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High frequency electromagnetic waves in a bounded, magnetized and warm plasma

A. K. T. Assis
Departamento de Raios Cósmicos e Cronologia, Instituto de Física, UNICAMP, Caixa Postal 6165, Campinas, 13081, SP, Brasil
and
P. H. Sakanaka
Departamento de Eletrônica Quântica, Instituto de Física, UNICAMP, Caixa Postal 6165, Campinas, 13081, SP, Brasil

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Abstract The electromagnetic waves in a uniform, magnetized plasma, bounded by a cylindrical waveguide are studied using warm plasma theory. In the case of a cold plasma, backward electromagnetic waves are obtained which exhibit the phenomenon of Faraday rotation, in contrast to the Trivelpiece and Gould electrostatic modes which do not show this characteristic. Numerical results are presented for the linear LISA machine. A generalization of the Ghosh and Pal dispersion relation for a warm, magnetized plasma is derived.

1. Introduction

The study of the characteristic modes of electromagnetic oscillation in waveguides has continued to be an important research topic in the last few years[1-3]. A global treatment of the problem is always necessary when the wavelength of the perturbation is of the same order as the dimensions of the system.

The purpose of the present paper is to include the electron temperature in the Trivelpiece and Gould problem[4,5]. Moreover, the study will not be restricted to the slow wave cases ($\omega^2/k^2 << c^2$) and a greater number of modes will be analysed. We study a particular case of a slow electromagnetic wave in a cold plasma and discuss its difference to the slow electrostatic wave obtained by Trivelpiece and Gould in the same range of frequencies. We also generalize the dispersion relation of Ghosh and Pal[6], which was obtained for a warm, magnetized plasma completely filling a cylindrical waveguide of circular cross section.

2. The Basic Equations and the Dielectric Tensor

The Trivelpiece and Gould problem is studied including the electron temperature and the perturbed part of the magnetic field. The plasma is then treated as an adiabatic fluid in which the ions are at rest. This approximation is valid in the highfrequency limit, $\omega >> \omega_{pi}$ and $\omega >> \omega_{ci}$, when the ions motion is completely negligible. The presence of a constant external magnetic field along the waveguide, $\vec{B}_0$, is included in the model. A linearization process is applied, where we assume small sinusoidal perturbations from steady state. This means that the perturbations have an $\exp(-i\omega t)$ time-dependence, where ω is the angular frequency of the electromagnetic field. The equations are obtained in the absence of an equilibrium electrostatic field, $\vec{E}_0 = 0$, and of an electron drift velocity, $\vec{u}_0 = 0$. The first order equations which describe the system are the equations of continuity, of momentum transfer and Maxwell's equations. With these assumptions they take the form, respectively[7-10]

$$i\omega p_1 = n_0 m U^2 \nabla \cdot \vec{u}_1, \tag{1}$$

$$i\omega n_0 m \vec{u}_1 = n_0 e(\vec{E}_1 + \vec{u}_1 \times \vec{B}_0) + \nabla p_1 \ , \tag{2}$$

$$\nabla \times \vec{E}_1 = i\omega\mu_0 \vec{H}_1 \ , \tag{3}$$

$$\nabla \times \vec{H}_1 = -i\omega\epsilon_0 \vec{E}_1 - n_0 e \vec{u}_1 \ , \tag{4}$$

where $p_1, n_0, m, U(= (\gamma k_B T_0/m)^{\frac{1}{2}})$, γ, k_B, T_0, $\vec{u}_1$, $-e$, $\vec{E}_1$, $\vec{H}_1$, μ_0 and ϵ_0 are, respectively, the perturbed pressure, fluid density, electron mass, electron thermal velocity, ratio of specific heats (usually $\gamma = 5/3$), Boltzmann's constant, electron temperature, perturbed fluid velocity, electron charge, perturbed electric and magnetic fields, vacuum magnetic permeability and vacuum dielectric constant. To obtain these equations we assumed also that the electron collision frequency is much smaller that the wave frequency ω.

In this work the propagation of electromagnetic waves in a plasma-filled cylindrical waveguide of circular cross section is studied. We assume that $\vec{B}_0$ is in the direction of the waveguide axis, Z axis of the coordinate system, and suppose a wave perturbation of the form $\exp(ikz - in\theta)$. Applying eq. (1) and eq. (2) and the result in eq. (4) yields

$$\nabla \times \vec{H}_1 = -i\omega \vec{\vec{\epsilon}} \cdot \vec{E}_1 \ , \tag{5}$$

where

$$(\vec{\vec{\epsilon}})_{11} = \frac{\epsilon_0}{\omega^2 - \omega_c^2}\left\{\omega^2 - \omega_p^2 - \omega_c^2 + U^2\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - n\frac{\omega_c}{\omega}\left(\frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\right)\right]\right\} ,$$

$$(\vec{\vec{\epsilon}})_{12} = \frac{i\epsilon_0}{\omega^2 - \omega_c^2}\left[\omega_p^2\frac{\omega_c}{\omega} - nU^2\left(\frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - \frac{n\omega_c}{r^2\omega}\right)\right],$$

$$(\vec{\vec{\epsilon}})_{13} = \frac{i\epsilon_0 kU^2}{\omega^2 - \omega_c^2}\left(\frac{d}{dr} - \frac{n\omega_c}{r\omega}\right),$$

$$(\vec{\vec{\epsilon}})_{21} = \frac{-i\epsilon_0}{\omega^2 - \omega_c^2}\left\{\omega_p^2\frac{\omega_c}{\omega} - U^2\left[\frac{\omega_c}{\omega}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) - n\left(\frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\right)\right]\right\},$$

$$(\vec{\vec{\epsilon}})_{22} = \frac{\epsilon_0}{\omega^2 - \omega_c^2}\left\{\omega^2 - \omega_p^2 - \omega_c^2 + nU^2\left[\frac{\omega_c}{\omega}\left(\frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) - \frac{n}{r^2}\right]\right\},$$

$$(\vec{\vec{\epsilon}})_{23} = \frac{-\epsilon_0 kU^2}{\omega^2 - \omega_c^2}\left(\frac{\omega_c}{\omega}\frac{d}{dr} - \frac{n}{r}\right) ,$$

$$(\vec{\vec{\epsilon}})_{31} = \frac{i\epsilon_0 kU^2}{r\omega^2}\left(r\frac{d}{dr} + 1\right),$$

$$(\vec{\vec{\epsilon}})_{32} = \frac{n\epsilon_0 kU^2}{r\omega^2},$$

$$(\vec{\vec{\epsilon}})_{33} = \epsilon_0\frac{\omega^2 - \omega_p^2 - k^2U^2}{\omega^2} .$$

