On the Equivalent of *Vis Viva*

Wilhelm Weber

Editor's Note: An English translation of Wilhelm Weber's 1874 paper "Ueber das Aequivalent lebendiger Kräfte".[1]

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[1] [Web74].

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Chapter 1

Translator's Introduction

Laurence Hecht[2]

This paper by Wilhelm Weber (1804-1891) continues the unique and still largely unrecognized approach to the atomic and sub-atomic realm that Weber and C. F. Gauss (1777-1855) had pioneered in their collaboration, beginning in 1830.

Here Weber draws on the discovery, first reported in his *Sixth Memoir* (1871),[3] that there can exist a stable state of aggregation of two similarly charged electrical particles, which he refers to here as a *particle pair*. In that memoir, he is able to derive a *critical length*, below which the similarly charged particles no longer repel, but rather attract each other. Weber's formula for *critical length* is expressed such that it applies to pairs of either negative or positive particles and also takes into account the possibility of different masses:

$$r < \frac{2}{c^2}\frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'}ee' \ , \tag{1.1}$$

where ε is the mass of the electrical particle, e is the charge in electrostatic units, and c the Weber constant.[4] (Weber later introduced the Greek letter ρ, rather than r, for his critical length.)

If one substitutes the values for electron and proton mass, which were derived many decades after Weber's death, into the Weber formula it results in a *critical length* of 5.6×10^{-13} *cm* (or twice the *classical electron radius*) for an electron pair, and a value 1836 times smaller for a proton pair. The later expression for the *classical electron radius* ($[e^2/(m_e c^2)]$ in the CGS system of units),[5] formulated well after Weber's death, is easily derived as a special case of Weber's *critical length*.

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[3][Web71] with English translations in [Web72] and [Web21b].

[4]The symbol c, as used by Weber, came to be known in mid 19th century physics as the *Weber constant*. Its value was derived by Weber and Kohlrausch in experiments reported in 1855 to 1857 on electrical discharge currents, comparing the force produced by a static to a moving charge, [Web55] with English translation in [Web21c]; [WK56] with Portuguese translation in [WK08] and English translation in [WK21]; [KW57] with English translation in [KW21]. As Weber saw it, c was the relative velocity between two like electrical particles at which the force between them would reduce to zero. It had a value $\sqrt{2}$ times light velocity. The modern application of the symbol c for light velocity appears to have originated later with Maxwell, $c_{Weber} = \sqrt{2} \cdot c_{Maxwell} = \sqrt{2} \cdot v_L$, where v_L = light velocity in vacuum, see [Ass21].

[5]Or $[\mu_o e^2/(4\pi m)]$ in the International System of Units MKSA.

Within a sphere of this diameter, the particles of Weber's pair will approach and recede from each other at a *relative velocity* (defined as the derivative with respect to time of the ever-changing distance between the two moving particles), which shall never exceed the value then referred to as the *Weber constant* (c). Interestingly, as Weber first notes in Section 2.3, this defines a motion, and consequent energy, independent of any fixed frame of reference and thus expressible without the need for spatial coordinates.

1.1 The Equivalent of *Vis Viva*

In this paper, Weber draws attention to the fact that his velocity-dependent force law, and the consequent existence of a boundary defined by the *critical length* (ρ), introduces a new consideration into the laws of motion and energy. The concept of a potential, which arose in the consideration of the laws of gravitation and electrostatics requires modification.

Potential was defined as the work that would be done to bring two interacting particles from a given distance apart to an infinite distance, or vice versa. While it had been sufficient to consider the sum of the *potential* and the *vis viva* (similar to the modern kinetic energy) as a constant, Weber recognizes that something new is required. He thus defines a new quantity, the *work capacity*, which measures the work done to bring two interacting particles from the critical length, ρ, to an infinite distance of separation. Like the potential, it is "an equivalent of *vis viva*" in that as one increases the other decreases, but it is evaluated independently of the existing distance r.

As an illustration of such a force, Weber cites the example of the elastic bar, a subject of careful analysis by early 19th century physicists, including his colleague Franz Neumann (1798-1895). In the theory of the elastic bar, the bar possesses a "natural length," independent of whatever length is observed at a particular time. The "equivalent of *vis viva*," in this case, is the restoring force which acts to bring the bar back to the natural length when the tension on it is released.

