The Unit of the Angle and the Angular Equations

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Abstract

It is shown that angle has unit like any other physical quantity. It is presented a procedure to write down all angular equations taking this into account and valid for degrees, radians and grades.

1 Unit of the Angle

It is usually stated in textbooks that angle has no units (or dimensions) as it is the ratio of two lengths. This is a wrong statement. To prove this fact it is necessary only one counter example, namely: What is the value of a right angle? Is it 90, $\pi/2$ or 100? If angle were a pure number, the numerical value of a right angle should be always the same, but this is not the case. If degrees, radians or grades are utilized, the value of the angle changes accordingly. A right angle can be expressed as 90^o, as $\pi/2\ rad$ or as $100\ grad$. Another simple way to prove this fact is to observe that $1^o \neq 1\ rad \neq 1\ grad$.

Pure numbers like $ln2 = 0.6931$, $\sqrt{2} = 1.4142$ or $e = 2.7183$ do not depend on the system of units. The numerical value of an angle, however, depends on the system of units as has been shown above.

How is possible to reply to the statement that "angle is a ratio of two lengths and as such it should have no units"? To answer this it is only necessary to remember that all physical quantities expressed in terms of units are the ratio of a property of the body and the standard utilized to measure it. For instance, it is possible to say that a specific body weighs 2 kg, 2000 g or 4.4092 pounds. In the first case its weight was divided by the weight of the standard kept in Paris and found a ratio of 2, in the second case its weight was divided by a standard of 1 g and found a ratio of 2000, while in the third case its weight was divided by a standard of 1 lb and found a ratio 4.4092. All these ratios are pure numbers. But in order to express how many times heavier is the body compared with the chosen standard, it is necessary to give not only this pure ratio but also a specific unit. In this way it is possible to know and to remember with what standard was it compared to. An equivalent procedure is valid for measurements of length, time, temperature, pressure etc.

As the same happens with the angle, there is no reason to say that it has no units.

2 Angular Equations

There are also practical subjects related to this topic. Usual statements such as $\sin\theta \approx \theta$ if $\theta \ll 1$, or:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + ... = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} , \qquad (1)$$

only make sense if θ is expressed in radians. For instance, $\sin(0.01\ rad) = 0.00999 \approx 0.01 = \theta$, while $\sin(0.01^o) = 0.00017 \approx 0.0002 \neq \theta$. That is, the usual series expansions of trigonometric functions work when the angles are expressed in radians, but not when they are expressed in degrees or in grades.

Suppose a point particle moving in a circular orbit of radius r, tangential velocity v_t and centripetal acceleration a_c. Physical expressions like $v_t = r\omega$ and $a_c = r\omega^2$ related to the angular velocity $\omega = d\theta/dt$ are valid only if θ is expressed in radians (that is, $\theta = s/r$, where s is the length of the circular arc inscribed in the angle θ). It is not possible to utilize these expressions with θ in degrees or grades. For instance, suppose a point particle moving in a circular orbit of radius $1\ m$ with a constant velocity, performing a complete revolution each second. Its linear velocity is $v_t = total\ length/period = 2\pi(1\ m)/(1\ s) = 6.28\ m/s$. This can be written as $r\omega$ if $\omega = d\theta/dt = 2\pi\ rad/(1\ s)$. It is not possible to utilize $v_t = r\omega$ if ω were defined in degrees/sec, like $\omega = (360^o)/(1\ s) = 360\ s^{-1}$. The application of $v_t = r\omega$ with this last ω would yield $v_t = 360\ m/s$, which is the wrong answer.

Other physical equations involving angle are usually only valid if the angle is expressed in radians.

We here present a procedure to generalize all angular equations so that they can be valid with all units (not only in radians but also in degrees and grades). We are here following the basic proposals of Brownstein and Lévy-Leblond, [1] and [2]. The idea is to substitute the angle θ in all equations by $\theta\Box$ or by $\theta/\triangleleft$. Here $\Box = 1\ rad^{-1} = (2\pi/360)\ deg^{-1} = (2\pi/400)\ grad^{-1}$ and $\triangleleft = 1\ rad = (360/2\pi)^o = (400/2\pi)\ grad$. This value of $\triangleleft$ is obtained observing that a complete circle contains an angle of 360^o, or 400 $grad$ or $2\pi\ rad$. If θ be expressed in radians, $\triangleleft$ must also be expressed in radians. If θ be expressed in degrees or in grades, $\triangleleft$ must also be expressed in degrees or in grades, respectively.

