Chapter 3
Newton and Inverse Problems

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3.1 Introduction

Isaac Newton (1642–1727) is one of the main scientists which ever lived. His two most important works were the *Principia*, first published in 1687, and the *Opticks*, first published in 1704. He always lived in England, entering Trinity College, in Cambridge, in 1661. He obtained the Bachelor of Arts degree in 1665, becoming in 1669 Lucasian Professor at Cambrigde University.

In this work we consider his research approach. We show that he always considered the inverse aspects of any problem. We show that this way of dealing with physics and mathematics is one of the sources of his immense creativity.

3.2 Inverse Problems in Mathematics

A very important period in Newton's life were the *anni mirabiles*, from 1664 to 1666, during which he obtained his first important results in mathematics and physics. In his own description of this period we can observe clearly his way of thinking (Westfall, 1990, p. 143)

> In the beginning of the year 1665 I found the Method of approximating series & the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of Tangents of Gregory & Slusius, & in November had the direct method of fluxions & the next year in January had the Theory of Colours & in May following I had entrance into y^e inverse method of fluxions. And the same year I began to think of gravity extending to y^e orb of the Moon & (having found how to estimate the force with w^{ch} [a] globe revolving within a sphere presses the surface of the sphere) from Keplers rule of the periodical times of the Planets being in sesquialterate proportion of their distances from the center of their Orbs, I deduced that the forces w^{ch} keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centers about w^{ch} they revolve: & thereby compared the force requisite to keep the Moon in her Orb with the force of gravity

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D. Krause, A. Videira (eds.), *Brazilian Studies in Philosophy and History of Science*, Boston Studies in the Philosophy of Science 290, DOI 10.1007/978-90-481-9422-3_3,

> at the surface of the earth, & found them answer pretty nearly. All this was in the two plague years of 1665–1666. For in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more then at any time since.

That is, he found direct and inverse methods of fluxions, which are the essence of our differential and integral calculus. From the method of tangents he could calculate derivatives and he could calculate areas by quadratures. His discovery of the fundamental theorem of the calculus linking integration as the inverse of differentiation also comes from his *anni mirabile*, [1, pp. 123–128].

3.3 Inverse Problems in Optics

In his book *Optics* Newton offered several examples of how he tackled opposite problems in physics, [2]. The work is divided into three books. The first book has two parts, dealing with the decomposition of white light into the colours of the spectrum after passing through a prism. The first part begins with eight definitions (of rays of light, their refrangibility and reflexibility etc.), eight axioms (the angle of reflexion is equal to the angle of incidence etc.), six theorems, two problems and sixteen experiments. The second part has five theorems, six problems and seventeen experiments. The second book deals with reflexions, refractions and colours of thin and thick transparent bodies (Newton's rings). The first part contains twenty-four observations. The second part has remarks upon the foregoing observations. The third part deals with the permanent colours of natural bodies and their analogy to colours of thin transparent plates, containing twenty propositions. The fourth part has thirteen observations concerning the reflexions and colours of thick transparent polished plates. The first part of the third book has eleven observations concerning the inflexions (difractions) of the rays of light and the colours made thereby. At the end of the book there are thirty-one Queries dealing with several aspects not only of optics, but also of mechanics, physics and philosophy in general. Although the structure of the book is somewhat similar to Euclid's *Elements*, the demonstrations of the propositions (also called theorems by Newton) are not based on pure logic as a set of constructions and reasonings following from the axioms. In the *Opticks* the proofs of the propositions are, in Newton's words, made "by experiments." This is a remarkable new feature introduced by Newton.

Let us see how the inverse aspects of the problems are handled by Newton in the field of optics. After presenting the definitions and axioms, he introduced a series of propositions, theorems and problems. His fourth proposition (also called the first problem) of the first part of book I runs as follows, [2, p. 64]: "To separate from one another the heterogeneous Rays of compound Light." He let the Sun's light into his darkened chamber through a small hole in his windowshut. About eleven feet from the window he placed a lens and after that a prism which separated the Sun's light into the colours of the spectrum upon a white paper. In the fifth proposition (also called theorem 4) of the second part of book I he explored the opposite effect, [2, p. 134]: "Whiteness and all grey Colours between white and black, may be coumpounded of Colours, and the whiteness of the Sun's Light is compounded of all the primary Colours mix'd in a due Proportion." In this case the proof is presented by six detailed experiments.