Analogously, Weber's critical length, ρ, is a sort of "natural length" which defines the behavior of the particle pair. For the case of two particles outside the critical length, at distances greater than ρ, a hidden force (the equivalent of *vis viva*, or *work capacity*) is depleted as the force of repulsion moves them apart. He defines the work capacity as the integral

$$U = \int_{\rho}^{\infty} R dr \ , \tag{1.2}$$

where R is the repulsive force between the two particles at the moment when their distance apart is equal to r.

The consequences are far-reaching. As Weber shows in Section 2.8 below, for two like electric particles, his integral expressing *work capacity* has the value $mc^2_{Weber}/4$ ($= mc^2_{Maxwell}/2$).[6]

In his *Sixth Memoir*, Weber had shown that for a pair of bound particles inside the critical length their maximum relative *vis viva* (when $(dr/dt)^2 = c^2$) attains the value $mc^2_{Weber}/4$ as well. Taken together, these two quantities are much like what came to be called the binding energy. But unlike the binding energy of the later nuclear physics, which was an *ad hoc* formulation to fit the experimental evidence, Weber's derives naturally from the velocity-

[6]Or, more precisely, $(1/4)mc^2_{Weber}\left[1 - v^2/c^2_{Weber}\right]$, where $v = dr/dt$.

dependent electrodynamic law first suggested by Gauss in 1835 and elaborated by Weber in his *First Memoir* of 1846.[7]

Interestingly, as the velocity c of Weber is $\sqrt{2}$ times greater than the velocity of light v_L (the c of Maxwell, Lorentz, and Einstein), or

$$c_{Weber} = \sqrt{2} \cdot c_{Maxwell} = \sqrt{2} \cdot v_L \ , \tag{1.3}$$

then

$$E = mc^2_{Maxwell} \tag{1.4}$$

becomes

$$E = \frac{mc^2_{Weber}}{2} \ , \tag{1.5}$$

retaining consistency between the mass-energy equation in Weber's formulation and the ordinary expression for kinetic energy or *vis viva*, $E = (mv^2/2)$. The two worlds, inner and outer, although distinct, obey the same natural laws.

The reader is warned that Weber's electrodynamics is an entirely different world than the one generally known today, which is based on the formulations of Faraday, Maxwell and their followers. It is not possible to simply carry over conceptions learned in college and graduate physics, and expect to grasp the alternative view that Gauss and Weber created. To have any deep comprehension of even this short paper requires a more thorough study of the fundamental conceptions presented in the First and Sixth Memoirs, than we can present here. Few are willing to undertake such a task. Yet we assure you, the hidden, and still scarcely explored, treasures that lie within are more than worth the effort.

Laurence Hecht
May 30, 2023

[7][Web46] with a partial French translation in [Web87] and complete English translations in [Web07] and [Web21a].

Chapter 2

On the Equivalent of *Vis Viva*

Wilhelm Weber[8,9,10,11]

Some of my earlier Treatises on electrodynamic measurements have already appeared in Volumes 73, 82, and 99 of the *Annalen*.[12,13] Here, only the principal subject of the last of these Treatises will be discussed in more detail, namely the connection of the established fundamental law of electrical action with the principle of the conservation of energy and with the equivalent living forces (*vires vivae*)[14] given by it.

As far as the objections raised against the aforementioned fundamental law of electrical action are concerned, lack of space compels me to confine myself here to the following general remarks.

The more recent mathematical investigations into electricity have primarily concerned far-action effects and have therefore, apart from electrostatics, mostly adhered to the integral and the elemental laws, in contrast to the point laws, as Neumann calls them.[15] From the

[8] [Web74].

[9] Translated by Laurence Hecht, larryhecht33@gmail.com, and edited by A. K. T. Assis, www.ifi.unicamp.br/~assis

[10] The Notes by Heinrich Weber, the Editor of Vol. 4 of *Wilhelm Weber's Werke*, are represented by [Note by HW:], the Notes by Laurence Hecht are represented by [Note by LH:], while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[11] [Note by HW:] *Annalen der Physik und Chemie*, Jubilee volume dedicated to the Editor of the *Annalen der Physik und Chemie*, J. C. Poggendorff, Leipzig 1874, pp. 199-213.

