}$$

Note our particular choice of expansion for z, Eq. (4). This choice satisfies the condition $G(\vec{x},\vec{x}'')=0$ in the upper and lower covers of the cylindrical box. The Green function can be expanded in a similar fashion:

$$G(\vec{x},\vec{x}'')=\frac{1}{\pi L}\left\{\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi'')}\left[\sum_{n=1}^{\infty}\sin\frac{n\pi z}{L}\,\sin\frac{n\pi z''}{L}g_m(k,\rho,\rho'')\right]\right\}, \tag{6}$$

where $k=n\pi/L$ and $g_m(k,\rho,\rho'')$ is the radial Green function to be determined. Substituting this expression into Eq. (2) and using (3)–(5) we obtain

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\,\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right)-\left(k^2+\frac{m^2}{\rho^2}\right)g_m=-\frac{4\pi}{\rho}\delta(\rho-\rho''). \tag{7}$$

For $\rho\neq\rho''$ the right-hand side of Eq. (7) is equal to zero. This means that g_m is a linear combination of modified Bessel functions, $I_m(k\rho)$ and $K_m(k\rho)$. Suppose that $\psi_1(k\rho)$ satisfies the boundary conditions for $\rho<\rho''$ and that $\psi_2(k\rho)$ satisfies the boundary conditions for $\rho>\rho''$:

$$\psi_1(k\rho_<)=AI_m(k\rho_<)+BK_m(k\rho_<), \tag{8}$$

$$\psi_2(k\rho_>)=CI_m(k\rho_>)+DK_m(k\rho_>). \tag{9}$$

Here, A, B, C and D are coefficients to be determined. The symmetry of the Green function in ρ and ρ'' requires that

$$g_m(k,\rho,\rho'')=\psi_1(k\rho_<)\psi_2(k\rho_>), \tag{10}$$

where $\rho_>$ and $\rho_<$ are, respectively, the larger and the smaller of ρ and ρ''. The potential must not diverge for $\rho\to 0$, so we must have $B=0$. The Green function must vanish at $\rho=a$, that is, $\psi_2(ka)=0$. This yields $C=-DK_m(ka)/I_m(ka)$. The function g_m can then be written as

$$g_m(k,\rho,\rho'') = HI_m(k\rho_<)\left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right]. \tag{11}$$

The normalization coefficient $H = AC$ is determined by the discontinuity implied by the delta function in Eq. (7):

$$\left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_+ - \left.\frac{\mathrm{d}g_m}{\mathrm{d}\rho}\right|_- = -\frac{4\pi}{\rho''} = kW[\psi_1,\psi_2], \tag{12}$$

where the $\pm$ signs mean evaluation at $\rho = \rho'' \pm \varepsilon$ and taking the limit $\varepsilon \to 0$. In the last equality, $W[\psi_1,\psi_2]$ is the Wronskian of ψ_1 and ψ_2. Substituting g_m into Eq. (12), and using that $W[I_m(k\rho''), K_m(k\rho'')] = -1/(k\rho'')$, we find $H = 4\pi$. The Green function for the problem of a finite conducting cylinder with a charge inside it can be finally written as

$$G(\vec{x},\vec{x}'') = \frac{4}{L}\left\{\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi'')}\left\{\sum_{n=1}^{\infty} \sin(kz)\sin(kz'')I_m(k\rho_<)\left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right]\right\}\right\}. \tag{13}$$

2.1. Cylinder held at zero potential

Consider the cylinder to be held at zero potential, namely, $\Phi(\vec{x}'') = 0$:

$$\Phi(a,\phi,0\leqslant z\leqslant L) = \Phi(\rho\leqslant a,\phi,L) = \Phi(\rho\leqslant a,\phi,0) = 0. \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (1) yields the potential inside the cylinder as (with $\rho(\vec{x}'') = q\delta(\vec{x}' - \vec{x}'')$)

$$\Phi(\vec{x},\vec{x}') = \frac{4\xi q}{L}\left\{\sum_{m=-\infty}^{\infty}\left\{\sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi\rho_<}{L}\right) \times\left[K_m\left(\frac{n\pi\rho_>}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)\frac{K_m\left(\frac{n\pi a}{L}\right)}{I_m\left(\frac{n\pi a}{L}\right)}\right]\right\}\right\}. \tag{15}$$

Here, $\rho_>$ ($\rho_<$) is the larger (smaller) of ρ and ρ'.

3. Infinite conducting cylinder with internal point charge

The solution for an infinite cylinder differs from the solution of the finite cylinder by changing essentially the expansion of the delta function in Eq. (4). In the infinite cylinder, there is no restriction on the choice of n (or k):

$$\delta(z - z'') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-z'')}\, dk = \frac{1}{\pi}\int_0^{\infty} \cos[k(z - z'')]\, dk. \tag{16}$$

The Green function can be written as

$$G(\vec{x}, \vec{x}'') = \frac{2}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi'')}\int_0^{\infty} \cos[k(z - z'')] I_m(k\rho_<) \times \left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right] dk\right\}. \tag{17}$$

Note that we can pass from Eq. (4) to Eq. (16) by transforming the Fourier series into the Fourier transform. That is, by letting $L \to \infty$, setting $n\pi/L = k$, $dk = \pi/L$, $z \to z + L/2$, $z'' \to z'' + L/2$ and by replacing the infinite sum by the integral over k.