In the fifth Query at the end of the *Opticks* we see once more Newton considering both aspects of a problem (Newton, 1979, p. 339):

> *Qu. 5.* Do not Bodies and Light act mutually upon one another; that is to say, Bodies upon Light in emitting, reflecting, refracting and inflecting it, and Light upon Bodies for heating them, and putting their parts into a vibrating motion wherein heat consists?

The last Queries of the *Optics*, 30 and 31, are other examples of this aspect of Newton's way of thinking (Newton, 1979, pp. 374–6):

> *Quest. 30.* Are not gross Bodies and Light convertible into one another, and may not Bodies receive much of their Activity from the Particles of Light which enter their Composition? For all fix'd Bodies being heated emit Light so long as they continue sufficiently hot, and Light mutually stops in Bodies as often as its Rays strike upon their Parts, as we shew'd above. I know no Body less apt to shine than Water; and yet Water by frequent Distillations changes into fix'd Earth, as Mr. *Boyle* has try'd; and then this Earth being enabled to endure a sufficient Heat, shines by Heat like other Bodies.
>
> The changing of Bodies into Light, and Light into Bodies, is very comformable to the Course of Nature, which seems delighted with Transmutations. [...]
>
> *Quest. 31.* Have not the small Particles of Bodies certain Powers, Virtues, or Forces, by which they act at a distance, not only upon the Rays of Light for reflecting, refracting, and inflecting them, but also upon one another for producing a great Part of the Phaenomena of Nature? [...]

3.4 Inverse Problems in Mechanics

We now consider Newton's masterpiece, the *Principia*, [3]. It begins with eight definitions (quantity of matter etc.), a Scholium about absolute motion, his three laws of motion (which he also called as axioms), six corollaries, followed by another Scholium where he discussed the laws of collision etc. The remainder of the work is divided into three books. The first one deals with the motion of bodies, containing ninety-eight propositions (50 theorems and 48 problems). The second book deals with the motion of bodies in resisting mediums, containing fifty-three propostions (41 theorems and 12 problems). The third book deals with the system of the world in mathematical treatment. It begins with four rules of reasoning in philosophy, followed by six celestial phenomena (the planets describe areas proportional to the times of description etc.). After this there come forty-two propositions (20 theorems and 22 problems). At the end of the book there is a famous General Scholium.

In several places of this work we can observe Newton dealing with opposite aspects of any mechanical problem. This is evident, for instance, already in his third axiom or law of motion (Newton, 1934, p. 13):

> *Law III: To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*
>
> Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a

> stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.

After the three laws of motion there are six corollaries. Then follows a Scholium where Newton demonstrates by pendulum experiments the validity of action and reaction in collisions. He also presents experiments showing that it is obeyed for magnetic attractions at a distance, (3, pp. 25–26):

> I made the experiment on the loadstone and iron. If these, placed apart in proper vessels, are made to float by one another in standing water, neither of them will propel the other; but, by being equally attracted, they will sustain each other's pressure, and rest at last in an equilibrium.

In the third book of the *Principia*, Newton presented six phenomena comprising Kepler's laws (Newton, 1934, pp. 401–405):

> Phenomenon I: *That the circumjovial planets, by radii drawn to Jupiter's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are as the 3/2th power of their distances from its centre.*
>
> [...]
>
> Phenomenon IV: *That the fixed stars being at rest, the periodic times of the five primary planets, and (whether of the sun about the earth, or) of the earth about the sun, are as the 3/2th power of their mean distances from the sun.*

From these phenomena he derived that the force of gravitation is inversely proportional to the distances (Newton, 1934, p. 406):

> Proposition I. Theorem I. *That the forces by which the circumjovial planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to Jupiter's centre; and are inversely as the squares of the distances of the places of those planets from that centre.*
>
> [...]
>
> Proposition II. Theorem II. *That the forces by which the primary planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to the sun; and are inversely as the squares of the distances of the places of those planets from the sun's centre.*

After arriving at this result, he begins the opposite process. That is, starting from a force of gravitation falling as $1/r^2$, he derives Kepler's laws. One example (Newton, 1934, p. 420):

> Proposition XIII. Theorem XIII. *The planets move in ellipses which have their common focus in the centre of the sun; and, by radii drawn to that centre, they describe areas proportional to the times of description.*
>
> We have discoursed above on these motion from the Phenomena. Now that we know the principles on which they depend, from those principles we deduce the motions of the heavens *a priori*.

