Surface Charges and Electric Field in a Two-Wire Resistive Transmission Line

A. K. T.Assis* and A. J. Mania

Instituto de Física 'Gleb Wataghin'

Universidade Estadual de Campinas - Unicamp

13083-970 Campinas, São Paulo, Brasil

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We consider a two-wire resistive transmission line carrying a constant current. We calculate the potential and electric field outside the wires showing that they are different from zero even for stationary wires carrying *dc* currents. We also calculate the surface charges giving rise to these fields and compare the magnetic force between the wires with the electric force between them. Finally we compare our calculations with Jefimenko's experiment.

I Introduction

One of the most important electrical systems is that of a two-wire transmission line, usually called twin-leads. We consider here homogeneous resistive wires fixed in the laboratory and carrying *dc* currents. Our goal is to calculate the electric field outside the wires. To this end we follow essentially the important works of Heald and Jackson, [1] and [2]. They call attention to the surface charges in a stationary resistive wire carrying a constant current. These authors have shown that the distribution of these net charges is constant in time if we have an stationary resistive wire with a *dc* current produced by a battery. These charges create not only the electric field inside the wire which opposes the resistive friction, but also an external electric field in the surrounding medium (air, for instance). This fact is not realized by most authors who consider only the magnetic field created by these currents. Heald, in particular, considered the case of a (two-dimensional) current loop and Jackson that of a coaxial cable of finite length with a return conductor of zero resistivity.

The case of twin-leads was first considered by Stratton, [3, p. 262]. Although he called attention to the electric field outside the transmission line, this has been forgotten by most authors as can be seen from the following quotation taken from Griffiths's book ([4, p. 196], our emphasys in boldface): "Two wires hang from the ceiling, a few inches apart. When I turn on a current, so that it passes up one wire and back down the other, the wires jump apart - they plainly repel one another. How do you explain this? Well, you might suppose that the battery (or whatever drives the current) is actually charging up the wire, so naturally the different sections repel. But this "explanation" is incorrect. **I could hold up a test charge near these wires and there would be no force on it, indicating that the wires are in fact electrically neutral. (It's true that electrons are flowing down the line - that's what a current *is* - but there are still just as many plus as minus charges on any given segment.)** Moreover, I could hook up my demonstration so as to make the current flow up *both* wires; in this case the wires are found to *attract!*"

In this work we will see that the wire is not electrically neutral on any given segment as there are surface charges distributed along its length. What creates the electric field anywhere along the transmission line are these surface charges and not the battery, although the battery is essential to maintain these surface charges in the case of constant current. As these surface charges create also an external electric field, a test charge placed near it will experience a force, contrary to Griffith's statement. The existence of this force has been confirmed by Jefimenko's experiments, [5] and [6]. Despite this fact we show here that the electrostatic force between two segments of the twin leads is many orders of magnitude smaller than the magnetic force between them. Our main goal is to call attention to the existence of the external electric field and to present analytical calculations which were not performed by Jefimenko.

II Two-Wire Transmission Line

The geometry of the system is given in Fig. 1. We have two equal straight wires of circular cross-sections

*E-mail: assis@ifi.unicamp.br; Web site: http://www.ifi.unicamp.br/assis. Also Collaborating Professor at the Department of Applied Mathematics, IMECC, State University of Campinas, 13081-970 Campinas, SP, Brazil.

of radii a and length ℓ, surrounded by air. Their axes are separated by a distance R and are parallel to the z axis, symmetrically located relative to the z and x axes. That is, the centers of the left and right wires are located at $(x,\ y,\ z) = (-R/2,\ 0, 0)$ and $(+R/2,\ 0,\ 0)$, respectively. The conductivity of the wires is g and their extremities are located at $z = -\ell/2$ and $z = +\ell/2$. Here we calculate the electric potential ϕ and the electric field $\vec{E}$ at a point $(x,\ y,\ z)$ such that $\ell \gg r = \sqrt{x^2+y^2+z^2}$. Moreover, we also assume that $\ell \gg R/2 > a$, so that we can neglect border effects.