Here ω_p and ω_c are, respectively, the electron plasma frequency and electron cyclotron frequency given by

$$\omega_p = \left(\frac{n_0 e^2}{\epsilon_0 m}\right)^{1/2}, \omega_c = \frac{eB_0}{m}. \tag{6}$$

In (5), $\vec{\vec{\epsilon}}$ is the warm plasma dielectric tensor. Due to the global treatment of the problem its elements have spatial derivatives which operate on the electric field components. The usual infinite warm plasma dielectric tensor is obained from (5) to (6) imposing $1/r \to 0$ and $d/dr \to ik_\perp$. The radial derivatives become important when the wavelength of the perturbation is of the same order as the radius of the waveguide. In this case a global treatment is necessary[11] and a plane wave propagation cannot occur.

3. Equations for the Field Components

With (1) and (4) we obtain p_1 and $\vec{u_1}$ in terms of $\vec{E}_1$ and $\vec{H}_1$. Applying this result in (2) yields

$$\nabla \times \vec{H}_1 = -\frac{ie}{m\omega}\left(\nabla \times \vec{H}_1\right) \times \vec{B}_0 - i\omega\epsilon_0 \frac{(\omega^2 - \omega_p^2)}{\omega^2}\vec{E}_1 + \frac{e\epsilon_0}{m}\vec{E}_1 \times \vec{B}_0 - i\frac{\epsilon_0 U^2}{\omega}\nabla\left(\nabla \cdot \vec{E}_1\right). \tag{7}$$

Applying (3) in (7) yields

$$GE_r = nA_1E_z + nB_1H_z + C_1\frac{dE_z}{dr} + D_1\frac{dH_z}{dr}\ , \tag{8}$$

$$iGE_\theta = \frac{nC_1}{r}E_z + \frac{nD_1}{r}H_z + rA_1\frac{dE_z}{dr} + rB_1\frac{dH_z}{dr}\ , \tag{9}$$

$$GH_r = nA_2E_z + nB_2H_z + C_2\frac{dE_z}{dr} + D_2\frac{dH_z}{dr}\ , \tag{10}$$

$$iGH_\theta = \frac{nC_2}{r}E_z + \frac{nD_2}{r}H_z + rA_2\frac{dE_z}{dr} + rB_2\frac{dH_z}{dr}\ , \tag{11}$$

where

$$G = k_e^4 - \frac{\omega_c^2}{\omega^2}k_f^4\ ,$$

$$k_f = \left(\frac{\omega^2}{c^2} - k^2\right)^{1/2}\ ,$$

$$k_e = \left(\frac{(\omega^2 - \omega_p^2)}{c^2} - k^2\right)^{1/2}\ ,$$

$$A_1 = -\frac{ik}{rc^2}\frac{\omega_c}{\omega}\left[\omega_p^2 + \frac{c^2U^2}{(c^2 - U^2)}\frac{k_f^2}{k^2}(\nabla_\perp^2 + k_e^2)\right]\ ,$$

$$B_1 = \mu_0\frac{\omega}{r}\left[k_e^2 - \left(\frac{\omega_c}{\omega}\right)^2 k_f^2\right]\ ,$$

$$C_1 = ik\left[k_e^2 - \left(\frac{\omega_c}{\omega}\right)^2 k_f^2 + \frac{U^2k_e^2}{k^2(c^2 - U^2)}(\nabla_\perp^2 + k_e^2)\right]\ ,$$

$$D_1 = -\frac{\mu_0\omega_c\omega_p^2}{c^2}\ ,$$

$$A_2 = -\frac{\epsilon_0(\omega^2-\omega_p^2)}{r\omega}\left[k_e^2 - \frac{\omega_c^2 k_f^2}{(\omega^2-\omega_p^2)} + \frac{k_e^2 c^2 U^2(\nabla_\perp^2 + k_e^2)}{(\omega^2-\omega_p^2)(c^2-U^2)}\right] ,$$

$$B_2 = -i\frac{k\omega_c\omega_p^2}{r\omega c^2} ,$$

$$C_2 = -\epsilon_0\omega_c\left[k_e^2 - k_f^2\frac{(\omega^2-\omega_p^2)}{\omega^2} - \frac{k_f^2 c^2 U^2(\nabla_\perp^2 + k_e^2)}{\omega^2(c^2-U^2)}\right] ,$$

$$D_2 = ik\left[k_e^2 - \left(\frac{\omega_c}{\omega}\right)^2 k_f^2\right] ,$$

$$\nabla_\perp^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} .$$

Applying (8) to (11) and (1) in (2) yields

$$u_r = \frac{-ie\omega}{m(\omega^2-\omega_c^2)} \times \left[E_r - i\frac{\omega_c}{\omega}E_\theta - i\frac{\omega_c}{\omega}\frac{nc^2U^2(\nabla_\perp^2+k_e^2)E_z}{rk\omega_p^2(c^2-U^2)} + i\frac{c^2U^2(\nabla_\perp^2+k_e^2)}{k\omega_p^2(c^2-U^2)}\frac{dE_z}{dr}\right] , \quad (12)$$

$$u_\theta = \frac{-ie\omega}{m(\omega^2-\omega_p^2)} \times \left[i\frac{\omega_c}{\omega}E_r + E_\theta + \frac{mc^2U^2(\nabla_\perp^2+k_e^2)}{rk\omega_p^2(c^2-U^2)}E_z - \frac{\omega_c}{\omega}\frac{c^2U^2(\nabla_\perp^2+k_e^2)}{k\omega_p^2(c^2-U^2)}\frac{dE_z}{dr}\right] , \quad (13)$$

$$u_z = -\frac{ie}{m\omega}\left[1 - \frac{c^2U^2(\nabla_\perp^2+k_e^2)}{\omega^2(c^2-U^2)}\right]E_z . \quad (14)$$

From (8)–(14) we see that all the transverse field components and all the components of the fluid velocity are obtained in terms of E_z and H_z.

