Electrodynamic Measurements, Especially on Electric Oscillations

Wilhelm Weber

Editor’s Note: English translation of Wilhelm Weber’s 1864 paper “Elektrodynamische Maassbestimmungen, insbesondere über elektrische Schwingungen”, [Web64]. This work is the fifth of Weber’s eight major Memoirs with the general title of Electrodynamic Measurements, “Elektrodynamische Maassbestimmungen”. Foreword in [Web63] with English translation in [Web21a].

Third version posted in February 2021 (first version posted in August 2020) at www.ifi.unicamp.br/~assis
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Introduction by the Translator

Dear Readers,

Who venture into studying the translation of Weber’s 1864 treatise “Electrodynamic Measurements, Especially on Electric Oscillations” which makes part of his “Electrodynamische Maassbestimmungen”:

If you have already studied Weber’s Introduction to this Treatise,\(^1\) you may be familiar with some special aspects, hopefully encouraging you to go on.

The present work by Wilhelm Eduard Weber on oscillations in conductors is an exhaustive and detailed description, in theory and practice, of a series of pioneering experiments by one of the prominent researchers of his time. The copy of the German original used here, accessible on internet, is from the library of the Deutsche Museum, München. The frontispice bears a hand-written personal dedication to Gustav Wiedemann, known from the Wiedemann-Franz rule on the thermal conductivity of metals (“Herrn Professor G. Wiedemann vom Verf.” - to prof. G. Wiedemann by the author).

The Treatise on oscillations consists of two main parts: Laws of motion (an Introduction followed by Sections (Weber calls them “Artikel”) 1 through 24) and Observations of Oscillations (Sections 25 through 36 followed by the list of contents).

Like its Introduction, the translated Treatise suggests some comments on Weber’s science, language, and presentation. With respective changes, the following remarks may also apply to other translated parts of his extensive work.

About Weber’s Science

Weber’s science rightfully receives revived attention. In the present Treatise Weber addresses what he calls a “second task” following the previous work (“first task”) on various forces investigated by Coulomb, Ampère, Faraday, Neumann for various experimental situations - static charges, current elements, induction, currents or circuits in motion, respectively. Its author considers this second task of electrodynamics, turning to the motions of electric masses due to the forces treated in the first task, as even more far-reaching, hitherto neglected, and hence all the more important.

Weber, a child of the 19th century, formulates very cautiously, one of the

\(^1\)Foreword in [Web63] with English translation in [Web21a].
reasons for the kind of language he uses (see below). For instance, he does not take the validity of Ohm’s law for granted when he turns to what we call alternating currents (“oscillations”). Clearly these investigations are of great importance for telegraphy, Weber’s pioneering subject in collaboration with Gauss, and for the later triumphant career of alternating currents in general. The validity of Ohm’s law for rapidly varying currents is questioned. Do we sense here an early taste of the skin effect in spite of the then still quite low frequencies?

His repeated and very meticulous experiments (assisted by Rudolf Kohlrausch) require a few remarks.

Of particular interest are his rotating magnet to produce well defined oscillations (Section 20) and his solution how to handle wires 5 miles long in a laboratory. Aiming at telegraphy, these experiments on long conductors called for the strategy to spool them as twin wires (bifilar winding) with opposite current directions to avoid inductive losses (Section 26).

Referring to Kirchhoff, Weber rightfully mentions the great importance of the agreement between the propagation velocity of electric waves and of light in free space.

The occurrence of a factor $\sqrt{2}$ when Weber addresses the propagation velocity of wave trains (Section 16) may appear strange; it has been commented in recent years by several scientists. We may speculate and seek its connection with Weber’s novel velocity dependent modification of the static Coulomb potential where the relative velocity $dr/dt = \dot{r}$ between two charges enters by means of the dynamic factor $(1 - \dot{r}^2/2c^2)$.

**About Weber’s Language**

Weber’s language is 19th century German, in his case with extensively long sentences, with technical terms not in use any more, with circumstantial formulations and intertwined syntax. The German grammar is suited to fill a person with awe when it comes to long chains of nouns linked together and to construct extended periods (just take a look at the very end of Section 36; yes — that is one single sentence!)

Yet this here is the attempt to convey the text in Weber’s spirit to the readers of our time, trying to stay close to the original at the cost of sounding strange. For instance, these many conjunctions (but, however, thus, (w)hence, finally, now, and more of the like) are indeed present in the original.

You may notice some compound nouns with a “German taste”, like rotation direction, oscillation phase, oscillation amplitude. They are to avoid some of the many “of - of - type” constructions for better readability.
The following glossary exemplifies Weber’s old-time German by means of some technical terms in 19th century formulation and spelling → here translated as:

Ablenkung → deflection (Weber uses Elongation, too).

Beharrlicher Strom → steady current (our DC, constant in density and direction; today beharrlich = tenacious or stubborn).

Beruhigung → damping (Weber also uses Absorption or Dämpfung; calling his device, intended to dampen, a “Beruhigungsmittel” — meaning tranquilizer today).

(Freie) Electricität → “(free) electricity” (corresponds to the scientific state of the art of the mid 19th century when Weber was among those scientists to already postulate a smallest indivisible charge long before the electron was identified as the mobile carrier; to be understood as (free) charge. Sometimes Weber also uses Ladung).

Kette → chain, translated as circuit.

Linearer Leiter → “linear”, or thin in the sense of 1-D, or line shaped, or straight conductor.

Maass → unit (this fits better than measure because Weber set great value on his “absolute units” as is also expressed in his use of Maasseinheit and Maassbestimmungen).

Säule → voltaic pile.

Schwingungsdauer → (oscillation) period.

Schwingungszahl → (oscillation) number.

Stärke der Ladung → amount of charge.

Strom or Strömung → current (Weber uses both).

Stromdichtigkeit or Stromintensität → current density (Weber does not clearly distinguish current density and intensity).

Tangentenbussole → tangent galvanometer (bussole means a compass with a graduated circle and a line of sighting).

Verhältniss → quotient in the sense of proportion or ratio.

Commutator, (electro)dynamometer, multiplier, magnetoelectric, electrodiamagnetic and others of the like have similar meanings in German and English; they are somewhat obsolete names, maybe not listed in a standard dictionary.

About Weber’s Presentation

Some additional illustrative figures of the experimental setup to assist or, better still, to reduce the text, would have been helpful. Weber contents himself to just 5 figures (see Section 25, The Commutators).

His nomenclature is not easy to follow, and so are some of his formulas.
His “partial differential equations” (e. g. Sections 17 and 20) are not formulated as “partial” according to our usage.

Weber’s treatment of a “line element”, $ds$, is different from our infinitesimal differentials: his $ds = l$ may be considered very small compared to the circumference of a conducting ring, yet large enough compared to its thickness, so as to consider the finite length $l$ as straight, “geradlinig” (Section 10).

You will definitely not fail to notice that the essence of this long treaty could be offered in a concise form. Brevity (“the wit of wisdom”), so it seems, was not in the spirit of Weber’s time. If interested to get just a taste of that spirit, feel free to resist the temptation to follow all the details.

Anyway, if you venture into the text and get acquainted with its peculiarities, you will find yourself rewarded entering a past world of great science. Enjoy!

Peter Marquardt$^2$

PS Apologies for all mistakes and linguistic lapses which have survived.

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Electrodynamic Measurements, Especially on Electric Oscillations

By Wilhelm Weber

The first task of these treatises on Electrodynamic Measurements has been to exactly and completely determinate the various forces exerted by electric masses. A fundamental law has been set up from which have been derived and determined first the forces of electrostatic interactions and their laws discovered by Coulomb, second the mutual electrodynamic forces between current elements and their laws discovered by Ampère, third the forces of induction discovered by Faraday (Volta-induction) — including a current co-moving with its carrier and a current changing in its stationary carrier, and also what Neumann first discovered and observed, namely, induction

[Note by AKTA:] Weber is referring to his force law presented in the first treatise on Electrodynamic Measurements, with partial French translation in [Web87] and a complete English translation in [Web07].

[Note by AKTA:] Charles Augustin de Coulomb (1736-1806). See [Cou88a] with English translation in [Cou12], [Cou88c], [Cou88d], [Cou88b], [Cou89], [Cou91], [Cou93] and [Pot84].

[Note by AKTA:] André-Marie Ampère (1775-1836), see [Amp23] and [Amp26] with a complete and commented English translation in [AC15].

[Note by AKTA:] Michael Faraday (1791-1867). The expression utilized by Weber, Volta-Induktion, had been first suggested by Faraday himself in paragraph 26 of his first paper on electromagnetic induction published in 1832, see [Far52], §26, p. 267 and [Far11], p. 159:

For the purpose of avoiding periphrasis, I propose to call this action of the current from the voltaic battery, volta-electric induction.

This phenomenon of Volta-induction is nowadays called Faraday’s law of induction.

[Note by AKTA:] Franz Ernst Neumann (1798-1895). See [Neu48] and [Neu49]. See also [Web49] with English translation in [Web21c].
with] the passage of a current through a sliding contact — and their laws.

Apart from these various forces due to purely electric interactions, also forces exerted by magnetism on electricity have been considered, namely electromagnetic and magnetoelectric induction forces due to magnetism moving relative to electric masses — including magnetism co-moving with its carrier and merely within its carrier. — The laws could also be derived for these forces beginning with the established fundamental electric law, namely when, following Ampère, molecular currents were substituted for molecular magnetism. The same was valid for electrodiamagnetic forces.

Finally, also the laws for the forces exerted by ponderable bodies on the electric masses moving within them and exerted on the latter, have been considered; and which are called the galvanic resistance forces of the ponderable bodies. On the basis of Ohm's law,\footnote{Note by AKTA: Georg Simon Ohm (1789-1854). Ohm’s law is from 1826: [Ohm26a], [Ohm26c], [Ohm26d], [Ohm26b] and [Ohm27] with French translation in [Ohm60] and English translation in [Ohm66].} established for steady currents, a more general fundamental law for these forces has been tentatively set up.

The investigation of these forces is closely tied to a second task of electrodynamics, namely the exploration of the motions of the electric masses driven by all these forces and the exploration of their laws in terms of these forces acting on the electric masses. Hence an exact and complete knowledge of all forces and their exploration is mandatory in order to determine these motions, the exploration of these forces may be considered as the means and the exploration of these motions as the aim to be arrived at in this way.

This second very general task of electrodynamics has found but little attention and we may rightfully ask why so little has been done to extend the foundation based on the knowledge of the forces? Obviously one can hesitate to consider this foundation as safely established and finished. The knowledge of all forces acting on the electric masses could be called in question, namely, whether some yet unknown co-acting electric molecular forces limited to immeasurably small scales, must be investigated besides the known purely electric forces acting at all distances, before one tried to develop the laws of motion depending on them. There was also some doubt about the reliability of the resistance law in ponderable conductors as applied for the development of the laws for high frequency electric motions, because Ohm's law was established for steady currents only and the generalization has been only been tentative. — To conclude, add to this that the knowledge of the forces is not the only necessary requisite to fulfill the second task, but moreover a more specific knowledge of the masses subject to motion besides other not yet sufficiently known details is required.
Nevertheless, Kirchhoff made a very thorough effort to fulfill the second task, in fact in such a comprehensive way as the conditions allowed, and published the results in Poggendorff’s *Annalen* 1857, Vols. 100 and 102. Irrespective of the above objections, this first attempt has rightfully received wide attention because the decision whether and, if so, how far the objections are justified can hardly be found by any way other than by experiment. — Kirchhoff indeed tried to set up a general theory on the motion of electricity in an infinitely long thin wire, however, indicating himself that he considered as generally valid some known facts which take place for constant currents, or for those whose intensity varies only slowly. His procedure will be considered in more detail in the following Section.

**Part I**

**Laws of Motion**

1 **Kirchhoff on the Propagation of Electricity in Conductors**

Let $x, y, z$ denote the rectangular co-ordinates of a point of the conductor and $u, v, w$ the $(x, y, z)$ components of the current densities which is present at time $t$ in this point of the conductor. — We understand current density here as the product of the velocity of the moving carriers times the amount of positive electricity per unit volume of the conductor. Assuming Ohm’s resistance law as generally valid, this is tantamount to the product of the electromotive force acting on the point $(x, y, z)$ under consideration times the specific conductivity of the metal conductor. Hence, if $A$ denotes the electromotive force at point $(x, y, z)$ — that is the difference of the forces acting on the unit measure of positive and negative electricity at the point $(x, y, z)$ —, and $\alpha, \beta, \gamma$ as the angles between this force and the three co-ordinate axes, and $k$ as the specific conductivity of the metal we have

$$u = A \cos \alpha \cdot k, \quad v = A \cos \beta \cdot k, \quad w = A \cos \gamma \cdot k,$$

[Note by AKTA: [Kir57b] and [Kir57c], with English translations in [Kir57a] and [GA94], respectively.]
where the mechanical measures\textsuperscript{14} always used by Kirchhoff are to be assumed for forces and conductivity.\textsuperscript{15}

The electromotive force, \( A \), however, originates partly from the free electricity distributed in the whole circuit and partly from the induction which acts in all parts of the conducting circuit due to the change of intensity of the current. To start with, we exclude all external electromotive forces, for example magnetolectric induction forces. Excluding resistive forces that could be taken into account, all other known forces acting on electric masses do not contribute to the electromotive force (if the resistive forces, that could be taken into account, are excluded), like for example the electrodynamic forces discovered by Ampère, resulting from interacting current elements,

\textsuperscript{14}[Note by AKTA:] mechanischen Maasse in the original. This expression can be translated as “mechanical measures”, “mechanical units” or “mechanical units of measures”.

\textsuperscript{15}[Note by WEW:] Let \( \xi, \eta, \zeta \) denote the displacement of an electric particle at point \((x, y, z)\) after time \( t \), hence \( d\xi/dt, d\eta/dt, d\zeta/dt \) the velocity components of the flowing electricity, then, when \( \mathcal{E} \) represents the amount of positive electricity in the unit volume of the conductor, we have according to the first relation:

\[ u = \mathcal{E} \frac{d\xi}{dt}, \quad v = \mathcal{E} \frac{d\eta}{dt}, \quad w = \mathcal{E} \frac{d\zeta}{dt}. \]

But, according to Ohm’s law for steady currents, the current intensity \( i \), if steady, in a linear conductor is proportional to, or, in mechanical measure, equal to the sum of all electromotive forces, that is \( \int A \, dl \) along the total length \( l \) of the conductor divided by the total resistance, that is \( \int dl/ks \), where \( s \) is the cross section and \( k \) the specific conductivity of the metal, hence we have \( i = \int A \, dl/\int dl/ks \). But in this form of Ohm’s law the current intensity \( i \) through the cross section \( s \) of the conducting wire is understood as the product of the velocity of the flowing electricity, that is \( d\sigma/dt \), and the amount \( \mathcal{E} \) of positive electricity in the volume unit of the conductor, hence putting

\[ \frac{d\xi}{dt} = \frac{d\sigma}{dt} \cdot \cos \alpha, \quad \frac{d\eta}{dt} = \frac{d\sigma}{dt} \cdot \cos \beta, \quad \frac{d\zeta}{dt} = \frac{d\sigma}{dt} \cdot \cos \gamma, \]

[one gets]

\[ \int \frac{A \, dl}{\int dl/ks} = \mathcal{E} \frac{d\sigma}{dt}. \]

Assuming that Ohm’s law holds in general for each length element in the circuit, one gets

\[ \frac{A \, dl}{\int dl/ks} = Ak = \mathcal{E} [d\sigma/dt], \]

or \( Ak = \mathcal{E} [d\sigma/dt] \), and from this [relation], decomposing for the coordinate axes

\[ A \cos \alpha \cdot k = \mathcal{E} \frac{d\xi}{dt} = u, \quad A \cos \beta \cdot k = \mathcal{E} \frac{d\eta}{dt} = v, \quad A \cos \gamma \cdot k = \mathcal{E} \frac{d\zeta}{dt} = w. \]
from which it is known that the difference between the forces acting on positive electricity and those acting on negative electricity is always zero, from which hence no electromotive force results.

The components of the first part of the electromotive force, which comes from the free electricity distributed in the conducting circuit, are represented by doubling the negative values of the partial differentials of $\Omega$, taken as the value of the potential function of the free electricity at point $(x, y, z)$, with respect to the three coordinate axes, that is these components are represented by

$$-2\frac{d\Omega}{dx}, -2\frac{d\Omega}{dy}, -2\frac{d\Omega}{dz},$$

as is readily realized taking into account that the electromotive force, that is the difference of the forces acting on the unit positive and negative electricities, is double the force acting only on one unit of positive electricity.

In order to determine the components of the second part of the electromotive force which comes from the induction caused by changes of current intensities in all parts of the conducting circuit, we denote the coordinates of a second point in the circuit by $x', y', z'$, further the values of $u, v, w$ at this point [are represented] by $u', v', w'$, and the distance between $(x, y, z)$ and $(x', y', z')$ by $r$.

From the fundamental law of electric action we get the electromotive force exerted by the electricity in volume element $dx' dy' dz'$, moving along the direction of the $x$ axis with velocity $d\xi'/dt$, remembering that, according to the previous footnote, $u' = \mathbf{E}[d\xi'/dt]$, [acting] at point $(x, y, z)$ along the $x$-axis, expressed in mechanical measure [as given by:]

$$= -\frac{8}{c^2} \cdot \frac{dx' dy' dz'}{r^3} \cdot (x - x')^2 \cdot \frac{du'}{dt}.$$

Hence, the electromagnetic force due to a current element of length $\alpha$ with its current intensity uniformly increasing by [a factor of] $i$ during the time $t$ that acts on a point at distance $r$ equals (see Electrodynamic Measurements, Vol. 5 of these Abhandlungen, p. 268, number 4)\textsuperscript{17,18}

\textsuperscript{16}[Note by AKTA:] Nowadays these partial derivatives would be written as:

$$-2\frac{\partial \Omega}{\partial x}, -2\frac{\partial \Omega}{\partial y}, -2\frac{\partial \Omega}{\partial z}.$$

In this English translation we are maintaining Weber’s original notation.

\textsuperscript{17}[Note by HW:] Wilhelm Weber’s Werke, Vol. III, p. 655.

\textsuperscript{18}[Note by AKTA:] [KW57, Section 18, number 4, p. 655] with English translation in [KW21, Section 18, number 4, p. 55].
= \frac{2\sqrt{2}}{c} \cdot \frac{\alpha}{r} \cdot \frac{i}{l} \cdot \cos \vartheta \cos \vartheta' ,

along the direction which makes an angle \vartheta' with the extension of \( r \), if \( \alpha \) makes an angle \( \vartheta \) with \( r \). Here, however, the current intensity \( i \), usually determined by means of a galvanometer, is to be expressed in terms of absolute magnetic measure; which can be replaced by the value expressed in mechanical measure if multiplied by \( 2\sqrt{2}/c \). In mechanical measure the above current intensity, on the other hand, is \( = u' dy' dz' \). Putting \( i = \left[ \frac{2\sqrt{2}}{c} \right] \cdot u' dy' dz' \), hence

\[
\frac{i}{l} = \frac{di}{dt} = \frac{2\sqrt{2}}{c} \cdot \frac{du'}{dt} \cdot dy' dz' ,
\]

and notice that in the above case \( \cos \vartheta = \cos \vartheta' = \frac{(x - x')}{r} \) and \( \alpha = dx' \), then we find the electromotive force we were looking for

\[
= \frac{2\sqrt{2}}{c} \cdot \frac{dx'}{r} \cdot \frac{2\sqrt{2}}{c} \cdot \frac{du'}{dt} \cdot dy' dz' \cdot \frac{(x - x')^2}{r^2} ,
\]

which equals the value given above.