We want to find the potential and electric field when a current I flows uniformly over the left wire in the direction $+\hat{z}$ and returns uniformly over the right wire in the direction $-\hat{z}$. The current densities in both wires are then given by $\vec{J} = (I/\pi a^2)\hat{z}$ and $\vec{J} = -(I/\pi a^2)\hat{z}$, respectively. As we are considering homogeneous wires with a constant resistivity g, Ohm's law yields the internal electric field in the wires as $\vec{E} = \pm(I/g\pi a^2)\hat{z}$. We don't need to consider in $\vec{E}$ the influence of the time variation of the vector potential as we are dealing with a dc current in stationary wires, so that $\partial\vec{A}/\partial t = 0$ everywhere. We can then write $\vec{E} = -\nabla\phi$. As we have a constant electric field in each wire, this implies that the potential is constant over each cross section and a linear function of z. In this work we consider a symmetrical situation for the potentials so that in the left wire the current flows from the potential ϕ_B at $z = -\ell/2$ to ϕ_A at $z = \ell/2$ and returns in the right wire from $-\phi_A$ at $z = \ell/2$ to $-\phi_B$ at $z = -\ell/2$, Figure 1 We can then write

$$\phi_L(z) = \frac{\phi_A - \phi_B}{\ell} z + \frac{\phi_A+\phi_B}{2} = \frac{I}{g\pi a^2} z + \frac{\phi_A+\phi_B}{2} \ , \tag{1}$$

$$\phi_R(z) = -\phi_L(z) \ . \tag{2}$$

In these equations $\phi_L(z)$ and $\phi_R(z)$ are the potentials as a function of z over the cross-section of the left and right conductors, respectively.

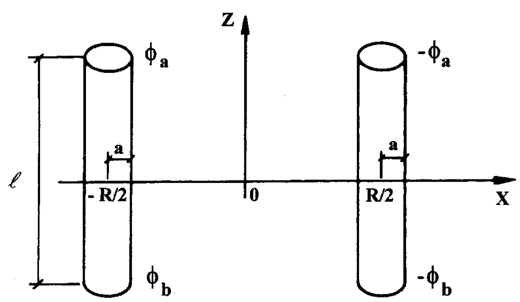

Figure 1. Two parallel wires of radii a separated by a distance R. The left wire carries a constant current I along the positive z direction while the right one carries the return current I along the negative z direction.

In this work we are neglecting the small Hall effect due to the poloidal magnetic field generated by these currents. This effect creates a redistribution of the current density within the wires, and modifies the surface charges also. As these are usually small effects, they will not be considered here.

We now find the potential in space supposing air outside the conductors. As the conductors are straight and the boundary conditions (the potentials over the surface of the conductors) are linear functions of z, the same must be valid everywhere, [7]. That is, $\phi = (Az + B)f(x,\ y)$, where A and B are constants and $f(x,\ y)$ is a function of x and y. This function can be found by the method of images imposing a constant potential ϕ_o over the left wire and $-\phi_o$ over the right one, [8, Section 2.1]. The final solution for ϕ and $\vec{E}$ satisfying the given boundary conditions, valid for the region outside the wires, is given by

$$\phi(x,\ y,\ z) = -\left(\frac{\phi_A-\phi_B}{\ell}z + \frac{\phi_A+\phi_B}{2}\right)\frac{1}{2\ln\frac{R-\sqrt{R^2-4a^2}}{2a}}$$

$$\times \ln\frac{(x-\sqrt{R^2-4a^2}/2)^2+y^2}{(x+\sqrt{R^2-4a^2}/2)^2+y^2} \ , \tag{3}$$