Applying the rotational operator to both sides of equation (7) and using (3) yields

$$(\nabla_\perp^2 + k_e^2)H_z = \frac{-i\epsilon_0 c^2\omega_c}{k(c^2-U^2)}\left[\frac{\omega^2-k^2U^2}{\omega^2}(\nabla_\perp^2+k_e^2) + \frac{\omega_p^2}{\omega^2}k^2\frac{c^2-U^2}{c^2}\right]E_z . \quad (15)$$

Applying the divergent operator to both sides of equation (7) and using the rotational operator of both sides of equation (3) yields

$$(\nabla_\perp^2 + k_f^2)H_z = \frac{-i}{\mu_0\omega_c}\Big[k\frac{c^2-U^2}{c^2}\nabla_\perp^2 + \frac{c^2}{k(c^2-U^2)} \times \left(\frac{U^2}{c^2}\nabla_\perp^2 + k_e^2\right)\left(\nabla_\perp^2 + \frac{\omega^2-\omega_p^2}{c^2} - k^2\frac{U^2}{c^2}\right)\Big]E_z\,. \qquad (16)$$

From (15) we obtain the equations for H_z when $B_0 = 0$ and also the equation for E_z when $B_0 \to \infty$, namely

$$\left(\nabla_\perp^2 + k_e^2\right)H_z = 0, \qquad (17)$$

$$\left(\nabla_\perp^2 + k_m^2\right)E_z = 0\,, \qquad (18)$$

where

$$k_m = \left[\frac{\left(\omega^2 - k^2c^2\right)\left(\omega^2 - k^2U^2 - \omega_p^2\right)}{c^2\left(\omega^2 - k^2U^2\right)}\right]^{\frac{1}{2}}\,.$$

From (16) we obtain the equation for H_z when $B_0 \to \infty$ and also the equation for E_z when $B_0 = 0$, namely

$$\left(\nabla_\perp^2 + k_f^2\right)H_z = 0\,, \qquad (19)$$

$$\left(\nabla_\perp^2 + k_e^2\right)\left(\nabla_\perp^2 + k_s^2\right)E_z = 0\,, \qquad (20)$$

where

$$k_s = \left(\frac{\omega^2 - \omega_p^2}{U^2} - k^2\right)^{\frac{1}{2}}\,.$$

In the case of a nonzero and finite magnetic field we can combine equations (15) and (16) to obtain

$$H_z = \frac{-ic^2U^2}{\mu_0 k\omega_c\omega_p^2(c^2-U^2)}\left(\nabla_\perp^2 + k_+^2\right)\left(\nabla_\perp^2 + k_-^2\right)E_z\,, \qquad (21)$$

where

$$k_\pm^2 = \frac{x \pm (x^2 - 4y)^{\frac{1}{2}}}{2}\,,$$

$$x = k_e^2 + k_s^2 - \frac{\omega_c^2}{\omega^2}\frac{\omega^2 - k^2U^2}{U^2}\,,$$

$$y = k_e^2k_s^2 - \left(\frac{\omega_c}{\omega}\right)^2\left[k_e^2\frac{\omega^2 - k^2U^2}{U^2} + k^2\omega_p^2\frac{c^2-U^2}{c^2U^2}\right]$$

Equation (21) shows that E_z and H_z are coupled in the situation of a finite magnetic field. As a consequence, the waves cannot be separated into TE and TM modes and only hybrid modes can propagate.

Applying the operator $(\nabla_\perp^2 + k_e^2)$ to both sides of equation (21) and using (15) yields

$$(\nabla_\perp^6 + b_1 \nabla_\perp^4 + b_2 \nabla_\perp^2 + b_3)E_z = 0, \tag{22}$$

where

$$b_1 = 2k_e^2 + k_s^2 - \left(\frac{\omega_c}{\omega}\right)^2 \frac{\omega^2 - k^2U^2}{U^2} ,$$
$$b_2 = k_e^4 + 2k_e^2k_s^2 - \left(\frac{\omega_c}{\omega}\right)^2 \left[(k_e^2 + k_f^2)\frac{\omega^2 - k^2U^2}{U^2} + k^2\omega_p^2\frac{c^2 - U^2}{c^2U^2}\right] ,$$
$$b_3 = k_e^4k_s^2 - \left(\frac{\omega_c}{\omega}\right)^2 k_f^2\left(k_e^2\frac{\omega^2 - k^2U^2}{U^2} + k^2\omega_p^2\frac{c^2 - U^2}{c^2U^2}\right) .$$

Equation (22) can also be obtained directly using (3) and (5). This sixth order equation for the longitudinal component of the electric field is more general than that obtained by Ghosh and Pal[6] . Those auhors, beginning from the same set of equations (1) to (4), arrived at a fourth order equation for E_z, due to simplifying assumptions (not specified). Moreover they only studied the circularly symmetric waves, $n = 0$, while the analysis of this paper is valid for any mode n.

Equation (22) can be written in the form

$$(\nabla_\perp^2 + k_1^2)(\nabla_\perp^2 + k_2^2)(\nabla_\perp^2 + k_3^2)E_z = 0 , \tag{23}$$

where k_1, k_2 and k_3 are analytic functions of b_1, b_2 and b_3, obtained by Cardan's formula. Cardan's formula gives algebraically the values of the roots of a cubic equation as a function of its coefficients[12].

4. Dispersion Relations

In order to obtain the dispersion relations we need to specify the boundary conditions. Assuming a perfectly conducting metallic cylinder of radius R limiting the plasma we have[13,14] :

$$E_z(R) = 0,\ E_\theta(R) = 0,\ H_r(R) = 0,\ u_r(R) = 0. \tag{24}$$

These are the boundary conditions for this problem. They require that the tangential components of the electric field and the normal component of the magnetic field vanish at the perfectly conducting wall. They also require that the normal component of the fluid velocity vanishes at the rigid metallic wall. It should be noted that the boundary condition for the magnetic field in equation (24) is different from the one utilized by Ghosh and Pal[6], namely, $H_z(R) = 0$. Our boundary condition, $H_r(R) = 0$, is justified because we have a metallic boundary[21].

