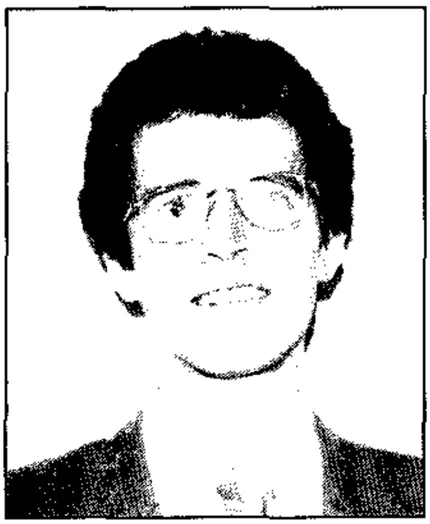
On the Velocity in the Lorentz Force Law

By A.K.T. Assis and F.M. Peixoto



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Classical electromagnetism is composed of three distinct parts, namely, (1) Maxwell's equations; (2) Constitutive relations depending on the medium (like Ohm's law $V = RI$, $\vec{D} = \varepsilon\vec{E}$, $\vec{J} = \sigma\vec{E}$, $\vec{B} = \mu\vec{H}$, etc.); and (3) the Lorentz force law. This last one states that a point charge q moving in an arbitrary electromagnetic field is acted on by a force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{1}$$

In this equation $\vec{E} = \vec{E}(\vec{r}, t)$ is the electric field at a point $\vec{r}$ where the charge q is located at the time t, and $\vec{B} = \vec{B}(\vec{r}, t)$ is the magnetic induction at the same point and at the same time.

The velocity $\vec{v}$ that appears in Eq. (1) is the instantaneous velocity of the test charge q. A fundamental question is: Velocity of q relative to what? Of course position, velocity, and acceleration are not intrinsic properties of any system, and any body can have *simultaneously* different velocities relative to different objects. What is the velocity of a man who is driving a car on a road at 80 km/h? Relative to his own car it is zero, relative to the Earth it is 80 km/h, relative to another car moving in the opposite direction at 60 km/h it is 140 km/h, relative to the Sun it is approximately 30 km/s, and so on.

Physically there are many meaningful possibilities: (A) The velocity of the charge q relative to a fixed ether in space, or relative to an ether at rest in the frame of the "fixed stars" (like the "aether" of Maxwell and Fresnel[1]); (B) Relative to the laboratory or to the Earth; (C) Relative to an inertial frame of reference; (D) Relative to an arbitrary observer, not necessarily an inertial one; (E) Relative to the macroscopic source of the magnetic field $\vec{B}$ (a magnet or a wire carrying a current I); (F) Relative to an average motion of the microscopic charges which generate $\vec{B}$, the electrons; and (G)Relative to the magnetic field. As a matter of fact, in the development of electrodynamics many force laws were proposed with different quantities being relevant to them. In Weber's electrodynamics, for instance, which is the oldest of all these models, only the relative velocities and accelerations between interacting charges were important, so that the force always had the same value for all observers.[2-9] In Clausius's theory, on the other hand, the force law called for the velocities of the charges relative to an ether.[10]

Standard Presentations

Curiously, when most textbooks introduce the Lorentz force law they do not state explicitly what the velocity $\vec{v}$ in Eq. (1) is relative to. Some examples in well-known works include:

- (I) Symon[11]: "The force exerted by a magnetic field on a charged particle at a point $\vec{r}$ depends on the velocity $\vec{v}$ of the particle, and is given in terms of the magnetic induction $\vec{B}\ (\vec{r}\ )$ by the equation $\vec{F} = q\vec{v} \times \vec{B}/c$."
- Feynman[12]: "We can write the force $\vec{F}$ on a charge q moving with velocity $\vec{v}$ as $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$." "The force on an electric charge depends not only on where it is, but also on how fast it is moving.... It is possible...to write the magnetic force as $q\vec{v} \times \vec{B}$."
- Jackson[13]: "Also essential for consideration of charged particle motion is the Lorentz force equation, $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$, that gives the force acting on a point charge q in the presence of electromagnetic fields." "The total electromagnetic force on a charge particle is $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$."
- Reitz and Milford[14]: "For the purpose of defining the magnetic induction it is convenient to define $\vec{F}_m$, the magnetic force (frequently called the Lorentz force), as that part of the force exerted on a moving charge which is neither electrostatic nor mechanical. The magnetic induction, $\vec{B}$, is then defined as the vector which satisfies $\vec{F}_m = q\vec{v} \times \vec{B}$, for all velocities."
- Sears[15]: "Force on a moving charge.... A positive charge q, moving with velocity v perpendicular to the direction of the induction, is found to experience a force F in the direction shown, perpendicular to its velocity v and to the induction B. The magnitude of this force is given by $F = qvB$.... Evidently, then, the force $\vec{F}$ on a charge q moving with velocity $\vec{v}$ in a magnetic field of flux density $\vec{B}$ is in vector notation, $\vec{F} = q\vec{v} \times \vec{B}$."
- Purcell[16]: "We say that an electric current has associated with it a magnetic field which pervades the surrounding space. Some other current, or any moving charged particle which finds itself in this field experiences a force proportional to the strength of the magnetic field in that locality. The force is always perpendicular to the velocity, for a charged particle. The entire force on a particle carrying charge q is given by $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$ where $\vec{B}$ is the magnetic field. We shall take Eq. (1) as the definition of $\vec{B}$."
- Panofsky and Phillips[17]: "According to Lorentz' electron theory, however, the only force which has physical significance is a resultant force which arises from the space-time forces acting on material charges and currents, namely, those obtained by averaging $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$."

After reading all these passages, a curious student could ask quite naturally and with complete reason, "But velocity of the charge q relative to what?"

In our opinion this lack of a clear initial statement on the meaning of the velocity in Eq. (1) is the reason for the confusion of students on this essential aspect of the theory. When we asked students who had taken courses of electromagnetism (undergraduate *or* graduate) to explain the meaning of $\vec{v}$ in Eq. (1) we received all sorts of answers from (A) to (G) above. At the conclusion of this article we present a short discussion of the origins and meanings of the expression $q\vec{v} \times \vec{B}$, where it is shown that even historically there have been different interpretations.

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The Meaning of the Velocity

Of course when we study any of the books cited, especially the sections dealing with the special theory of relativity, we grasp the correct answer according to the standard electromagnetic theory, namely, in Eq. (1) $\vec{v}$ is the velocity of the charge q relative to an observer or frame of reference. If we apply the Lorentz force law together with Newton's second law of motion ($\vec{F} = m\vec{a}$) or with its analogous relativistic generalization ($\vec{F} = d\vec{p}/dt$, with $\vec{p} = m_o\vec{v}/\sqrt{1 - v^2/c^2}$), then the observer or reference frame needs to be an inertial one. If this were stated more clearly when the Lorentz force law is presented, students could understand much more easily the interdependence and mutual transformation of electric and magnetic fields. This interconnection of $\vec{E}$ and $\vec{B}$ appears only due to this meaning of the velocity $\vec{v}$ in Eq. (1). Since this velocity is understood as a velocity relative to an inertial observer, fields $\vec{E}$ and $\vec{B}$ must also be understood like that, so their values must be different in different inertial frames. We can see the necessity of this if we imagine an inertial frame S in which there is no electric field but only a magnetic field, and that a charge q moves relative to this frame S. According to the Lorentz force law, it should feel a force given by $\vec{F} = q\vec{v} \times \vec{B}$. Relative to another inertial frame S'—which moves relative to S with the same velocity $\vec{v}$, so that in this frame the charge q is at rest ($\vec{v}\,' = 0$)—the force would be zero if there were no transformation of the fields. But the force must exist in all inertial frames, which means that somehow there must exist an electric field $\vec{E}\,'$ in the frame S' that will exert a force on q. We present here the general Lorentz transformations of the fields: If in an inertial frame S we have the electric and magnetic fields given by $\vec{E}$ and $\vec{B}$, then in an inertial frame S', which moves relative to S with a velocity $\vec{v}$, the fields will be $\vec{E}\,'$ and $\vec{B}\,'$, and these are related to the previous fields by[18]:

$$E'_{||} = E \tag{2}$$

$$B'_{||} = B \tag{3}$$

$$E'_{\perp} = \frac{(\vec{E} + \vec{v} \times \vec{B})}{\sqrt{1 - v^2/c^2}}\perp \tag{4}$$

$$B'_{\perp} = \frac{(\vec{B} - \vec{v} \times \vec{E}/c^2)}{\sqrt{1 - v^2/c^2}}\perp \tag{5}$$

In these equations the subscripts $||$ and $\perp$ mean the components of the fields parallel and perpendicular to the direction of the velocity $\vec{v}$, respectively.