$$\vec{E} = -\left(\frac{\phi_A-\phi_B}{\ell}z + \frac{\phi_A+\phi_B}{2}\right)\frac{\sqrt{R^2-4a^2}}{\ln\frac{R+\sqrt{R^2-4a^2}}{2a}}$$

$$\times \frac{(x^2 - y^2 + a^2 - R^2/4)\hat{x} + 2xy\hat{y}}{x^4 + y^4 + R^4/16 + a^4 + 2x^2y^2 - R^2x^2/2 + 2a^2x^2 + R^2y^2/2 - 2a^2y^2 - R^2a^2/2}$$

$$+ \; \frac{\phi_A - \phi_B}{\ell} \frac{1}{2\ln\frac{R-\sqrt{R^2-4a^2}}{2a}} \ln\frac{(x - \sqrt{R^2-4a^2}/2)^2 + y^2}{(x + \sqrt{R^2-4a^2}/2)^2 + y^2}\hat{z} \; . \quad (4)$$

The equipotentials at $z = 0$ are plotted in Fig. 2.

It is also relevant to express these results in cylindrical coordinates (ρ, φ, z) centered on the left and right wires, see Fig. 3. For the left wire this can be accomplished replacing x by $\rho_L\cos\varphi_L - R/2$, y by $\rho_L\sin\varphi_L$, $\hat{x}$ by $\hat{\rho}_L\cos\varphi_L - \hat{\varphi}_L\sin\varphi_L$ and $\hat{y} = \hat{\rho}_L\sin\varphi_L + \hat{\varphi}_L\cos\varphi_L$, yielding

$$\phi(\rho_L,\ \varphi_L,\ z) = -\left(\frac{\phi_A - \phi_B}{\ell}z + \frac{\phi_A + \phi_B}{2}\right)\frac{1}{2\ln\frac{R-\sqrt{R^2-4a^2}}{2a}}$$

$$\times \ln\sqrt{\frac{\rho_L^2 - \rho_L\cos\varphi_L(R + \sqrt{R^2-4a^2}) + R^2/2 - a^2 + R\sqrt{R^2-4a^2}/2}{\rho_L^2 - \rho_L\cos\varphi_L(R - \sqrt{R^2-4a^2}) + R^2/2 - a^2 - R\sqrt{R^2-4a^2}/2}} \; , \quad (5)$$

$$\vec{E} = -\left(\frac{\phi_A - \phi_B}{\ell}z + \frac{\phi_A + \phi_B}{2}\right)\frac{\sqrt{R^2-4a^2}}{\ln\frac{R+\sqrt{R^2-4a^2}}{2a}}$$

$$\times \frac{(\rho_L^2\cos\varphi_L - \rho_L R + a^2\cos\varphi_L)\hat{\rho}_L + \sin\varphi_L(\rho_L^2 - a^2)\hat{\varphi}_L}{\rho_L^4 - 2\rho_L^3R\cos\varphi_L + \rho_L^2R^2 + a^4 + 2\rho_L^2a^2(\cos^2\varphi_L - \sin^2\varphi_L) - 2\rho_L Ra^2\cos\varphi_L}$$

$$+ \; \frac{\phi_A - \phi_B}{\ell}\frac{1}{2\ln\frac{R-\sqrt{R^2-4a^2}}{2a}}\hat{z}$$

$$\times \ln\frac{\rho_L^2 - \rho_L\cos\varphi_L(R + \sqrt{R^2-4a^2}) + R^2/2 - a^2 + R\sqrt{R^2-4a^2}/2}{\rho_L^2 - \rho_L\cos\varphi_L(R - \sqrt{R^2-4a^2}) + R^2/2 - a^2 - R\sqrt{R^2-4a^2}/2} \; . \quad (6)$$