3.1. Cylinder held at zero potential

Consider the cylinder to be held at zero potential, namely, $\Phi(a, \phi, z) = 0$. Substituting Eq. (17) into Eq. (1), the potential inside the cylinder can be written as (with $\rho(\vec{x}'') = q\delta(\vec{x}' - \vec{x}'')$)

$$\Phi(\vec{x}, \vec{x}') = \frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} \cos[k(z - z')] I_m(k\rho_<) \times \left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right] dk\right\}. \tag{18}$$

Once more $\rho_>$ ($\rho_<$) is the larger (smaller) of ρ and ρ'.

The electric field is given by $\vec{E} = -\nabla\Phi$, with components

$$E_\rho(\rho < \rho') = -\frac{\partial\Phi}{\partial\rho} = -\frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} k\cos[k(z - z')] I'_m(k\rho) \times \left[K_m(k\rho') - I_m(k\rho')\frac{K_m(ka)}{I_m(ka)}\right] dk\right\}, \tag{19}$$

$$E_\rho(\rho > \rho') = -\frac{\partial\Phi}{\partial\rho} = -\frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} k\cos[k(z - z')] I_m(k\rho') \times \left[K'_m(k\rho) - I'_m(k\rho)\frac{K_m(ka)}{I_m(ka)}\right] dk\right\}, \tag{20}$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = \frac{4\xi q}{\pi\rho}\left\{\sum_{m=1}^{\infty} m\sin[m(\phi-\phi')]\int_0^\infty \cos[k(z-z')]I_m(k\rho_<)\right.$$
$$\left.\times\left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right]\mathrm{d}k\right\}, \tag{21}$$

$$E_z = -\frac{\partial \Phi}{\partial z} = \frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\int_0^\infty k\sin[k(z-z')]I_m(k\rho_<)\right.$$
$$\left.\times\left[K_m(k\rho_>) - I_m(k\rho_>)\frac{K_m(ka)}{I_m(ka)}\right]\mathrm{d}k\right\}. \tag{22}$$

The force $\vec{F} = q\vec{E}(\vec{x}')$ acting upon the charge q is given by Eq. (19) at $\vec{x} = \vec{x}'$ without the first term between brackets (it is the field generated by the charge q itself). There is only a radial component when the cylinder has an infinite length

$$\vec{F}(\vec{x}') = \frac{2\xi q^2}{\pi}\left\{\sum_{m=-\infty}^{\infty}\int_0^\infty kI_m(k\rho')I'_m(k\rho')\frac{K_m(ka)}{I_m(ka)}\,\mathrm{d}k\right\}\hat{\rho}$$
$$= -\frac{\xi q^2}{\pi\rho'^2}\left\{\sum_{m=-\infty}^{\infty}\int_0^\infty I_m^2(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{K_m(xa/\rho')}{I_m(xa/\rho')}\right]\mathrm{d}x\right\}\hat{\rho}. \tag{23}$$

In the last equation we integrated by parts. We plotted in Fig. 2 the force of Eq. (23), normalized by $F_0 \equiv \xi q^2/a^2$, as a function of ρ'/a. This force goes to zero when $\rho'/a = 0$ and diverges when $\rho' \to a$, as expected.

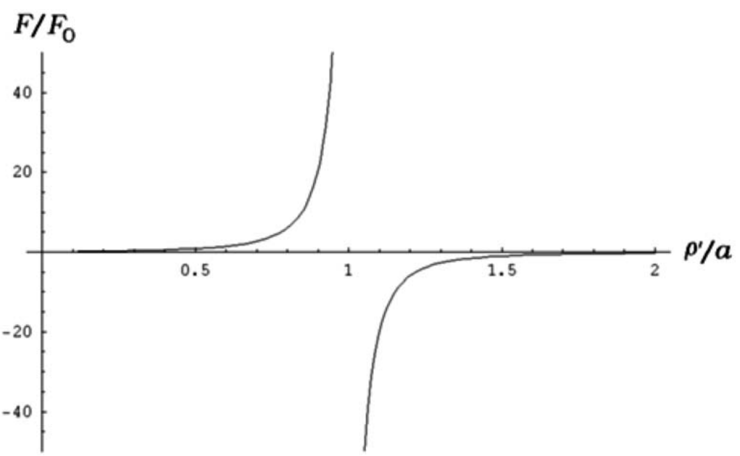

Fig. 2. Force F between an infinite grounded conducting cylinder of radius a centered on the z axis and a point charge q at a distance ρ' from the z axis, normalized by $F_0 = \xi q^2/a^2$.