Considering the motion of electricity in element \( dx' dy' dz' \) along the \( y \) or \( z \) axis instead of along the \( x \) axis, the value of \( \cos \vartheta \) is given by \( \frac{(y - y')}{r} \) or \( \frac{(z - z')}{r} \) instead of \( \frac{(x - x')}{r} \), and \( \frac{dv'}{dt} \) or \( \frac{dw'}{dt} \) instead of \( \frac{du'}{dt} \), from which the total electromotive force exerted by the element \( dx' dy' dz' \) at point \( (x, y, z) \) along the \( x \) axis results as equal to

\[
= -\frac{8}{c^2} \cdot \frac{dx' dy' dz'}{r^3} \cdot (x - x') \left( \frac{dv'}{dt}(x - x') + \frac{dv'}{dt}(y - y') + \frac{dw'}{dt}(z - z') \right) .
\]

Likewise, putting \( \frac{(y - y')}{r} \) or \( \frac{(z - z')}{r} \) instead of \( \frac{(x - x')}{r} \) for \( \cos \vartheta' \), one finds the total electromotive force exerted along the \( y \) or \( z \) axis as equal to

\[
= -\frac{8}{c^2} \cdot \frac{dx' dy' dz'}{r^3} \cdot (y - y') \left( \frac{dv'}{dt}(x - x') + \frac{dv'}{dt}(y - y') + \frac{dw'}{dt}(z - z') \right) ,
\]

or

\[
= -\frac{8}{c^2} \cdot \frac{dx' dy' dz'}{r^3} \cdot (z - z') \left( \frac{dv'}{dt}(x - x') + \frac{dv'}{dt}(y - y') + \frac{dw'}{dt}(z - z') \right) .
\]

Putting for brevity
\[ U = \int \int \int \frac{dx' dy' dz'}{r^3} (x - x')(u'(x - x') + v'(y - y') + w'(z - z')), \quad (1) \]

\[ V = \int \int \int \frac{dx' dy' dz'}{r^3} (y - y')(u'(x - x') + v'(y - y') + w'(z - z')), \quad (2) \]

\[ W = \int \int \int \frac{dx' dy' dz'}{r^3} (z - z')(u'(x - x') + v'(y - y') + w'(z - z')), \quad (3) \]

one gets the components of the second part of the electromotive force which comes from the induction as a result from changes of current intensities in all parts of the conducting circuit equal to

\[ = -\frac{8}{c^2} \frac{dU}{dt}, \quad = -\frac{8}{c^2} \frac{dV}{dt}, \quad = -\frac{8}{c^2} \frac{dW}{dt}. \]

However, the above components of the total electromotive force have been expressed as

\[ A \cos \alpha, \quad A \cos \beta, \quad A \cos \gamma, \]

thus one gets

\[ A \cos \alpha = -2 \left( \frac{d\Omega}{dx} + \frac{4}{c^2} \cdot \frac{dU}{dt} \right), \]

\[ A \cos \beta = -2 \left( \frac{d\Omega}{dy} + \frac{4}{c^2} \cdot \frac{dV}{dt} \right), \]

\[ A \cos \gamma = -2 \left( \frac{d\Omega}{dz} + \frac{4}{c^2} \cdot \frac{dW}{dt} \right). \]

Finally, putting these values into the above equations for the current densities \( u, v, w \) at point \((x, y, z)\), one gets the following equations

\[ u = -2k \left( \frac{d\Omega}{dx} + \frac{4}{c^2} \cdot \frac{dU}{dt} \right), \quad (4) \]

\[ v = -2k \left( \frac{d\Omega}{dy} + \frac{4}{c^2} \cdot \frac{dV}{dt} \right), \quad (5) \]

\[ w = -2k \left( \frac{d\Omega}{dz} + \frac{4}{c^2} \cdot \frac{dW}{dt} \right), \quad (6) \]
As a special prerequisite to determine the value $\Omega$ of the potential function at point $(x, y, z)$ of the total free electricity distributed in the whole circuit, the density of free electricity in the inner part of the conductor where current motions take place must not be equated to zero, as for a conductor with electricity at rest. Denoting by $\varepsilon'$ the non-zero [volume] density of free electricity at point $(x', y', z')$, if inside the conductor, and by $e'$, when located on the surface element $dS'$, thus denoting the [surface] density of free electricity in the surface element $dS'$ by $e'$, we then get the following value of $\Omega$, namely

$$\Omega = \int \int \int \frac{dx'dy'dz'}{r} \cdot \varepsilon' + \int \int \frac{dS'}{r} \cdot e'.$$  

(7)

In addition, the distribution of free electricity inside the whole conducting circuit as well as at its surface which is determined by the values of $\varepsilon'$ and $e'$, may indeed change with time, but these changes depend on the motion of electricity in the circuit, whence there must be two equations to represent the partial differential coefficients of $\varepsilon'$ and $e'$ with respect to time in their dependence on the motion of electricity.

The difference between the positive electricity leaving the element $dx'dy'dz'$ along the $x, y, z$ axis during the time element $dt$ and the electricity entering it is

$$dx'dy'dz' \cdot \frac{du'}{dx'}dt, \quad dx'dy'dz' \cdot \frac{dv'}{dy'}dt, \quad dx'dy'dz' \cdot \frac{dw'}{dz'}dt.$$

The sum of these differences yields the decrease of the free electricity $dx'dy'dz'$ of $\varepsilon'$ contained in $dx'dy'dz'$ during the time element $dt$ which is produced by motion of the positive electricity. There is, however, another equal decrease resulting from the opposite motion of negative electricity during the same time element $dt$; consequently this sum equals half of the total decrease of the free electricity contained in the element $dx'dy'dz'$ during the time element $dt$, that is half of $= -dx'dy'dz' \cdot [d\varepsilon' / dt] \cdot dt$, hence we have

$$\frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'} = -\frac{1}{2} \frac{d\varepsilon'}{dt}.$$  

(8)

Finally, denoting the angles between the inward normal on the surface element $dS'$ and the $x, y, z$ axes by $(N', x'), (N', y'), (N', z')$, the amount of positive electricity flowing back inwards from the surface element $dS'$ during the time element $dt$ equals
\[ = \left( u' \cos(N', x') + v' \cos(N', y') + w' \cos(N', z') \right) dS' \cdot dt , \]

and, because an equal amount of negative electricity flows from the interior towards the surface element \( dS' \) during the same time, this amount equals half of the total decrease of free electricity \( e'dS' \) at the surface element \( dS' \) during the time element \( dt \), that is, \( = -\frac{1}{2} \frac{de'}{dt} \cdot dS' dt \), hence we have

\[ u' \cos(N', x') + v' \cos(N', y') + w' \cos(N', z') = -\frac{1}{2} \frac{de'}{dt} . \quad (9) \]

As general as this derivation of the equations of motion of electricity in any conductor by Kirchhoff may be in other respects, it is based on three limiting assumptions, namely:

1. the assumption that the value of the electromotive force in a point, as was done above, may simply be determined by doubling the force exerted on the positive electricity, that is, assuming equal amounts of positive and negative electricity in all parts of the conductor, or, more precisely, that this would strictly mean that the densities \( \varepsilon' \) and \( e' \) of free electricity inside and on the surface of the conductor would always and everywhere be equal to zero, which is not the case, that at least the present free electricity may be considered as vanishingly small compared with the amount of a neutral mixture of both electricities at the same position;

2. the assumption that always equal amounts of positive and negative electricity pass through each cross section simultaneously in opposite directions, which is only justified, when in addition we can assume everywhere an arbitrary motion of the neutral fluid, on the grounds that such an added motion of a neutral fluid, if it were really present, would have no influence at all on the observations;

3. the assumption of a more general validity of Ohm's law which, as is to be shown later, may be reduced to the assumption that the mass of the electric fluid would vanish everywhere compared with the mass of its ponderable carrier, which, however, is usually assumed in general.
2 Derivation of the Expression for the Electromotive Force Exerted by the Free Electricity and by the Electric Motions in a Small Piece of the Conducting Wire, Considered as a Cylinder, on Any Point of the Middle Circular Cross Section of This Piece

The more precise determination of the electromotive force acting in any point of the cross section of the conducting wire suggests to divide the latter into two parts, namely the part that comes from the element of the conducting wire which contains the point under consideration, and the part that comes from all other elements that lie in greater measurable distances from the point under consideration.\(^{19}\)

Let the element of the conducting wire which contains the point under consideration be a cylinder whose radius is very small compared with its length. Kirchhoff assumes the distribution of free electricity as well as that of the electric motions in that cylinder as *symmetric with respect to the cylinder axis*. With respect to the coordinates, let the \(x\) axis coincide with the cylinder axis and put

\[
y = \rho \cos \varphi, \quad y' = \rho' \cos \varphi',
\]

\[
z = \rho \sin \varphi, \quad z' = \rho' \sin \varphi'.
\]

Further, distinguishing the current densities parallel to the cylinder axis and perpendicular to the cylinder axis, the latter is everywhere *radial* under the assumption of the symmetry of the motions, that is, in any point its direction coincides with the cylinder radius through that point. Hence, denoting \(\sigma\) this *radial current density* at point \((x, y, z)\) and \(\sigma'\) at point \((x', y', z')\), it follows that

\(^{19}\) [Note by AKTA:] Weber will calculate the electromotive force at a point \(P\) inside the wire. He then divides the whole circuit \(AD\) below into two parts.

The first part, \(BC\), contains all points closed to the point \(P\) and will be considered as cylindrical. The point \(P\) is located in the cross section of the wire located in the middle of \(BC\). The second part, composed of pieces \(AB\) and \(CD\), contains the points which are at great distances from \(P\).
\[ v = \sigma \cos \varphi, \quad v' = \sigma' \cos \varphi', \]
\[ w = \sigma \sin \varphi, \quad w' = \sigma' \sin \varphi', \]
where the values of \( \sigma \) and \( \sigma' \) are independent of \( \varphi \) and \( \varphi' \).

Substituting these values in the expressions of \( \Omega \) and \( U \) in the previous Section and taking \( \alpha \) as the radius of the cylinder, we get

\[ \Omega = \int \int \int \frac{dx' \cdot \rho' d\rho' d\varphi'}{r} \cdot \varepsilon' + \alpha \int \int \frac{dx' d\varphi'}{r} \cdot \varepsilon', \]  
(1)

\[ U = \int \int \int \frac{dx' \cdot \rho' d\rho' d\varphi'}{r^3} (\mathbf{x} - \mathbf{x}') \left( u'(\mathbf{x} - \mathbf{x}') + \sigma'(\rho \cos(\varphi - \varphi') - \rho') \right), \]  
(2)

Further, this substitution yields

\[ \frac{dv'}{dy'} = \frac{d \cdot \sigma' \cos \varphi'}{dy'}, \]

where \( \sigma' \) depends only on the variable \( \rho' \) for a given value of \( x' \). Hence putting \( \sigma' = f(\rho') = f\left(\sqrt{y'^2 + z'^2}\right) \), we get

\[ \frac{dv'}{dy'} = \frac{d}{dy'} \left( \frac{y' f\left(\sqrt{y'^2 + z'^2}\right)}{\sqrt{y'^2 + z'^2}} \right) = \frac{y'^2}{\rho'^2} \cdot \frac{d\sigma'}{d\rho'} + \frac{\rho'^2 - y'^2}{\rho'^3} \cdot \sigma'. \]

Likewise, we get

\[ \frac{dw'}{dz'} = \frac{z'^2}{\rho'^2} \cdot \frac{d\sigma'}{d\rho'} + \frac{\rho'^2 - z'^2}{\rho'^3} \cdot \sigma'. \]

hence

\[ \frac{dv'}{dy'} + \frac{dw'}{dz'} = \frac{d\sigma'}{d\rho'} + \frac{\sigma'}{\rho'} = \frac{1}{\rho'} \frac{d \cdot \rho' \sigma'}{d\rho'}. \]

Adding \( du'/dx' \) and substituting the respective value for the sum \( du'/dx' + dv'/dy' + dw'/dz' \) in Equation (8) of the preceding Section, yields the equation

\[ \frac{du'}{dx} + \frac{1}{\rho'} \frac{d \cdot \rho' \sigma'}{d\rho'} = -\frac{1}{2} \frac{d\varepsilon'}{dt}. \]  
(3)
Finally one finds the following values for the angles between the normal of the surface element \( dS' \) pointing inwards and the directions of the three coordinate axes:

\[
(N', \ x') = \frac{\pi}{2}, \quad (N', \ y') = \varphi' + \pi, \quad (N', \ z') = \varphi' + \frac{\pi}{2};
\]

hence we have\(^{20}\)

\[
u' \cos(N', \ x') = 0, \\
v' \cos(N', \ y') = -\sigma' \cos^2 \varphi', \\
w' = \cos(N', \ z') = -\sigma' \sin^2 \varphi'.
\]

Substituting these values in Equation (9) of the preceding Section then yields

\[
\sigma' = \frac{1}{2} \frac{d\epsilon'}{dt}. \tag{4}
\]

Putting for brevity

\[
x' - x = \lambda, \quad \text{hence} \quad dx' = d\lambda,
\]

\[
\rho^2 + \rho'^2 - 2\rho \rho' \cos(\varphi - \varphi') = \beta^2, \quad \text{hence} \quad r^2 = \beta^2 + \lambda^2,
\]

then, according to Equations (1) and (2), we have

\[
\Omega = \int \int \rho' d\rho' d\varphi' \int_{-1/2}^{1/2} \frac{\epsilon' d\lambda}{\sqrt{\beta^2 + \lambda^2}} + \alpha \int d\varphi' \int_{-1/2}^{1/2} \frac{\epsilon' d\lambda}{\sqrt{\beta^2 + \lambda^2}};
\]

\(^{20}\)Note by AKTA:] The second and third equation below were written in the original text as, respectively:

\[
v' \cos(N', \ y') = -\sigma' \cos^2 \varphi', \\
w' = \cos(N', \ z') = -\sigma' \sin^2 \varphi'.
\]

In order to avoid confusion, in this translation we are replacing the notations \( \sin \varphi'^2 \) and \( \cos \varphi'^2 \) used in Weber's time by their modern counterparts, namely, \( \sin^2 \varphi' \) and \( \cos^2 \varphi' \), respectively.
\[ U = \int \int \rho \, d\rho \, d\varphi \int_{-l/2}^{l/2} \frac{u' \lambda^2 d\lambda}{(\beta^2 + \lambda^2)^{3/2}} \]
\[ + \int \int \rho^2 \left( 1 - \frac{\rho}{\rho'} \cos(\varphi - \varphi') \right) \, d\rho \, d\varphi' \int_{-l/2}^{l/2} \frac{\sigma' \lambda d\lambda}{(\beta^2 + \lambda^2)^{3/2}}. \]

where \( l \) denotes the length of the cylinder and the point \((x, y, z)\) lies on the cross section that cuts this length in half.\(^{21}\)

Calculating the series expansion of \( \varepsilon' \) and \( e' \) in powers of \( \lambda \) in the first equation, namely
\[ e' = e + \frac{de}{dx} \cdot \lambda + \frac{1}{1 \cdot 2} \cdot \frac{d^2 e}{dx^2} \cdot \lambda^2 + \ldots, \]
\[ \varepsilon' = \varepsilon'_0 + \frac{d\varepsilon'_0}{dx} \cdot \lambda + \frac{1}{1 \cdot 2} \cdot \frac{d^2 \varepsilon'_0}{dx^2} \cdot \lambda^2 + \ldots. \]

where \( \varepsilon'_0 \), apart from time, depends only on the variable \( \rho' \), then for very small values of \( \beta^2/l^2 \) which follow with necessity from small values of \( \alpha/l \) because \( \beta^2 \) can never be greater than \( 4\alpha^2 \), we may put\(^{22}\)
\[ \int_{-l/2}^{l/2} \frac{d\lambda}{\sqrt{\beta^2 + \lambda^2}} = 2 \log \frac{1}{\beta}, \]
\[ \int_{-l/2}^{l/2} \frac{\lambda d\lambda}{\sqrt{\beta^2 + \lambda^2}} = 0, \]
\[ \int_{-l/2}^{l/2} \frac{\lambda^2 d\lambda}{\sqrt{\beta^2 + \lambda^2}} = \frac{l^2}{4}, \]

whence for small values of \( l \) we get
\[ \Omega = \int \int \rho' d\rho' d\varphi' \left( 2\varepsilon'_0 \log \frac{l}{\beta} + \frac{1}{8} \frac{d^2 \varepsilon'_0}{dx^2} l^2 \right) + \alpha \int d\varphi' \left( 2 \varepsilon \log \frac{l}{\beta} + \frac{1}{8} \frac{d^2 e}{dx^2} l^2 \right). \]

The integration is to be carried out from \( \varphi' = 0 \) to \( \varphi' = 2\pi \) and from \( \rho' = 0 \) to \( \rho' = \alpha \), whence we get

\(^{21}\)Note by AKTA: In footnote 15 on page 12 of Section 1 Weber had called \( l \) the total length of the curved and thin conductor. He is now representing by the same letter the length of the small cylinder \( BC \) represented in footnote 19 on page 18.

\(^{22}\)Note by AKTA: What Weber represents by the symbol \( \log \) in the next equations should be understood as the natural logarithm represented nowadays as \( \ln \).
\[ \Omega = 2\pi \int_0^\alpha \rho' d\rho' \left( 2\varepsilon_0' \log l + \frac{1}{8} \frac{d^2\varepsilon_0'}{dx^2} \right)^2 \right) + 2\pi \alpha \left( 2e \log l + \frac{1}{8} \frac{d^2e}{dx^2} \right) 
- 2 \int_0^\alpha \rho' d\rho' \cdot \varepsilon_0' \int_0^{2\pi} d\varphi' \cdot \log \beta - 2\alpha e \int_0^{2\pi} d\varphi' \cdot \log \beta . \]

Considering that

\[ \int_0^{2\pi} d\varphi' \cdot \log \beta = \frac{1}{2} \int_0^{2\pi} d\varphi' \cdot \log \left( \rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') \right) \]

either equals \( 2\pi \log \rho' \) when \( \rho' > \rho \), or equals \( 2\pi \log \rho \) when \( \rho > \rho' \), then we get for the part referring to the surface for which \( \rho' = \alpha \),

\[ -2\alpha e \int_0^{2\pi} d\varphi' \cdot \log \beta = -4\pi \alpha e \log \alpha . \]

The part referring to the interior [of the conductor] is decomposed into two parts, namely

\[ -2 \int_0^\alpha \rho' d\rho' \cdot \varepsilon_0' \int_0^{2\pi} d\varphi' \cdot \log \beta = -4\pi \log \rho \int_0^\rho \rho' d\rho' \cdot \varepsilon_0' - 4\pi \int_0^\alpha \rho' d\rho' \cdot \varepsilon_0' \log \rho' , \]

which hence reduces to

\[ -4\pi \log \alpha \cdot \int_0^\alpha \rho' d\rho' \cdot \varepsilon_0' , \]

in the limit where \( \rho = \alpha \), and [reduces] to

\[ -4\pi \int_0^\alpha \rho' d\rho' \cdot \varepsilon_0' \log \rho' , \]

in the other limit where \( \rho = 0 \), which expressions differ the less from each other the smaller [the value of] \( \alpha \), so that, with sufficient precision for very small values of \( \alpha \) one may put

\[ ^{23}[\text{Note by WEW:}] \text{Under the assumption of the symmetric distribution of free electricity in the wire, } \varepsilon_0' \text{ approaches a constant with decreasing values of } \rho'. \text{ Thus, if for small values of } \alpha \text{ it is allowed to put } \varepsilon_0' \text{ equal to a constant for all values of } \rho' < \alpha, \text{ the value found for the first limit turns into} \]

\[ -4\pi \varepsilon_0' \log \alpha \int_0^\alpha \rho' d\rho' = -2\pi \alpha^2 \varepsilon_0' \log \alpha , \]
\[2 \int_0^\alpha \rho' \, d\rho' \cdot \varepsilon'_0 \int_0^{2\pi} \, d\varphi' \cdot \log \beta = -4\pi \log \alpha \cdot \int_0^\alpha \rho' \, d\rho' \cdot \varepsilon'_0\]

for the part referring to the interior; hence

\[\Omega = 2\pi \int_0^\alpha \rho' \, d\rho' \left(2 \varepsilon'_0 \log l + \frac{1}{8} \frac{d^2 \varepsilon'_0}{dx^2} l^2 \right) + 2\pi \alpha \left(2e \log l + \frac{1}{8} \frac{d^2 e}{dx^2} l^2 \right)
- 4\pi \log \alpha \cdot \left(\alpha e + \int_0^\alpha \rho' \, d\rho' \cdot \varepsilon'_0 \right),\]

or, more concisely

\[\Omega = 4\pi \log \frac{l}{\alpha} \cdot \left(\alpha e + \int_0^\alpha \rho' \, d\rho' \cdot \varepsilon'_0 \right) + \frac{1}{4} \pi l^2 \cdot \left(\alpha \frac{d^2 e}{dx^2} + \int_0^\alpha \rho' \, d\rho' \cdot \frac{d^2 \varepsilon'_0}{dx^2} \right).\]

Finally, putting

\[2\pi \alpha e + 2\pi \int_0^\alpha \rho' \, d\rho' \cdot \varepsilon'_0 = E,\]

that means, denoting the amount of free electricity contained in the conductor element \(dx\), partly at its surface, partly in the interior, by \(Edx\), then differentiating twice one gets

\[2\pi \alpha \cdot \frac{d^2 e}{dx^2} + 2\pi \int_0^\alpha \rho' \, d\rho' \cdot \frac{d^2 \varepsilon'_0}{dx^2} = \frac{d^2 E}{dx^2},\]

hence

\[\Omega = 2E \log \frac{l}{\alpha} + \frac{1}{8} \frac{d^2 E}{dx^2} \cdot l^2. \quad (5)\]

Likewise, the values of \(u'\) and \(\sigma'\) may be developed in the above equation for \(U\) by a series expansion in powers of \(\lambda\), namely

\[u' = u'_0 + \frac{du'_0}{dx} \cdot \lambda + \frac{1}{1 \cdot 2} \frac{d^2 u'_0}{dx^2} \cdot \lambda^2 + \ldots,\]

and the one for the latter limit [turns] into

\[\int_0^\alpha \rho' \, d\rho' \log \rho' = -2\pi \alpha^2 \varepsilon'_0 \left(\log \alpha - \frac{1}{2} \right),\]

which expressions differ the less, the smaller \(\alpha\).

\[24\] [Note by AKTA:] Therefore the magnitude \(E\) means the linear charge density of the wire, that is, the amount of free charge per unit length.
\[ \sigma' = \sigma'_0 + \frac{d\sigma'_0}{dx} \cdot \lambda + \frac{1}{1 \cdot 2} \frac{d^2\sigma'_0}{dx^2} \cdot \lambda^2 + \ldots, \]

where, apart from the time, \( u'_0 \) and \( \sigma'_0 \) depend only on the variable \( \rho' \) for a given value of \( x' \).