The density of surface charges over the left and right wires, σ_L and σ_R, can then be found by $\varepsilon_o = 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}$ times the radial component of the electric field over the surface of each cylinder, yielding (ε_o is the vacuum permittivity):

$$\sigma_L = \left(\frac{\phi_A - \phi_B}{\ell}z + \frac{\phi_A + \phi_B}{2}\right)\frac{\varepsilon_o\sqrt{R^2-4a^2}}{2a\ln\frac{R+\sqrt{R^2-4a^2}}{2a}}\frac{1}{R/2 - a\cos\varphi_L} \; , \quad (7)$$

$$\sigma_R = -\left(\frac{\phi_A - \phi_B}{\ell}z + \frac{\phi_A + \phi_B}{2}\right)\frac{\varepsilon_o\sqrt{R^2-4a^2}}{2a\ln\frac{R+\sqrt{R^2-4a^2}}{2a}}\frac{1}{R/2 + a\cos\varphi_R} \; , \quad (8)$$

In order to check our results we calculated the potential ϕ inside each wire and in space beginning with these surface charges densities and utilizing

$$\phi(x,\ y,\ z) = \frac{1}{4\pi\varepsilon_o}\left(\int_{z'=-\ell/2}^{\ell/2}\int_{\varphi_L'=0}^{2\pi}\frac{\sigma_L(\varphi_L')ad\varphi_L'dz'}{|\vec{r} - \vec{r}\,'|}\right.$$

$$\left. + \int_{z'=-\ell/2}^{\ell/2}\int_{\varphi_R'=0}^{2\pi}\frac{\sigma_R(\varphi_R')ad\varphi_R'dz'}{|\vec{r} - \vec{r}\,'|}\right) \; . \quad (9)$$

Here we integrate over the surfaces of the left and right cylinders, S_L and S_R, respectively. We could then check our results assuming the correctness of the method of images for the electrostatic problem and utilizing the approximations $\ell \gg |\vec{r}|$ and $\ell \gg R/2 > a$.

The magnetic field of each wire surrounded by air can be easily obtained by the circuital law $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o I_C$, where I_C is the current flowing through the closed circuit C and $\mu_o = 4\pi \times 10^{-7} kgmC^{-2}$ is the vacuum permeability. For a long straight wire of radius a carrying a total current I we obtain: $B(\rho < a) = \mu_o I \rho / 2\pi a^2$ and $B(\rho > a) = \mu_o I / 2\pi\rho$, both in the poloidal direction. Adding the magnetic field of both wires taking into account that they carry currents in opposite directions yields the magnetic field anywhere in space (in this approximation that $\ell \gg r$).

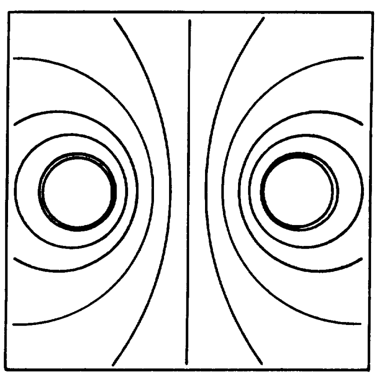

Figure 2. Equipotentials in the plane $z = 0$ given by Eq. (3).

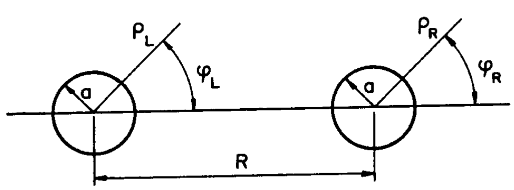

Figure 3. Left (L) and right (R) cylindrical coordinates for the left and right wires, respectively.