The surface charges can be calculated using Gauss' law, yielding

$$\sigma(a,\phi,z) = \frac{E_\rho(a,\phi,z)}{4\pi\xi}$$
$$= -\frac{q}{2\pi^2 a}\left[\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty \cos[k(z-z')]\frac{I_m(k\rho')}{I_m(ka)}\,dk\right]. \tag{24}$$

The surface charges per unit of length $\lambda(z)$ is given by

$$\lambda(a,z) = \int_0^{2\pi}\sigma(a,\phi,z)a\,d\phi = -\frac{q}{\pi}\int_0^\infty \cos[k(z-z')]\frac{I_0(k\rho')}{I_0(ka)}\,dk. \tag{25}$$

The total charge induced in the cylinder supposing $z'=0$ can be obtained integrating Eq. (25) from $z=-\infty$ to ∞. Utilizing

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^\infty \cos(kz)\,dz = \frac{1}{\pi}\int_0^\infty \cos(kz)\,dz, \tag{26}$$

this yields

$$Q = \int_{-\infty}^\infty \lambda(a,z)\,dz = -q. \tag{27}$$

4. Infinite conducting cylinder with external point charge

Suppose that the point charge q is located at $\vec{x}' = (\rho',\phi',z')$, with $\rho' > a$. Green's function can be written analogously in this case as

$$G(\vec{x},\vec{x}'') = \frac{1}{2\pi^2}\left[\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi'')}\int_0^\infty \cos[k(z-z'')]g_m(k,\rho,\rho'')\,dk\right], \tag{28}$$

where g_m can be written as the product of $\psi'_1(\rho<\rho'')\psi'_2(\rho>\rho'')$. The functions ψ'_1 and ψ'_2 satisfy the modified Bessel equation. They can be written as a linear combination of the possible solutions

$$\psi'_1(k\rho_<) = A' I_m(k\rho_<) + B' K_m(k\rho_<), \tag{29}$$

$$\psi'_2(k\rho_>) = C' I_m(k\rho_>) + D' K_m(k\rho_>). \tag{30}$$

For $\rho\to\infty$ Green's function must remain finite. This means that $C'=0$. Additionally, Green's function must be zero at the boundary surface. That is, $G=0$ at the surface of the cylinder $\rho=a$. This yields

$$\psi'_1(a) = A' I_m(ka) + B' K_m(ka) = 0 \quad \to \quad B' = -A'\frac{I_m(ka)}{K_m(ka)}. \tag{31}$$

In order to obtain the function g_m we still have to find the constant H':

$$g_m(k,\rho,\rho'') = H'\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>), \quad (32)$$

where $\rho_>$ ($\rho_<$) is the larger (smaller) of ρ and ρ''.

From Eq. (12) we have that $H' = 4\pi$:

$$g_m(k,\rho,\rho'') = 4\pi\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>). \quad (33)$$

The Green function is then given by

$$G(\vec{x},\vec{x}'') = \frac{2}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi'')}\int_0^\infty \cos[k(z-z'')] \times\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>)\,dk\right\}. \quad (34)$$

4.1. Cylinder held at zero potential

Suppose that the surface of the cylinder is held at zero potential, namely

$$\Phi(a,\phi,z) = 0. \quad (35)$$

Applying Eqs. (34) and (35) in Eq. (1) with $\rho(\vec{x}'') = q\delta(\vec{x}' - \vec{x}'')$ yield

$$\Phi(\vec{x},\vec{x}') = \frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty \cos[k(z-z')] \times\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>)\,dk\right\}. \quad (36)$$

Here, $\rho_>$ ($\rho_<$) is the larger (smaller) of ρ and ρ'.

Far from the origin, ρ is much larger than ρ', hence we can express Eq. (36) in approximate form. The first term that appears between brackets is given by $I_m(k\rho_<)K_m(k\rho_>)$, with $\rho_< = \rho'$ and $\rho_> = \rho$. Note the presence of the term $K_m(k\rho)$, with $\rho \gg \rho'$, which decays rapidly for increasing k. This implies that the main contribution of the integrand is in the region $0 < k < 1/\rho$. Then we can approximate $I_m(k\rho')$ for small arguments, that is, for $k\rho' \ll 1$, yielding $I_m(k\rho') \approx (k\rho'/2)^m/m!$. From this we can see that the most relevant term is the first one, $m = 0$. The integral of the first term between brackets in Eq. (36) is then given by

$$\Phi_1(\rho \gg \rho') \approx \frac{2\xi q}{\pi}\int_0^\infty \cos[k(z-z')]K_0(k\rho)\,dk = \frac{\xi q}{\rho}, \quad (37)$$

where we have used in the last equation the identity: $(2/\pi)\int_0^\infty \cos(xt)K_0(yt)\,dt = 1/\sqrt{x^2+y^2}$, [2, Problem 11.5.11]. The second term that appears between brackets in Eq. (36) can be treated in a similar way. The main contribution of the integrand is in the region $0 < k < 1/\rho$. Again, the most relevant term is the first one. Accordingly, we approximate the function $K_0(k\rho')$ for small arguments: $K_0(k\rho') \approx -\ln(k\rho')$. This yields

$$\Phi_2(\rho \geqslant \rho') \approx -\frac{2\xi q}{\pi}\int_0^{\infty} \cos[k(z-z')]\frac{\ln(k\rho')}{\ln(ka)}K_0(k\rho)\,\mathrm{d}k. \tag{38}$$

