Potential, electric field and surface charges for a resistive long straight strip carrying a constant current

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Abstract

We consider a long resistive straight strip carrying a constant current. We calculate the potential and electric field everywhere in space and also the density of surface charges along the strip. We compare the calculations with experimental results.

1 The Problem

Here we consider a case which has not yet been treated in the literature: a resistive straight strip with a constant current flowing over its surface. The strip is in the $y = 0$ plane localized in the region $-a < x < a$ and $-\ell < z < \ell$, such that $\ell \gg a > 0$. The medium around the strip is taken to be air or vacuum. The constant currente I flows uniformly along the positive z direction with a surface current density given by $\vec{K} = I\hat{z}/2a$.

Our goal is to find the potential ϕ and the electric field $\vec{E}$ everywhere in space and the surface charge distribution σ along the strip which creates this electric field. The problem can be solved by finding the solution of Laplace's equation $\nabla^2\phi = 0$ in empty space and applying a linear potential along the strip as a boundary condition.

2 The Solution

Due to the symmetry of the problem it is convenient to utilize elliptic-cylindrical coordinates (η, φ, z) [1, page 17], defined by:

$$\left\{ \begin{array}{rcl} x & = & a\cosh\eta\cos\varphi, \\ y & = & a\sinh\eta\sin\varphi, \\ z & = & z, \end{array} \right. \tag{1}$$

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The solution of Laplace's equation for the electric potential ϕ in these coordinates is given by $\phi(\eta,\varphi,z) = H(\eta)\Phi(\varphi)Z(z)$, where the funcions H, Φ and Z satisfy the following general equations (with α_2 and α_3 constants):

$$\left\{\begin{array}{rcl} H'' - (\alpha_2 + \alpha_3 a^2\cosh^2\eta)H &=& 0, \\ \Phi'' + (\alpha_2 + \alpha_3 a^2 \cos\varphi)\Phi &=& 0, \\ Z'' + \alpha_3 Z &=& 0. \end{array}\right. \tag{2}$$

In the case of a long straight strip being considered here, it is possible to neglect border effects near $z = \ell$ and $z = -\ell$: the potential must be a linear function of z not only over the strip but also all over space [2], which means that $\alpha_3 = 0$. As the potential along the strip doesn't depend on x, it mustn't depend on φ as well. This gives us also $\alpha_2 = 0$. The solution to this problem becomes:

$$\phi(\eta, z) = (A_1\eta - A_2)(A_3 z - A_4), \tag{3}$$

where $\eta = \tanh^{-1}\sqrt{(x^2-y^2-a^2+\Omega)/2x^2}$, $\Omega = \sqrt{(x^2+y^2+a^2)^2 - 4a^2x^2}$ and A_i are constants, with $i = 1\ldots 4$.

The electric field can be obtained from the electric potential by calculating the gradient:

$$\vec{E} = -A_1\left(\frac{|x|\sqrt{x^2-y^2-a^2+\Omega}}{x\sqrt{2\Omega}}\hat{x} + \frac{|x|y\sqrt{2}}{\Omega\sqrt{x^2-y^2-a^2+\Omega}}\hat{y}\right)(A_3 z - A_4) - A_3\left(A_1\eta - A_2\right)\hat{z}. \tag{4}$$

The surface charge density σ can be obtained from Gauss' Law $\oint_S \vec{E}\cdot d\vec{a} = Q/\epsilon_0$, where ϵ_0 is the vacuum permittivity, $d\vec{a}$ is a surface area element pointing outwards normal to the surface in each point and Q is the total charge inside the closed surface S. It is then found to be given by:

$$\sigma(x,z) = -\frac{2\epsilon_0 A_1(A_3 z - A_4)}{\sqrt{a^2-x^2}}. \tag{5}$$

3 Discussion and Conclusion

In the plane $y = 0$ the current in the strip will create a magnetic field $\vec{B}$ pointing along the positive (negative) y direction for $x > 0$ ($x < 0$). The magnetic field will act on the conduction electrons moving with drifting velocity $\vec{v}_d$ with a force given by $q\vec{v}_d \times \vec{B}$. This will cause a redistribution of charges along the x direction, with negative charges concentrating along the center of the strip and positive charges at the extremities $x = \pm a$. In the steady-state situation this redistribution of charges will create an electric field along the x direction, E_x, which will balance the magnetic force. In this work we are disregarding this Hall electric field as it is usually much smaller than the electric field giving rise to the current, [4].