III Discussion and Conclusions

The first aspect to be discussed here is the qualitative interpretation of these results. In all this Section we will assume $\phi_A = 0$ in order to simplify the analysis. The distribution of surface charges for a given z is similar to the distribution of charges in the electrostatic problem given the potentials ϕ_o and $-\phi_o$ at the left and right wires, without current. That is, $\sigma_L(\varphi_L) > 0$ for any φ_L and its maximum value is at $\varphi_L = 0$. The density of surface charges at the right wire, σ_R, has the same behaviour of σ_L with an overall change of sign, with its maximum magnitude happening at $\varphi_R = \pi$. A qualitative plot of the surface charges at $z = 0$ is given in Fig. 4. A quantitative plot of σ_L is given in Fig. 5 supposing $R/2a = 10/3$ and normalizing the surface charge density by the value of σ_L at $\varphi_L = \pi$. It should also be remarked that for a fixed φ_L the surface density decreases linearly from $z = -\ell/2$ to $z = \ell/2$, the opposite happening with σ_R for a fixed φ_R.

Figure 4. Qualitative distribution of surface charges for the two parallel wires at $z = 0$.

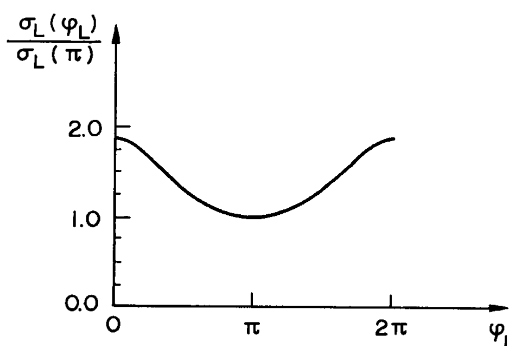

Figure 5. Surface charge density at the left wire in $z = 0$ for $R/2a = 10/3$ as a function of φ_L, normalized by its value at $\varphi_L = \pi$: $\sigma_L(\varphi_L)/\sigma_L(\pi) \times \varphi_L$.

We can integrate the surface charges over the periphery of each wire obtaining the integrated charge per unit length $\lambda(z)$ as:

$$\lambda_L(z) = \int_{\varphi_L=0}^{2\pi} a\sigma_L(\varphi_L) d\varphi_L$$

$$= -\frac{2\pi\varepsilon_o}{\ln\left((R - \sqrt{R^2 - 4a^2})/2a\right)} \left(\frac{\phi_A - \phi_B}{\ell} z + \frac{\phi_A + \phi_B}{2}\right) . \quad (10)$$

$$\lambda_R(z) = -\int_{\varphi_R=0}^{2\pi} a\sigma_R(\varphi_R) d\varphi_R = -\lambda_L(z) . \quad (11)$$

One important aspect to discuss is the experimental relevance of these surface charges in terms of forces. That is, as the wires have a net charge in each section, there will be an electrostatic force acting on them. We can then compare this force with the magnetic one. This last one is given essentially by (force per unit length)

$$\frac{dF_M}{dz} = \frac{\mu_o I^2}{2\pi R} , \tag{12}$$

where we are supposing $R/2 \gg a$. We now calculate the electric force per unit length on the left wire integrating the force over its periphery. We consider a typical region in the middle of the wire, around $z = 0$, and once more suppose $R/2 \gg a$:

$$\frac{d\vec{F}_E}{dz} = \int_{\varphi_L=0}^{2\pi} a\sigma_L(\varphi_L)\vec{E}(\rho_L = a, \varphi_L, z = 0)d\varphi_L \approx \frac{\pi\varepsilon_o\phi_B^2}{\ln^2 R/a}\left(\frac{\hat{x}}{R} + \frac{\hat{z}}{\ell}\right) . \tag{13}$$

From Eqs. (12) and (13) the ratio of the magnetic to the radial electric force is given by (with Ohm's law $\phi_B^2/I^2 = R_o^2 = (\ell/g\pi a^2)^2$, R_o being the resistance of each wire):

$$\frac{F_M}{F_E} \approx \frac{\mu_o/\varepsilon_o}{2R_o^2}\ln^2\frac{R}{a} . \tag{14}$$

As $\mu_o/\varepsilon_o = 1.4 \times 10^5\Omega^2$ this ratio will be usually many orders of magnitude greater than 1. This would be of the order of 1 when $R_o \approx 370\Omega$ (supposing $\ln R/a \approx 1$). This is a very large resistance for homogeneous wires.