From Eq. (36) the electric field is given by $\vec{E} = -\nabla\Phi$, with components

$$E_\rho(\rho<\rho') = -\frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\int_0^{\infty} k\cos[k(z-z')] \times\left[I'_m(k\rho) - K'_m(k\rho)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho')\,\mathrm{d}k\right\}, \tag{39}$$

$$E_\rho(\rho>\rho') = -\frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\int_0^{\infty} k\cos[k(z-z')] \times\left[I_m(k\rho') - K_m(k\rho')\frac{I_m(ka)}{K_m(ka)}\right]K'_m(k\rho)\,\mathrm{d}k\right\}, \tag{40}$$

$$E_\phi = \frac{4\xi q}{\pi\rho}\left\{\sum_{m=1}^{\infty} m\sin[m(\phi-\phi')]\int_0^{\infty} \cos[k(z-z')] \times\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>)\,\mathrm{d}k\right\}, \tag{41}$$

$$E_z = \frac{2\xi q}{\pi}\left\{\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')}\int_0^{\infty} k\sin[k(z-z')] \times\left[I_m(k\rho_<) - K_m(k\rho_<)\frac{I_m(ka)}{K_m(ka)}\right]K_m(k\rho_>)\,\mathrm{d}k\right\}. \tag{42}$$

The force $\vec{F} = q\vec{E}(\vec{x}')$ acting upon the charge q is given by Eq. (39) at $\vec{x} = \vec{x}'$ without the first term between brackets (it is the field generated by the charge q itself). There is only a radial component

$$\vec{F}(\vec{x}') = \frac{2\xi q^2}{\pi}\left[\sum_{m=-\infty}^{\infty}\int_0^{\infty} kK_m(k\rho')K'_m(k\rho')\frac{I_m(ka)}{K_m(ka)}\,\mathrm{d}k\right]\hat{\rho} = -\frac{\xi q^2}{\pi\rho'^2}\left\{\sum_{m=-\infty}^{\infty}\int_0^{\infty} K_m^2(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{I_m(ax/\rho')}{K_m(ax/\rho')}\right]\mathrm{d}x\right\}\hat{\rho}, \tag{43}$$

In the last equation we integrated by parts. We plot the force of Eq. (43) in Fig. 2, normalized by $F_0 \equiv \xi q^2/a^2$, as a function of ρ'/a. This force goes to zero when $\rho'/a \to \infty$ and diverges when $\rho' \to a$, as expected.

The surface charges can be calculated using Gauss' law, yielding

$$\sigma(a,\phi,z) = \frac{E_\rho(a,\phi,z)}{4\pi\xi} = -\frac{q}{2\pi^2 a}\left[\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^\infty \cos[k(z-z')]\frac{K_m(k\rho')}{K_m(ka)}\,\mathrm{d}k\right]. \tag{44}$$

The surface charge per unit of length $\lambda(z)$ is given by

$$\lambda(a,z) = \int_0^{2\pi} \sigma(a,\phi,z)a\,\mathrm{d}\phi = -\frac{q}{\pi}\int_0^\infty \cos[k(z-z')]\frac{K_0(k\rho')}{K_0(ka)}\,\mathrm{d}k. \tag{45}$$

It is interesting to obtain the behaviour of λ for a thin wire, far from z' ($|z-z'| \gg \rho' \gg a$). Utilizing Eq. (3.150) of [1] we obtain

$$\lambda \approx -\frac{q}{2\ln(|z|/a)}\frac{1}{\sqrt{\rho'^2+z^2}}. \tag{46}$$

The total charge induced in the cylinder can be obtained integrating Eq. (45) from $z=-\infty$ to ∞. Utilizing Eq. (26) this yields

$$Q = \int_{-\infty}^{\infty} \lambda(a,z)\,\mathrm{d}z = -q. \tag{47}$$

A plot of $\lambda(a,z)$ as a function of z, with $z'=0$ and normalized by q/ρ', is given in Fig. 3. The maximum value of $\lambda(a,z)$ is given at $z=z'$, as expected. In Fig. 4 we plot $\lambda_{\max}$ as a function of ρ'/a, normalized by q/ρ'. From this figure we can see that

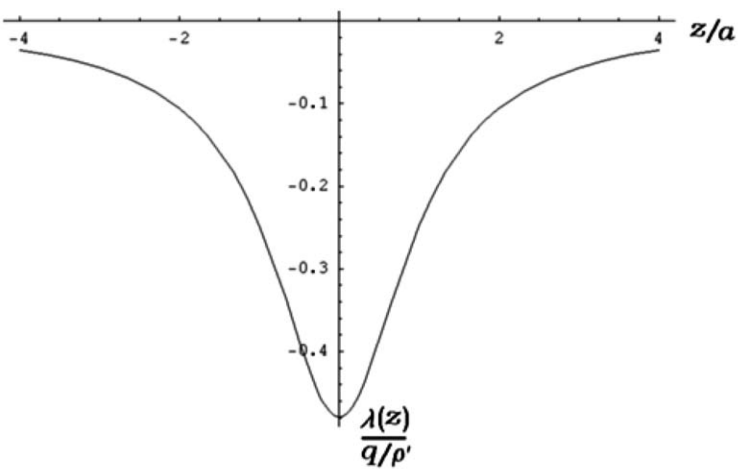

Fig. 3. Induced linear charge density λ on the conducting cylinder with a point charge outside it, Eq. (45), as a function of z/a. We utilized $z'=0$, $\rho'/a=2$ and normalized by q/ρ'.