The dispersion relations obtained in this paper are for the situation of a finite magnetic field or for the situation when the magnetic field goes to zero. The dispersion relation for the case of infinite magnetic field and a cold or warm plasma can be easily obtained using (18) and (24). This is a known result[15], and will not be presented here.

4.1 Case of Zero Magnetic Field

There are three regions in the ω versus k plane. Region I: $k_s^2 > 0$ and $k_e^2 > 0$. Region II: $k_s^2 > 0$ and $k_e^2 < 0$. Region III: $k_s^2 < 0$ and $k_e^2 < 0$. These regions are presented in **Figure 1.** The solutions of (20) and (17) which are finite at the axis are

$$\text{Region I:} E_z = A_{1n} J_n(rk_e) + B_{1n} J_n(rk_s) \ ,$$
$$H_z = C_{1n} J_n(rk_e) \ , \tag{25}$$
$$\text{Region II:} E_z = A_{2n} I_n(rk_{e2}) + B_{2n} J_n(rk_s) \ ,$$
$$H_z = C_{2n} I_n(rk_{e2}) \ , \tag{26}$$
$$\text{Region III:} E_z = A_{3n} I_n(rk_{e2}) + B_{3n} I_n(rk_{s2}) \ ,$$
$$H_z = C_{3n} I_n(rk_{e2}), \tag{27}$$

where

$$k_{e2} = \left(k^2 - \frac{\omega^2 - \omega_p^2}{c^2}\right)^{\frac{1}{2}} \quad , \quad k_{s2} = \left(k^2 - \frac{\omega^2 - \omega_p^2}{U^2}\right)^{\frac{1}{2}} \ ,$$

and where $J_n(x)$ and $I_n(x)$ are, respectively, th nth-order Bessel function and modified Bessel function of first kind.

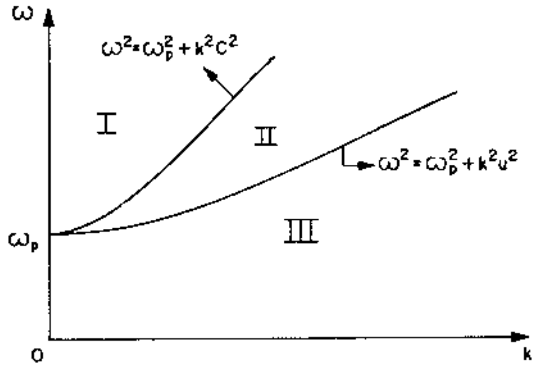

Fig. 1 - Distinct regions in the ω versus k diagram. Case of zero magnetic field.

Application of the boundary conditions yields the dispersion relations

Region I:

$$\omega^2 \frac{J_n'(Rk_s)J_n'(Rk_e)}{J_n(Rk_s)J_n(Rk_e)} + \frac{k^2\omega_p^2}{k_s k_e}\frac{J_n'^2(Rk_e)}{J_n^2(Rk_e)} - \frac{n^2(\omega^2-\omega_p^2)\omega_p^2}{k_s k_e^3 c^2 R^2} = 0 \ , \qquad (28)$$

Region II:

$$\omega^2 \frac{J_n'(Rk_s)I_n'(Rk_{e2})}{J_n(Rk_s)I_n(Rk_{e2})} - \frac{k^2\omega_p^2}{k_s k_{e2}}\frac{I_n'^2(Rk_{e2})}{I_n^2(Rk_{e2})} + \frac{n^2(\omega^2-\omega_p^2)\omega_p^2}{k_s k_{e2}^3 c^2 R^2} = 0 \ , \qquad (29)$$

Region III:

$$\omega^2 \frac{I_n'(Rk_{s2}I_n'(Rk_{e2})}{I_n Rk_{s2}I_n(Rk_{e2})} - \frac{k^2\omega_p^2}{k_{s2} k_{e2}}\frac{I_n'^2(Rk_{e2})}{I_n^2(Rk_{e2})} + \frac{n^2(\omega^2-\omega_p^2)\omega_p^2}{k_{s2} k_{e2}^3 c^2 R^2} = 0 \ , \qquad (30)$$

where $J_n'(x)$ and $I_n'(x)$ mean derivatives with respect to the argument. Figures (2) and (3) show the graphs of frequency versus wavenumber for (28) to (30). The values of the density, guide radius and plasma temperature are those of the linear LISA machine, of Universidade Federal Fluminense, Brazil[16,17]. In regions I and II there are infinitely many curves and from Figure 2 we see that they pass smoothly from region I to region II. In region III there is only one dispersion curve for each temperature, which tends asymptotically to the plasma frequency for $k \rightarrow \infty$.

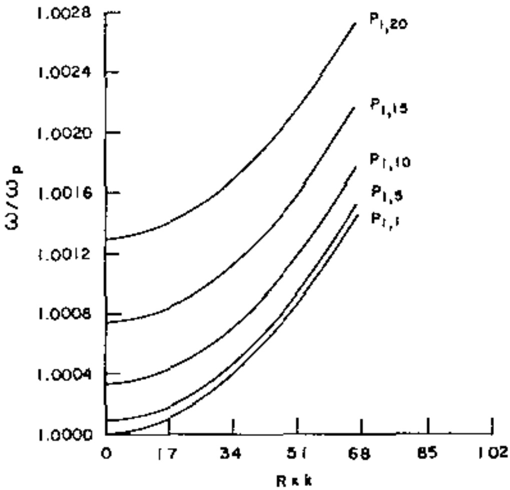

Fig. 2 - Dispersion relation when $B_0 = 0$, $n = 0$, $k_BT_0 = 40eV$, $\omega_p = 5.040 \times 10^{10}\ s^{-1}$, $R = 0.085m$. Modes $p_{1,1}$; $p_{1,5}$; $p_{1,10}$; $p_{1,15}$ and $p_{1,20}$. Initial points obtained from $J_1(Rk_s) = 0$. Region $\omega > \omega_p$.