In order to compare this force with the magnetic one we suppose typical copper wires of conductivities $g = 5.7 \times 10^7 m^{-1}\Omega^{-1}$, lengths $\ell = 1m$, separated by a distance $R = 6mm$ and diameters $2a = 1mm$. This means that by Ohm's law $\phi_B^2/I^2 = R_o^2 \approx 5 \times 10^{-4}\Omega^2$. With these values the ratio of the longitudinal electric force to the magnetic one is of the order of 7×10^{-11}, while the ratio of the radial electric force to the magnetic one is of the order of 1×10^{-8}. That is, the electric force between the wires due to these surface charges is typically 10^{-8} times smaller than the magnetic one. This shows that we can usually neglect these electric forces.

Despite this fact it should be remarked that while the magnetic force is repulsive in this situation (parallel wires carrying currents in opposite directions), the radial electric force is attractive, as we can see from the charges of Fig. 4.

It must be stressed that the surface charges are essential for understanding the origins of the electric field driving the current. The role of the battery is to separate the charges and keep this distribution of charges fixed in time for *dc* currents. But what creates the electric field inside and outside the wires is not the battery but these surface charges. Moreover, this external electric field can also be seen and measured if we have a dielectric material which can be polarized by the electric field, but which is not influenced by the magnetic field. This was the technique employed by Jefimenko, [5] and [6, Section 9-6 and Plate 6]. In his experiment he obtained the lines of electric field utilizing grass seeds, in a similar way that we obtain the lines of magnetic field utilizing iron fillings. The situation described in this paper is very similar to the experiment performed by Jefimenko whose results are presented in Fig. 5 of [5] or in Plate 6 and Fig. 9.13 of [6]. We can compare his experiment with our theoretical calculations by plotting the equipotentials obtained here. Jefimenko did not give the dimensions of his experiment but from Fig. 5 of [5] or from Plate 6 of [6] we can estimate the ratio of the important distances as $R/2a \approx 10/3$, $\ell/R \approx 5/2$ and $\ell/2a \approx 50/6$. With these values and $\varphi_A = 0$ and $\varphi_B = 1V$ we obtain the equipotentials given by Eq. (3) at $y = 0$, Fig. 6.

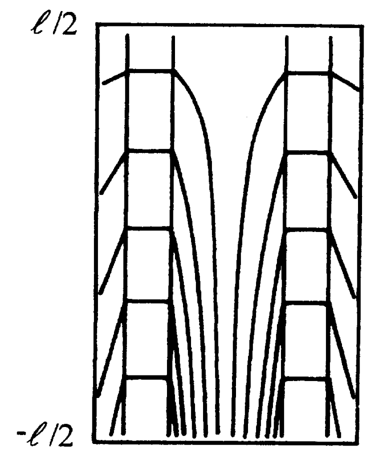

Figure 6. Equipotentials in the plane $y = 0$ given by Eqs. (1), (2) and (3) with the dimensions corresponding to Jefimenko's experiment, from $z = -\ell/2$ to $\ell/2$.

These lines can also be interpreted as lines of Poynting field $\vec{S} = \vec{E} \times \vec{B}/\mu_o$, where $\vec{B}$ is the magnetic field. That is, they may also represent the energy flow from the battery (at $z = -\ell/2$) to the wires given by Poynting vector throughout the space. This has been pointed out in general by Heald in his important work, [1].