Now, for very small values of \( \beta^2/l^2 \) corresponding to very small values of \( \alpha^2/l^2 \), one can put

\[
\int_{-l/2}^{l/2} \frac{\lambda d\lambda}{(\beta^2 + \lambda^2)^{3/2}} = 0, \\
\int_{-l/2}^{l/2} \frac{\lambda^2 d\lambda}{(\beta^2 + \lambda^2)^{3/2}} = 2 \left( \log \frac{l}{\beta} - 1 \right) = 2 \log \frac{l}{e\beta}, \\
\int_{-l/2}^{l/2} \frac{\lambda^3 d\lambda}{(\beta^2 + \lambda^2)^{3/2}} = 0, \\
\int_{-l/2}^{l/2} \frac{\lambda^4 d\lambda}{(\beta^2 + \lambda^2)^{3/2}} = \frac{1}{4} l^2,
\]

where \( e \) is the base of the natural logarithms. Hence one gets the following equation for \( U \):

\[ U = \int \int \rho' d\rho' d\varphi' \left( 2u'_0 \cdot \log \frac{l}{e\beta} + 1 \cdot \frac{1}{8} \frac{d^2u'_0}{dx^2} \cdot l^2 \right) + \int \int \rho'^2 \left( 1 - \frac{\rho}{\rho'} \cos(\varphi - \varphi') \right) d\rho' d\varphi' \left( \frac{2}{24} \frac{d^3u'_0}{dx^3} \cdot \log \frac{l}{e\beta} + 1 \cdot \frac{1}{8} \frac{d^2u'_0}{dx^2} \cdot l^2 \right). \]

The latter part of this value of \( U \) may be considered very small when \( \alpha \) is very small, because the integration for \( \rho' \) has to be carried out from \( \rho' = 0 \) to \( \rho' = \alpha \), hence

\[ U = \int \int \rho' d\rho' d\varphi' \left( 2u'_0 \cdot \log \frac{l}{e\beta} + 1 \cdot \frac{1}{8} \frac{d^2u'_0}{dx^2} \cdot l^2 \right) \]

where the integration has to be carried out from \( \varphi' = 0 \) to \( \varphi' = 2\pi \) and from \( \rho' = 0 \) to \( \rho' = \alpha \), thus

\[ U = 2\pi \int_0^{\alpha} \rho' d\rho' \cdot \left( 2u'_0 \cdot \log \frac{l}{e\beta} + 1 \cdot \frac{1}{8} \frac{d^2u'_0}{dx^2} \cdot l^2 \right) - 2 \int_0^\alpha \rho' d\rho' \cdot u'_0 \int_0^{2\pi} d\varphi' \log \beta. \]

As
\[
\int_0^{2\pi} d\varphi' \cdot \log \beta = \frac{1}{2} \int_0^{2\pi} d\varphi' \cdot \log \left( \rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') \right)
\]
equals either \(2\pi \log \rho'\) if \(\rho' > \rho\), or equals \(2\pi \log \rho\) if \(\rho > \rho'\), one gets

\[
U = 2\pi \int_0^\alpha \rho' d\rho' \cdot \left( 2u_0' \log \frac{l}{e} + \frac{1}{8} d^2 u_0' dx^2 \right)
- 4\pi \log \rho \int_0^\rho \rho' d\rho' \cdot u_0' - 4\pi \int_\rho^\alpha \rho' d\rho' \cdot u_0' \log \rho',
\]
for which one can also write

\[
U = 4\pi \log \frac{l}{e\alpha} \int_0^\alpha \rho' d\rho' \cdot u_0' + \frac{1}{4} \pi l^2 \int_0^\alpha \rho' d\rho' \cdot \frac{d^2 u_0'}{dx^2}
+ 4\pi \log \alpha \cdot \int_0^\rho \rho' d\rho' \cdot u_0' + 4\pi \int_\rho^\alpha \rho' d\rho' \cdot u_0' \log \frac{\alpha}{\rho}.
\]

But now, when \(\alpha\) is very small, the latter two parts of this value of \(U\) may be considered as vanishing, then one may put

\[
U = 4\pi \log \frac{l}{e\alpha} \cdot \int_0^\alpha \rho' d\rho' \cdot u_0' + \frac{1}{4} \pi l^2 \cdot \int_0^\alpha \rho' d\rho' \cdot \frac{d^2 u_0'}{dx^2}.
\]

Finally putting

\[
2\pi \int_0^\alpha \rho' d\rho' \cdot u_0' = i,
\]
that means, denoting the amount of positive electricity flowing through the cross section of the conducting wire during the time element \(dt\) by \(idt\), where \(i\) expresses the current intensity in mechanical measure, then differentiating twice yields

\[
2\pi \int_0^\alpha \rho' d\rho' \cdot \frac{d^2 u_0'}{dx^2} = \frac{d^2 i}{dx^2},
\]
hence

\[
U = 2i \log \frac{l}{e\alpha} + \frac{1}{8} \frac{d^2 i}{dx^2} \cdot l^2.
\]

Hereafter the electromotive force exerted by the free electricity in a small piece of the conducting wire, considered as a cylinder, on any point of the
middle part of this piece, is determined more precisely, namely, from the value of $\Omega$, [through]

$$-2 \frac{d\Omega}{dx} = -4 \frac{dE}{dx} \cdot \log \frac{l}{\alpha} - \frac{1}{4} \frac{d^3E}{dx^3} \cdot l^2,$$

and likewise the electromotive force exerted by induction of the electric motions in the same piece on the same point [is determined], namely, from the value of $U$, [through]

$$-\frac{8 dU}{c^2 dt} = -\frac{16}{c^2} \frac{di}{dt} \cdot \log \frac{l}{e\alpha} - \frac{1}{c^2} \cdot \frac{d^3i}{dx^2 dt} \cdot l^2.$$

Finally, assuming a very large number for the value of $\log[l/\alpha]$ as Kirchhoff did, one may put

$$-2 \frac{d\Omega}{dx} = -4 \frac{dE}{dx} \cdot \log \frac{l}{\alpha},$$

$$-\frac{8 dU}{c^2 dt} = -\frac{16}{c^2} \frac{di}{dt} \cdot \log \frac{l}{e\alpha},$$

or, when 1 vanishes completely compared with $\log[l/\alpha]$,

$$-\frac{8 dU}{c^2 dt} = -\frac{16}{c^2} \frac{di}{dt} \cdot \log \frac{l}{\alpha}.$$

3 Simplification of the General Equations

Following a more exact determination of the electromotive forces acting on a point $(x, y, z)$ of the conducting wire, that come partly from free electricity, partly from the electric motions in a small part of the conducting wire to be considered as a cylinder, Kirchhoff has tried to simplify the general equations presented in Section 1 under the following conditions, namely

1. that the radius of the conducting wire, $\alpha$, be so small compared to the length $l$ of its element, considered as cylindrical, that $\log[l/\alpha]$ represents a very large number as was assumed already in the preceding Section for the simplification of the expression of the electromotive forces;

2. that the electromotive forces acting on a point $(x, y, z)$ in such a thin conducting wire originating from the free electricity and from the electric motions, except from the single small piece considered as a cylinder whose central cross section contains the point $(x, y, z)$, be vanishingly small compared to the electromotive forces acting on the same point.
originating from the electric motions in this small piece. — In addition, we have the assumption already used also for the development of the general equations in Section 1;

3. that Ohm’s law be separately valid for all current elements, even if the current intensities in these elements are very different and vary rapidly.

If now, according to the first assumption, \( \log[l/\alpha] \) is a very large number and if, according to the second assumption, only the electromotive forces determined more precisely in the preceding Section are taken into consideration, compared to which the others due to the more distant parts of the conducting wire are vanishingly small, one finds after the conclusion of the preceding Section the complete expression of the electromotive force along the axis of the conducting wire, [namely]

\[
-2 \left( \frac{d\Omega}{dx} + \frac{4}{c^2} \frac{dU}{dt} \right) = -4 \log \frac{l}{\alpha} \left( \frac{dE}{dx} + \frac{4}{c^2} \frac{di}{dt} \right).
\]

If this formula is now the expression of the total electromotive force, then it yields according to Section 1, multiplied by the specific conductivity \( k \), in accordance with the third assumption, the current density \( u \) along the direction of the conducting wire in the point \( (x, y, z) \), namely

\[
u = -4k \log \frac{l}{\alpha} \cdot \left( \frac{dE}{dx} + \frac{4}{c^2} \frac{di}{dt} \right).
\]

Considering, finally, that the current density at the point \( (x, y, z) \), hereafter independent of \( \rho \) and, consequently, equal for all points of the cross section of the wire, multiplied by the wire cross section \( \pi \alpha^2 \), yields therefore the current intensity \( i \), then, multiplying the previous equation by \( \pi \alpha^2 \) one gets the following equation derived from the seven first general equations of Section 1:

\[
i = -4\pi \alpha^2 k \log \frac{l}{\alpha} \cdot \left( \frac{dE}{dx} + \frac{4}{c^2} \frac{di}{dt} \right).
\]

Hence there only remain the last two of the general equations derived in Section 1, which have been reduced in Section 2 to

\[
\frac{du}{dx} + \frac{1}{\rho} \cdot \frac{d}{dp} \cdot \rho \sigma = -\frac{1}{2} \frac{d\varepsilon}{dt},
\]

\[
\sigma = \frac{1}{2} \frac{de}{dt}.
\]
Multiplying the first equation by $\rho d\rho d\varphi$ and integrating over the total cross section of the conducting wire, and finally subtracting the second equation multiplied by $2\pi \alpha$, one gets

$$
\pi \alpha^2 \frac{du}{dx} = -\pi \alpha \frac{de}{dt} - \pi \int_0^\alpha \rho d\rho \cdot \frac{d\varepsilon}{dt}.
$$

But now, according to Section 2, for $\rho' = \rho$ we have

$$
2\pi \alpha c + 2\pi \int_0^\alpha \rho d\rho \cdot \varepsilon = E,
$$

whence

$$
2\pi \alpha \frac{de}{dt} + 2\pi \int_0^\alpha \rho d\rho \cdot \frac{d\varepsilon}{dt} = \frac{dE}{dt},
$$

and so, because we had $\pi \alpha^2 u = i$, from which follows $\pi \alpha^2 \cdot (du/dx) = di/dx$, the two last equations of Section 1 yield the following [equation]:

$$
\frac{di}{dx} = -\frac{1}{2} \frac{dE}{dt}.
$$

Based on this reduction from nine general equations to two, namely

$$
i = -4\pi \alpha^2 k \log \frac{l}{\alpha} \cdot \left( \frac{dE}{dx} + \frac{4}{c^2} \frac{di}{dt} \right),
$$

$$
\frac{di}{dx} = -\frac{1}{2} \frac{dE}{dt},
$$

we can, eliminating $i$, finally derive the law that allows to determine the distribution of free electricity, $E$, in the circuit for any moment, namely

$$
\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2/2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{8\pi \alpha^2 k ln \frac{l}{\alpha}} \frac{\partial E}{\partial t}.
$$

Weber and Kohlrausch had measured Weber’s constant $c$ in 1855-6. They obtained $c = 4.39450 \times 10^8$ m/s, [KW57] with English translation in [KW21]. Therefore $c/\sqrt{2} = 3.1 \times 10^8$ m/s, essentially the value of light velocity in vacuum, as pointed out by Kirchhoff in 1857, [Kir57b] with English translation in [Kir57a].

Therefore the next equation presented by Weber is the modern telegraphy equation describing the propagation of an electric disturbance along a resistive wire. When the resistance is negligible, we obtain the wave equation for a signal propagating at light velocity $v_L = c/\sqrt{2}$, namely
\[
\frac{d^2 E}{dt^2} - \frac{c^2}{2} \frac{d^2 E}{dx^2} + \frac{c^2}{16\pi\alpha^2 k \log \frac{L}{\alpha}} \cdot \frac{dE}{dt} = 0,
\]
or we can, eliminating \( E \), derive the law that allows to determine the current intensity, \( i \), for any point of the circuit and for any moment, namely

\[
\frac{d^2 i}{dt^2} - \frac{c^2}{2} \frac{d^2 i}{dx^2} + \frac{c^2}{16\pi\alpha^2 k \log \frac{L}{\alpha}} \cdot \frac{di}{dt} = 0.
\]

As is easy to see, the distribution of free electricity as well as the current intensities in all parts, however, would have followed all by itself from the motions of all electric particles in the conducting wire, were the law of the latter known. Vice versa the latter law is easily derived from the known law of the distribution and the current intensities where it suffices to formulate it for the motions of all positive electric particles in the conducting wire, because the opposite motions of all negative electric particles follow all by themselves.

Let \( s \) denote any point of the conducting wire\(^{26}\) and \( E ds \) the total amount of positive electricity which is contained in the length element, \( ds \), of the conducting wire, and further \( \sigma \) the displacement of one particle of this positive electricity after time \( t \) from its initial equilibrium, thus \( d\sigma/dt \) the velocity of this particle in the conducting wire and \( d\sigma/ds \) the dilution of the positive electricity at the point \( s \) of the conducting wire at the end of time \( t \), which always corresponds to an equally great dilution of negative electricity; then the current intensity \( i \) at the point \( s \) of the conducting wire at the end of time \( t \) equals the product \( E d\sigma/dt \), and the [linear] density \( E \) of free electricity, that is the surplus of positive electricity over negative in the element \( ds \) at the end of time \( t \), equals double the negative product \( E d\sigma/ds \), thus

\[
\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2/2} \frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} - \frac{1}{c_L^2} \frac{\partial^2 E}{\partial t^2} = 0.
\]

\(^{26}\)Note by AKTA:] In Section 1 Weber had considered \( s = \pi\alpha^2 \) as the area of the curved and thin wire with a circular cross section of radius \( \alpha \) and total length \( l \). Now he will consider \( s \) as the position of a point along the curved axis of this thin measured from a given origin \( 0 \), as represented below.

(a) \hspace{2in} (b)
\[ i = \mathbf{E} \cdot \frac{d\sigma}{dt}, \]
\[ E = -2\mathbf{E} \cdot \frac{d\sigma}{ds}. \]

However, substituting these values in the preceding equations we obtain the two equations
\[ \frac{d^3 \sigma}{dt^3} - \frac{c^2}{2} \frac{d^3 \sigma}{ds^3} \frac{d}{dt} + \frac{c^2}{16\pi\alpha^2 k \log \frac{L}{\alpha}} \cdot \frac{d^2 \sigma}{dt^2} = 0, \]
\[ \frac{d^3 \sigma}{ds dt^2} - \frac{c^2}{2} \frac{d^3 \sigma}{ds^3} + \frac{c^2}{16\pi\alpha^2 k \log \frac{L}{\alpha}} \cdot \frac{d^2 \sigma}{ds dt} = 0, \]
whence, taking into consideration that \( \sigma \) was equal to zero in the whole conducting wire during the initial equilibrium of electricity, the law of the motion of all positive electric particles in the conducting wire follows, namely
\[ \frac{d^2 \sigma}{dt^2} - \frac{c^2}{2} \frac{d^2 \sigma}{ds^2} + \frac{c^2}{16\pi\alpha^2 k \log \frac{L}{\alpha}} \cdot \frac{d\sigma}{dt} = 0. \]

4 Test of the Prerequisites Made in the Previous Section

At the beginning of the previous Section the prerequisites for the simplification of the equations have been compiled, from which already one was the basis for the development presented in Section 1. Concerning now the assumption added for the first simplification, namely the assumption of a very thin conducting wire, that seems to be so natural that it barely needs any further test, but goes without saying if it is a question of simplification; upon closer inspection, however, it is easily seen that the fineness of the conducting wire is and must be demanded here to such a degree that is never met in reality, so that any practical application of its consequences becomes questionable. In addition yet we have the special concern whether this prerequisite may not come into conflict with the prerequisite for the development of Section 1 concerning Ohm’s law, because the latter must apparently be limited to less fine conducting wires.

If namely there is no objection against considering the thickness of the conducting wire compared to its total length for linear conductors as vanishingly small, the consideration of this thickness as vanishing compared to a single element, still considered as straight, of the conducting wire is more
far reaching; and still more far reaching is the assumption of the logarithm of the ratio of the length of such a small element to that thickness as a large number, against which the number one is considered as insignificant, as was done in that prerequisite. For, taking just 20 as such a large number, would demand a wire whose smallest piece, still considered as straight, would have to be longer than thick by a factor of more than 200 millions, which does not exist.

More important, however, is the other objection whether the assumption of such a thin conduction wire, if it existed, would come into conflict with the prerequisite concerning Ohm’s law. In any case, it must be called to doubt whether the latter prerequisite is generally and strictly valid or whether it holds approximately for less fine wires, and, as is easily seen, this doubt can only be remedied by a development of the laws of motion independent of this very prerequisite. We shall try to present such a development, at least in so far as seems to be necessary for the test of the indicated doubt, preliminarily sticking to the first prerequisite, namely a wire so thin, that the logarithm of the ratio of the length of the elements, still considered as straight, to their thickness be so large as to neglect the number one by comparison. This development rests on the following consideration.

Were all forces really known which act on the electric particles in the conducting wire and were all these forces expressed exactly in known mechanical measures, then it would be self evident that a development of the laws of motion of these electric particles in the conducting wire is possible quite independent of the prerequisite of Ohm’s law; for the resultant of all forces acting on any particle divided by the acceleration of that particle must, as with all bodies, yield always the same quotient, which in mechanics is usually called the mass of the particle.

5 On the Derivation of the Equation of Motion Independent of Assuming Ohm’s Law

Hence we first try to enumerate all forces acting on an electric particle in the conducting wire and to express them by mechanical measure, namely

1. those electric forces, already determined by Kirchhoff, acting at small distances from which, under the assumption of the fineness of the wire, yielded the electromotive force

\[ E_{\text{dev}} = -4 \log \frac{l}{\alpha} \cdot \left( \frac{dE}{ds} + 4 \frac{di}{c^2 dt} \right) \]
for a point $s$ of the conducting wire expressed by mechanical measure. This electromotive force is the difference between the two forces that act on the positive and on the negative electric unit of measure (as is defined in electrostatics) if they are present in that point. As these two forces are equal, apart from their opposite directions, it follows that half of this electromotive force, namely

$$\frac{l}{\alpha} \cdot \left(\frac{dE}{ds} + \frac{4 \, di}{c^2 \, dt}\right),$$

is the force acting on any positive electric unit of measure at the point $s$. But the number of positive electric units of measure contained in the length element, $ds$, of the conducting wire has been denoted earlier in Section 3 by $\mathcal{E}ds$, where it has been noted that $\mathcal{E}d\sigma/dt = i$ and $-2\mathcal{E}d\sigma/ds = E$. Multiplying the above force by the number $\mathcal{E}ds$ and substituting the above values we get the force acting on the positive electricity in the element $ds$ expressed in mechanical units, namely

$$= 4\mathcal{E}^2 \log \frac{l}{\alpha} \cdot \left(\frac{d^2\sigma}{ds^2} - \frac{2 \, d^2\sigma}{c^2 \, dt^2}\right) \cdot ds.$$

In addition to these previously determined forces we have to add

2. the forces exerted by the ponderable conductor particles on the positive electricity in the element $ds$ which we try to determine as follows.

According to Ohm’s law, established for steady currents as shown in the footnote of Section 1,27 the electromotive force in one point of the conductor = $u/k$, which is independent of the ponderable particles of the conducting wire, or = $i/\pi\alpha^2k$, because $\pi\alpha^2u = i$ according to Section 3. But the steadiness of the current, that is the constant velocity of the electric particles in the conducting wire, proves that, apart from this electromotive force independent of ponderable particles, a second electromotive force of equal value and of opposite direction must exist which must obviously originates from the action of ponderable conductor particles on the electricity in the conductor, which hence is given by

$$= -\frac{i}{\pi\alpha^2k}.$$
then is, as is clear from the previous statement, the force which is exerted by the ponderable conductor particles on each positive electric unit of measure in the point \( s \) under consideration. Hence multiplying this force by the number of positive units of measure \( E ds \) contained in the element \( ds \) and substituting also here as before \( E d\sigma/dt \) by \( i \), we find the force exerted on the positive electricity contained in the element \( ds \) expressed by mechanical measure, namely

\[
= -\frac{i}{2\pi\alpha^2 k},
\]

Considering finally that the cases of non-steady currents differ from those of steady currents only regarding situations coming from different interactions between the electric particles, wherein the forces due to the interaction between ponderable conductor particles on the electric particles have no direct dependence, it seems justified to assume that the presented law for the determination of the latter forces, when it is valid for all cases of steady currents, holds in general, also in the cases of non-steady currents.

In order to take into account all forces which act on the electric particle under consideration, we finally summarize

3. all forces acting from a distance, wherever they may originate, and understand among them in particular also all forces originating from the interaction of the electricity [located on distant points of the conductor and acting] on the electricity in the point under consideration, apart from those [originating from electric particles located] in the element \( ds \) itself which contains the point under consideration, and which Kirchhoff has assumed as vanishingly small. We denote by \( S \) the electromotive force originating from this at point \( s \) in mechanical measure, half of which multiplied by \( E ds \) yields the force exerted on the positive electricity in element \( ds \), expressed in mechanical measure,

\[
= \frac{1}{2} E S ds.
\]

As all these forces, expressed in mechanical measure, that means by parts of that force which conveys the unit of velocity (one millimeter during one second) during the time unit (during the time of a second) to the ponderable...
unit of mass (the mass of one milligram), [then] it follows, according to the law of motion valid for all bodies, that the quotient of the sum of all these unidirectional forces and the acceleration, that is of the velocity conveyed by the sum of forces due to the positive electricity in the element $ds$ acting on them during the unit time, namely

$$\frac{d^2\sigma}{dt^2},$$

yields the definition of the mass of the positive electricity contained in the element $ds$, expressed in the unit mass (milligram) defined for all bodies.

It is remarkable that one is led hereby to a new kind of absolute determination of an amount of electricity, about which the following remark, for comparison of this new method of absolute determination with the already known methods may find room here for the application to the present consideration.