[12] [Note by HW:] *Wilhelm Weber's Werke*, Vol. III, pp. 215, 276, 597.

[13] [Note by AKTA:] [Web48] with English translations in [Web52], [Web66], [Web19] and [Web21e]; [Web51] with English translations in [Web61] and [Web21d]; and [WK56] with Portuguese translation in [WK08] and English translations in [WK03] and [WK21].

[14] [Note by LH and AKTA:] We translate the German *lebendige Kraft*, literally "living force," by the Latin term *vis viva* (plural *vires vivae*) also meaning "living force." Originated by Gottfried Leibniz (1646-1716) in the 17th century, the *vis viva* of a body of mass m moving with velocity v relative to an inertial frame of reference was defined as mv^2, that is, twice the modern kinetic energy. During the XIXth century many authors, including Hermann von Helmholtz (1821-1894) and Wilhelm Weber, defined the *vis viva* as $mv^2/2$, that is, like the modern kinetic energy.

[15] [Note by LH and AKTA:] In a paper in 1872, Carl Neumann (1832-1925), professor of mathematical-physics at Leipzig University, categorized the existing theories of electrodynamics based on far-action or action at a distance into three types. These were the point laws (*Punktgezetze* in German) of the type employed by Coulomb and Weber, based on the assumption of forces between electrical particles or mass points; elemental laws (*Elementargesetze* in German) referring to Ampère's approach using mathematically described elements of current and the force between them; and integral laws (*Integralgesetze* in German) of

physical point of view, on the other hand, the effects of electricity on the bodies through which it flows, particularly the thermal and chemical effects, have become so important and significant that the physicist cannot avoid turning his attention to the point laws, which alone can provide insight into the inner connection between electricity and heat, as well as into the inner mechanics of chemical processes.

These thermal and the chemical effects are, above all, what make it necessary to ascribe a *molecular constitution* to bodies in general. In its molecular constitution, however, every molecule, like a celestial body, is a world in motion by itself, whose inner relationships and motions are not directly observed. But these inner relationships and motions play a great part in all thermal and chemical processes, so that it has become necessary to assume a *molecular constitution for a body*, in both the *mechanical theory of heat* and the *chemical theory of the atom*.

In the case of a *molecularly constituted body*, however, as is easily seen, one cannot posit any arbitrary mass distribution; for the atomic masses in the molecules are masses of a *given magnitude*, which always remain *distinct*. They start out at definite distances from one another and in definite relative motions with respect to each other, and always maintain such mutual interactions so as to remain distinct and apart from each other. It is thus easy to understand that for such atoms *not every conceivable mass accumulation* will occur, and arbitrary assumptions about such mass accumulations can lead to contradictions. And even if all the laws of interaction of the atoms were known, these more precise details could not be determined so long as the three-body problem remains unsolved; because knowledge of the laws of motion of *two* mutually interacting bodies is not sufficient for this.

The foregoing objections must therefore be left unanswered for the time being, even if they are otherwise well-founded, because one does not know whether to place the blame for the contradictions on the contested Fundamental Law, or instead on those arbitrarily assumed mass accumulations.

2.1 Principle of Conservation of Energy

The law of inertia already prescribes that *if there is no external influence on a body, its vis viva remains unchanged*.

The principle of the conservation of *vis viva* follows from this: *for a system of bodies without external influence, the sum of its vires vivae remains constant so long as the relative positions of the bodies are the same*.

This principle of the conservation of *vis viva* was finally extended to the Law of the Conservation of Energy: *for a system of bodies without external influence, if the relative positions of the bodies vary, it is not the sum of their vires vivae alone, but rather this sum added to the work that would be done as a result of the interaction of the bodies if they were displaced to an infinite distance of separation, that remains constant at any given time*. That sum of the *vires vivae* is called the *kinetic energy*, and the work that would be done is called the *potential energy* of the system of bodies, the sum of which always remains the same when external influences are excluded.