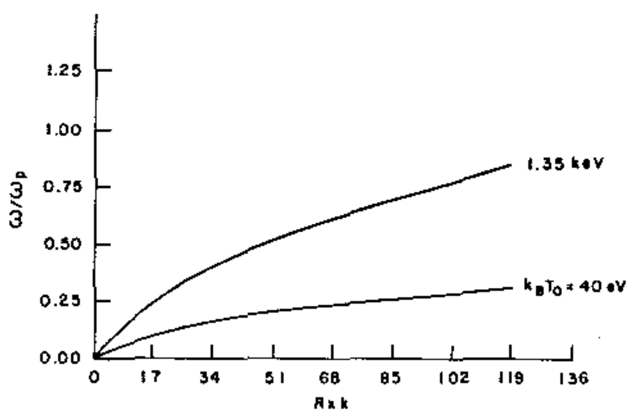

Fig. 3 - Dispersion relation when $B_0 = 0$, $n = 0$, $\omega_p = 5.040 \times 10^{10}\ s^{-1}$,$R = 0.085m$. Region $\omega < \omega_p$. The lower curve is for the LISA temperature: $k_BT_0 = 40eV$. The upper curve is for $k_BT_0 = 1.37keV$.

Results (28) to (30) are a generalization of the results obtained by Ghosh and Pal[6]. Their dispersion relation is only valid for the lowest circularly symmetric mode, $n = 0$. When $n \neq 0$ only hybrid modes can propagate because the TE and TM modes cannot satisfy simultaneously all the boundary conditions (24). This case of zero DC magnetic field was also studied by Azakami, Narita and Aye Thein[18], who showed that when $n = 0$ the waves can be separated into transverse electric and transverse magnetic modes.

4.2 Case of a Cold Magnetized Plasma

From (21) and (22) we obtain, in the limit $T_0 \to 0$:

$$H_z = -i\frac{(\omega^2 - \omega_p^2 - \omega_c^2)}{\mu_0 k w_c \omega_p^2}(\nabla_\perp^2 + k_R^2)E_z, \tag{31}$$

$$(\nabla_\perp^2 + k_A^2)(\nabla_\perp^2 + k_B^2)E_z = 0, \tag{32}$$

where

$$k_R = \left(\frac{\omega^2 - \omega_p^2}{\omega^2}\frac{\omega^2 k_e^2 - \omega_c^2 k_f^2}{(\omega^2 - \omega_p^2 - \omega_c^2)}\right)^{\frac{1}{2}} ,$$

$$k_A = \left[\frac{A - D^{\frac{1}{2}}}{F}\right]^{\frac{1}{2}} ,$$

$$k_B = \left[\frac{A + D^{\frac{1}{2}}}{F}\right]^{\frac{1}{2}} ,$$

$$A = -\omega_c^2\omega_p^2(\omega^2 + k^2c^2) + 2\omega^2(\omega^2 - \omega_p^2 - k^2c^2)(\omega^2 - \omega_p^2 - \omega_c^2) ,$$

$$D = \omega_p^4\omega_c^2[\omega_c^2(\omega^2 - k^2c^2)^2 + 4\omega^2k^2c^2(\omega^2 - \omega_p^2)] ,$$

$$F = 2c^2\omega^2(\omega^2 - \omega_p^2 - \omega_c^2) .$$

Accepting complex arguments, the solution of (32) which is finite at the axis is given by

$$E_z = A_n J_n(rk_A) + B_n J_n(rk_B) . \tag{33}$$

Applying (24) in (33), (9) and (12) yields the dispersion relation

$$k_A\frac{J_n'(Rk_A)}{J_n(Rk_A)} - k_B\frac{J_n'(Rk_B)}{J_n(Rk_B)} - \frac{nD^{\frac{1}{2}}}{\omega\omega_p^2\omega_c Rk^2c^2} = 0 . \tag{34}$$

This is the dispersion relation for a hybrid electromagnetic mode. As this is an odd function of n, Faraday rotation of the plane of polarization will happen. This is due to the different phase velocities of the $n = +N$ and $n = -N$ modes, where N is any natural number. A superposition of these two modes yields a composite wave in which the direction of polarization will be rotated as a function of distance along the guide[19]. Trivelpiece concluded that an electrostatic wave does not present Faraday rotation when excited in a cold plasma-filled waveguide, although it will present this rotation if the cold plasma only fills the waveguide partially[5]. Here we see that an electromagnetic wave in a cold plasma-filled waveguide presents Faraday rotation.

In the limit of slow waves ($\omega/k << c$), eq. (34) yields

$$T\frac{J_n'(RT)}{J_n(RT)} - k\frac{I_n'(Rk)}{I_n(Rk)} - \frac{n\omega_c}{R\omega} = 0, \tag{35}$$

where

$$T = \left[k^2\frac{(\omega^2-\omega_p^2)(\omega_c^2-\omega^2)}{\omega^2(\omega^2-\omega_p^2-\omega_c^2)}\right]^{\frac{1}{2}} .$$

Here we see that Faraday rotation will also be present for these slow electromagnetic waves. A qualitative graph of this dispersion relation is presented in Fig. 4 for te case $\omega_c < \omega_p$. There we can see that backward electromagnetic waves are predicted in the region $\omega_p < \omega < \omega_{UH}$, where $\omega_{UH} = (\omega_p^2 + \omega_c^2)^{\frac{1}{2}}$. This is confirmed in the numerical computation presented in Figs. 5 and 6 with the values of the density, guide radius and DC magnetic field of the linear machine LISA[16,17]. There are infinitely many curves in the region $\omega < \omega_c$ and in the region $\omega_p < \omega < \omega_{UH}$, but we only present 5 curves in Fig. 5 and 3 curves in Fig. 6. If $\omega_p < \omega_c$ then the backward waves will be in the region $\omega_c < \omega < \omega_{UH}$, while the passband for the forward waves will be in the region $0 < \omega < \omega_p$.