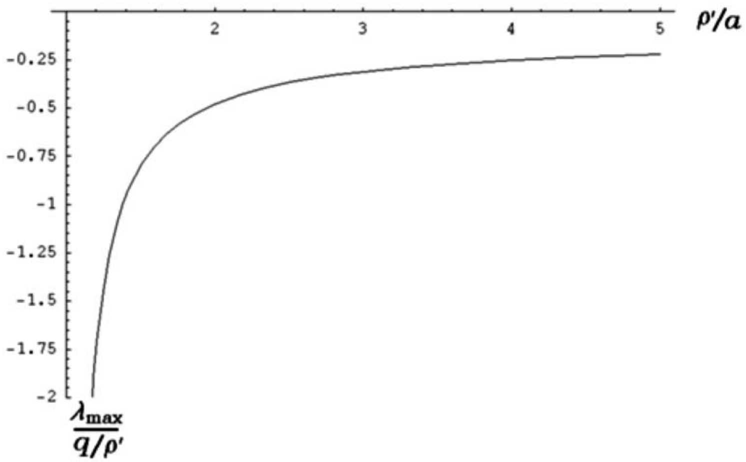

Fig. 4. Maximum induced linear charge density $\lambda_{\max}(z=z')$ on the conducting cylinder with a point charge outside it, Eq. (45), as a function of ρ'/a. We normalized the plot by q/ρ'.

$\lambda_{\max} \to 0$ when $\rho'/a \to \infty$, that is, for a conducting cylinder of zero thickness, a simple conducting straight line.

4.2. Thin wire held at zero potential

Consider that the grounded conducting cylinder is very thin, i.e., $a \ll \rho'$. The modified Bessel functions can be approximated for small argument by [3, Section 8.44]

$$I_m(y \ll 1) \approx \frac{1}{m!}\frac{y^m}{2^m}, \tag{48}$$

$$K_m(y \ll 1) \approx \frac{(m-1)!2^{m-1}}{y^m}, \quad m>0, \tag{49}$$

$$K_0(y \ll 1) \approx -\ln\frac{y}{2} - \gamma. \tag{50}$$

Here $\gamma = 0.577$ is the Euler–Mascheroni constant.

The term between brackets in Eq. (43) for $m=0$ and for $m>0$ can be approximated by, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{1}{-\ln(ax/2\rho')-\gamma}\right] \approx -\frac{1}{\ln(a/\rho')},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\,\frac{1}{m!}\,\frac{(ax/\rho')^m}{2^m}\,\frac{(ax/\rho')^m}{(m-1)!2^{m-1}}\right] \approx \frac{(2m+1)x^{2m}(a/\rho')^{2m}}{m!(m-1)!2^{2m-1}}.$$

The most relevant term for $\rho' \gg a$ is therefore $m = 0$. Using the identity $\int_0^\infty K_0^2(x)\,\mathrm{d}x = \pi^2/4$ we have the force acting upon the charge q as given by

$$\vec{F}(\rho' \gg a) \approx -\frac{\xi q^2}{\pi\rho'}\int_0^\infty K_0^2(x)\frac{\ln(2\rho'/xa) - \gamma + 1}{[\gamma - \ln(2\rho'/xa)]^2}\,\mathrm{d}x\hat{\rho}$$
$$\approx -\frac{\xi q^2}{\pi\rho'^2\ln(\rho'/a)}\int_0^\infty K_0^2(x)\,\mathrm{d}x\hat{\rho} = -\frac{\xi q^2\pi}{4\rho'^2\ln(\rho'/a)}\,\hat{\rho}. \tag{51}$$

Alternatively, another expression for the force can be found by integrating the force exerted by the linear charge density of a thin cylinder, $\lambda(a,z)$ of Eq. (45), acting upon the point charge q. Utilizing that

$$\int_{-\infty}^{\infty}\frac{\rho'\cos[k(z-z')]}{[\rho'^2 + (z-z')^2]^{3/2}}\,\mathrm{d}z = 2kK_1(k\rho')$$

we obtain

$$\vec{F}(\vec{x}') = -\xi q\int_{-\infty}^{\infty}\frac{\rho'}{\sqrt{\rho'^2+z^2}}\frac{\lambda(a,z)}{\rho'^2+z^2}\,\mathrm{d}z\hat{\rho} = -\frac{2\xi q^2}{\pi\rho'^2}\int_0^\infty x\,\frac{K_0(x)K_1(x)}{K_0(xa/\rho')}\,\mathrm{d}x\hat{\rho}. \tag{52}$$