As can be seen, however, this statement of the law of conservation of energy contains various interconnected principles which are better separated from one another: *firstly* prin-

the type developed by his father, physicist Franz Neumann (1798-1895), related to closed electric circuits, [Neu72]. We thank Kirk McDonald for bringing C. Neumann's paper to our attention.

ciples which are properties of individual bodies or particles, and *secondly* principles which relate to the properties of particle pairs.

2.2 Distinction of the Properties of an Individual Body or Particle, from Those of a Particle Pair

1. *Every body or particle* (every physical point or atom), considered by itself alone, possesses properties which completely determine its behavior in space and time, if the forces acting on it at any given instant and at all subsequent instants are given. These properties of the individual particles are *inertia* and *mass*, and there is no difference in these properties possessed by different particles other than the amount of their mass.

2. *Each pair of particles* has properties that are quite independent of the properties of the individual particles. These properties of the *pairs* are the reason that *work* is done with every change in the distance between the two particles. And the reason for a force of mutual repulsion or attraction lies in the properties of those *pairs*, which, multiplied by the change in distance between the two particles, gives the *work* done by the *pair*.

3. A system of *three* or more particles has *no* properties that are not already contained in the properties of the individual particles and the particle pairs.

This restriction of the nature of bodies to the properties of the *individual particles* and *particle pairs* leads to the important result that the investigation of the nature of all bodies can be greatly simplified by reducing them to the study of the individual pairs, which can be considered independently of one another. It would not be possible to so reduce the examination of the nature of all bodies, if the *totality of all particles* possessed unique properties that were not already contained in the properties of the *individual particles* or *particle pairs.* It is very important, in any case, that all the elements essential for the consideration of general processes of nature are already completely contained in the properties of *individual particles* and *particle pairs.*

2.3 Characteristics of a Fundamental Law of Interaction

Once the properties of *individual* particles were completely determined by *general mechanics*, *physics* essentially had only to determine the properties of particle pairs. For this determination nothing more need be considered than the *nature* and *mutual relations* of the particles forming a pair, and *the work produced by their interaction at every change in their distance of separation.* The *relationships* between these two particles are given by their *distance of separation* and the *velocity at which it changes.* — The fixed spatial *coordinate system*, which is required for a complete determination of the position and motion of particles, is completely unnecessary for the determination of the distance and relative velocity of the particles in a pair, and does not come into consideration at all here. It must therefore be possible to represent the *fundamental laws of particle interaction* for a *particle pair* without the aid of spatial coordinates.[16]

[16]The concept of representing *relative vis viva* in a system free of fixed coordinates is mentioned by Carl Jacobi (1804-1851) in his *Lectures on Dynamics*, Fourth Lecture, "The Principle of Conservation of *vis viva*," edited by A. Clebsch in 1866 from notes provided by C.W. Brockardt who attended Jacobi's lectures at the

The only variables required to represent the *fundamental law of particle interaction* are the *relative distance* (r) between the two particles, their *relative velocity* (dr/dt) and functions of these two magnitudes. Functions of the spatial coordinates are not needed.

None of the propositions listed above under Section 2.1, not even the last one (the proposition advanced under the name of the principle of the conservation of energy), corresponds to these requirements for a fundamental law of particle interaction. This is because those propositions are not simply about the properties of a particle pair, but rather more generally about the properties of a system of bodies (a system of particle pairs), namely a property which this system does not always have, but only possesses so long as no outside influence is present. And further because the sum, which according to the theorem in Section 2.1 should remain constant, is essentially a function of the space coordinates x, y, z.

2.4 Two Kinds of Equivalents of *Vis Viva*

A pair of particles, ε and ε', moving away from or approaching each other with the velocity dr/dt, possesses a *relative vis viva* which is determined by the product of the square dr^2/dt^2 multiplied by a factor dependent upon the masses ε and ε',[17] namely

$$\frac{1}{2}\left(\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\right) .$$ [18]

We need not consider here the mean *absolute vis viva* of the pair (attributed to their center of gravity), namely

$$\frac{1}{2}\left[\frac{(\varepsilon\alpha+\varepsilon'\alpha')^2}{\varepsilon+\varepsilon'}+(\varepsilon+\varepsilon')\gamma^2\right] ,$$

where α and α' signify the velocities of the two particles in the direction r, and γ the velocity of the center of gravity perpendicular to r.