Waves in these ranges of frequencies were detected experimentally by Trivelpiece and Gould[4,5]. They interpreted their result as being electrostatic waves satisfying the dispersion relation

$$J_n(RT) = 0, \tag{36}$$

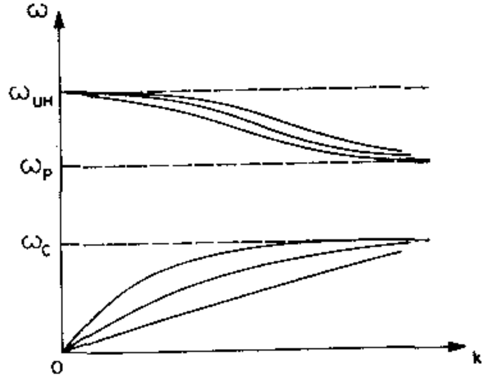

Fig. 4 - Dispersion relation for slow electromagnetic waves in a cold plasma-filled waveguide. $\omega_{UH} = \left(\omega_p^2 + \omega_c^2\right)^{\frac{1}{2}}$. Case in which $\omega_c < \omega_p$. Whistler waves in the region $\omega < \omega_c$ and backward waves in the region $\omega_p < \omega < \omega_{UH}$.

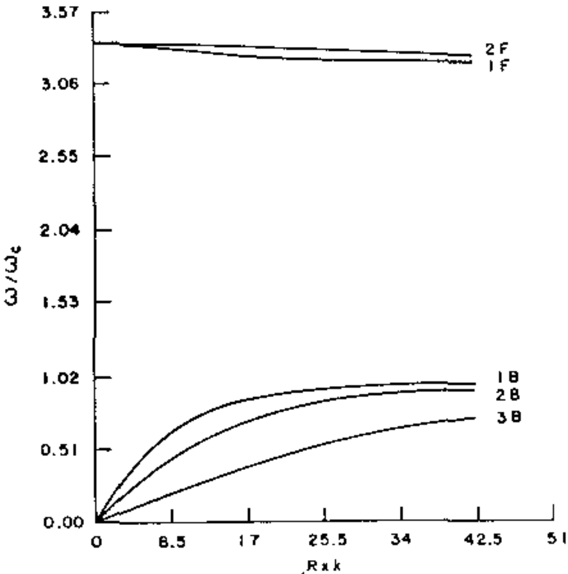

Fig. 5 - Dispersion relation for $k_B T_0 = 0$, $R = 0.085m$, $\omega_c = 1.57 \times 10^{10}\ s^{-1}$, $\omega_p = 5.04 \times 10^{10}\ s^{-1}$, $\omega_{UH} = 5.28 \times 10^{10}\ s^{-1}$, $n = 0$.

that is, $RT = p_{n\nu}$. But some remarks should be made. The first is that (36) was obtained by Trivelpiece applying the boundary condition $\phi_1(R) = 0$, where ϕ_1 is the electrostatic potential (for electrostatic waves the electric field is derived from

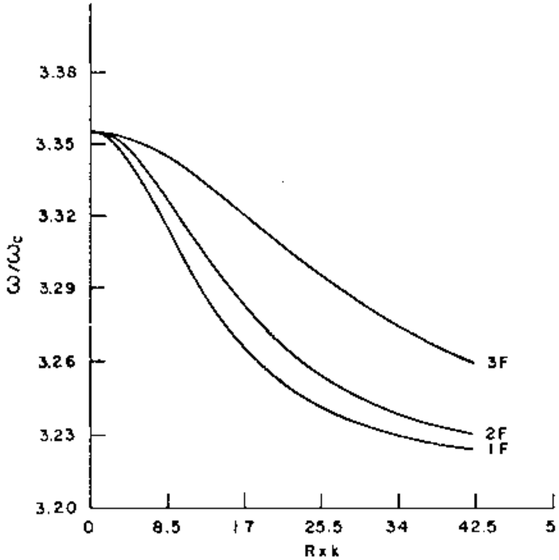

Fig. 6 - Backward waves for $k_B T_0 = 0$, $R = 0.085m$, $n = 0$, $\omega_p = 5.04 \times 10^{10}\ s^{-1}$, $\omega_{UH} = 5.28 \times 10^{10}\ s^{-1}$.

a scalar potential). This yields $E_z(R) = 0$ and $E_\theta(R) = 0$, as can be seen from equations (III.23) to (III.26) of Trivelpice's work (1967). But $\phi_1(R) = 0$ does not yield $u_r(R) = 0$ as can be seen from equation (III.6) of his work. However, it was shown by Ferrari[20] that the electrostatic approximation is reasonably accurate and can describe the waves very well.

Our interpretation is that the waves detected were slow electromagnetic waves satisfying (36) and not slow electrostatic waves satisfying (35). Eq. (35) satisfies all the boundary conditions (24), while (36) does not. We suggest a way to resolve this question: the measurement of Faraday rotation for the backward and forward waves. Trivelpiece and Gould did not report any measurement of this kind in the experiment in which the plasma completely fills the cylindrical waveguide. The dispersion relation (36) does not predict any Faraday rotation when the modes $n = \pm 1$ are excited simultaneously. But according to (35) this should happen. However, the magnitude of this Faraday rotation, if it exists, should be quite small in most cases.

4.3 Case of a Warm Magnetized Plasm

The solution of (23) which is finite at the axis is

$$E_z = A_n J_n(rk_1) + B_n J_n(rk_2) + C_n J_n(rk_3) \ . \tag{37}$$

Applying (37) in (21) yields

$$H_z = \frac{-ic^2U^2}{\mu_0 k\omega_c\omega_p^2(c^2-U^2)}\Big[A_n(k_+^2-k_1^2)(k_-^2-k_1^2)J_n(rk_1) + B_n(k_+^2-k_2^2)(k_-^2-k_2^2)J_n(rk_2) + C_n(k_+^2-k_3^2)(k_-^2-k_3^2)J_n(rk_3)\Big] \ . \tag{38}$$