Arranging the different methods of absolute determinations of an amount of electricity according to the exactness they allow in practice, the absolute determinations by the galvanometric method have to be placed topmost by which the amount of electricity, present as part of the neutral fluid, is obtained expressed as part of the amount of electricity which passes during the unit of time through the cross section of the conductor with the unit of current intensity determined galvanometrically. — Then follow the absolute determinations by means of electrostatic measurement, by which an existing amount of free electricity is obtained expressed as part of that amount of electricity which exerts the unit of force on the same amount [of free electricity] from a unit distance according to the electrostatic law. This determination is applied only to small amounts of electricity occurring as free in comparison with the large amounts of electricity in the neutral fluid determined galvanometrically. — Especially important is the knowledge of the ratio of the units of measure determined by the two methods, obtained by measuring twice one and the same amount of electricity, galvanometrically as well as by the electrostatic method, namely the ratio $155\ 370 \cdot 10^6 : 1$ (see the previous Abhandlung, Vol. V, p. 261)\textsuperscript{28,29} — To these two absolute methods one may now add as third one that by which an existing amount of electricity is to be expressed by its mass in parts of the unit of mass (milligram) determined for all bodies; here, however, we have to remark that until now no existing amount of electricity has been expressed by this method because no way has


\textsuperscript{29}[Note by AKTA:] [KW57, Section 15, p. 649 of Weber’s Werke] with English translation in [KW21, Section 15, p. 48].
yet been discovered which would just approximately lead to such a knowledge. Consequently, there is a complete lack of knowledge concerning the ratio of the units of measure due to the latter method and that due to the previous method because no measurement of one and the same amount of electricity could be carried out according to these different methods. Were this ratio \( r : 1 \) known, then the mass of this amount of electricity in milligrams expressed as \( = \frac{1}{r} \cdot E \Delta s \) could be obtained from the number \( E \Delta s \) of electrostatic units of measure of positive electricity contained in the conductor element \( ds \).

Introducing this expression for the mass and equating it with the above quotient, one obtains the following equation:

\[
\frac{1}{r} \cdot E \Delta s = \frac{1}{r} \cdot E \Delta s ,
\]

or, arranging and putting

\[
\frac{c^2}{8 \log \frac{1}{\alpha} \cdot r \mathcal{E}} = \lambda ,
\]

one obtains

\[
\frac{d^2 \sigma}{dt^2} - \frac{c^2}{2(1 + \lambda)} \cdot \frac{d^2 \sigma}{ds^2} + \frac{c^2}{16 \pi \alpha^2 k \log \frac{1}{\alpha} \cdot (1 + \lambda)} \cdot \frac{d \sigma}{dt} = \frac{c^2}{16 \mathcal{E} \log \frac{1}{\alpha} \cdot (1 + \lambda)} \cdot S .
\]

### 6 Comparison of the Results

This more general equation is seen to contain Kirchhoff’s above equation, namely under the two assumptions that \( S = 0 \) and \( \lambda = 0 \); then we have

\[
\frac{d^2 \sigma}{dt^2} - \frac{c^2}{2} \cdot \frac{d^2 \sigma}{ds^2} + \frac{c^2}{16 \pi \alpha^2 k \log \frac{1}{\alpha}} \cdot \frac{d \sigma}{dt} = 0 ,
\]

in total agreement with the equation developed at the end of Section 3.

Here it may be remarked that Poggendorff’s Note to Kirchhoff’s treatise in the 1857 *Annalen*, Vol. 100, p. 351, refers to this more general equation,
just derived, and to its agreement with Kirchhoff’s equation.\textsuperscript{30,31}

The assumption that $S = 0$ does not just contain Kirchhoff’s previously made assumption, that no external electromotive force shall act on the electricity in the conducting wire, but especially also the second assumption from the three made at the beginning of the third Section, namely that all electromotive forces originating from the free electricity and from the electric motions in the whole conducting wire, apart from the small piece considered as cylindrical with the point under consideration in its center, are vanishingly small compared to those electromotive forces acting on the same point originating from the free electricity and from the electric motions in the cylindrical small piece itself.

The assumption that $\lambda = 0$, on the other hand, agrees with Kirchhoff’s assumption of the general validity of Ohm’s law. It may, however, seem that $\lambda = c^2/[8r \cdot \mathcal{E} \log(l/\alpha)]$ vanishes for $\log(l/\alpha) = \infty$, and that the assumption $\lambda = 0$ would approximately be fulfilled by Kirchhoff’s assumption that $\alpha$ vanishes compared to $l$; but this is not the case, [we have] rather $\lambda = \infty$ when $\alpha$ vanishes, as is easily seen because the number, $= \mathcal{E}$, of positive units of measure contained in the unit length of the conducting wire is proportional to the square of the radius $\alpha$, and, denoting the constant number of positive electric units of measure contained in the unit volume of the conducting wire by $\mathcal{E}_0$, is given by

$$\mathcal{E} = \pi \alpha^2 \cdot \mathcal{E}_0,$$

whence it follows that the product

$$\mathcal{E} \log \frac{l}{\alpha} = \pi \mathcal{E}_0 \cdot \alpha^2 \log \frac{l}{\alpha}$$

vanishes together with $\alpha$ and thus

$$\lambda = \frac{c^2}{8r \cdot \mathcal{E} \log \frac{l}{\alpha}}$$

becomes infinite.

Hence it follows that in \textit{thicker} conducting wires, with larger values of $\alpha$, Ohm’s law indeed could approximately hold in general as assumed by Kirchhoff, namely for a very small value of the constant quotient $c^2/[r \mathcal{E}_0]$; that, on the other hand, Ohm’s law would lose this more general validity for \textit{thinner} conducting wires, particularly if this refinement is to be pushed so

\textsuperscript{30}[Note by HW:] This Note can be found at the end of this paper under number VI.

\textsuperscript{31}[Note by AKTA:] See [Pog57], reprinted in [Web94b, Paper number VI, p. 242 of Weber’s \textit{Werke}]. English translation in [Pog21].
far as to make \( \log(l/\alpha) \) a very large number, whence the explicit objection about the incompatibility of the two assumptions presented under (1) and (3) at the beginning of Section 3 seems to be well founded.

On the other hand, if by observation cases of thinner conducting wires could be demonstrated where Ohm’s law does not receive this more general validity, but measurable deviations became obvious from which \( \lambda \) could be determined, it would yield the knowledge of the constant quotient

\[
\frac{c^2}{rE_0} = 8\pi \alpha^2 \log \frac{l}{\alpha} \cdot \lambda ,
\]

and the knowledge of the ratio \( r : 1 \), that is the number of electrostatic units of measure per milligram, would merely depend on the exploration of the number of electrostatic units of measure, \( E_0 \), which are contained in 1 cubic millimeter of the conductor.

7 Development of the Expression for the Electromotive Force which is Exerted by the Free Electricity and by the Electric Motions in the Whole Conductor on One Point of a Closed Thin Conductor, apart from that Element which Contains the Point under Consideration

If the forces which could not be determined, acting on a point \( s \) of the conducting wire from a distance including those which act from more distant parts of the conducting wire itself and those acting from outside were put equal = 0, then according to the developments of the preceding Section one obtains the following partial differential equation for the displacement \( \sigma \) of the positive particle in the point \( s \):\(^{32}\)

\[
\frac{d^2 \sigma}{dt^2} - a \frac{d^2 \sigma}{ds^2} + b \frac{d\sigma}{dt} = 0 ,
\]

where the only difference was that the meaning of the constant coefficients \( a \) and \( b \) in this equation after Section 3 was a bit different from that used in Section 6, a difference which possibly does not need to be considered,\(^{32}\)

\(^{32}\)[Note by AKTA:] We are maintaining Weber’s notation for partial derivatives, see footnote 16 on page 13.
namely if the experience should show that the quotient in the previous Section denoted by \( c^2/[r_0 E_0] \) had a vanishingly small value for all kinds of conductors.

This agreement, however, does not at all make the above equation suited to really determine the motions of electricity in a conducting wire; even if there were cases with no external electromotive forces acting on the electricity in the conducting wire, there would be no case where also no electromotive forces would be acting [originating] from the more distant parts of the conducting wire itself, if any disturbance of the equilibrium of the electricity has happened. Thus, in order to arrive at an equation which would really serve to determine the motion of electricity in a conducting wire, the development of the electromotive forces from Section 2, exerted by the free electricity and by the electric motions of the single element \( ds \) containing the point \( s \) is not sufficient, but also those electromotive forces remain to be determined which are exerted by the free electricity and by the electric motions in all remaining parts of the conducting wire on the point \( s \). Therefore, Kirchhoff’s conclusions from the above equation remain to be tested with respect to the influence of these latter forces.

As the dimensions of the elements in question, \( ds \) and \( ds' \), vanish compared to their distance, it suffices indeed for the development of these forces to consider only the total values of the [linear] densities of free electricity and the current intensities \( E, E', i, i' \), respectively, for the total cross section which are merely functions of \( s \) and \( t \) or [functions of] \( s' \) and \( t' \). But it is self evident that, unlike in Section 2, these functions cannot be expanded into Taylor series\(^{33}\) because the same can be arbitrarily given in the first moment \( t = 0 \); instead one must try to represent them in terms of sine and cosine series.

Hence putting for a closed conducting wire of length \( 2\pi a \)

\[
E' = \sum \left( a_n \sin \frac{ns'}{a} + b_n \cos \frac{ns'}{a} \right),
\]

\[
i' = \sum \left( c_n \sin \frac{ns'}{a} + \partial_n \cos \frac{ns'}{a} \right),
\]

where \( n \) takes all successive integer numbers, and denoting the distance between the points \( s \) and \( s' \) by \( r \) and the angles which \( ds \) and \( ds' \) form with \( r \) by \( \vartheta \) and \( \vartheta' \), we get according to Section 1

\[
\Omega = \int \frac{E'ds'}{r} = \int \frac{ds'}{r} \sum \left( a_n \sin \frac{ns'}{a} + b_n \cos \frac{ns'}{a} \right),
\]

\(^{33}\)Note by AKTA:] Series named after the English mathematician Brook Taylor (1685-1731).
\[ U = \int \frac{ds'}{r} \cos \vartheta \cos \vartheta' \cdot i' = \int \frac{ds'}{r} \cos \vartheta \cos \vartheta' \cdot \sum \left( c_n \sin \frac{ns'}{a} + \partial_n \cos \frac{ns'}{a} \right). \]

In addition, we still have the equation found in Section 3,
\[ \frac{dl'}{ds'} = -\frac{1}{2} \frac{dE'}{dt}, \]

or expressed in terms of sine and cosine series,
\[ \frac{1}{a} \sum n \left( c_n \cos \frac{ns}{a} - \partial_n \sin \frac{ns}{a} \right) = -\frac{1}{2} \sum \left( \frac{da_n}{dt} \cdot \sin \frac{ns}{a} + \frac{db_n}{dt} \cos \frac{ns}{a} \right). \]

Hence it follows, because this equation is to be valid for all values of \( s' \)
\[ c_n = -\frac{a}{2n} \frac{db_n}{dt}, \]
\[ \partial_n = +\frac{a}{2n} \frac{da_n}{dt}. \]

Aiming now to determine the electromotive force acting on the point \( s \) of the closed conducting wire from the obtained expressions for \( \Omega \) and \( U \), one may put \( s' - s = \sigma \), and substitute \( s + \sigma \) for \( s' \) and \( d\sigma \) for \( ds' \) in the expressions for \( \Omega \) and \( U \). This then yields
\[ \Omega = \sum \int \frac{d\sigma}{r} \left( a_n \sin \left( \frac{n\sigma}{a} + \frac{ns}{a} \right) + b_n \cos \left( \frac{n\sigma}{a} + \frac{ns}{a} \right) \right), \]
\[ U = \sum \int \frac{d\sigma}{r} \cos \vartheta \cos \vartheta' \cdot \left( c_n \sin \left( \frac{n\sigma}{a} + \frac{ns}{a} \right) + \partial_n \cos \left( \frac{n\sigma}{a} + \frac{ns}{a} \right) \right). \]

Expanding the sum in terms of sine and cosine, one gets
\[ \Omega = \sum \left( a_n \cos \frac{ns}{a} - b_n \sin \frac{ns}{a} \right) \cdot \int \frac{\sin \frac{n\sigma}{a} \cdot d\sigma}{r} \]
\[ + \sum \left( a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a} \right) \cdot \int \frac{\cos \frac{n\sigma}{a} \cdot d\sigma}{r}. \]

\[ U = \sum \left( c_n \cos \frac{ns}{a} - \partial_n \sin \frac{ns}{a} \right) \cdot \int \frac{\cos \vartheta \cos \vartheta' \sin \frac{n\sigma}{a} \cdot d\sigma}{r} \]
\[ + \sum \left( c_n \sin \frac{ns}{a} + \partial_n \cos \frac{ns}{a} \right) \cdot \int \frac{\cos \vartheta \cos \vartheta' \cos \frac{n\sigma}{a} \cdot d\sigma}{r}. \]
Here, $r$, $\cos \vartheta$ and $\cos \vartheta'$ are functions of $\sigma$ which result from the equation of the curve of the conducting wire. It follows that for each consecutive number $n$ the four integrals to be taken between the limits from $\sigma = \frac{1}{2}$ to $\sigma = 2\pi a - \frac{1}{2}$ (where $l$ is the length of the same piece of the conducting wire as in Section 2)

\[
\int \sin \frac{na}{a} d\sigma, \quad \int \cos \frac{na}{a} d\sigma, \quad \int \cos \vartheta \cos \vartheta' \sin \frac{na}{a} d\sigma, \quad \int \cos \vartheta \cos \vartheta' \cos \frac{na}{a} d\sigma
\]

are given and determined by the shape of the conductor, the values of which shall be denoted by

$N, N', M, M'$.

Then one has

\[
\Omega = \sum \left( (a_n N' - b_n N) \sin \frac{ns}{a} + (a_n N + b_n N') \cos \frac{ns}{a} \right),
\]

\[
U = \sum \left( (c_n M' - \partial_n M) \sin \frac{ns}{a} + (c_n M + \partial_n M') \cos \frac{ns}{a} \right),
\]

from which now the electromotive forces can be determined, namely

\[
-2 \frac{d\Omega}{ds} = -2a \sum n \left( (a_n N' - b_n N) \cos \frac{ns}{a} - (a_n N + b_n N') \sin \frac{ns}{a} \right),
\]

\[
-\frac{8}{c^2} \cdot \frac{dU}{dt} = -\frac{8}{c^2} \sum \left[ \left( \frac{dc_n}{dt} \cdot M' - \frac{d\partial_n}{dt} \cdot M \right) \sin \frac{ns}{a} + \left( \frac{dc_n}{dt} \cdot M + \frac{d\partial_n}{dt} \cdot M' \right) \cos \frac{ns}{a} \right],
\]

or, substituting the above values of $c_n$ and $\partial_n$ in the latter equation

\[
-\frac{8}{c^2} \cdot \frac{dU}{dt} = +\frac{4a}{c^2} \sum \frac{1}{n} \left[ \left( \frac{d^2b_n}{dt^2} \cdot M' + \frac{d^2a_n}{dt^2} \cdot M \right) \sin \frac{ns}{a} + \left( \frac{d^2b_n}{dt^2} \cdot M - \frac{d^2a_n}{dt^2} \cdot M' \right) \cos \frac{ns}{a} \right].
\]
8 Equation of Motion of the Electricity in a Closed Conductor

In order to formulate the equation of motion of the electricity in a closed conductor according to the method presented in Sections 4 to 5, at first all forces have to be taken into account which act on the positive electricity in an element $ds$ of the conducting wire and [it is necessary] to express the value of these forces by mechanical measure.

1. At the end of Section 2 the electromotive forces acting in the vicinity of the point $s$ of the conducting wire have been found:

$$-2 \frac{d\Omega}{ds} = -4 \frac{dE}{ds} \cdot \log \frac{l}{\alpha} - \frac{1}{4} \frac{d^3E}{ds^3} \cdot t^2,$$

$$-8 \frac{dU}{c^2 \, dt} = - \frac{16}{c^2} \frac{di}{dt} \cdot \log \frac{l}{e\alpha} - \frac{1}{c^2} \frac{d^3i}{ds^2 dt} \cdot t^2.$$

But here we can substitute according to the previous Section

$$E = \sum \left( a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a} \right),$$

$$i = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \cdot \sin \frac{ns}{a} - \frac{da_n}{dt} \cdot \cos \frac{ns}{a} \right),$$

hence

$$-2 \frac{d\Omega}{ds} = -4 \frac{dE}{ds} \cdot \log \frac{l}{\alpha} - \frac{1}{16} \frac{n^3 l^2}{a^2} \left( a_n \cos \frac{ns}{a} - b_n \sin \frac{ns}{a} \right),$$

$$-8 \frac{dU}{c^2 \, dt} = \frac{8}{c^2} \sum \left( \frac{a}{n} \log \frac{l}{e\alpha} - \frac{1}{16} \frac{n l^2}{a} \right) \left( \frac{d^2b_n}{dt^2} \cdot \sin \frac{ns}{a} - \frac{d^2a_n}{dt^2} \cdot \cos \frac{ns}{a} \right).$$

2. At the end of the previous Section the electromotive forces acting from a distance on the point $s$ of the conducting wire have been found

$$-2 \frac{d\Omega}{ds} = -\frac{2}{a} \sum n \left[ (a_n N' - b_n N) \cos \frac{ns}{a} - (a_n N + b_n N') \sin \frac{ns}{a} \right],$$

$$-8 \frac{dU}{c^2 \, dt} = \frac{8}{c^2} \sum \left( \frac{a}{n} \log \frac{l}{e\alpha} - \frac{1}{16} \frac{n l^2}{a} \right) \left( \frac{d^2b_n}{dt^2} \cdot \sin \frac{ns}{a} - \frac{d^2a_n}{dt^2} \cdot \cos \frac{ns}{a} \right).$$
\[-\frac{8}{c^2} \frac{dU}{dt} = + \frac{4a}{c^2} \sum_{n} \frac{1}{n} \left[ \left( \frac{d^2 b_n}{dt^2} \cdot M' + \frac{d^2 a_n}{dt^2} \cdot M \right) \cdot \frac{n s}{a} \right].\]

Thus, putting
\[N' + 2 \log \frac{l}{\alpha} - \frac{1}{8} \frac{n^2 l^2}{a^2} = N'',\]
\[M' + 2 \log \frac{l}{c \alpha} - \frac{1}{8} \frac{n^2 l^2}{a^2} = M'',\]
the electromotive forces acting from the vicinity and from a distance taken together are
\[-2 \frac{d\Omega}{ds} = - \frac{2}{a} \sum_{n} a \left[ (a_n N'' - b_n N) \cos \frac{n s}{a} - (a_n N + b_n N'') \sin \frac{n s}{a} \right],\]
\[-\frac{8}{c^2} \frac{dU}{dt} = + \frac{4a}{c^2} \sum_{n} \frac{1}{n} \left[ \left( \frac{d^2 b_n}{dt^2} \cdot M'' + \frac{d^2 a_n}{dt^2} \cdot M \right) \cdot \frac{n s}{a} \right].\]

Now these electromotive forces are the differences of those forces which act on the positive and the negative electric unit of measure at the point \(s\). As, however, the force acting on the positive unit of measure equals that acting on the negative unit of measure, apart from the opposite direction, it follows that half of these electromotive forces are those acting on each positive unit of measure in the point \(s\). But the number of the positive units of measure contained in the length element, \(ds\), of the conducting wire has been denoted by \(E ds\) in Section 3; multiplying half of the above electromotive forces by \(E ds\), one finds the forces which act on the positive electricity in the element \(ds\), expressed in mechanical measure,
\[
\sum n \left[ (a_nN'' - b_nN) \cos \frac{ns}{a} - (a_nN + b_nN'') \sin \frac{ns}{a} \right] \\
+ \frac{2a\mathcal{E} ds}{c^2} \sum \frac{1}{n} \left[ \left( \frac{d^2b_n}{dt^2} \cdot M'' + \frac{d^2a_n}{dt^2} \cdot M \right) \sin \frac{ns}{a} \right. \\
\left. + \left( \frac{d^2b_n}{dt^2} \cdot M - \frac{d^2a_n}{dt^2} \cdot M'' \right) \cos \frac{ns}{a} \right].
\]

3. The resistive force originating from the *ponderable conductor particles* acting on the positive electricity in the element \(ds\) was found in Section 5 and is given by, expressed in mechanical measure

\[
= -\frac{1}{2\pi\alpha^2 k} \cdot \mathcal{E}^2 \cdot \frac{d\sigma}{dt} ds,
\]

wherein

\[
\frac{\mathcal{E} d\sigma}{dt} = i = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \cdot \sin \frac{ns}{a} - \frac{da_n}{dt} \cdot \cos \frac{ns}{a} \right),
\]

which yields this force as

\[
= +\frac{a\mathcal{E} ds}{4\pi\alpha^2 k} \cdot \sum \frac{1}{n} \left( \frac{db_n}{dt} \cdot \sin \frac{ns}{a} - \frac{da_n}{dt} \cdot \cos \frac{ns}{a} \right).
\]

In addition, finally, we have

4. the force acting *from outside* on the positive electricity in the element \(ds\) which, according to Section 5 (3),\(^{34}\) yields

\[
= +\frac{1}{2} \mathcal{E} S ds,
\]

where \(S\) denotes here only the external electromotive force acting on the point \(s\). Expanding now \(S\) in sine and cosine series

\[
S = \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right),
\]

then this force is represented by

\[
= +\frac{1}{2} \mathcal{E} ds \cdot \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right).
\]

\(^{34}\) [Note by AKTA:] That is, item 3 in Section 5.
As now all these forces are expressed in mechanical measure, that is in parts of that force which conveys the unit of velocity to the unit of ponderable mass (milligrams) during the unit of time (second), it follows that, according to the well known law of motion, valid for all bodies, the quotient of the sum of all these forces and of the acceleration, \( d^2\sigma/dt^2 \), conveyed to the positive electricity in the element \( ds \) on which they act, expresses the definition of the mass of this amount of electricity, where the measure of mass (milligram) has been denoted by \( [1/r]\mathcal{E}ds \) milligram in Section 5. Multiplying the equation thus obtained by \( [1/r]\mathcal{E}ds \cdot [d^2\sigma/dt^2] \) and putting

\[
\mathcal{E}d^2\sigma = \frac{di}{dt} = \frac{-a}{2} \sum \frac{1}{n} \left( \frac{d^2b_n}{dt^2} \sin \frac{ns}{a} - \frac{d^2a_n}{dt^2} \cos \frac{ns}{a} \right),
\]

one gets the desired equation of motion of the electricity in a closed conducting wire as follows:

\[
-\frac{1}{a} \sum n \left[ (a_n N'' - b_n N) \cos \frac{ns}{a} - (a_n N + b_n N'') \sin \frac{ns}{a} \right] + \frac{2a}{c^2} \sum \frac{1}{n} \left[ \left( \frac{d^2b_n}{dt^2} \cdot M'' + \frac{d^2a_n}{dt^2} \cdot M \right) \sin \frac{ns}{a} + \left( \frac{d^2b_n}{dt^2} \cdot M - \frac{d^2a_n}{dt^2} \cdot M'' \right) \cos \frac{ns}{a} \right] + \frac{a}{4\pi\alpha^2 k} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) + \frac{1}{2} \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)
= \frac{-a}{2\pi} \mathcal{E} \cdot \sum \frac{1}{n} \left( \frac{d^2b_n}{dt^2} \sin \frac{ns}{a} - \frac{d^2a_n}{dt^2} \cos \frac{ns}{a} \right).
\]

As \( N, N'', M, M'' \) depend only on the equation of the shape of the conductor, they can be presented as function of \( s \). In the single case where this shape is a circle, each of these quantities has the same value for all points \( s \) and then the above equation can be split into the two simpler equations, namely, putting \( c^2/[4M''\mathcal{E}r] = \lambda \),

\[
\frac{d^2a_n}{dt^2} + \frac{c^2}{8\pi\alpha^2 k M''(1 + \lambda)} \cdot \frac{da_n}{dt} + \frac{n^2c^2N''}{2a^2M''(1 + \lambda)} \cdot a_n - \frac{nc^2}{4aM''(1 + \lambda)} \cdot g_n = \frac{M}{M''(1 + \lambda)} \cdot \frac{d^2b_n}{dt^2} + \frac{n^2c^2N}{2a^2M''(1 + \lambda)} \cdot b_n,
\]

\[35\text{[Note by AKTA:] That is, a constant force increasing in 1 mm/s the velocity of the mass under consideration.}\]
Hereby the treatment of the case of a conductor of circular shape is considerably simplified and deserves to be considered in particular. In all other cases $N, N'', M, M''$ as functions of $s$ would have to be expanded further in series of sine and cosine whereby the equations would considerably lose their simplicity.