Nor do we need to consider the *vis viva* arising from the motion of the particles *around each other*, namely[19]

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\frac{ds^2}{dt^2} ,$$

where [ds/dt] represents the velocity with which the two particles move towards or away from each other in the direction perpendicular to r.

University of Königsberg in the winter semester of 1842-43, [Cle09]. Jacobi's idea derives from earlier ideas of Lagrange (1736-1813) and Hamilton (1805-1865) describing a system by its conservation of *vis viva*.

[17] [Note by AKTA:] Weber's expression dr^2/dt^2 should be understood as $(dr/dt)^2$. For a detailed deduction of the following formulas of this Section see, in particular, Section 4 of Weber's Sixth Memoir, [Web71] with English translations in [Web72] and [Web21b].

[18] [Note by LH:] Weber here employs the concept of *reduced mass* ($mm'/(m+m')$), still used to describe a two-body system in celestial mechanics. In the case that the electrical particles, ε and ε', are of equal mass, the *reduced mass* is equal to one-half the mass of either particle.

[19] [Note by AKTA:] The next equation should be understood as

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\left(\frac{ds}{dt}\right)^2 .$$

However, the *relative vis viva* belonging to the system of both particles varies with time, such that as one part of it is lost, a new part is added. But such a change does not take place without something else simultaneously being changed, and when that other thing is restored, the lost part of the *vis viva* is also restored. The part of the *vis viva* which has been temporarily lost is said to have been replaced by something else, and this thing that replaces it is described as the *equivalent* of the part of the *vis viva* that was lost.

It has become a major focus of physics to establish *laws for this equivalent*, according to which it can be determined from *measurable quantities*.

Now, this *equivalent* can *either* be the motion of a body, *or* not. In the *first* case, the equivalent would also be a *vis viva*, like that which it replaces, and would differ from it only in belonging to other particle pairs. But it could also happen that it would become imperceptible to us as motion, as in the case of heat. In all these cases, where the equivalent of the lost *vis viva* is also a *vis viva* and, in fact, of the same magnitude, the sum of the *vires vivae* remains unchanged and it is merely a question of an altered distribution of it, the explanation for which is to be sought in the laws of motion.

The *other* case is completely different from this, where the lost *vis viva* is not directly replaced by *vis viva*, but by something else different from *vis viva*. The investigation of this later kind of equivalents for the lost parts of the relative *vis viva* of the particle pair requires closer examination, both as to their *nature* and as to the laws of their determination from other *measurable quantities*.

2.5 The Second Kind of Equivalent

1. *Vis viva* is something *real* and always *positive*, just like *mass*. It follows from this that the *potential of certain forces* (electrical forces, for example) which can be either positive or negative, *cannot be an equivalent of vis viva* in the real sense. Also, the potential is not something really *present*, as it is the work which would be done if the two particles were brought from an infinite distance to their distance of separation, r. Now, if instead of the work that would be done, one could put into the definition of potential the work that *was done* by moving the two particles from an infinite distance to the distance r, then *this work* itself would either consist of a change in the *relative vis viva of both particles*, or in the *cancellation of other work*. But *cancelled work* is also not something that really exists any more than a cancelled force. And a *change in the vis viva* is already contained as part of the existing *vis viva*, and therefore cannot be counted as being present alongside the existing *vis viva*.

2. Still, even such an *imaginary form of work*[20] as is the potential, which is the work that would be done if the two particles were brought *from a finite to an infinite distance of separation*, or vice versa, might serve as the *definition and magnitude* of something actually *existing*, namely as the definition of an actually existing *property* which the system of two particles possesses. Except that in the definition of such a property, the *presently existing* distance, r, should not be taken as the finite distance of separation from which the two particles are to be carried to an infinite distance of separation. Rather, that finite distance must be determined *completely independently of the existing distance*, r; for the property should hold *for all values of* r, without distinction.