The lines of electric field orthogonal to the equipotentials can be obtained by the procedure described in Sommerfeld's book, [9, p. 128]. We are looking for a function $\xi(x, y=0, z)$ such that

$$\nabla\xi(x,0,z)\cdot\nabla\phi(x,0,z)=0 \ . \qquad (15)$$

The equipotential lines can be written as $z_1(x) = K_1$, where K_1 is a constant (for each constant we have a different equipotential line). Analogously, the lines of electric force will be given by $z_2(x) = K_2$, where K_2 is another constant (for each K_2 we have a different line of electric force). From Eq. (15) we get $dz_2/dx = -1/(dz_1/dx) = (\partial\phi/\partial z)/(\partial\phi/\partial x)$. Integrating this equation we obtain $\xi(x,0,z)$. This yields the following solutions in the plane $y=0$ outside the wires:

$$\xi^{out}(x,0,z) = -2Bz + A\left[\frac{x(x^2-3x_o^2)}{6x_o}\ln\frac{(x-x_o)^2}{(x+x_o)^2}\right.$$

$$\left. + \frac{x_o^2}{3}\ln[(x-x_o)^2(x+x_o)^2] - x^3/3 - z^2\right] \ , \qquad (16)$$

where $A = (\phi_A - \phi_B)/\ell$, $B = (\phi_A + \phi_B)/2$ and $x_o = \sqrt{R^2 - 4a^2}/2$.

The lines of electric field inside the left and right wires can be written as, respectively:

$$\xi^L(x,0,z) = -Ax \ , \qquad (17)$$

$$\xi^R(x,0,z) = Ax \ , \qquad (18)$$

The lines of electric field are then plotted imposing $\xi(x,0,z) = constant$. With Jefimenko's dimensions for R, a and ℓ we obtain the lines of force by these equations as given in Fig. 7. This numerical plot is extremely similar to Jefimenko's experiment as presented in Fig. 5 of [5] or in Plate 6 of [6]. Although our calculation is strictly valid only for $r \ll \ell$, our numerical plot goes from $z = -\ell/2$ to $\ell/2$. As the result is in very good agreement with Jefimenko's experiment, we conclude that the exact boundary conditions at $z = \pm\ell/2$ are not very important in this particular configuration. Our work might be considered as a complementation of Jefimenko's one, as he realized the experiment but made no theoretical calculations for the transmission line considering straight cylindrical wires. The only calcuations he presented in [6, Section 9-6] were restricted to the current flowing over one surface of a resistive capacitor plate and returning through the other. He didn't consider twin-leads nor cylindrical conductors.

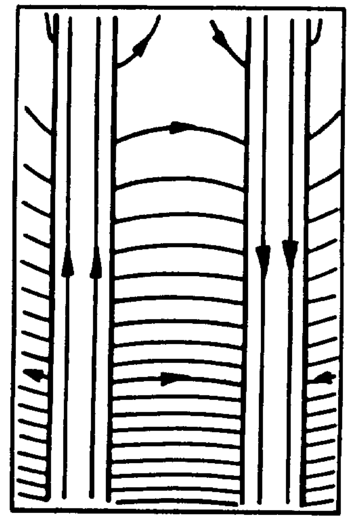

Figure 7. Lines of electric field in the plane $y = 0$ given by Eqs. (16), (17) and (18) with the dimensions corresponding to Jefimenko's experiment, from $z = -\ell/2$ to $\ell/2$.

We can also estimate the ratio of the radial component of the electric field to the axial one just outside the wire. We consider the left wire at three different heights: $z = -\ell/2$, $z = 0$ and $z = \ell/2$. The axial component E_z is constant over the cross section and does not depend on z. On the other hand the radial component E_x is a linear function of z and also depends on φ_L. In this comparison we consider $\varphi_L = 0$. With these values and Jefimenko's data in Eq. (4) we obtain $E_x/E_z \approx 12$ at $z = -\ell/2$, 6 at $z = 0$ and 0 at $z = \ell/2$. That is, the radial component of the electric field just outside the wire is typically one order of magnitude larger than the axial electric field responsible for the current. Jefimenko's experiment gives a clear confirmation of this fact.

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