To compare Eqs. (51) and (52) we can expand the latter using the approximation $\rho' \gg a$. Using that $K_1(x) = -\mathrm{d}K_0/\mathrm{d}x$, integrating by parts, and $K_0(xa/\rho') \approx -\ln(xa/2\rho') - \gamma \approx \ln(\rho'/a)$ we obtain

$$\vec{F} = \frac{2\xi q^2}{\pi\rho'^2}\int_0^\infty x\frac{K_0(x)(\mathrm{d}K_0/\mathrm{d}x)}{K_0(xa/\rho')}\,\mathrm{d}x\hat{\rho} \approx -\frac{\xi q^2}{\pi\rho'}\int_0^\infty K_0^2(x)\frac{\ln(2\rho'/xa)-\gamma+1}{[\gamma-\ln(2\rho'/xa)]^2}\,\mathrm{d}x\hat{\rho}$$
$$\approx -\frac{\xi q^2\pi}{4\rho'^2\ln(\rho'/a)}\,\hat{\rho}, \tag{53}$$

which is exactly Eq. (51).

4.3. Infinite cylinder held at constant potential

Suppose that the conducting cylinder is held at a constant potential, $\Phi(a,\phi,z) = \Phi_0$. From Eq. (34) we obtain (with $n'' = \rho_<$ and $\rho_> = \rho$)

$$\left.\frac{\partial G}{\partial n''}\right|_{\rho''=a} = -\frac{2}{\pi}\left\{\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi'')}\int_0^\infty k\cos[k(z-z'')]\frac{K_m(k\rho)}{K_m(ka)}\right.$$
$$\left.\times[I_m'(k\rho'')K_m(ka) - I_m(ka)K_m'(k\rho'')]\,\mathrm{d}k\right\}_{\rho''=a}$$
$$= \frac{2}{\pi a}\left\{\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi'')}\int_0^\infty\cos[k(z-z'')]\frac{K_m(k\rho)}{K_m(ka)}\,\mathrm{d}k\right\}. \tag{54}$$

In the last equality, we used the Wronskian relation $W[I_m(k\rho''), K_m(k\rho'')] = -1/(k\rho'')$.

The second term given by Eq. (1) can be written as

$$\begin{aligned}\Phi^{+} = & -\frac{1}{4\pi}\oiint_S \Phi(\vec{x}'')\frac{\partial G}{\partial n''}\,\mathrm{d}a'' = \frac{\Phi_0}{2\pi^2 a}\int_{-\infty}^{\infty} a\,\mathrm{d}z''\int_0^{2\pi}\mathrm{d}\phi'' \\ & \times\left\{\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\phi')}\int_0^{\infty}\cos[k(z-z')]\frac{K_m(k\rho)}{K_m(ka)}\,\mathrm{d}k\right\} \\ = & \frac{\Phi_0}{\pi}\int_{-\infty}^{\infty}\mathrm{d}z''\int_0^{\infty}\cos[k(z-z')]\frac{K_0(k\rho)}{K_0(ka)}\,\mathrm{d}k \\ = & \frac{2\Phi_0}{\pi}\int_0^{\infty}\mathrm{d}z''\int_0^{\infty}\cos[k(z-z')]\frac{K_0(k\rho)}{K_0(ka)}\,\mathrm{d}k. \end{aligned} \tag{55}$$

In the last equality, we changed the limits of the integral over z''.

In order to calculate the last integral, we utilize Eq. (16). Changing variables, we have

$$\int_0^{\infty}\cos[k(z-z')]\,\mathrm{d}z' = \pi\delta(k). \tag{56}$$

The approximation for small argument of $K_0(y)$, namely, $K_0(y) \approx -\ln y$, is in this case inappropriate, because the term $\lim_{k\to 0} K_0(k\rho)/K_0(ka) \to 1$ for any ρ. This is true for an infinite cylinder, but gives no physical insight into the behaviour of the potential as a function of ρ. We should use instead $k \ll 1/\rho < 1/a$, yielding

$$\Phi^{+} \approx \Phi_0\frac{\ln(k\rho)}{\ln(ka)} \quad \text{for } k \ll 1/\rho < 1/a. \tag{57}$$

The potential outside an infinite conducting cylinder with an external charge, held at a constant potential Φ_0, is then given by the summation of Eqs. (36) and (57).

We can find the potential of a cylinder held at a constant potential Φ_0 by a different method. Suppose we have a long straight line of length ℓ along the z axis, uniformly charged with a linear charge density λ. The potential at a distance ρ from the z-axis, for $\ell \gg \rho$, is given by

$$\Phi_{\text{line}} \approx 2\xi\lambda\ln\frac{\ell}{\rho}. \tag{58}$$

At a distance $\rho = a$ from the z-axis, we have a constant potential $\Phi_0 = 2\lambda\ln(\ell/a)$, which is the same boundary condition as before. This implies that the solution is the same. Substituting λ, we obtain the potential as given by

$$\Phi_{\text{line}} = \Phi_0\frac{\ln(\ell/\rho)}{\ln(\ell/a)}. \tag{59}$$

Note that Eq. (57) with $k \ll 1/\rho < 1/a$ and Eq. (59) with $\ell \gg a > \rho$ are essentially the same. Henceforth, we utilize Eq. (59) as the solution for a long cylinder held at a constant potential.