As an illustration of how such *imaginary work* can serve to define a property that actually

[20] [Note by AKTA:] In German: *gedachte Arbeit*.

exists, the example of *elastic rods* can be cited, wherein a very specific finite distance between the two ends of the rod is determined by the so-called *natural rod length* ρ, *completely independent of the existing rod length* r. For such a rod, a merely *imaginary work* is used (namely, that which would be performed by the rod if it were brought from twice its natural length to its simple natural length, i.e. from 2ρ to ρ) as the *definition and measure* of something actually existing in the rod, namely its actually existing property of *elasticity*.

Thus, if the *tension* corresponding to the rod length r is denoted by R, then the *elasticity* of the rod is expressed by $\int_{2\rho}^{\rho} Rdr$. In the same way, the expression $\int_{\rho}^{\infty} Rdr$ can serve as a definition of the property of the two electric particles given above, if R designates their repulsive force at the moment when, during the transfer from the finite distance ρ[21,22] (determined independently of r) to an infinite distance, their distance has become *equal* to r.

— If this property of the two particles is to apply in general, then R must be a function of time t, in order to always correctly represent the repulsive force, which can be different at different times at the same distance.

— The property of the two electric particles defined by $\int_{\rho}^{\infty} Rdr$ shall be called their *work capacity*.[23,24]

3. Since the relative *vis viva* of two particles is something completely independent of their distance of separation, r, the equivalent of this *vis viva* must also be independent of r. If, therefore, the equivalent, like the potential, were represented by the integral of a function of r, this integral would have to be a *definite integral* taken between limits which are completely independent of r, just like the property just defined of two electric particles, which has been called its work capacity.

4. Finally, it is implicit in the notion of the equivalent of *vis viva* (because *vis viva* is unchangeable so long as no part of it is replaced by an equivalent), that the equivalent, expressed in proper units, of the lost portion of *vis viva* added to the existing part of the *vis viva* must form a *constant*. However, because of the *finite value* of this constant (as the sum of two positive magnitudes), a *limit* is given which cannot be exceeded by either of these two magnitudes, neither by the *vis viva*, nor by its equivalent. From this it is evident that wherever the relative *vis viva* of two particles could grow infinitely, beyond any limit, there could exist no equivalent of *vis viva*, and thus also no principle of conservation of energy for which the existence of an equivalent of *vis viva* is a condition.

But a *vis viva* constrained by such a *limit*, or a limited relative velocity of the particles, might seem to conflict with the principles of *general mechanics*, where all forces are taken as given without asking their *origin*, just as the initial distribution of the masses and their velocities are taken as given without asking how they came about. In fact, if in *general me-*

[21] [Note by LH:] Weber's use of the symbol ρ in this development of the illustration of an elastic rod draws on a conception developed a few years earlier in his *Sixth Memoir*. In his derivation there of the *critical length* for an electrical particle pair, ρ, Weber recognized that the force between two similarly charged electric particles would reverse at this length. In the case cited here ($\int_{\rho}^{\infty} Rdr$), Weber is discussing the repulsive force at distances greater than the critical length. However, within a sphere of diameter ρ, the particle pair would be in a bound state oscillating back and forth from a distance of separation $r = 0$ to $r = \rho$. (Compare Weber's *Sixth major Memoir on Electrodynamic Measurements*, Sections 2, 8, 10, and 11).

[22] [Note by AKTA:] See footnote 3.

[23] [Note by LH:] In German: *Arbeitsfähigkeit*. The more literal translation "ability to do work" is often used in science instruction as a first definition of energy. We take the term *work capacity*, from the first English translation of Weber's Sixth Memoir.

[24] [Note by AKTA:] [Web71, p. 267 of Weber's *Werke*], [Web72, p. 19] and [Web21b, p. 83].

chanics by the "given forces" we were to understand *any forces*, the possibility of producing any relative velocity between two particles would be self-evident. This possibility is by no means plausible when one goes back to the *origin of the forces*, as in physics, and all forces are derived from the *lawful* interaction of bodies. Here one sees that, *in principle*, nothing stands in the way of such a *limit*, but the existence or non-existence of such a limit can only be decided from the *laws* of interaction of the bodies themselves.