9 Equation for the Mean Values of the Electromotive Forces and Current Intensities in Closed Conductors with Arbitrary Shape

Considerations and applications of closed circuits often occur which do not demand the knowledge of the electromotive forces and current intensities in individual points of the circuit, but where the knowledge of their mean values for the total length of the conducting wire suffices. Hence before entering the special development of the laws of motion of the electricity in a circular conductor, the laws just found shall be applied in order to derive from them the equation for the mean values of the electromotive forces and current intensities in closed conductors of arbitrary shape.

This equation results when the terms of the equation found in the previous Section are multiplied by $ds$ and are integrated from $s = 0$ to $s = 2\pi a$. It is considerably simplified because first, according to a known theorem, the value of the integral of the electromotive forces originating from the free electricity in the conducting wire is always equal to zero,[36] and because second the integral value of the external electromotive forces can usually be considered as given. Hence one obtains first

\[ \oint_C \vec{E} \cdot d\vec{r} = 0, \]

where $\vec{E}$ is the electric field due to free charges, $C$ is a closed circuit of arbitrary shape and $d\vec{r}$ is an infinitesimal element of length.

[36] Note by AKTA:] That is, in modern terms,
\[ \int_0^{2\pi a} \frac{ds}{a} \sum n \left( (a_n N'' - b_n N) \cos \frac{ns}{a} - (a_n N + b_n N'') \sin \frac{ns}{a} \right) = 0 , \]

second, denoting by \( S \) the integral value of the external electromotive forces,

\[ \int_0^{2\pi a} ds \sum (f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a}) = S . \]

As now further, putting

\[ i_n = -\frac{a}{2n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) , \]

we had \( i = \sum i_n \); hence one gets

\[ \int ds \cdot \frac{2a}{c^2} \sum \frac{1}{n} \left[ \left( \frac{d^2 b_n}{dt^2} M'' + \frac{d^2 a_n}{dt^2} M \right) \sin \frac{ns}{a} \right. \]
\[ + \left. \left( \frac{d^2 b_n}{dt^2} M - \frac{d^2 a_n}{dt^2} M'' \right) \cos \frac{ns}{a} \right] \]
\[ = -\frac{4}{c^2} \int ds \sum M'' \frac{d i_n}{dt} - \frac{4a}{c^2} \sum \frac{1}{n} \int \frac{d^2 i_n}{ds dt} M ds . \]

Now one has

\[ \int \frac{d^2 i_n}{dt^2} M ds = M \frac{d i_n}{dt} - \int \frac{d i_n}{dt} \frac{dM}{ds} ds ; \]

hence

\[ \int_0^{2\pi a} \frac{d^2 i_n}{ds dt} M ds = -\int_0^{2\pi a} \frac{d i_n}{dt} \frac{dM}{ds} ds , \]

thus also

\[ \int_0^{2\pi a} ds \cdot \frac{2a}{c^2} \sum \frac{1}{n} \left[ \left( \frac{d^2 b_n}{dt^2} M'' + \frac{d^2 a_n}{dt^2} M \right) \sin \frac{ns}{a} \right. \]
\[ + \left. \left( \frac{d^2 b_n}{dt^2} M - \frac{d^2 a_n}{dt^2} M'' \right) \cos \frac{ns}{a} \right] \]
\[ = -\frac{4}{c^2} \int_0^{2\pi a} ds \sum M'' \frac{d i_n}{dt} + \frac{4a}{c^2} \sum \frac{1}{n} \int_0^{2\pi a} \frac{d i_n}{dt} \frac{dM}{ds} ds . \]

Finally adding that
\[
\int \frac{ads}{4\pi\alpha^2 k} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) = -\frac{1}{2\pi\alpha^2 k} \cdot \int ids ,
\]

\[
\int \frac{ads}{2r\mathcal{E}} \sum \frac{1}{n} \left( \frac{d^2 b_n}{dt^2} \sin \frac{ns}{a} - \frac{d^2 a_n}{dt^2} \cos \frac{ns}{a} \right) = -\frac{1}{r\mathcal{E}} \cdot \int \frac{di}{dt} ds ,
\]

one gets the following equation for the mean values of the electromotive forces and current intensities, \(\frac{1}{2\pi a} \cdot S\) and \(\frac{1}{2\pi a} \cdot \int_0^{2\pi a} ids\), respectively:

\[
S = \frac{1}{\pi\alpha^2 k} \cdot \int_0^{2\pi a} ids + \frac{8}{c^2} \int_0^{2\pi a} ds \sum M'' \frac{d_i}{dt} d_i + \frac{8a}{c^2} \int_0^{2\pi a} dM \frac{d_i}{dt} ds + \frac{2}{r\mathcal{E}} \int_0^{2\pi a} \frac{di}{dt} ds .
\]

Now these mean values obviously come into primary consideration when there is no difference at all for the electric motions in the different elements of the conducting wire, or so small that it can be totally neglected. Thus in all these cases \(i\) and \(di/dt\) are quantities independent of \(s\), and one may put \(i = i_0\) and \(di/dt = di_0/dt\), hence \(di_n/dt = 0\) for \(n > 0\), from which

\[
S = \frac{2\pi a}{\pi\alpha^2 k} i_0 + \left( \frac{8}{c^2} \int_0^{2\pi a} M'' ds + \frac{4\pi a}{r\mathcal{E}} \right) \frac{di_0}{dt} ,
\]

where \(2\pi a/\pi\alpha^2 k = w\) is the resistance of the whole circuit. Putting

\[
\frac{8}{c^2} \int_0^{2\pi a} M'' ds + \frac{4\pi a}{r\mathcal{E}} = p ,
\]

and writing \(i\) for \(i_0\), one gets

\[
S = wi + p \frac{di}{dt} ,
\]

wherein \(S, i\) and \(di/dt\) are only functions of \(t\). Integrating one gets

\[
i = \frac{1}{p} e^{-wt/p} \cdot \int e^{wt/p} \cdot S dt .
\]
10 Laws of Motion of the Electricity in a Circular Conducting Wire

When the shape of a closed conductor is given, [then] the values of $N$, $N'$, $M$, $M'$, that is the values of the definite integrals

\[
\int_{l/2}^{2\pi a - l/2} \frac{\sin \frac{n\sigma}{a} d\sigma}{r}, \quad \int_{l/2}^{2\pi a - l/2} \frac{\cos \frac{n\sigma}{a} d\sigma}{r},
\]

\[
\int_{l/2}^{2\pi a - l/2} \frac{\cos \vartheta \cos \vartheta' \sin \frac{n\sigma}{a} d\sigma}{r}, \quad \int_{l/2}^{2\pi a - l/2} \frac{\cos \vartheta \cos \vartheta' \cos \frac{n\sigma}{a} d\sigma}{r},
\]

can be determined for any point $s$ of the conductor. As an example take a conductor with the shape of a circle with radius $a$.

With this circular shape the distance $r$ between two point $s$ and $s'$ equals the chord of the arc $(s' - s)/a = \sigma/a$; hence we have

\[r = 2a \sin \frac{\sigma}{2a} .\]

Furthermore, the angle $\vartheta$ between the element $ds$ and $r$ equals the angle $\vartheta'$ between the element $ds'$ and $r$, and both equal the angle between the tangent to the circle at the point $s$ and the chord of the arc $\sigma/a$, that is

\[\vartheta = \vartheta' = \frac{\sigma}{2a} .\]

Hence we have

\[N = \frac{1}{2a} \int_{l/2}^{2\pi a - l/2} \frac{\sin \frac{n\sigma}{a} d\sigma}{\sin \frac{\sigma}{2a}} ,\]

\[\text{[Note by AKTA:] Weber will consider a conductor in the shape of a ring with larger radius $a$, smaller radius $\alpha$, with $s$ being the arc length of a specific point $P$ measured from a fixed origin $0$:}

\[
\begin{align*}
\text{Diagram:} & \quad S \\
\text{Path:} & \quad \gamma \\
\text{Radius:} & \quad a \\
\text{Angle:} & \quad 2\alpha
\end{align*}
\]
\[
N' = \frac{1}{2a} \int_{1/2}^{2\pi a - 1/2} \cos \frac{n\sigma}{a} d\sigma \sin \frac{\sigma}{2a},
\]

\[
M = \frac{1}{2a} \int_{1/2}^{2\pi a - 1/2} \left( \cos \frac{\sigma}{2a} \right)^2 \cdot \sin \frac{2\sigma}{a} d\sigma \sin \frac{\sigma}{2a} = N - \frac{1}{2a} \int_{1/2}^{2\pi a - 1/2} \sin \frac{n\sigma}{a} \sin \frac{\sigma}{2a} d\sigma,
\]

\[
M' = \frac{1}{2a} \int_{1/2}^{2\pi a - 1/2} \left( \cos \frac{\sigma}{2a} \right)^2 \cdot \cos \frac{2\sigma}{a} d\sigma = N' - \frac{1}{2a} \int_{1/2}^{2\pi a - 1/2} \cos \frac{n\sigma}{a} \sin \frac{\sigma}{2a} d\sigma.
\]

Putting now \( \sigma/2a = z \), hence

\[
N = \int_{1/(4a)}^{\pi - 1/(4a)} \frac{\sin 2nz \cdot dz}{\sin z},
\]

\[
N' = \int_{1/(4a)}^{\pi - 1/(4a)} \frac{\cos 2nz \cdot dz}{\sin z},
\]

\[
M = N - \int_{1/(4a)}^{\pi - 1/(4a)} \sin 2nz \cdot \sin z dz,
\]

\[
M' = N' - \int_{1/(4a)}^{\pi - 1/(4a)} \cos 2nz \cdot \sin z dz,
\]

and considering that

\[
\int \frac{\sin 2nz \cdot dz}{\sin z} = +2 \int \cos(2n - 1)z \cdot dz + 2 \int \cos(2n - 3)z \cdot dz + \ldots + 2 \int \cos zdz,
\]

\[
\int \frac{\cos 2nz \cdot dz}{\sin z} = -2 \int \sin(2n - 1)z \cdot dz - 2 \int \sin(2n - 3)z \cdot dz - \ldots - 2 \int \sin zdz + \int \frac{dz}{\sin z},
\]

then one finds, taking all integrals between \( z = l/(4a) \) and \( z = \pi - [l/(4a)] \),

\[
N = 0,
\]
\[ N' = -4 \left( \cos \frac{l}{4a} + \frac{1}{3} \cos \frac{3l}{4a} + \ldots + \frac{1}{2n-1} \cos \frac{(2n-1)l}{4a} \right) - 2 \log \tan \frac{l}{8a} . \]

Furthermore, as

\[
\int \sin 2nz \cdot \sin zdz = \frac{1}{2(2n-1)} \sin(2n-1)z - \frac{1}{2(2n+1)} \sin(2n+1)z ,
\]

\[
\int \cos 2nz \cdot \sin zdz = \frac{1}{2(2n-1)} \cos(2n-1)z - \frac{1}{2(2n+1)} \cos(2n+1)z ,
\]

taking also these integrals between the limits from \( z = l/(4a) \) to \( z = \pi - [l/(4a)] \), one finds

\[ M = 0 , \]
\[ M' = N' + \frac{1}{2n+1} \cos(2n+1) \frac{1}{4a} - \frac{1}{2n-1} \cos(2n-1) \frac{l}{4a} . \]

Hence follows finally, according to Section 8,

\[ N'' = -4 \left( \cos \frac{l}{4a} + \frac{1}{3} \cos \frac{3l}{4a} + \ldots + \frac{1}{2n-1} \cos \frac{(2n-1)l}{4a} \right) \]
\[ - 2 \log \tan \frac{l}{8a} + 2 \log \frac{1}{\alpha} - \frac{1}{8} \frac{n^2 l^2}{a^2} , \]

\[ M'' = -4 \left( \cos \frac{l}{4a} + \frac{1}{3} \cos \frac{3l}{4a} + \ldots + \frac{1}{2n-1} \cos \frac{(2n-1)l}{4a} \right) \]
\[ - 2 \log \tan \frac{l}{8a} + 2 \log \frac{l}{e\alpha} - \frac{1}{8} \frac{n^2 l^2}{a^2} \]
\[ + \frac{1}{2n+1} \cos(2n+1) \frac{l}{4a} - \frac{1}{2n-1} \cos(2n-1) \frac{l}{4a} . \]

But here \( l \) denotes the length of the conductor element \( ds \), considered as linear, with the point under consideration in its center. Between certain limits, this length is arbitrary, its choice is only limited by the quantities \( \alpha/l \) and \( l/a \) being considered as vanishingly small values which must be the case if the conductor is to be considered as linear. The difference in the values of
l, which are possible within these limits, does not have a noticeable influence on the values of \(N''\) and \(M''\). We may therefore put

\[ l = \sqrt{a\alpha}, \]

because for every conductor to be considered as linear, this value must lie between the specified limits. It also becomes clear that then \(\tan(l/8a)\) may be replaced by \(l/8a\). Putting for brevity

\[ \frac{n^2\alpha}{8a} = 2\log \nu, \]

[yields]

\[ N'' = -4 \left( \cos \frac{1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{3} \cos \frac{3}{4} \sqrt{\frac{\alpha}{a}} + ... + \frac{1}{2n-1} \cos \frac{2n-1}{4} \sqrt{\frac{\alpha}{a}} \right) + 2\log \frac{8a}{\nu \alpha}, \]

\[ M'' = -4 \left( \cos \frac{1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{3} \cos \frac{3}{4} \sqrt{\frac{\alpha}{a}} + ... + \frac{1}{2n-1} \cos \frac{2n-1}{4} \sqrt{\frac{\alpha}{a}} \right) + 2\log \frac{8a}{\mu e \alpha}. \]

Now substituting the values of \(N, N'', M, M''\) found for a circular conductor into the equations found at the end of the Section 8 one gets the following two equations for the motions of the electricity in a circular conductor

\[
\frac{d^2a_n}{dt^2} + \frac{c^2}{8\pi \alpha^2 k M''(1 + \lambda)} \cdot \frac{da_n}{dt} + \frac{n^2c^2N''}{2a^2 M''(1 + \lambda)} \cdot a_n - \frac{nc^2}{4a M''(1 + \lambda)} \cdot g_n = 0,
\]

\[
\frac{d^2b_n}{dt^2} + \frac{c^2}{8\pi \alpha^2 k M''(1 + \lambda)} \cdot \frac{db_n}{dt} + \frac{n^2c^2N''}{2a^2 M''(1 + \lambda)} \cdot b_n + \frac{nc^2}{4a M''(1 + \lambda)} \cdot f_n = 0,
\]

where \(N''\) and \(M''\) have the above values.
11 Equilibrium of Electricity in a Circular Conductor

For the case of equilibrium of the electricity one has in all parts of the conductor

\[ i = 0 \quad \text{and} \quad \frac{di}{dt} = 0 . \]

Putting the value for \( i \) from Section 8 (3)\(^{38} \) one gets

\[ -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) = 0 , \]

\[ -\frac{a}{2} \sum \frac{1}{n} \left( \frac{d^2b_n}{dt^2} \sin \frac{ns}{a} - \frac{d^2a_n}{dt^2} \cos \frac{ns}{a} \right) = 0 , \]

whence follows

\[ \frac{da_n}{dt} = 0, \quad \frac{db_n}{dt} = 0, \quad \frac{d^2a_n}{dt^2} = 0, \quad \frac{d^2b_n}{dt^2} = 0 , \]

where it has to be added that also for \( n = 0 \) one need to have

\[ \frac{1}{n} \frac{da_n}{dt} = 0, \quad \frac{1}{n} \frac{d^2a_n}{dt^2} = 0 . \]

The equations of motion, established at the end of the preceding Section, then turn into the following \textit{equations of equilibrium}, namely, when \( n > 0 , \)

\[ \frac{nN''}{a} \cdot a_n - \frac{1}{2} g_n = 0 , \]

\[ \frac{nN''}{a} \cdot b_n + \frac{1}{2} f_n = 0 , \]

where \( g_0 = 0 \) still has to be added. Hence follows as equilibrium condition for the electricity that the sum of all external electromotive forces acting on the circular conductor, must equal

\[ S = \int ds \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) = 0 , \]

in complete agreement with the known Ohm’s law according to which the current intensity is proportional to the sum of all these forces and hence can only be zero together with this sum.

\(^{38}\) [Note by AKTA:] That is, from item 3 of Section 8.
12 Steady Currents of Electricity in a Cicular Conductor

The motion of electricity in a conductor is called a *steady* current if it stays constant in any point of the conductor. In the case of such a steady current one has for all points of the closed conductor

\[ i = \text{constant}, \]

hence

\[ \frac{di}{dt} = -\frac{a}{2} \sum_{n} \frac{1}{n} \left( \frac{d^2 b_n}{dt^2} \sin \frac{ns}{a} - \frac{d^2 a_n}{dt^2} \cos \frac{ns}{a} \right) = 0, \]

whence

\[ \frac{d^2 a_n}{dt^2} = 0, \quad \frac{d^2 b_n}{dt^2} = 0, \]

where it still has to be added that one must have \( [1/n][d^2 a_n/dt^2] = 0 \), also for \( n = 0 \).

The equations of motion given at the end of Section 10 then turn into the following equations of motion for *steady* currents, namely when \( n > 0 \)

\[ \frac{1}{4\pi \alpha^2 k} \cdot \frac{da_n}{dt} + \frac{n^2 N'}{a^2} \cdot a_n - \frac{n}{2a} g_n = 0, \]

\[ \frac{1}{4\pi \alpha^2 k} \cdot \frac{db_n}{dt} + \frac{n^2 N'}{a^2} \cdot b_n + \frac{n}{2a} f_n = 0, \]

where also \( [1/n][da_n/dt] = \text{constant for } n = 0, \) hence \( a_0 = \text{constant has to be added. It follows that for *steady* current the sum of all external electromotive forces acting on the circular conductor must be given by} \]

\[ S = \int ds \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) \]

\[ = \frac{2}{a} \int ds \sum nN' \left( a_n \cos \frac{ns}{a} - b_n \sin \frac{ns}{a} \right) \]

\[ - \frac{a}{2\pi \alpha^2 k} \int ds \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right); \]

hence, because

\[ -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) = i, \]
and because

\[
\int ds \sum nN'' \left( a_n \cos \frac{ns}{a} - b_n \sin \frac{ns}{a} \right) = 0 ,
\]
as is easily seen considering that \( a_0 \) has a constant value and hence \( na_n = 0 \) for \( n = 0 \),

\[
S = \frac{1}{\pi \alpha^2 k} \cdot \int ids .
\]

But now \( [1/(2\pi a)] \cdot \int ids = J \) is the mean value of the current intensity in the whole conductor, and \( 2\pi a/|\pi \alpha^2 k| = w \) is the resistance of the whole conductor; thus \( S = Jw \), that is, the sum of the external electromotive forces in the whole conductor must be equal the product of the resistance and the average current intensity of the whole conductor, quite in agreement with Ohm’s law that yields the electromotive force of the circuit as the product of the resistance and the current, which is identical with the above results when it is assumed that there are no differences of the current intensities in the various points of the conductor. This need not be the case according to the above theory; but should there be current intensities in various points differing from the steady current in any single point, then the electromotive forces acting from outside must change in proportion to time, a case that does not occur in reality and therefore has been left out of the consideration of Ohm’s law that is founded on experience. It is, namely, clear that if

\[
i = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right)
\]
is to have different values in different parts of the conductor, either \( da_n/dt \) or \( db_n/dt \) must have a non-zero value = \( A \) for a non-zero value of \( n \), whence follows either \( a_n = At + B \) or \( b_n = At + B \). Substituting in one case \( At + B \) for \( a_n \) in the first of the above equations for the condition for steady currents, one gets

\[
\frac{1}{4\pi \alpha^2 k} \cdot A + \frac{n^2 N''}{a^2} (At + B) - \frac{n}{2a} g_n = 0 ,
\]
whence follows that \( g_n \) changes in proportion to time. Substituting in the other case \( At + B \) for \( b_n \) in the second conditional equation, it follows in a similar way that \( f_n \) changes in proportion to time. Hence in both cases also the electromotive force

\[
S_n = f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a}
\]
would change in proportion to time.
The theory of the motion of the electricity left to itself after an arbitrary disturbance in a conductor comprises the important science of propagation, in particular the following questions, whether the propagation of motion in conductors by electricity is mediated by waves, just as the propagation of light in the luminiferous ether or the propagation of sound in air, furthermore what is the velocity of these waves and finally which laws at all are valid for this propagation of waves. The initial disturbance may in fact be restricted to a small part of the conductor, and if subsequently similar disturbances of the equilibrium occur without external influence successively in all other parts of the conductor all by themselves, this transmission is given the name propagation, and the disturbance is given the name wave.