The final potential of the problem of a long conducting cylinder held at a constant potential Φ_0 with an external charge q is given by

$$\Phi(\vec{x}, \vec{x}') = \frac{2\xi q}{\pi} \left\{ \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^{\infty} \cos[k(z-z')] K_m(k\rho_>) \times \left[I_m(k\rho_<) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho_<) \right] \mathrm{d}k \right\} + \Phi_0 \frac{\ln(\ell/\rho)}{\ln(\ell/a)}. \quad (60)$$

The electric field, the force exerted on q, the surface charge density and the linear charge density are given by, respectively,

$$E_\rho(\rho < \rho') = -\frac{2\xi q}{\pi} \left\{ \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^{\infty} k \cos[k(z-z')] \times \left[I'_m(k\rho) - \frac{I_m(ka)}{K_m(ka)} K'_m(k\rho) \right] K_m(k\rho') \,\mathrm{d}k \right\} + \frac{\Phi_0}{\rho \ln(\ell/a)}, \quad (61)$$

$$E_\rho(\rho > \rho') = -\frac{2\xi q}{\pi} \left\{ \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^{\infty} k \cos[k(z-z')] \times \left[I_m(k\rho') - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho') \right] K'_m(k\rho) \,\mathrm{d}k \right\} + \frac{\Phi_0}{\rho \ln(\ell/a)}, \quad (62)$$

$$E_\phi = \frac{4\xi q}{\pi\rho} \left\{ \sum_{m=1}^{\infty} m \sin[m(\phi-\phi')] \int_0^{\infty} \cos[k(z-z')] \times [I_m(k\rho_<) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho_<)] K_m(k\rho_>) \,\mathrm{d}k \right\}, \quad (63)$$

$$E_z = \frac{2\xi q}{\pi} \left\{ \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^{\infty} k \sin[k(z-z')] \times \left[I_m(k\rho_<) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho_<) \right] K_m(k\rho_>) \,\mathrm{d}k \right\}, \quad (64)$$

$$\vec{F}(\vec{x}') = -\frac{\xi q^2}{\pi \rho'^2} \left\{ \sum_{m=-\infty}^{\infty} \int_0^{\infty} K_m^2(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[x \frac{I_m(ax/\rho')}{K_m(ax/\rho')} \right] \mathrm{d}x \right\} \hat{\rho} + \frac{q\Phi_0}{\rho' \ln(\ell/a)} \hat{\rho}, \quad (65)$$

$$\sigma(a, \phi, z) = -\frac{q}{2\pi^2} \left\{ \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi-\phi')} \int_0^{\infty} k \cos[k(z-z')] \frac{K_m(k\rho')}{K_m(ka)} \,\mathrm{d}k \right\} + \frac{\Phi_0}{4\pi\xi a \ln(\ell/a)}, \quad (66)$$

$$\lambda(a,z) = -\frac{q}{\pi}\int_0^\infty \cos[k(z-z')]\frac{K_0(k\rho')}{K_0(ka)}\,\mathrm{d}k + \frac{\Phi_0}{2\xi\,\ln(\ell/a)}. \tag{67}$$

From Eq. (67) we can calculate the total charge on the cylinder:

$$\begin{aligned} Q &= \int_{-\infty}^{\infty}\lambda(a,z)\,\mathrm{d}z = -\frac{q}{\pi}\int_{-\infty}^{\infty}\mathrm{d}z\int_0^\infty \cos[k(z-z')]\frac{K_0(k\rho')}{K_0(ka)}\,\mathrm{d}k + \frac{\ell\Phi_0}{2\xi\,\ln(\ell/a)} \\ &= -\frac{q}{\pi\rho'^2}\lim_{k\to 0}\frac{\ln(k\rho')}{\ln(ka)} + \frac{\ell\Phi_0}{2\ln(\ell/a)} = -q + \frac{\ell\Phi_0}{2\xi\,\ln(\ell/a)}. \end{aligned} \tag{68}$$