Applying (37), (38) and (8)-(14) in (24) yields

$$n^2[F_1(L_2-L_3)+F_2(L_3-L_1)+F_3(L_1-L_2)] + n[P_1(F_3-F_2)+Q_1(L_2-L_3)] \times \frac{J_n'(Rk_1)}{J_n(Rk_1)} + n[P_2(F_1-F_3)+Q_2(L_3-L_1)]\frac{J_n'(Rk_2)}{J_n(Rk_2)} + n\,[P_3(F_2-F_1)+Q_3(L_1-L_2)]\,\frac{J_n'(Rk_3)}{J_n(Rk_3)} + (Q_1P_2-P_1Q_2)\frac{J_n'(Rk_1)J_n'(Rk_2)}{J_n(Rk_1)J_n(Rk_2)} + (Q_2P_3-P_2Q_3)\frac{J_n'(Rk_2)J_n'(Rk_3)}{J_n(Rk_2)J_n(Rk_3)} + (Q_3P_1-P_3Q_1)\frac{J_n'(Rk_3)J_n'(Rk_1)}{J_n(Rk_3)J_n(Rk_1)} = 0 \ , \tag{39}$$

where

$$F_j = \frac{U^2[k_e^2(k_e^2-k_j^2)+(k_+^2-k_j^2)(k_-^2-k_j^2)]}{Rk(c^2-U^2)} \ ,$$

$$P_j = \frac{-k_j}{k\omega^2\omega_p^2G(c^2-U^2)}[\omega^2U^2(k_e^2-k_j^2)(c^2G+k_e^2\omega_p^2) + k^2\omega_p^2(c^2-U^2)(\omega^2k_e^2-\omega_c^2k_f^2) + U^2\omega^2\omega_p^2(k_+^2-k_j^2)(k_-^2-k_j^2)] \ ,$$

$$L_j = \frac{U^2}{Rk\omega\omega_c\omega_p^2G(c^2-U^2)} \times [\omega_c^2(k_e^2-k_j^2)(c^2G+k_f^2\omega_p^2) + c^2(\omega^2k_e^2-\omega_c^2k_f^2)(k_+^2-k_j^2)(k_-^2-k_j^2)] \ ,$$

$$Q_j = \frac{-k_j}{c^2(c^2 - U^2)\omega_p^2 k\omega\omega_c} \times [k^2\omega_c^2\omega_p^4(c^2 - U^2) + \omega_c^2 k_f^2 U^2 \omega_p^2 c^2 (k_e^2 - k_j^2) + c^4 U^2 (\omega^2 k_e^2 - \omega_c^2 k_f^2)(k_+^2 - k_j^2)(k_-^2 - k_j^2)] ,$$

and where $j = 1, 2$ or 3.

This is the dispersion relation for a warm magnetized plasma completely filling a cylindrical waveguide. Eq. (39) is the most general result of this paper.

As this equation has odd powers of n the phenomenon of Faraday rotation appears again. Another point to note is that this dispersion relation refers to the hybrid modes due to the coupling between H_z and E_z in eq. (21).

This dispersion relation is more general than that obtained by Ghosh and Pal[6]. Eq. (39) is valid for any integer n and was obtained without further simplifications besides those required by the model. Eqs. (1) to (4), solution (37) and boundary conditions (24) yield, after a long algebraic manipulation, the dispersion relation (39. No other simplifications were made. Applying the limit $T_0 \to 0$ in (39) yields (34), as expected.

The main point of this section was to obtain the general dispersion relation eq. (39). In Figures figs. 7 to 9 we present the dispersion relations, eq. (39), for modes with $n = 0$, $n = 1$, $n = -1$, respectively. We present in each figure the six lowest modes in each case. The parameter ℓ indicates the number of times the component $E_z(r)$ goes to zero for $0 < r < R$. We utilized the following parameters: $k_B T_0 = 40\ eV$, $\omega_p = 1.20 \times 10^{10}\ s^{-1}$, $\omega_c = 1.50 \times 10^{10}\ s^{-1}$.

In Fig. 10 we plotted the Faraday rotation for the case $n = \pm 1$, eq. (39). This was obtained for $\ell = 1$ (see above). From it we can see that although "the electrostatic theory is reasonably accurate"[20] , our model indicates that Faraday rotation at this temperature can be detected in the laboratory. For instance, for a frequency $\omega \simeq 1.14\ \omega_c$ we expect a Faraday rotation of $\simeq 3\ rad$ if the waveguide length is ten times its radius.

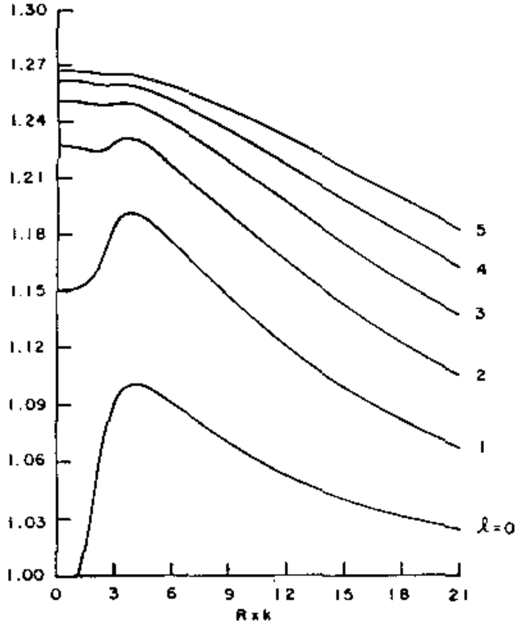

Fig. 7 -Backward waves for $k_B T_0 =$ 40 eV, $R = 0.085m$, $n = 0$, $\omega_p = 1.20 \times 10^{10}\ s^{-1}$, $\omega_c = 1.50 \times 10^{10}\ s^{-1}$. The parameter ℓ indicates the number of times $E_z(r) = 0$ for $0 < r < R$.