If the electricity in the conductor is to be left to itself, all external forces that would act on the electricity in the conductor have to be put equal = 0. Hence for this case one obtains the equations of motion, putting

\[ f_n = 0 \quad \text{and} \quad g_n = 0 , \]

in the equations at the end of Section 10, obtaining the following equations:

\[
\frac{d^2 a_n}{dt^2} + \frac{c^2}{8\pi \alpha^2 k M''(1 + \lambda)} \cdot \frac{da_n}{dt} + \frac{n^2 c^2 N''}{2a^2 M''(1 + \lambda)} \cdot a_n = 0 ,
\]

\[
\frac{d^2 b_n}{dt^2} + \frac{c^2}{8\pi \alpha^2 k M''(1 + \lambda)} \cdot \frac{db_n}{dt} + \frac{n^2 c^2 N''}{2a^2 M''(1 + \lambda)} \cdot b_n = 0 .
\]

Setting

\[
\frac{c^2}{16\pi \alpha^2 k M''(1 + \lambda)} = \varepsilon ,
\]

[and]

\[
\frac{n^2 c^2 N''}{2a^2 M''(1 + \lambda)} = m^2 + \varepsilon^2 ,
\]

then one obtains from these two equations by integration

\[ a_n = Ae^{-\varepsilon t} \cdot \sin (t - A') , \]
\[ b_n = Be^{-ct} \cdot \sin m(t - B') , \]

where \( A, A', B, B' \) are constants of integration, to be determined from the given initial disturbance.

If the original distribution of free electricity in the conductor is given by the following equation, where \( E_0 \) denotes the value of the [linear charge] density \( E \) for \( t = 0 \), namely

\[ E_0 = \sum \left( a_n^0 \sin \frac{ns}{a} + b_n^0 \cos \frac{ns}{a} \right) , \]

and if the original currents in all parts of the conductor, where \( i_0 \) denotes the value of the current intensity \( i \) for \( t = 0 \), [is given] by the following equation

\[ i_0 = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n^0}{dt} \sin \frac{ns}{a} - \frac{da_n^0}{dt} \cos \frac{ns}{a} \right) , \]

where \( a_n^0, b_n^0, \frac{da_n^0}{dt}, \frac{db_n^0}{dt} \) have known values, inserting these values for \( t = 0 \) into the above equation yields

\[ a_n^0 = -A \sin mA' , \]
\[ b_n^0 = -B \sin mB' , \]

and, after differentiating the above equations,

\[ \frac{da_n^0}{dt} = mA \cos mA' - \varepsilon a_n^0 , \]
\[ \frac{db_n^0}{dt} = mB \cos mB' - \varepsilon b_n^0 . \]

These four equations yield the following values of the constants of integration:

\[ A = \sqrt{a_n^0 \cdot 2 + \frac{1}{m^2} \left( \varepsilon a_n^0 + \frac{da_n^0}{dt} \right)^2} , \]
\[ B = \sqrt{b_n^0 \cdot 2 + \frac{1}{m^2} \left( \varepsilon b_n^0 + \frac{db_n^0}{dt} \right)^2} , \]
\[ A' = -\frac{1}{m} \arcsin \frac{a_n^0}{A} , \]
\[ B' = -\frac{1}{m} \arcsin \frac{b_n^0}{B} . \]
Inserting the latter two values into the above equations one gets

\[ a_n = Ae^{-\varepsilon t} \sin \left( mt + \arcsin \frac{a_0}{A} \right), \]

\[ b_n = Be^{-\varepsilon t} \sin \left( mt + \arcsin \frac{b_0}{B} \right), \]

and hence the distribution law of free electricity in the conductor [is given by]:

\[
E = \sum e^{-\varepsilon t} \left[ A \sin \frac{ns}{a} \sin \left( mt + \arcsin \frac{a_0}{A} \right) + B \cos \frac{ns}{a} \sin \left( mt + \arcsin \frac{b_0}{B} \right) \right].
\]

or, expanding the sine of the sum of two arcs:

\[
E = \sum e^{-\varepsilon t} \left( a_n^0 \sin \frac{ns}{a} \cos mt + \sqrt{B^2 - b_n^0} \cos \frac{ns}{a} \sin mt \\
+ b_n^0 \cos \frac{ns}{a} \cos mt + \sqrt{A^2 - a_n^0} \sin \frac{ns}{a} \sin mt \right).
\]

Now putting

\[
a_n^0 = p + q, \quad b_n^0 = p' + q',
\]

\[
\sqrt{B^2 - b_n^0} = p - q, \quad \sqrt{A^2 - a_n^0} = p' - q',
\]

whereby \( p, q, p', q' \) are determined, namely,

\[
p = \frac{1}{2} \left( a_n^0 + \frac{1}{m} \left( \varepsilon b_n^0 + \frac{db_n^0}{dt} \right) \right),
\]

\[
q = \frac{1}{2} \left( a_n^0 - \frac{1}{m} \left( \varepsilon b_n^0 + \frac{db_n^0}{dt} \right) \right),
\]

\[
p' = \frac{1}{2} \left( b_n^0 + \frac{1}{m} \left( \varepsilon a_n^0 + \frac{da_n^0}{dt} \right) \right),
\]

\[
q' = \frac{1}{2} \left( b_n^0 - \frac{1}{m} \left( \varepsilon a_n^0 + \frac{da_n^0}{dt} \right) \right).
\]
thus one obtains

\[
E = \sum e^{-\varepsilon t} \cdot \left( q \sin \left( \frac{ns}{a} - mt \right) + p' \cos \left( \frac{ns}{a} - mt \right) \right) + \sum e^{-\varepsilon t} \cdot \left( p \sin \left( \frac{ns}{a} + mt \right) + q' \cos \left( \frac{ns}{a} + mt \right) \right),
\]

or, alternatively,

\[
E = \sum \sqrt{p'^2 + q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{ns}{a} - mt + \arctan \frac{p'}{q} \right) + \sum \sqrt{p^2 + q'^2} \cdot e^{-\varepsilon t} \sin \left( \frac{ns}{a} + mt + \arctan \frac{q'}{p} \right).
\]

Similarly one finds the law of the current of electricity in the conductor, namely:

\[
i = \sum \sqrt{P'^2 + Q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{ns}{a} - mt + \arctan \frac{P'}{Q} \right) + \sum \sqrt{P^2 + Q'^2} \cdot e^{-\varepsilon t} \sin \left( \frac{ns}{a} + mt + \arctan \frac{Q'}{P} \right),
\]

where \( P, Q, P', Q' \) have the following values:

\[
P = -a \frac{1}{4n} \left( \frac{da_0^0}{dt} + \frac{1}{m} \left( m^2 + \varepsilon^2 \right) a_0^0 + \varepsilon \frac{da_0^0}{dt} \right),
\]

\[
Q = -a \frac{1}{4n} \left( \frac{db_0^0}{dt} - \frac{1}{m} \left( m^2 + \varepsilon^2 \right) a_0^0 + \varepsilon \frac{da_0^0}{dt} \right),
\]

\[
P' = +a \frac{1}{4n} \left( \frac{da_0^0}{dt} + \frac{1}{m} \left( m^2 + \varepsilon^2 \right) b_0^0 + \varepsilon \frac{db_0^0}{dt} \right),
\]

\[
Q' = +a \frac{1}{4n} \left( \frac{db_0^0}{dt} - \frac{1}{m} \left( m^2 + \varepsilon^2 \right) b_0^0 + \varepsilon \frac{db_0^0}{dt} \right).
\]

14 Comparison with Ohm’s Law

It has already been discussed in Section 6 on which grounds Ohm’s law, formulated for steady currents, may also be applied to variable currents. This [application] depends on the value of
\[ \lambda = \frac{c^2}{8r\mathcal{E}\log\frac{1}{\alpha}}. \]

Wherever this quantity comes into question and its value does not vanish, Ohm’s law does not apply at all or only approximately. This magnitude \( \lambda \) has been considered in more detail in Section 8, with respect to the influence of more distant parts of the circuit not yet considered in Section 6, namely

\[ \lambda = \frac{c^2}{4M''r\mathcal{E}}. \]

where the value \( M'' \) put for \( 2\log\left[\frac{l}{\alpha}\right] \) in Section 6 has been defined exactly and determined for a circular conductor in Section 10. The magnitude \( \lambda \), or, as the value of the factor \( c^4/[4M''] \) may be considered as constant, the value of the product \( r\mathcal{E} \) decides on the applicability of Ohm’s law, thereby gaining particular importance for the theory of motion of the electricity in conductors, whose reason is easily seen from the physical meaning of the product \( r\mathcal{E} \).

In particular, the amount of positive electricity contained in the unit length of the conductor has been denoted by \( \mathcal{E} \), expressed in the unit of measure determined from the electrostatic law, and its mass in milligrams has been put equal to \( [1/r]\mathcal{E} \). From the definition of the unit of measure established for the electrostatic law (where the amount of electricity is taken as the unit of measure which exerts on an equal [amount of electricity] the unit force at unit distance according to the electrostatic law, that is, a force that produces the unit velocity during the unit time on one milligram), it follows that \( r^2 \) is the force exerted by one milligram of positive or negative electricity on an equal milligram of electricity at the unit distance. Whence it follows that the product \( r\mathcal{E} \) means the force that the positive electricity contained in the unit length of the conductor, if concentrated in one point, would exert on one milligram of positive electricity at unit distance.

Now the influence of this magnitude \( \lambda \) or that of the product \( r\mathcal{E} \) will be determined in more detail on the basis of the development of the laws of motion of the electricity in a closed conductor as given in Sections 8 and the following. To begin with, from Sections 11 and 12 it follows that the laws of equilibrium and of steady currents of the electricity in conductors are in complete agreement with Ohm’s law because the magnitude \( \lambda \) or \( r\mathcal{E} \) does not come into question here, while it follows from Section 13 that the laws of propagation, or generally the laws of all changes of motion effective after a disturbance of equilibrium, vitally depend above all on the values of \( m \) and \( \varepsilon \) and hence indirectly on \( \lambda \) or \( r\mathcal{E} \). Therefore it follows that the magnitude \( \lambda \) or the product \( r\mathcal{E} \) (hence indirectly the total mass of the amount of electricity,
in milligrams, contained in the conductor if the amount of electricity per unit length of a conductor were known in *electrostatic measure*) can only be known by means of observations which disclose certain deviations from *Ohm’s law* in the *changes of motion* of the electricity in conductors after a disturbance of the equilibrium.

The importance thereby gained due to *more detailed observations on the changes of motion or on the propagation of motion through electricity in conductors* is clear; if these observations really allowed to detect any deviation *from Ohm’s law*, then this result would disclose the value of the product \( rE \), that is, the number of electrostatic units of measure which make *one milligram* of electricity when the *number of electrostatic units of measure* per unit length of the conductor is known.

To begin with, the laws of *electric wave motions* in circular conductors according to Section 13 shall be developed in more detail in order to test whether it could yield a certain guide on how to conduct such observations; then, if this were not the case, the reason for this shall be searched and if there were other motions in circular conductors which are better suited than *wave motion*.

### 15 Electric Wave Motions in a Circular Conductor

From the laws developed in Section 13 it follows that all motions of the electricity left to itself in a circular conductor after an arbitrary disturbance turn out to be a series of *wave trains* propagating *forward* and a series of *wave trains* propagating *backwards*, whose *first wave train* consists of two waves in each series, namely, one positive and one negative, which together cover the whole circular periphery; the *second wave train* of each series consists of four alternately positive and negative waves which together fill out the whole circle; the *third wave train* consists of six waves and so on.

Breaking up into their terms the sums, which in Section 13 represented the *linear* density of the free electricity \( E \) and the current intensity \( i \), and denoting these terms by \( E_n \) and \( i_n \) according to their place number, \( n \), then one gets

\[
E_1 = \sqrt{p'^2 + q'^2} \cdot e^{-\epsilon t} \sin \left( \frac{s}{a} - mt + \arctan \frac{p'}{q} \right)
\]
\[
+ \sqrt{p^2 + q^2} \cdot e^{-\epsilon t} \sin \left( \frac{s}{a} + mt + \arctan \frac{q'}{p} \right)
\]
\[ i_1 = \sqrt{P^2 + Q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{s}{a} - mt + \arctan \frac{P'}{Q} \right) \\
+ \sqrt{P^2 + Q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{s}{a} + mt + \arctan \frac{Q'}{P} \right), \]

wherein the first parts, containing the sine of an arc that change in proportion to \((s - a m t)\), represent the first wave train propagating forward, [and] the latter parts containing the sine of an arc that change in proportion to \((s + a m t)\), [represent] the first wave train propagating backward. But the first wave train propagating forward consists of a positive wave which extends from \(s = 0\) to \(s = \pi a\) at the moment \(t = [1/m] \arctan[p'/q]\), where the wave produces a charge of the conductor with positive free electricity, and [consists] of a negative wave which extends from \(s = \pi a\) to \(s = 2\pi a\) at the same moment, where the wave produces a charge of the conductor with negative free electricity. But both waves together cover the total circular periphery. The same holds for the first wave train propagating backward, consisting of a positive wave extending from \(s = 0\) to \(s = \pi a\) at the moment \(t = -[1/m] \arctan[q'/p]\), and of a negative wave extending from \(s = \pi a\) to \(s = 2\pi a\) at the same moment.

Furthermore we have

\[ E_2 = \sqrt{p'^2 + q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{2s}{a} - mt + \arctan \frac{p'}{q} \right) \\
+ \sqrt{p^2 + q'^2} \cdot e^{-\varepsilon t} \sin \left( \frac{2s}{a} + mt + \arctan \frac{q'}{p} \right), \]

\[ i_2 = \sqrt{P'^2 + Q^2} \cdot e^{-\varepsilon t} \sin \left( \frac{2s}{a} - mt + \arctan \frac{P'}{Q} \right) \\
+ \sqrt{P^2 + Q'^2} \cdot e^{-\varepsilon t} \sin \left( \frac{2s}{a} + mt + \arctan \frac{Q'}{P} \right), \]

where the first parts, containing the sine of an arc that changes in proportion to \((s - a m t/2)\), represent the second wave train propagating forward, [and] the latter parts, containing the sine of an arc that changes in proportion to \((s + a m t/2)\), represent the second wave train propagating backward. This wave train propagating forward consists of 4 waves whose first positive one extends from \(s = 0\) to \(s = \pi a/2\), the second negative from \(s = \pi a/2\) to \(s = \pi a\), the third positive from \(s = \pi a\) to \(s = 3\pi a/2\), and the fourth negative
from \( s = 3\pi a/2 \) to \( s = 2\pi a \) at the moment \( t = [1/m] \arctan[p'/q] \). The same holds for the 4 waves of the wave train propagating backward at the moment \( t = -[1/m] \arctan[q'/p] \).

Similarly, the third wave trains of both series result from \( E_3 \) and \( i_3 \), and so on.

The **intensities** of the various wave trains, which equal \( i^2 \) according to the rules of wave theory, decrease while propagating, in fact by a factor of

\[
1 : e^{-2\varepsilon t},
\]
during the time \( t \). Because the value of \( \varepsilon \) changes with the value of \( n \), this decrease varies with the place number \( n \) of the wave trains; for we had

\[
\varepsilon = \frac{c^2}{16\pi\alpha^2 k M''(1 + \lambda)},
\]

\[
\lambda = \frac{c^2}{4M''t^2},
\]

wherein, according to Section 10,

\[
M'' = -4 \left( \cos \frac{1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{3} \cos \frac{3}{4} \sqrt{\frac{\alpha}{a}} + \ldots + \frac{1}{2n - 1} \cos \frac{2n - 1}{4} \sqrt{\frac{\alpha}{a}} \right) + 2 \log \frac{8\alpha}{e\alpha} - \frac{1}{8} \frac{n^2\alpha}{a} + \frac{1}{2n + 1} \cos \frac{2n + 1}{4} \sqrt{\frac{\alpha}{a}} - \frac{1}{2n - 1} \cos \frac{2n - 1}{4} \sqrt{\frac{\alpha}{a}},
\]

whence if \( \alpha/a \) is very small

for \( n = 1 \), \( M'' = 2 \log \frac{8\alpha}{\alpha} - 6.666\ldots \),

for \( n = 2 \), \( M'' = 2 \log \frac{8\alpha}{\alpha} - 7.466\ldots \)

and so on.

Letting \( w' = 1/[\pi\alpha^2 k] \) denote the resistance of the unit length of the conductor, and putting \( \lambda = 0 \), that is, restricting to those cases in which Ohm’s law applies, we get a decrease of intensity in the unit of time in the proportion

\[
= 1 : e^{-w'c^2/16 \log \frac{8\alpha}{\alpha} - 53.33\ldots}
\]

for the first wave trains with \( n = 1 \), [and in the proportion]
for the second wave trains with \( n = 2 \), and so on.

Hence one sees that the decrease is the faster, the greater the resistance per unit length of the conductor, the thicker the conductor compared to its length, and the larger the place number \( n \) of the wave train, that means, the shorter the waves.

16 Propagation Velocity of the Wave Trains in a Circular Conductor

From Section 13 it follows, as shown above, that, after each disturbance of the equilibrium, the motions of electricity in a circular conductor can be split into wave trains whose propagation is determined by simple laws, as is the case for many other bodies. For some bodies like air in a circular tube, in addition, these wave trains are not altered at all by the propagation, that specifically no decrease of the intensity takes place, and that furthermore all wave trains are propagated at equal velocity, whence it follows that all wave trains propagating forward combine to a single wave train which in turn is propagated unaltered and at the same velocity like the single wave trains of which it consists. Such a combined wave train, however, consists of combined waves which can largely differ in size, form, and intensity. Such combined waves, remaining coherent due to the same velocity of all its constituents, have a particular physical meaning as objects of observation and are called waves in the strict sense of the word.

Thus in this more strict sense electric waves in a circular conductor where electric equilibrium has been disturbed would not exist, already because of the different decreases of the intensities of the various elementary wave trains, even less, however, if the various elementary wave trains had different propagation velocities.

Where waves exist in the more strict sense, the propagation velocity is of utmost importance for the knowledge about the medium of propagation, therefore this question concerning electricity has awakened particular interest and therefore the respective results from Section 13 shall be considered more closely.

The propagation velocities of the various elementary wave trains from the formulas developed in Section 13 were found to be equal to the increase or decrease which \( s \) must get if, when \( t \) increases by 1 [unit] in the values of \( E_n \) and \( i_n \), the values of the arcs under the sines shall remain unaltered, that is
\[m = \frac{ma}{n},\]
or, inserting the value of \(m\) from Section 13
\[m = \sqrt{\frac{n^2c^2N''}{2a^2M''(1 + \lambda)} - \left(\frac{c^2}{16\pi a^2kM''(1 + \lambda)}\right)^2},\]

[we get]
\[= \frac{c}{\sqrt{2}} \cdot \sqrt{\frac{N''}{M''(1 + \lambda)} - \frac{a^2c^2w'^2}{128n^2M''^2(1 + \lambda)^2}},\]

wherein \(w' = 1/\pi a^2k\) is put as above. Restricting ourselves to the cases where we may put \(\lambda = 0\), that is, where Ohm’s law applies, then the expression for this propagation velocity reduces to
\[= \frac{c}{\sqrt{2}} \cdot \sqrt{\frac{N''}{M''} - \frac{a^2c^2w'^2}{128n^2M''^2}},\]

wherein the values of \(N''\) and \(M''\) are determined as follows
\[N'' = 2 \log \frac{8a}{\alpha} - 4 \left(\cos \frac{1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{3} \cos \frac{3}{4} \sqrt{\frac{\alpha}{a}} + ... + \frac{1}{2n - 1} \cos \frac{2n - 1}{4} \sqrt{\frac{\alpha}{a}} - \frac{n^2\alpha}{8a}\right),\]
\[M'' = 2 \log \frac{8a}{\alpha} - 4 \left(\cos \frac{1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{3} \cos \frac{3}{4} \sqrt{\frac{\alpha}{a}} + ... + \frac{1}{2n - 1} \cos \frac{2n - 1}{4} \sqrt{\frac{\alpha}{a}} - \frac{n^2\alpha}{8a} - 2 - \frac{1}{2n - 1} \cos \frac{2n - 1}{4} \sqrt{\frac{\alpha}{a}} + \frac{1}{2n + 1} \cos \frac{2n + 1}{4} \sqrt{\frac{\alpha}{a}}\right).\]

Thus it follows that the propagation velocity is different for the various wave trains according to their different place numbers \(n\), and it only remains the question whether, under certain conditions, the differences of the various propagation velocities would not be so small as to consider them approximately as vanishing, and what would then be the limit to be approached by all these propagation velocities.