Here we illustrate this with some basic equations. If s is the length of an arc of circle or radius r, the angle θ contained in this arc is given by

$$\frac{\theta}{\triangleleft} = \frac{s}{r} \, . \tag{2}$$

In this equation we always utilize s and r with the same units (meter and meter, or inch and inch, for instance). Obviously the same must be applied to θ and $\triangleleft$ (degree and degree, radian and radian, or grade and grade). In this way both sides of this equation will always yield pure numbers. And it will be possible to utilize it not only with θ expressed in radians, as usual, but also for the angle expressed in degrees or grades. For instance, if $r = 1\ m$ and $s = 3\ m$, the angle $\theta = (s/r)\triangleleft$ will be given by: $\theta = (3\ m/1\ m)(1\ rad) = 3\ rad$ or $\theta = (3\ m/1\ m)(360/2\pi)^o = (540/pi)^o = 171.887^o$.

In the circular motion of a particle around a circle its tangential velocity $v_t \equiv ds/dt$ and tangential acceleration $a_t \equiv d^2s/dt^2$ will be given by (from the previous equation and the usual definitions of angular velocity $\omega \equiv d\theta/dt$ and angular acceleration $\alpha \equiv d^2\theta/dt^2$):

$$\frac{\omega}{\triangleleft} = \frac{v_t}{r} \, , \tag{3}$$

$$\frac{\alpha}{\triangleleft} = \frac{a_t}{r} \, . \tag{4}$$

Once more both sides of these equations have the same unit ($time^{-1}$ and $time^{-2}$, respectively) and will be valid for radians, grades and degrees.

For this reason the angular momentum L_z about the z axis due to the motion of a particle in the xy plane will be given by

$$L_z = (\vec{r} \times m\vec{v})_z = mrv_t = mr^2 \frac{\omega}{\triangleleft} \, . \tag{5}$$

In this way the angle will not appear in the unit of L and this expression will be valid not only for radians but also for degrees and grades.

Analogously, the centripetal acceleration must be written as $a_c = r(\omega/\triangleleft)^2$.

The same should be applied to other physical equations involving angle.

As regards Eq. (1), the procedure is exactly the same, namely, replace θ by $\theta/\triangleleft$:

$$\sin\frac{\theta}{\triangleleft} = \frac{\theta}{\triangleleft} - \frac{1}{3!}\left(\frac{\theta}{\triangleleft}\right)^3 + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta}{\triangleleft}\right)^{2n+1} \, . \tag{6}$$

This is valid for all units of angles and allows the expression $\sin(\theta/\triangleleft) \approx \theta/\triangleleft$ if $\theta/\triangleleft \ll 1$. For instance, $\sin(0.01\ rad/\triangleleft) = \sin 0.01 = 0.00999 \approx 0.01$ and $\sin(0.01^o/\triangleleft) = \sin(0.01/(360/2\pi)) = \sin 0.0001745 \approx 0.0001745$ (although, as we had seen, $\sin 0.01^o$ is not ≈ 0.01 but ≈ 0.0002).

Another advantage of Eq. (6) as regards Eq. (1) is that the argument of sin becomes really unitless, as it should. The same should be applied, obviously, to all equations involving $\cos\theta$, $\tan\theta$ etc. That is, we should always replace θ by $\theta/\triangleleft$.

3 Conclusion

This work has proven that angle is not a unitless quantity. It has unit of angle. Moreover, a procedure has been suggested to take this into account in all mathematical and physical equations involving angles: to replace the angle θ by $\theta/\triangleleft$, where the constant $\triangleleft$ is given by $\triangleleft = 1\ rad = (360/2\pi)^o = (400/2\pi)\ grad$.

It seems to us that Brownstein's and Lévy-Leblond's proposal should be utilized in textbooks, classrooms and scientific papers.

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References

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