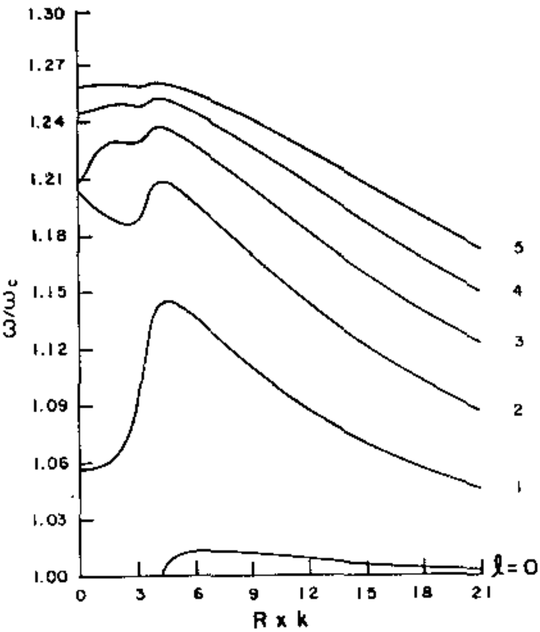

Fig. 8 - Backward waves for $k_B T_0 =$ 40 eV, $R = 0.085m$, $n = 1$, $\omega_p = 1.20 \times 10^{10}\ s^{-1}$, $\omega_c = 1.50 \times 10^{10}\ s^{-1}$. The parameter ℓ indicates the number of times $E_z(r) = 0$ for $0 < r < R$.

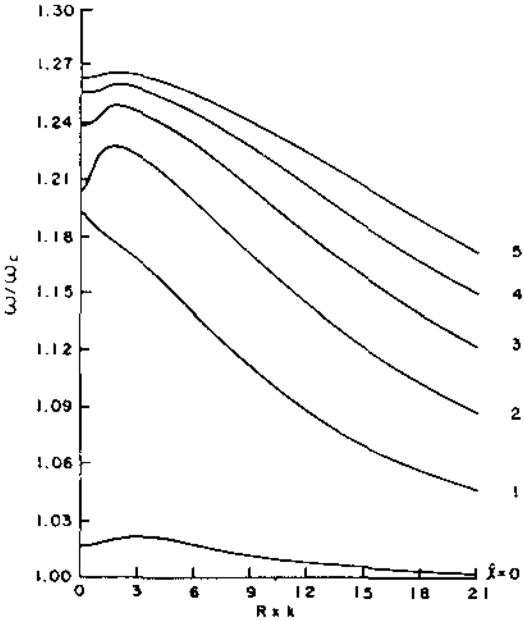

Fig. 9 - Backward waves for $k_BT_0 = 40\ eV$, $R = 0.085m$, $n = -1$, $\omega_p = 1.20 \times 10^{10}\ s^{-1}$, $\omega_c = 1.50 \times 10^{10}\ s^{-1}$. The parameter ℓ indicates the number of times $E_z(r) = 0$ for $0 < r < R$.

Conclusions

In this paper we studied the propagation of electromagnetic waves in a plasma-filled cylindrical waveguide of circular cross section. We obtained the global dielectric tensor of a warm magnetized plasma and showed that its elements have spatial derivatives which operate on the electric field components. With the solution of the equations for E_z and H_z, together with the appropriate boundary conditions we obtained the dispersion relations in several situations.

In the case of zero magnetic field we concluded that only hybrid modes can propagate when $n \neq 0$. In the case of a cold magnetized plasma we obtained a dispersion relation which is an odd function of n, indicating Faraday rotation for the electromagnetic waves. For slow waves we arrived at two passbands where excitation of the modes can occur: if $\omega_c < \omega_p$ when the passbands are $\omega < \omega_c$ and $\omega_p < \omega < \omega_{UH}$; if $\omega_p < \omega_c$ then the passbands are $\omega < \omega_p$ and $\omega_c < \omega < \omega_{UH}$. We

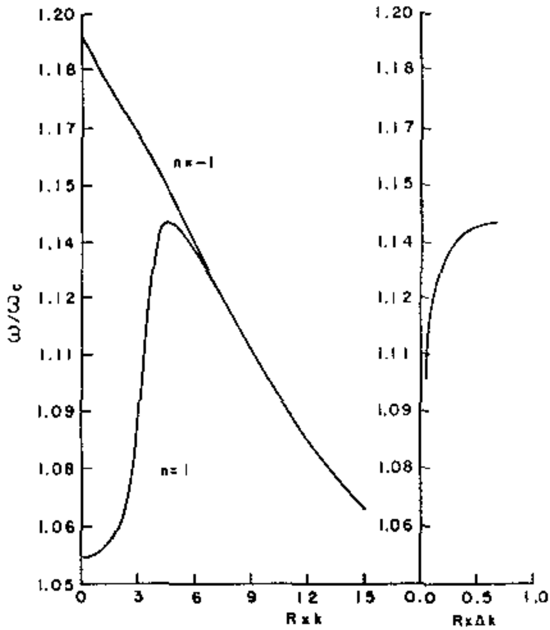

Fig. 10 - Faraday rotation for $k_B T_0 = 40\ eV$, $R = 0.085m$, $n = \pm 1$, $\omega_p = 1.20 \times 10^{10}\ s^{-1}$, $\omega_c = 1.50 \times 10^{10}\ s^{-1}$, $\ell = 1$.

then showed numerically that in the upper passbands we have backward waves. To distinguish what the waves detected experimentally by Trivelpiece and Gould in this range of frequencies were, i.e., to determine if they were electrostatic or electromagnetic in nature, we propose the measurement of the Faraday rotation of these waves. A rotation of the plane of polarization would indicate that they were electromagnetic waves while a fixed polarization would indicate that they were electrostatic waves. Finally, in the case of a warm magnetized plasma we generalized the results of Ghosh and Pal. The general dispersion relation came from a sixth order equation and is valid for any angular mode n.

We presented three curves for this general dispersion relation and also one curve for Faraday rotation indicating its magnitude in a typical situation.

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Resumo

São estudadas as ondas eletromagnéticas em um plasma uniforme, magnetizado e limitado por uma guia de onda cilíndrica usando a teoria de plasma morno. No caso de um plasma frio, são obtidas ondas eletromagnéticas retrógadas que exibem o fenômeno de rotação de Faraday, em contraste com os modos eletrostáticos de Trivelpiece e Gould que não apresentam esta característica. Resultados numéricos são apresentados para a máquina linear LISA. É derivada uma generalização da relação de dispersão de Ghosh e Pal para um plasma magnetizado e morno.