From the values presented it follows indeed that, as long as the place number, \(n\), does not exceed those values for which \(n^2\alpha/a\) may be considered as vanishing compared to 1, we may put
\[ \frac{N''}{M''} = 1 + \frac{8n^2}{(4n^2 - 1)M''} . \]

For large values of \( M'' \) for which the fraction \( \frac{8n^2}{(4n^2 - 1)M''} \) vanishes compared to 1, and for small values of the resistance for which the fraction \( \frac{a^2c^2w^2}{128n^2M''^2} \) vanishes compared to 1,\(^{39}\) then \( c/\sqrt{2} \) is the desired limit which is approached by all propagation velocities, and, for the given value\(^{40}\) \( c = 439\,450 \cdot 10^6 \) millimeter/second, this limit equals

\[ \frac{c}{\sqrt{2}} = 310\,740 \cdot 10^6 \text{ millimeter/second}, \]

that is, a velocity of 41 950 miles/second.

Already Kirchhoff has found this velocity and remarked:\(^{41}\)

“that it is independent of the cross section, of the conductivity of the wire, also, finally, of the density of the electricity: its value is 41 950 German miles in a second, hence very nearly equal to the velocity of light \( \textit{in vacuo}. \)"

Could this close coincidence of the propagation velocity of electric waves with that of the light be considered as a hint to the inner relationship of both theories, then it would demand the greatest interest considering the great importance of the investigation of such a relationship. But it is clear that above all, the true meaning of this velocity with respect to electricity must be considered, but it is not of the kind encouraging great expectations.

For the approximation of the true propagation velocity to this limiting value which coincides with the velocity of light presupposes, as just demonstrated, a conducting wire not only very thin compared to its length, but also that this long and thin conducting wire would have a very small resistance. Hence it is clear that the close approach to this limit will occur only rarely, larger deviations from it very frequently. A corresponding survey is best obtained giving examples.

As examples we choose three circular copper wires with respective radii

\[ a = 1000, \quad 1\,000\,000, \quad 1\,000\,000 \text{ millimeter}, \]

\(^{39}\)Note by WEW:] The fraction \( \frac{a^2c^2w^2}{128n^2M''^2} \) can be considered as vanishing compared with 1 when, for large values of \( M'' \), that velocity expressing the resistance of the whole conductor in \textit{absolute magnetic measure of resistance}, that is, \( \frac{\pi c/4}{acw'} \), is very small compared to the velocity \( c. \)

\(^{40}\)Note by AKTA:] See [KW57, Section 17, p. 652 of Weber’s \textit{Werke}] with English translation in [KW21, Section 17, p. 52].

\(^{41}\)Note by AKTA:] [Kir57b, pp. 209-210] with English translation in [Kir57a, p. 406].
and respective cross sections

\[ \pi a^2 = 1, \quad 1, \quad \frac{1}{10} \text{ square millimeter}. \]

The resistance of these wires, as found by measurement in absolute magnetic measure of resistance, can be put in rounded figures equal to

\[ W = \frac{2\pi a}{\pi a^2} \cdot 2 \cdot 10^6, \]

(see the Abhandlungen der Königl. Gesellschaften der Wissenschaften zu Göttingen, Vol. 5, Section 9).\textsuperscript{42,43} But, according to the known relation between magnetic and mechanical measures of resistivity, we have

\[ W = \pi c^2 a w'/4 \text{ or } a^2 c^2 w^2/128 = W^2/\left[8\pi^2 c^2\right], \]

after what\textsuperscript{44}

\[ \frac{c}{\sqrt{2}} \sqrt{\frac{N_n}{M^n} - \frac{a^2 c^2 w^2}{128n^2 M^n}} = \frac{c}{\sqrt{2}} \sqrt{\frac{N_n}{M^n} - \frac{W^2}{8\pi^2 c^2 n^2 M^n}} = \frac{c'}{\sqrt{2}}. \]

The following table is calculated on this basis.

\textsuperscript{43}[Note by AKTA:] [Web53c, Section 9, pp. 315-319 of Weber’s Werke], see also [Web53a] and [Web53b].
\textsuperscript{44}[Note by AKTA:] Weber is defining the magnitude $c'$ by the following equation.
<table>
<thead>
<tr>
<th>n</th>
<th>First wire</th>
<th></th>
<th>Second wire</th>
<th></th>
<th>Third wire</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a = 1000)</td>
<td>(a = 1000000)</td>
<td>(a = 1000000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\pi\alpha^2 = 1)</td>
<td>(\pi\alpha^2 = 1)</td>
<td>(\pi\alpha^2 = 1/10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(W = 4 \cdot 10^9 \cdot \pi)</td>
<td>(W = 4 \cdot 10^{112} \cdot \pi)</td>
<td>(W = 4 \cdot 10^{13} \cdot \pi)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(N'' = 15.119)</td>
<td>(= 28.935)</td>
<td>(= 31.605)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M'' = 12.452)</td>
<td>(= 25.268)</td>
<td>(= 28.938)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(N''/M'' = 1.214)</td>
<td>(= 1.145)</td>
<td>(= 1.092)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{W^2}{8\pi^2c_n^2M''^2} = 145970000)</td>
<td>(= 0.0166)</td>
<td>(= 1.2364)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c^2/c^2 = 1.214)</td>
<td>(= 1.128)</td>
<td>(= -0.0443)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(N'' = 13.786)</td>
<td>(= 27.601)</td>
<td>(= 31.062)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M'' = 11.652)</td>
<td>(= 25.468)</td>
<td>(= 28.928)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(N''/M'' = 1.183)</td>
<td>(= 1.084)</td>
<td>(= 1.074)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{W^2}{8\pi^2c_n^2M''^2} = 52450000)</td>
<td>(= 0.00408)</td>
<td>(= 0.3093)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c^2/c^2 = 1.183)</td>
<td>(= 1.080)</td>
<td>(= 0.7644)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(N'' = 12.986)</td>
<td>(= 26.801)</td>
<td>(= 30.262)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M'' = 10.929)</td>
<td>(= 24.747)</td>
<td>(= 28.205)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(N''/M'' = 1.188)</td>
<td>(= 1.083)</td>
<td>(= 1.073)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{W^2}{8\pi^2c_n^2M''^2} = 103800000)</td>
<td>(= 0.00192)</td>
<td>(= 0.1446)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c^2/c^2 = 1.188)</td>
<td>(= 1.081)</td>
<td>(= 0.9283)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(N'' = 12.414)</td>
<td>(= 26.230)</td>
<td>(= 29.690)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M'' = 10.383)</td>
<td>(= 24.198)</td>
<td>(= 27.659)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(N''/M'' = 1.196)</td>
<td>(= 1.084)</td>
<td>(= 1.073)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{W^2}{8\pi^2c_n^2M''^2} = 10620000)</td>
<td>(= 0.00113)</td>
<td>(= 0.0846)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c^2/c^2 = 1.197)</td>
<td>(= 1.083)</td>
<td>(= 0.9889)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(N'' = 11.970)</td>
<td>(= 25.785)</td>
<td>(= 29.246)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M'' = 9.950)</td>
<td>(= 23.765)</td>
<td>(= 27.226)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(N''/M'' = 1.203)</td>
<td>(= 1.085)</td>
<td>(= 1.074)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{W^2}{8\pi^2c_n^2M''^2} = 23900000)</td>
<td>(= 0.00075)</td>
<td>(= 0.0559)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c^2/c^2 = 1.203)</td>
<td>(= 1.084)</td>
<td>(= 1.0183)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The values of \(c^2/c^2\) in the above Table which give the squares of the propagation velocities \(c'/\sqrt{2}\) in parts of the square of the limiting value \(c/\sqrt{2}\) disclose essential differences already among the first five wave trains to which the Table is restricted; for the third wire \(c^2/c^2\) even has a negative value for \(n = 1\); hence here the expression for the propagation velocity of the first wave
train becomes *imaginary* and therefore the laws of the changes of motion in this wire after a disturbance of the equilibrium cannot at all be interpreted in terms of propagating waves trains, but they require a different form which represents the changes of motion as a pure approximation to the state of equilibrium which may be called *absorption*, and which deserves particular attention because of its special importance for long and thin conducting wires with large resistance, namely telegraph wires.

### 17 Damping of Electric Motions in a Circular Conductor

In Section 13, in the integration of the two partial differential equations for the motion of the electricity left to itself in a circular conductor, namely the equations

\[
\frac{d^2a_n}{dt^2} + 2\varepsilon \frac{da_n}{dt} + (m^2 + \varepsilon^2)a_n = 0,
\]

\[
\frac{d^2b_n}{dt^2} + 2\varepsilon \frac{db_n}{dt} + (m^2 + \varepsilon^2)b_n = 0,
\]

it has been assumed for the two expressions set up for \(a_n\) and \(b_n\),

\[
a_n = Ae^{-\varepsilon t} \sin m(t - A'),
\]

\[
b_n = Be^{-\varepsilon t} \sin m(t - B'),
\]

that \(m\) would be a real value which, however, is not always the case. Because, putting \(1/[\pi \alpha^2 k] = w'\), we had

\[
m = \frac{n}{a} \cdot \frac{c}{\sqrt{2}} \sqrt{\frac{N''}{M''(1 + \lambda)}} - \frac{a^2 c^2 w'^2}{128 n^2 M''(1 + \lambda)^2},
\]

namely this assumption can also be formulated so that

\[
\frac{a^2 c^2 w'^2}{128 n^2 M''(1 + \lambda)^2} < \frac{N''}{M''(1 + \lambda)},
\]

should hold, or, if \(\lambda = 0\),

\[
\frac{a^2 c^2 w'^2}{128 n^2 M''} < \frac{N''}{M''}.
\]
On the other hand, the example of the third wire in the preceding Section shows that with long and thin conducting wires also the case may occur in which

\[
\frac{a^2 c^2 w^2}{128 n^2 M''^2} > \frac{N''}{M''},
\]

whence it becomes clear that the integration of the above differential equations becomes illusive and hence must be sought in a different form.

Therefore, putting for this purpose

\[
m = \frac{n}{a} \cdot \frac{c}{\sqrt{2}} \sqrt{\frac{a^2 c^2 w^2}{128 n^2 M''^2 (1 + \lambda)^2} - \frac{N''}{M'' (1 + \lambda)}},
\]

then the two differential equations get the following form, namely

\[
\begin{align*}
\frac{d^2 a_n}{dt^2} + 2\varepsilon \frac{da_n}{dt} + (\varepsilon^2 - m^2)a_n &= 0, \\
\frac{d^2 b_n}{dt^2} + 2\varepsilon \frac{db_n}{dt} + (\varepsilon^2 - m^2)b_n &= 0,
\end{align*}
\]

from which by integration we get

\[
\begin{align*}
a_n &= Ae^{-\varepsilon t} \cdot \left( e^{\varepsilon(t-A')} - e^{-\varepsilon(t-A')} \right), \\
b_n &= Be^{-\varepsilon t} \cdot \left( e^{\varepsilon(t-B')} - e^{-\varepsilon(t-B')} \right).
\end{align*}
\]

As performed in Section 13, the constants of integration \( A, A', B, B' \) will be found from the values of \( a^0_n, b^0_n, da^0_n/dt, db^0_n/dt \) given for \( t = 0 \) by means of which the original distribution of the free electricity in the conductor and the original currents are expressed. In this way one obtains

\[
\begin{align*}
Ae^{-mA'} &= \frac{1}{2m} \left( (\varepsilon + m)a^0_n + \frac{da^0_n}{dt} \right), \\
Ae^{+mA'} &= \frac{1}{2m} \left( (\varepsilon - m)a^0_n + \frac{da^0_n}{dt} \right), \\
Be^{-mB'} &= \frac{1}{2m} \left( (\varepsilon + m)b^0_n + \frac{db^0_n}{dt} \right), \\
Be^{+mB'} &= \frac{1}{2m} \left( (\varepsilon - m)b^0_n + \frac{db^0_n}{dt} \right).
\end{align*}
\]
Substituting these values, one obtains the following two equations

\[
a_n = \frac{1}{2m} \left[ \left( \varepsilon + m \right) a_n^0 + \frac{da_n^0}{dt} \right] e^{-(\varepsilon-m)t} - \left( \varepsilon - m \right) a_n^0 + \frac{da_n^0}{dt} \right] e^{-(\varepsilon+m)t} ,
\]

\[
b_n = \frac{1}{2m} \left[ \left( \varepsilon + m \right) b_n^0 + \frac{db_n^0}{dt} \right] e^{-(\varepsilon-m)t} - \left( \varepsilon - m \right) b_n^0 + \frac{db_n^0}{dt} \right] e^{-(\varepsilon+m)t} .
\]

Finally inserting these values of \(a_n\) and \(b_n\) into the equations

\[
E = \sum \left( a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a} \right) ,
\]

\[
i = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{dh_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) ,
\]

one finds the laws of distribution of the free electricity and of the currents in the circular conductor for the cases considered here.

Now such a case occurs with each circular conductor, namely when the given original distribution of the free electricity and of the currents is such that the value of \(b_n^0\) or \(1/n \cdot |da_n^0/dt|\) is not zero for \(n = 0\), and therefore this case has been excluded from the consideration in Section 15, where only those values of \(E_n\) and \(i_n\) have been discussed which are valid for \(n = 1, \ 2, \ 3 ...\) Is in fact \(n = 0\), then it becomes clear that

\[
\frac{a^2 c^2 w'^2}{128 n^2 M''^2 (1 + \lambda)^2} > \frac{N''}{M''(1 + \lambda)}
\]

holds and that consequently we have to put

\[
m = \frac{1}{a} \cdot \frac{c}{\sqrt{2}} \sqrt{\frac{a^2 c^2 w'^2}{128 M''^2 (1 + \lambda)^2}} .
\]

But we had

\[
\varepsilon = \frac{c^2 w'}{16 M''(1 + \lambda)} ,
\]

whence it follows that, for \(n = 0\), we have to put

\[
m = \varepsilon .
\]

Now substituting this value of \(m\) into the above values of \(a_n\) and \(b_n\), we get
\[ a_0 = a_0^0 + \frac{1}{2\varepsilon} \frac{da_0}{dt} - \frac{1}{2\varepsilon} \frac{da_0}{dt} \cdot e^{-2\varepsilon t}, \]
\[ b_0 = b_0^0 + \frac{1}{2\varepsilon} \frac{db_0}{dt} - \frac{1}{2\varepsilon} \frac{db_0}{dt} \cdot e^{-2\varepsilon t}, \]
whence by differentiation
\[ \frac{da_0}{dt} = \frac{da_0^0}{dt} \cdot e^{-2\varepsilon t}, \]
\[ \frac{db_0}{dt} = \frac{db_0^0}{dt} \cdot e^{-2\varepsilon t}. \]
Now inserting these values into the equations
\[ E_n = a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a}, \]
\[ i_n = -a \frac{1}{2n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right), \]
for \( n = 0 \), one finds
\[ E_0 = b_0^0 + \frac{1}{2\varepsilon} \frac{db_0^0}{dt} \left( 1 - e^{-2\varepsilon t} \right), \]
\[ i_0 = \frac{s}{2} \frac{db_0^0}{dt} e^{-2\varepsilon t} + \frac{a}{2} \left( \frac{1}{n} \frac{da_n}{dt} \right)_0, \]
where
\[ \left( \frac{1}{n} \frac{da_n}{dt} \right)_0 \]
denotes the value of \( ([1/n] \cdot [da_n/dt]) \) for \( n = 0 \); hence, as
\[ \left( \frac{1}{n} \frac{da_n}{dt} \right)_0 = \left( \frac{1}{n} \frac{da_n^0}{dt} \right)_0 \cdot e^{-2\varepsilon t}, \]
and as the coefficients of \( \sin(ns/a) \) and \( \cos(ns/a) \) are to have finite values in the equation
\[ i_n = -a \frac{1}{2n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right), \]
whence, for \( n = 0 \), we should have
\[
\frac{da_0}{dt} = 0 \quad \text{and} \quad \frac{db_0}{dt} = 0,
\]

[therefore,]

\[E_0 = b_0^0,\]

\[i_0 = \frac{a}{2} \left( \frac{1}{n} \frac{da_n^0}{dt} \right)_0 \cdot e^{-2\varepsilon t}.\]

Whence it follows that, if a circular conductor is originally homogeneously charged with free electricity along its total length, so that each unit length contains the same amount of free electricity \( b_0^0 \), then this charge does not change with time \( t \), which becomes clear all by itself. But in addition to that, if there is originally the same current in all parts of this conductor, that this current does not vanish at the moment after which the electricity is left to itself, but gradually decreases following the law of a geometric series as time \( t \) increases arithmetically. If here also the necessity of gradual vanishing becomes clear a priori, it can not be easily seen a priori how fast this should happen and what differences in this speed should take place between different conductors.

If a current with certain intensity \( i \) flows in a closed conductor at the very moment after which the electricity is left to itself because no external electromotive force acts on it, as is for example the case when an inductive magnet moving with respect to a conductor is suddenly stopped in this motion by pushing against it — then it is interesting for some practical questions to determine the amount of positive or negative electricity which still passes after this moment through each cross section of the conductor; and then further to determine the time that has to pass after the same moment until the current intensity \( i \) has decreased to \( i/2 \).

If \( i = (a/2)([1/n] \cdot [da_n^0/dt])_0 \) is given for that moment \( t = 0 \), then the current intensity after time \( t \) [is given by]

\[= i \cdot e^{-2\varepsilon t},\]

which, expressed in mechanical measure, denotes the amount of positive electricity that would pass through the cross section of the conductor in the unit of time for this current intensity. Hence, the amount of positive electricity passing through the cross section of the conductor during the time element, \( dt \), equals

\[= i \cdot e^{-2\varepsilon t} dt,\]
and the integral value hereof, taken from \( t = 0 \) to \( t = \infty \), yields the total amount of positive electricity which at all passes through any cross section of the conductor after the moment considered, namely

\[
i \int_0^\infty e^{-2\varepsilon t} dt = \frac{1}{2\varepsilon} \cdot i .
\]

The amount of negative electricity passing through the cross section in opposite direction is equally large.

Furthermore, the following equation yields the time \( t \) during which the current intensity decays to half of its value:

\[
e^{-2\varepsilon t} = \frac{1}{2},
\]

hence

\[
t = \frac{1}{2\varepsilon} \log \frac{2}{\varepsilon} .
\]

Now we had \( \varepsilon = c^2 w'[2^2 M''/(1 + \lambda)] \), wherein we have to put \( M'' = 2 \log(8a/\alpha) \) for \( n = 0 \); hence, taking \( \lambda = 0 \), that amount of electricity passing through the cross section of the conductor equals

\[
\frac{1}{2\varepsilon} \cdot i = \frac{16}{c^2 w'} \cdot \log \frac{8a}{\alpha} \cdot i = \frac{2}{W'} \cdot \log \frac{8a}{\alpha} \cdot i ,
\]

when \( W' = [c^2/8] \cdot w' \) denotes the resistance in magnetic measure in unit length of the conductor.

The time during which the current intensity decays to half of its value is then, expressed in seconds

\[
\frac{1}{2\varepsilon} \cdot \log 2 = \frac{16}{c^2 w'} \cdot \log \frac{8a}{\alpha} \cdot \log 2 = \frac{2}{W'} \cdot \log \frac{8a}{\alpha} \cdot \log 2 .
\]

Hence we get the following values for the wires exemplified in Section 16:

<table>
<thead>
<tr>
<th>First wire</th>
<th>Second wire</th>
<th>Third wire</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2\varepsilon} )</td>
<td>( \frac{1}{104607} )</td>
<td>( \frac{1}{60726} )</td>
</tr>
<tr>
<td>( \frac{\log 2}{2\varepsilon} )</td>
<td>( \frac{1}{150916} )</td>
<td>( \frac{1}{87609} )</td>
</tr>
</tbody>
</table>

However small may be here the fraction \( 1/2\varepsilon \) of the amount which would pass through the cross section of the conductor with the original current, produced by the positive electricity transported by the vanishing current
through the cross section in the unit of time of the conductor, yet this amount of electricity could produce a very strong charge of the conductor if it were used for this purpose. If, for example, the current intensity originally present were equal to the magnetic unit of measure (which decomposes 1 milligram of water during \(106^{2/3}\) seconds), then the positive amount of electricity passing through the cross section of the conductor in the unit of time at this current would amount to \(155,370 \times 10^6\) electrostatic units of measure, and, as the current disappeared in the first wire, still \(\frac{155,370}{104,607} \times 10^6\), that is almost \(1\frac{1}{3}\) million electrostatic units of measure of positive electricity would be carried through any cross section of the conductor, that is about the 24th part of the weakest, or the 33rd part of the strongest charge of the small Leiden jar which have been used for the experiment described in Vol. 5 of the previous Abhandlung, where these charges have been determined in more detail on p. 254.

It is easily seen that a similar vanishing of the current flowing in a closed conductor occurs at the moment when the circuit of a galvanic current is disrupted and that the positive amount of electricity then carried through the center cross section by the decaying current in fact contributes to charging the first half of the conductor, and likewise the amount of negative electricity oppositely carried through the same cross section contributes to charging the second half of the conductor, and that the opposite charges produce the spark of disruption at the place where the circuit has been disrupted, where it is interesting to learn about the amounts of electricity discharged by the spark of disruption.

Likewise the importance is clear to further develop the laws of the current decay for the determination of the inductive forces thus exerted on other conductors, especially for the theory of the Rühmkorff type and other similar inductive machines which hereby is given its foundation.