But beyond this, Newton derived a whole set of new results beginning from a force of gravitation proportional to the product of the interacting masses and falling as the inverse square of the distance between them. As an example we have his Prop. XIX, Problem III: "*To find the proportion of the axis of a planet to the diameters perpendicular thereto.*" That is, he calculated the flattening of the planets at their poles. He also derived from his law of gravitation the motion of the Moon around the Earth. In Prop. XXIV, Theorem XIX, he began the explanation of a whole new set of phenomena based on the gravitational attraction, namely, [3, p. 435]: "*That the flux and reflux of the sea arise from the actions of the sun and moon.*" Another result which he could explain appears in Prop. XXXIX, Problem XX: "*To find the precession of the equinoxes.*" In the next few propositions he derived the motion of the comets around the Sun.

In essence, Newton began with Kepler's laws of planetary motion in order to derive his law of universal gravitation. He then applied this law to deduce Kepler's laws and a whole series of new phenomena.

3.5 Inverse Problems in Philosophy

In the last Query of the *Optics* Newton presented his general view on how to proceed in natural philosophy (Newton, 1979, pp. 404–405):

> As in Mathematicks, so in Natural Philosophy, the Investigation of difficult Things by the Method of Analysis, ought ever to precede the Method of Composition. This Analysis consists in making Experiments and Observations, and in drawing general Conclusions from them by Induction, and admitting of no Objections against the Conclusions, but such as are taken from Experiments, or other certain Truths. For Hypotheses are not to be regarded in experimental Philosophy. And although the arguing from Experiments and Observations by Induction be no Demonstration of general Conclusions; yet it is the best way of arguing which the Nature of Things admits of, and may be looked upon as so much the stronger, by how much the Induction is more general. And if no Exception occur from Phaenomena, the Conclusion may be pronounced generally. But if at any time afterwards any Exception shall occur from Experiments, it may then begin to be pronounced with such Exceptions as occur. By this way of Analysis we may proceed from Compounds to Ingredients, and from Motions to the Forces producing them, and from particular Causes to more general ones, till the Argument end in the most general. This is the Method of Analysis: And the Synthesis consists in assuming the Causes discover'd, and establish'd as Principles, and by them explaining the Phaenomena proceeding from them, and proving the Explanations.
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> In the first two Books of these Opticks, I proceeded by this Analysis to discover and prove the original Differences of the Rays of Light in respect of Refrangibility, Reflexibility, and Colour, and their alternate Fits of easy Reflexion and easy Transmission, and the Properties of Bodies, both opake and pellucid, on which their Reflexions and

> Colours depend. And these Discoveries being proved, may be assumed in the Method of Composition for explaining the Phaenomena arising from them: An Instance of which Method I gave in the End of the first Book.

Newton formalized his general approach in science at the Preface of the first edition of his *Principia* (Newton, 1934, p. xvii):

> I offer this work as the mathematical principles of philosophy, for the whole burden of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena; and to this end the general propositions in the first and second Books are directed. In the third Book I give an example of this in the explication of the System of the World; for by the propositions mathematically demonstrated in the former Books, in the third I derive from the celestial phenomena the forces of gravity with which bodies tend to the sun and the several planets. Then from these forces, by other propositions which are also mathematical, I deduce the motions of the planets, the comets, the moon, and the sea.

At the beginning of the third book of the *Principia* he presented a similar approach, namely (Newton, 1934, p. 397):

> In the preceding books I have laid down the principles of philosophy, principles not philosophical but mathematical; such, namely, as we may build our reasonings upon in philosophical inquiries. These principles are the laws and conditions of certain motions, and powers or forces, which chiefly have respect to philosophy; but, lest they should have appeared of themselves dry and barren, I have illustrated them here and there with some philosophical scholiums, giving an account of such things as are of more general nature, and which philosophy seems chiefly to be founded on; such as the density and resistance of bodies, spaces void of all bodies, and the motion of light and sounds. It remains that, from the same principles, I now demonstrate the frame of the System of the World.

3.6 Conclusion

In this work we have shown how Newton always considered inverse aspects in all branches of knowledge. This includes mathematics, mechanics, optics and philosophy. This certainly was one of the main sources of his powerful creativity in science.

Acknowledgements The author wishes to thank the Institute for the History of Natural Sciences of Hamburg University and the Alexander von Humboldt Foundation of Germany for a research fellowship during which this work was completed.

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