For a neutral charged cylinder, i.e., $Q=0$, we can relate the constant potential Φ_0 with the charge q by

$$\Phi_0 = \frac{2\xi q\,\ln(\ell/a)}{\ell}. \tag{69}$$

5. Discussion

We can express the force exerted by the grounded conducting infinite cylinder of radius a upon the external point charge q at a distance ρ' from the axis of the cylinder as given by

$$\vec{F} = -\alpha_L\frac{\xi q^2}{\rho'^2}\,\hat{\rho}, \tag{70}$$

where α_L is a dimensionless parameter. In this work, we obtained three different expressions for this force, namely, Eqs. (43), (51) and (52). The parameter α_L for these three cases is given by, respectively,

$$\alpha_L = \frac{1}{\pi}\left\{\sum_{m=-\infty}^{\infty}\int_0^\infty K_m^2(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x\,\frac{I_m(ax/\rho')}{K_m(ax/\rho')}\right]\mathrm{d}x\right\}, \tag{71}$$

$$\alpha_L \approx \frac{1}{\pi}\int_0^\infty K_0^2(x)\,\frac{\ln(2\rho'/xa)-\gamma+1}{[\gamma-\ln(2\rho'/xa)]^2}\,\mathrm{d}x \approx \frac{\pi}{4\ln(\rho'/a)}, \tag{72}$$

$$\alpha_L = \frac{2}{\pi}\int_0^\infty x\,\frac{K_0(x)K_1(x)}{K_0(xa/\rho')}\,\mathrm{d}x. \tag{73}$$

We plot these three values of α_L as functions of a/ρ' in Figs. 5 and 6. We can see that these three values of α_L converge to one another as $a/\rho' \to 0$. This was expected because Eq. (43) is valid for a cylinder of finite thickness with arbitrary value of a/ρ', while Eqs. (51) and (52) are valid only for a thin cylinder, that is, for $a/\rho' \to 0$.

From Eq. (72) we can see that when $a/\rho' \ll 1$, the parameter α_L behaves as $\pi/[4\ln(\rho'/a)]$. That is, it goes to zero when $a/\rho' \to 0$. According to these calculations we conclude that there is no force between a point charge and an idealized grounded conducting line (of zero thickness). One of the authors (AKTA) had expected $0<\alpha_L<1$ [4], not specifically for a grounded conducting line, but for a conducting line with zero total charge. In particular, he expected that $0.1<\alpha_L<0.9$, by guessing

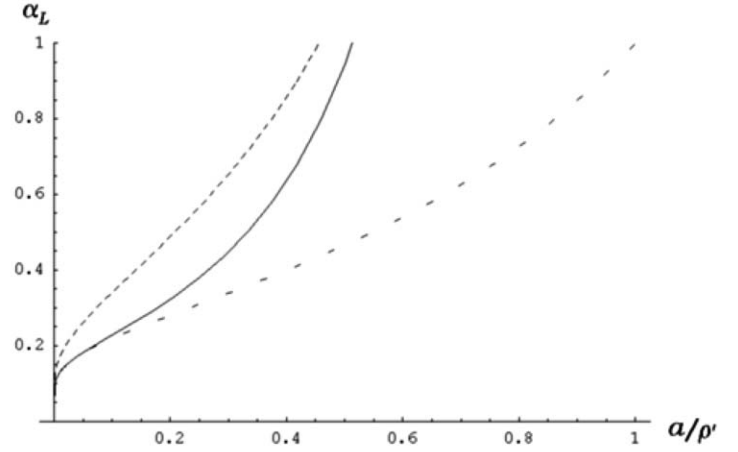

Fig. 5. Dimensionless parameter α_L given by Eq. (70) as a function of a/ρ'. The continuous line represents the parameter from Eq. (71); the tight-dashed line that of Eq. (72); and the light-dashed line that of Eq. (73).

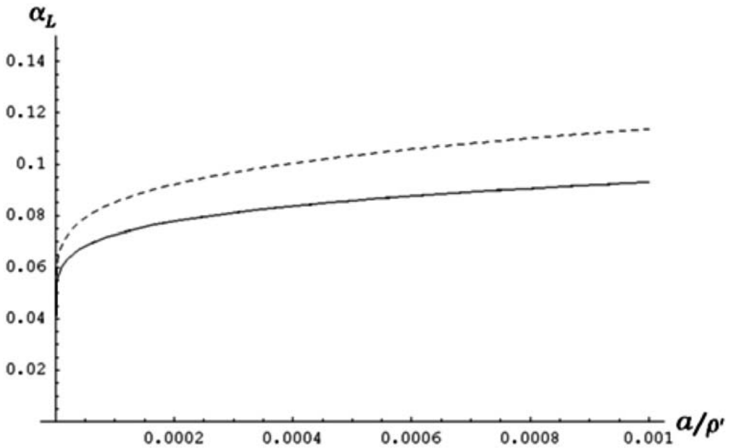

Fig. 6. Dimensionless parameter α_L given by Eq. (70) as a function of a/ρ', for the region $a/\rho' \ll 1$. The continuous line represents the parameter from Eq. (71); the tight-dashed line that of Eq. (72); and the light-dashed line that of Eq. (73).

the result based on dimensional analysis and in analogy with the case of a point charge q at a distance ρ' from an infinite conducting plane. In this last case, the net force upon the test charge is given by $\alpha_P \xi q^2/\rho'^2$, with $\alpha_P = \frac{1}{4}$. The results of the calculations presented here, on the other hand, indicate that $\alpha_L = 0$ when $a/\rho' = 0$ (in the case of a grounded infinite line). This is an interesting result indicating that the existence of a force upon the external test charge requires not only that it is at a finite distance to the cylinder, but also the existence of a surface area different from zero in the conductor with which it is interacting.

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