2.6 The Law of Work Capacity under the Assumption that it is the Equivalent of *Vis Viva*

It was explained at the beginning of the previous Section that only an actually existing and positive magnitude can be the sought for equivalent of *vis viva*. It then turned out that the potential of two particles $\int_\infty^r Rdr$, (i.e. *the work* that would be done if the two particles were brought from an infinite distance to the existing distance r) would not be such a magnitude and therefore could not be the sought-after equivalent of living force. On the other hand, $\int_\rho^\infty Rdr$, i.e. *the work* that would be done if both particles were brought from a finite distance ρ (which can be determined quite independently of the existing distance r) to an infinite distance, could probably serve as the *definition of such a magnitude*, namely as the definition of an actually existing property of the system of both particles in their current state, which was called the system's existing *work capacity*. *This work capacity* will now be determined more precisely, on the assumption that it is the sought for *equivalent of vis viva*.

If we call U the *work capacity* of two electrical particles (ε and ε') at any given time, and x the *relative vis viva of these particles* at this same time, then it follows that the *work capacity* U (*as the equivalent*), added to the *vis viva* x, is equal to a constant, a:

$$x + U = a \ , \qquad \text{or} \qquad U = a\left(1 - \frac{x}{a}\right) \ .$$

If now, following Section 2.4, we set the *vis viva* as:

$$x = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \ ,$$

and, for the case of a disappearing U (where $x = a$), we set $dr^2/dt^2 = c^2$,[25] it follows that:

$$U = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'}c^2\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) \ .$$

[25][Note by LH:] Hence the *relative vis viva*, x, for two like electrical particles when $dr^2/dt^2 = c^2$ is $mc^2_{Weber}/4 = mc^2_{Maxwell}/2$. Weber here describes the case where the particles are beyond the critical length.

The relative velocity c occurs naturally inside the *critical length*, where the bound particle pair oscillates along the line connecting them from zero relative velocity at $r = \rho$ to a maximum relative velocity c at the center of the sphere, whence the *relative vis viva*, becomes also equal to $mc^2_{Weber}/4 = mc^2_{Maxwell}/2$. When $dr^2/dt^2 = c^2$, the sum of the values for *vis viva* both within and without the critical length becomes $mc^2_{Weber}/2 = mc^2_{Maxwell}$.

Note that Weber's c is a *relative velocity*, *i.e.* the rate of change of the distance *between* the particles, not their velocity as seen by a stationary observer in the laboratory frame of reference. For the observer in laboratory of two like, bound particles approaching each other at *relative velocity* $= c_{Weber}$, each particle will have the velocity $c_{Weber}/2 = (\sqrt{2}c_{Maxwell})/2$. There is no mass increase in Weber's formulation. Rather, the decrease in *charge-to-mass ratio* of high-velocity charged particles such as in the experiments reported by Kaufmann in 1901, had already been theoretically described in Weber's earliest electrodynamic work as due to a decrease in the *force* between electrical particles as their relative velocity increased.

In this expression for the work capacity U, however, the masses ε and ε' are, as is well known, quantities which cannot be determined by measurement. Therefore, two electric particles are commonly denoted not by the immeasurable masses, ε and ε', but rather by the measurable forces e^2 and e'^2, which each of these two particles exerts individually on the other equal one when *at relative rest*, at a distance $= 1$, or rather by the square roots of these values e and e', which are proportional to the immeasurable masses.[26]

In order to now substitute into the expression for *work capacity*

$$U = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}c^2\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}\right)$$

the *measurable* quantities e and e' for the *immeasurable* ε and ε', we must have reference to the limiting value of U for vanishing values of the velocity dr/dt, or for vanishing values of the *vis viva* ($x = [1/2][\varepsilon\varepsilon'/(\varepsilon+\varepsilon')][dr^2/dt^2]$) which may be denoted by U_0, namely:

$$U_0 = a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}c^2 \ .$$

In this case, the *electrostatic* law applies, according to which the work that would be done during a virtual change in distance dr is $= [ee'/r^2]dr$. If one understands by U_0 *that work* which would be performed as a result of this *electrostatic* interaction, if both particles were brought from a finite distance ρ (which can be determined quite independently of the existing distance r) to an infinite distance, then one obtains

$$U_0 = a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}c^2 = \int_\rho^\infty \frac{ee'}{r^2}dr = \frac{ee'}{\rho} \ .$$