As a matter of fact, if $\vec{v}$ were, for instance, the velocity of q relative to the macroscopic source of $\vec{B}$ (let us suppose a magnet) then the magnetic force would not change from an inertial frame S to another inertial frame S' moving with velocity $\vec{v}$ relative to S. If this were the case the magnetic component of Eq. (1) would read $\vec{F}_m = q(\vec{v}_q - \vec{v}_m) \times \vec{B}_m$, where $\vec{v}_q$ and $\vec{v}_m$ are the velocities of the charge and magnet, respectively, relative to an inertial frame of reference. Moreover $\vec{B}_m$ would be the magnetic field at the position of the charge q, *relative to the magnet* (and therefore having the same value to all observers since it would depend only on the position of q relative to the magnet). If in frame S we had $\vec{v}_m = 0$ (which means that the magnet is at rest in this frame), then according to our last expression the magnetic force would be $\vec{F}_m = q\vec{v}_q \times \vec{B}_m$. Let us suppose now that the frame S' is moving relative to the frame S with a velocity $\vec{v} = \vec{v}_q$, so that in this frame S' the charge q would be seen at rest, namely, $\vec{v}\,'_q = 0$. Moreover $\vec{v}\,'_m = -\vec{v}_q$, so that the magnetic force in this frame S' would be, according to our last expression, $\vec{F}\,'_m = q'[0 - (-\vec{v}_q)] \times \vec{B}\,'_m$. Since $q' = q$ and $\vec{B}_m = \vec{B}\,'_m$ we then would get $\vec{F}\,'_m = \vec{F}_m$.

This indicates clearly that the transformation properties of $\vec{E}$ and $\vec{B}$ into $\vec{E}\,'$ and $\vec{B}\,'$, and vice versa, arise (are necessary) only because $\vec{v}$ in the Lorentz force law has different values in different inertial frames.

History of the Magnetic Force

According to Whittaker (pp. 306–310)[10] the first to arrive at an expression like this (except for the factor 1/2) was J.J. Thomson in 1881, two years after Maxwell's death. Thomson arrived at the expression $q\vec{v} \times \vec{B}/2$ when studying theoretically the force of a magnet on a charge q moving through a medium characterized by a dielectric constant ε and magnetic permeability μ. According to him the velocity $\vec{v}$ in this expression (which he called the "actual velocity" of the charge) was not relative to the magnet or to the observer, but relative to the medium. In his own words: "It must be remarked that what we have for convenience called the actual velocity of the particle is, in fact, the velocity of the particle relative to the medium through which it is moving...[a] medium whose magnetic permeability is μ."[19] Eight years later Heaviside, in another theoretical paper, corrected the factor 1/2 of Thomson's work. He did not comment on Thomson's meaning of the velocity $\vec{v}$, so we can assume that he accepted Thomson's interpretation. This is even more evident by the title of his work: On the electromagnetic effects due to the motion of electrification through a dielectric.[20] A detailed and careful analysis of the works of Thomson and Heaviside can be found in Buchwald's book.[21]

Theoretical physicist Lorentz presented his force law, Eq. (1), for the first time in 1895.[22] Contrary to the interpretations of Thomson and Heaviside, and also contrary to our present-day interpretation, Lorentz maintained that the velocity $\vec{v}$ in Eq. (1) was the velocity of the charge q relative to the ether, which according to him was in a state of absolute rest relative to the frame of the fixed stars.[1] This can also be seen in his most famous book, *The Theory of Electrons*,[23] and also in his *Lectures on Theoretical Physics*.[24] The present-day interpretation that the velocity $\vec{v}$ in Eq. (1) is to be understood as the velocity of the charge q *relative to an inertial frame of reference*, the same being true for the electric fields $\vec{E}$ and $\vec{B}$, appeared for the first time in Einstein's paper of 1905.[25] There he presents the difference between the old paradigm of electromagnetism and the new one based on his theory of relativity. It seems that later on Lorentz accepted this interpretation by Einstein.

We think that it is very instructive to see this conceptual change in the meaning of one of the most utilized expressions of physics. Perhaps this is one of the reasons for the lack of clarity in the major part of textbooks when presenting the Lorentz force law.

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