A particular case is the electric potential calculated very close to the strip, near its center. In this case we have $a^2 \gg r^2 = x^2 + y^2$, $\Omega \approx a^2 + y^2 - x^2 + 2x^2y^2/a^2$ and $\eta \approx |y|/a$, such that:

$$\phi \approx \left(A_1 \frac{|y|}{a} - A_2 \right) (A_3 z - A_4). \tag{6}$$

This coincides with a simpler case we had considered before, [3]. Another case of interest is the electric potential very far from the strip, $r^2 \gg a^2$:

$$\phi \approx \left(A_1 \ln \frac{r}{a} - A_2 \right) (A_3 z - A_4). \tag{7}$$

This is in agreement with Eq. (8) of [4], as it should. A plot of the electric field lines for the electric field (4) and electric potential (3) are presented in Figures 1 and 2, overlaid on the experimental results of Jefimenko [5, Plate 6] and Jefimenko, Barnett and Kelly [6], respectively.

In the first experiment, Jefimenko mapped the lines of electric field in the plane of the strip by spreading grass seeds above and around the two dimensional conducting strip painted on glass plates. The seeds are polarized in the presence of an electric field and align themselves with it. The second experiment shows the equipotential lines around a rectangular hollow chamber with electrodes for end walls and semi-conducting side walls carrying uniform currents. They applied 80 volts to the electrodes and mapped the equipotential lines utilizing a radioactive alpha-source to ionize the air at the points where the field was to be measured.

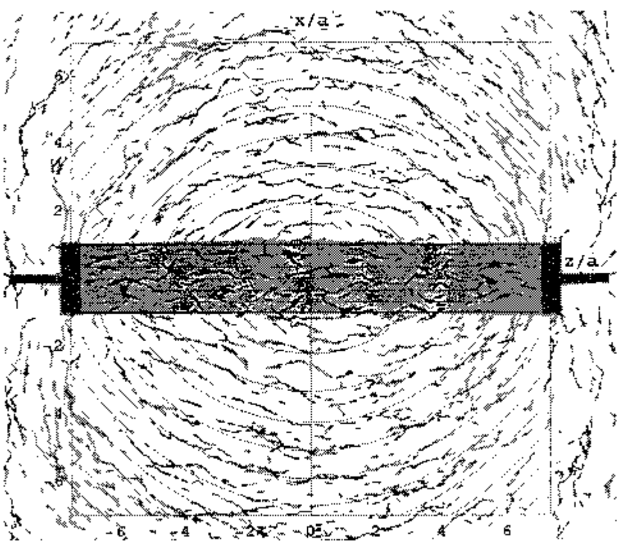

Figure 1: Theoretical electric field lines overlaid on Jefimenko's experiment of a conducting plate with constant current. Grass seeds align themselves along the electric field lines.

Our theoretical results represent well the results obtained experimentally by Jefimenko and co-authors. This work shows the existence of electric fields outside the conductors, and complements Jefimenko's experiments.

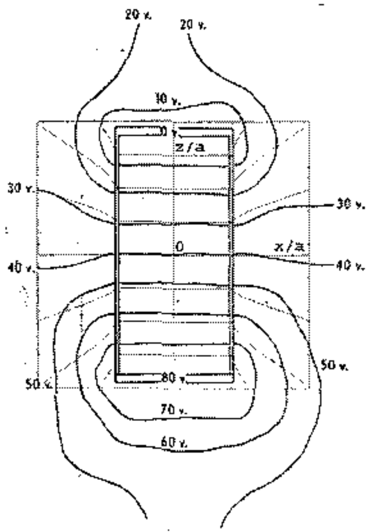

Figure 2: Equipotentials of a conducting strip overlaid on Jefimenko's experiment which measured the electric potential on a conducting chamber.

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