From this it follows: *first*, that for every system of two electrical particles there really is a finite distance $\rho = ee'/a$ that can be determined quite independently of the existing distance r; *second*, that if one substitutes ee'/ρ in place of a in the equation $U = a(1-[1/c^2][dr^2/dt^2])$, *the sought for work capacity* or the sought for *equivalent of vis viva becomes*:

$$U = \frac{ee'}{\rho}\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}\right) \ .$$

2.7 Derivation of the Potential from the Work Capacity

In the previous Section we defined the work capacity, U, as the work that would be done if both particles were brought from the finite distance of separation, ρ, which can be determined quite independently of the existing distance r, to an infinite distance from each other, i.e.

$$U = \int_\rho^\infty R dr \ .$$

However, we found at the end of the previous Section

[26][Note by AKTA:] While ε and ε' are the values of the inertial masses of the two particles, e and e' are the values of their electric charges. The electrostatic force R between two point charges e and e' when they are separated by a distance r is given by $R = ee'/r^2$ in Gauss and Weber's absolute system of units.

$$U = \frac{ee'}{\rho}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) ,$$

which can also be written as

$$U = \int_{\rho}^{\infty} \frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) dr .$$

from which it follows that the repulsive force resulting from the interaction of both particles is

$$R = \frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) .$$

According to the definition of the potential, V, as *the work* which would be done as a result of the interaction of both particles when they are brought from an infinite distance apart to the existing distance, r, we get, finally:

$$V = \int_{\infty}^{r} Rdr = \int_{\infty}^{r} \frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) dr ,$$

or,

$$V = \frac{ee'}{r}\left(\frac{1}{c^2}\frac{dr^2}{dt^2} - 1\right) .$$

2.8 General Application

Finally, if we consider the masses m and m' of *any* two particles whose relative distance apart is r, and whose relative *vis viva* is

$$\frac{1}{2}\frac{mm'}{m+m'}\frac{dr^2}{dt^2} ,$$

and if the interaction occurring between these two particles when *at relative rest* is such that the incremental change in distance, dr, corresponds to the work $[kmm'/r^n]dr$, then, under the same assumption about the existence of an equivalent of *vis viva* for the previously considered *electrical* particles, there results for these particles a *work capacity* U such that:

$$U = \frac{1}{2}\frac{mm'}{m+m'}c^2\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) .$$

Further, if the *constant sum* of the *vis viva* and work capacity, i.e.

$$\frac{1}{2}\frac{mm'}{m+m'}c^2 ,$$

for these particles is equated with the *limiting value* of the work capacity for the case of a vanishing *vis viva* (where the law of interaction valid for the case of *relative rest* applies), that is, if

$$\frac{1}{2}\frac{mm'}{m+m'}c^2 = \int_{\rho}^{\infty} \frac{kmm'}{r^n}dr = \frac{k}{n-1}\cdot\frac{mm'}{\rho^{n-1}} ,$$

it results that

$$U = \frac{k}{n-1} \cdot \frac{mm'}{\rho^{n-1}} \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) \ ,$$

which can also be written:

$$U = k \int_\rho^\infty \frac{mm'}{r^n} \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{(n-1)c^2}\frac{d^2r}{dt^2}\right) dr \ .$$

It follows from this that the *repulsive force* R resulting from the interaction of the particles (which, for the case of *relative rest*, is assumed to be $= kmm'/r^n$) , becomes for the case of relative motion, that is when the *vis viva* $= [1/2][mm'/(m+m')][dr^2/dt^2]$:

$$R = \frac{kmm'}{r^n} \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{(n-1)c^2}\frac{d^2r}{dt^2}\right) \ .$$

Finally, we get from this the expression for the potential, V:

$$V = \int_\infty^r R dr = \int_\infty^r \frac{kmm'}{r^n} \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{(n-1)c^2}\frac{d^2r}{dt^2}\right) dr \ ,$$

or,

$$V = \frac{k}{n-1} \cdot \frac{mm'}{r^{n-1}} \cdot \left(\frac{1}{c^2}\frac{dr^2}{dt^2} - 1\right) \ .$$

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