18 Reference to Heat Conduction

For increasing values of \(t\) where eventually \(e^{-2mt}\) vanishes in comparison with 1, the two equations found for \(a_n\) and \(b_n\) in the preceding Section, namely

\[\text{Note by AKTA:} \text{[Web41b] and [Web42] with English translation in [Web20].}\]

\[\text{Note by HW:} \text{Wilhelm Weber’s Werke, Vol. III, p. 641.}\]

\[\text{Note by AKTA:} \text{[KW57, Section 12, p. 641 of Weber’s Werke] with English translation in [KW21, Section 12, pp. 36-40].}\]

\[\text{Note by AKTA:} \text{Heinrich Daniel Rühmkorff (1803-1877) was a German instrument maker who commercialised the induction coil known by his name.}\]
\[ a_n = \frac{1}{2m} \left[ \left( (\varepsilon + m)a_n^0 + \frac{da_n^0}{dt} \right) e^{-(\varepsilon - m)t} - \left( (\varepsilon - m)a_n^0 + \frac{da_n^0}{dt} \right) e^{-(\varepsilon + m)t} \right] , \]

\[ b_n = \frac{1}{2m} \left[ \left( (\varepsilon + m)b_n^0 + \frac{db_n^0}{dt} \right) e^{-(\varepsilon - m)t} - \left( (\varepsilon - m)b_n^0 + \frac{db_n^0}{dt} \right) e^{-(\varepsilon + m)t} \right] , \]

turn into the simpler equations:

\[ a_n = \frac{1}{2m} \left( (\varepsilon + m)a_n^0 + \frac{da_n^0}{dt} \right) e^{-(\varepsilon - m)t} , \]

\[ b_n = \frac{1}{2m} \left( (\varepsilon + m)b_n^0 + \frac{db_n^0}{dt} \right) e^{-(\varepsilon - m)t} , \]

and inserting these values of \( a_n \) and \( b_n \) into the equations

\[ E = \sum \left( a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a} \right) , \]

\[ i = -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right) , \]

we get the following distribution laws of the free electricity and the currents in the circular conductor:

\[ E = \sum \frac{1}{2m} \left[ \left( (\varepsilon + m)a_n^0 + \frac{da_n^0}{dt} \right) \sin \frac{ns}{a} + \left( (\varepsilon + m)b_n^0 + \frac{db_n^0}{dt} \right) \cos \frac{ns}{a} \right] e^{-(\varepsilon - m)t} , \]

\[ i = \frac{a}{4} \sum \frac{\varepsilon - m}{mn} \left[ \left( (\varepsilon + m)b_n^0 + \frac{db_n^0}{dt} \right) \sin \frac{ns}{a} - \left( (\varepsilon + m)a_n^0 + \frac{da_n^0}{dt} \right) \cos \frac{ns}{a} \right] e^{-(\varepsilon - m)t} . \]

Here it is easily seen that in all cases where \( \varepsilon - m/[n^2] = \beta \) is a coefficient independent of \( n \) we get

\[ i = -\frac{a^2 b}{2} \frac{dE}{ds} , \]
\[
\frac{di}{ds} = -\frac{1}{2} \frac{dE}{dt},
\]
whence it follows by eliminating \(i\)
\[
\frac{dE}{dt} = a^2 \beta \frac{d^2 E}{ds^2},
\]
an equation having the same form as the equation for the heat conduction in solid bodies.

But in the preceding Section we had put
\[
m = \frac{n}{a} \cdot \frac{c}{\sqrt{2}} \sqrt{\frac{a^2 c^2 w^2}{128 n^2 M''(1 + \lambda)^2} - \frac{N''}{M''(1 + \lambda)}},
\]
wherein \(c^2 w' /[16 M''(1 + \lambda)] = \varepsilon\), thus
\[
m = \varepsilon \sqrt{1 - \frac{128 N'' M''(1 + \lambda)}{a^2 c^2 w'^2} \cdot n^2}.
\]
Now in all cases where the values of \(n^2/[a^2 c^2 w'^2]\) and \(\alpha/a\) are very small, we may put instead
\[
m = \varepsilon \left(1 - \frac{256 (\log \frac{8a}{\alpha})^2 \cdot (1 + \lambda)}{a^2 c^2 w'^2} \cdot n^2\right),
\]
from which \(\varepsilon - m = [8/a^2 w'] n^2 \log(8a/\alpha)\), therefore
\[
\beta = \frac{8}{a^2 w'} \cdot \log \frac{8a}{\alpha}
\]
is a coefficient independent of \(n\).

Hence for the changes of motion of electricity in the cases just described, this yields laws similar to those for the heat conduction in solids as has already been demonstrated by Thomson and Kirchhoff.\(^{49}\) Even if the expression for the propagation velocity of the longer wave trains, that is for smaller values of \(n\), become imaginary and hence demand other laws for this part of the motion which approach the laws of heat conduction in solid bodies, a still remaining part of the motion deserves particular attention which yields shorter wave trains for which greater values of \(n\) are valid, for which the expression for the propagation velocity stays real and thus the laws developed in Section 13 remain valid. After a disturbance of the equilibrium, thus there

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\(^{49}\) [Note by AKTA:] See [Tho56a] and [Tho56b] with Portuguese translation in [TBA18]; [Kir57b] and [Kir57c], with English translations in [Kir57a] and [GA94], respectively.
are always *wave trains* in such a conductor propagating at certain *velocities*, however, there is no *pure wave motion*, but it is mixed with other motions which are governed by laws similar to those when heat is conducted.

Now considering all conditions resulting from such a mixture of motions subject to quite different laws, it becomes immediately clear that Wheatstone’s observation\(^{50}\) of the *non-synchronicity of the sparks* at two mutually very distant places where the long conducting wire is disrupted does not at all allow to conclude a definite propagation velocity, and that Wheatstone’s way of observation, as meaningful it may be and as valuable its results for other contexts may be, if they could really be guaranteed exactly, yet it is not suited directly for the purpose in question, like it will generally be not possible at all to succeed in finding such ways of observation by means of which the laws of all changes of motion of electricity in a conductor after a disturbance of the equilibrium may be founded on *pure experience*. Hence the aim of the *observations* will here be restricted to *test* the laws derived up to now from our previous knowledge about electricity. Therefore it was necessary, as has been tried in the previous Sections, to treat the derivation of the laws before the observations to be carried out for their test, even more because the laws thus formulated have to be used as a *guide* in the search of the *most suitable ways of observation* to be applied for the test.

### 19 Oscillations of Electricity in a Circular Conductor

As regards the *most appropriate ways of observation* to test the laws of electric motions, it becomes automatically clear from the laws developed so far that, considering the extremely high *velocity* of most of the *electric wave trains* in good conductors according to these laws and the quick *damping* of these wave trains resulting from the same laws, with the limits imposed on all observations by the sensory tools, it would barely be possible to *conduct exact observations and measurements for direct testing of these laws*. An exact performance of measurements always demands a certain expenditure of time which is appropriate for such non-persistent phenomena. Considering therefore that the finest measurements of physics are those concerning either *equilibrium phenomena*, or *uniform motions*, or *periodically recurring phenomena*, as for example oscillations of a pendulum, then this suggests to base a testing method also for these laws on observations of the motion of electricity in conductors, apart from *constant currents*, on *periodically recurring*\(^{50}\)\footnote{Note by AKTA:} [Whe34].
phenomena, supposing that methods will be found for the fine performance of such observations.

But periodically occurring motions of the electricity in a conductor cannot exist all by themselves, but only due to continuous excitation by external electromotive forces and their production suggests the fast rotation of a small magnet about an axis perpendicular to its magnetic axis as the simplest and, for finer observations and measurements, most practicable method. In order to obtain a guide to practical equipments for exact observations of periodically occurring motions or oscillations of the electricity in a conductor thus produced, we shall first try to develop the laws of such electric oscillations in a circular conductor from the partial differential equations formulated in Section 10.

20 Oscillations Due to Induction by a Rotating Magnet

The electromotive force exerted by fast rotation of a small magnet in the vicinity of the circular conductor on any point, \( s \), of the conductor in a certain moment of time may be represented, if \( a \) denotes the radius [of the conductor], by

\[ \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) , \]

where \( f_n \) and \( g_n \) depend only on the place number, \( n \). But due to the uniform rotation of the magnet, all these forces acting on different points, \( s \), of the conductor are subjected to a regular periodic change, and in effect, with a suitable set up, they are proportional to the sine of an angle uniformly growing with time. For an arbitrary moment, all these forces may be represented at the end of time \( t \) by

\[ \sin \mu t \cdot \sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) . \]

Inserting \( f_n \sin \mu t \) and \( g_n \sin \mu t \) into the two partial differential equations at the end of Section 10 in place of \( f_n \) and \( g_n \) which denoted arbitrary functions of the time there, where now \( f_n \) and \( g_n \) have values independent of time, one gets the following two partial differential equations\(^{51}\)

\[ \frac{d^2 a_n}{dt^2} + \frac{c^2}{8\pi \alpha^2 k M''(1 + \lambda)} \frac{da_n}{dt} + \frac{n^2 c^2 N''}{2a^2 M''(1 + \lambda)} a_n - \frac{nc^2}{4aM''(1 + \lambda)} \cdot g_n \sin \mu t = 0 , \]

\(^{51}\)Note by PM and AKTA: The following equations are ordinary differential equations.
\[
\frac{d^2b_n}{dt^2} + \frac{c^2}{8\pi\alpha^2kM''(1 + \lambda)} \frac{db_n}{dt} + \frac{n^2c^2N''}{2a^2M''(1 + \lambda)} b_n + \frac{nc^2}{4aM''(1 + \lambda)} f_n \sin\mu t = 0.
\]

Now one sees easily that, putting

\[a_n = p \sin(\mu t + \rho),\]
\[b_n = q \sin(\mu t + \rho),\]

[then] \(p, q\) and \(\rho\) can be determined so that the values of \(a_n\) and \(b_n\) thus obtained satisfy the two partial differential equations. Inserting the above values, namely \(a_n\) and \(b_n\), and the values derived from them,

\[\frac{da_n}{dt} = p\mu \cos(\mu t + \rho),\]
\[\frac{db_n}{dt} = q\mu \cos(\mu t + \rho),\]
\[\frac{d^2a_n}{dt^2} = -p\mu^2 \sin(\mu t + \rho),\]
\[\frac{d^2b_n}{dt^2} = -q\mu^2 \sin(\mu t + \rho),\]

into the equations above, one obtains, putting \(1/[\pi\alpha^2k] = w'\) for brevity, and either \(\lambda = 0\) according to Ohm’s law, or \(M''\) for \(M''(1 + \lambda)\),

\[\]

\[-p\mu^2 \sin(\mu t + \rho) + \frac{p\mu c^2 w'}{8M''} \cos(\mu t + \rho) + \frac{pm c^2 N''}{2a^2M''} \sin(\mu t + \rho) - \frac{c^2 n}{4aM''} g_n \sin\mu t = 0,\]

\[-q\mu^2 \sin(\mu t + \rho) + \frac{q\mu c^2 w'}{8M''} \cos(\mu t + \rho) + \frac{qN c^2 N''}{2a^2M''} \sin(\mu t + \rho) + \frac{c^2 n}{4aM''} f_n \sin\mu t = 0.\]

Expanding the sine and cosine of the sum in terms of sine and cosine of the parts, one gets

\[
\left(\frac{\mu c^2 w'}{8M''} \cdot p \sin \rho + \left(\mu^2 - \frac{n^2 c^2 N''}{2a^2M''}\right) \cdot p \cos \rho + \frac{c^2 n}{4aM''} \cdot g_n\right) \sin\mu t \\
+ \left(\left(\mu^2 - \frac{n^2 c^2 N''}{2a^2M''}\right) \cdot p \sin \rho - \frac{\mu c^2 w'}{8M''} \cdot p \cos \rho\right) \cos\mu t = 0,
\]
\[
\left( \frac{\mu c^2 w'}{8M''} \cdot q \sin \rho + \left( \mu^2 - \frac{n^2 c^2 N''}{2a^2 M''} \right) \cdot q \cos \rho - \frac{c^2 n}{4aM''} \cdot f_n \right) \sin \mu t \\
+ \left( \mu^2 - \frac{n^2 c^2 N''}{2a^2 M''} \right) \cdot q \sin \rho - \frac{\mu c^2 w'}{8M''} \cdot q \cos \rho \right) \cos \mu t = 0 .
\]

If these equations are to be valid for any value of \( t \), one obtains for \( \cos \mu t = 0 \) the two equations

\[
\frac{\mu c^2 w'}{8M''} \cdot p \sin \rho + \left( \mu^2 - \frac{n^2 c^2 N''}{2a^2 M''} \right) \cdot p \cos \rho + \frac{c^2 n}{4aM''} \cdot g_n = 0 ,
\]

\[
\frac{\mu c^2 w'}{8M''} \cdot q \sin \rho + \left( \mu^2 - \frac{n^2 c^2 N''}{2a^2 M''} \right) \cdot q \cos \rho - \frac{c^2 n}{4aM''} \cdot f_n = 0 ,
\]

and for \( \sin \mu t = 0 \) also the third equation, namely

\[
\left( \mu^2 - \frac{n^2 c^2 N''}{2a^2 M''} \right) \sin \rho - \frac{\mu c^2 w'}{8M''} \cdot \cos \rho = 0 ,
\]

from which \( p, q \) and \( \rho \) are be determined so that the two partial differential equations are satisfied by the values of \( a_n \) and \( b_n \) thus determined. One gets in fact

\[
\rho = \arctan \frac{\mu a^2 c^2 w'}{4(2\mu^2 a^2 M'' - n^2 c^2 N'')} ,
\]

\[
p = - \frac{2ac^2 n}{2(2\mu^2 a^2 M'' - n^2 c^2 N'')} \cdot g_n \cos \rho \\
= - \frac{2ac^2 n}{\sqrt{16(2\mu^2 a^2 M'' - n^2 c^2 N'')^2 + \mu^2 a^4 c^4 w'^2}} \cdot g_n ,
\]

\[
q = + \frac{2ac^2 n}{2(2\mu^2 a^2 M'' - n^2 c^2 N'')} \cdot f_n \cos \rho \\
= + \frac{2ac^2 n}{\sqrt{16(2\mu^2 a^2 M'' - n^2 c^2 N'')^2 + \mu^2 a^4 c^4 w'^2}} \cdot f_n .
\]

Adding the values of \( a_n \) and \( b_n \), found in Section 13 for the case where \( f_n = 0 \) and \( g_n = 0 \), to these special values of \( a_n \) and \( b_n \) which satisfy the partial differential equations, then the two sums yield the complete integral values of \( a_n \) and \( b_n \), namely
\begin{align*}
a_n &= p \sin(\mu t + \rho) + Ae^{-\varepsilon t} \cdot \sin \left( mt + \arcsin \frac{\alpha_n^0}{A} \right), \\
b_n &= q \sin(\mu t + \rho) + Be^{-\varepsilon t} \cdot \sin \left( mt + \arcsin \frac{\beta_n^0}{B} \right),
\end{align*}

wherein \( A \) and \( B \) as well as \( \varepsilon \) and \( m \) have the meaning as given in Section 13. If \( m \) has an imaginary value, then the values of \( a_n \) and \( b_n \) developed in Section 17 replace the added terms. But it becomes clear that the added terms decrease for increasing values of \( t \) and that they, as shown in Section 17, may be considered as vanishing already after a very small portion of a second has passed, thus from then on the motion of electricity in the circular conductor becomes uniform and periodic, the production of which was the aim of the described method with the rotating magnet.

Omitting the terms vanishing with time and inserting these values of \( a_n \) and \( b_n \) into the equations

\begin{align*}
E &= \sum \left( a_n \sin \frac{ns}{a} + b_n \cos \frac{ns}{a} \right), \\
i &= -\frac{a}{2} \sum \frac{1}{n} \left( \frac{db_n}{dt} \sin \frac{ns}{a} - \frac{da_n}{dt} \cos \frac{ns}{a} \right),
\end{align*}

yields the following laws of the distribution of free electricity and of the currents in the circular conductor for the regularly ongoing electric oscillation:

\begin{align*}
E &= \sum \sin(\mu t + \rho) \left( p \sin \frac{ns}{a} + q \cos \frac{ns}{a} \right), \\
i &= -\frac{a\mu}{2} \sum \frac{1}{n} \cos(\mu t + \rho) \left( q \sin \frac{ns}{a} - p \cos \frac{ns}{a} \right),
\end{align*}

where \( p \), \( q \) and \( \rho \) have the above values. From these values, however, we get

\begin{align*}
p &= -\frac{2n}{\mu aw'} \sin \rho \cdot g_n, \\
q &= +\frac{2n}{\mu aw'} \sin \rho \cdot f_n.
\end{align*}

Substituting these values of \( p \) and \( q \) in both equations, we get

\begin{align*}
E &= \frac{2}{\mu aw'} \sum n \sin \rho \sin(\mu t + \rho) \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right), \\
i &= -\frac{1}{w'} \sum \sin \rho \cos(\mu t + \rho) \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right),
\end{align*}
or, expanding \( \sin(\mu t + \rho) \) and \( \cos(\mu t + \rho) \)\(^{52}\)

\[
E = \frac{2}{\mu a w'} \sin \mu \cdot \sum n \sin \rho \cos \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \\
+ \frac{2}{\mu a w'} \cos \mu t \cdot \sum n \sin^2 \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right),
\]

\[
i = \frac{1}{w'} \sin \mu t \cdot \sum \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)
- \frac{1}{w'} \cos \mu t \cdot \sum \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right).
\]

Finally, herein putting

\[
\frac{\sum \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)}{\sum \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)} = \tan \gamma,
\]

\[
\frac{\sum n \sin^2 \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right)}{\sum n \sin \rho \cos \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right)} = \tan \gamma',
\]

\[
\left( \sum \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) \right)^2
+ \left( \sum \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) \right)^2 = k^2,
\]

\[
\left( \sum n \sin^2 \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right)^2
+ \left( \sum n \sin \rho \cos \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right)^2 = k'^2,
\]

we get

\[
E = \frac{2}{\mu a w'} \cdot k' \sin(\mu t + \gamma'),
\]

\[
i = -\frac{1}{w'} \cdot k' \cos(\mu t + \gamma).
\]

But putting

\(^{52}\)Note by AKTA: In the next equations we replaced Weber’s notation \( \sin^2 \rho \) by \( \sin^2 \rho \).
\[
\sum \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) = f, \\
\sum \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) = g,
\]
\[
\frac{df}{ds} = f', \\
\frac{dg}{ds} = g',
\]
we get
\[
E = \frac{2}{\mu w} \sqrt{f'^2 + g'^2} \cdot \sin \left( \mu t + \arctan \frac{f'}{g'} \right),
\]
\[
i = -\frac{1}{w} \sqrt{f^2 + g^2} \cdot \cos \left( \mu t + \arctan \frac{f}{g} \right),
\]
whence it is easy to derive the equation
\[
\frac{di}{ds} = -\frac{1}{2} \frac{dE}{dt}.
\]

21 Equality of Phases and Amplitudes of Electric Oscillations in Circular Conductors

Considering that the electromotive force exerted on the whole conducting wire by the rotating magnet is represented by
\[
\sin \mu t \cdot \int ds \cdot \sum (f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a}),
\]
and that \( g_0 \) must have a specific finite value if this whole force should not be zero, then the value found for \( i \) can be presented more clearly if, in the given values of \( \tan \gamma \) and \( k^2 \), the first terms of the series, namely the terms corresponding to the place number \( n = 0 \), are separated in the following way, denoting the value of \( \rho \) for \( n = 0 \) by \( \rho_0 \):
\[
\tan \gamma = \frac{g_0 \sin^2 \rho_0 + \sum_1^\infty \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)}{g_0 \sin \rho_0 \cos \rho_0 + \sum_1^\infty \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)},
\]
\[ k^2 = g_0^2 \sin^2 \rho_0 + 2g_0 \sin \rho_0 \cos \rho_0 \cdot \sum_{n=1}^{\infty} \sin \rho \cos \rho \left( f_n \sin \frac{na}{a} + g_n \cos \frac{na}{a} \right) \\
+ 2g_0 \sin^2 \rho_0 \cdot \sum_{n=1}^{\infty} \sin^2 \rho \left( f_n \sin \frac{na}{a} + g_n \cos \frac{na}{a} \right) \\
+ \left( \sum_{n=1}^{\infty} \sin \rho \cos \rho \left( f_n \sin \frac{na}{a} + g_n \cos \frac{na}{a} \right) \right)^2 \\
+ \left( \sum_{n=1}^{\infty} \sin^2 \rho \left( f_n \sin \frac{na}{a} + g_n \cos \frac{na}{a} \right) \right)^2. \]

As now herein we had

\[
\sin \rho = \frac{\mu a^2 c^2 w'}{\sqrt{16(2\mu^2 a^2 M'' - n^2 c^2 N'')^2 + \mu^2 a^4 c^4 w'^2}},
\]
\[
\cos \rho = \frac{4(2\mu^2 a^2 M'' - n^2 c^2 N'')}{\sqrt{16(2\mu^2 a^2 M'' - n^2 c^2 N'')^2 + \mu^2 a^4 c^4 w'^2}},
\]

and, denoting the value of \( M'' \) for \( n = 0 \) by \( M''_0 \), we get the values of \( \sin \rho_0 \) and \( \cos \rho_0 \) [as given by]

\[
\sin \rho_0 = \frac{c^2 w'}{\sqrt{64\mu^2 M''_0^2 + c^4 w'^2}},
\]
\[
\cos \rho_0 = \frac{8\mu M''_0}{\sqrt{64\mu^2 M''_0^2 + c^4 w'^2}}.
\]

Considering furthermore that the ratios \( \mu a^2 w'/N'' \) and \( \mu a/c \) are very small fractions also for very long and thin conductors and for the greatest accessible velocity of rotation of the small magnet, then it is clear that we may put with sufficient approximation for all values \( n > 0 \)

\[
\sin \rho = \frac{\mu a^2 w'}{4n^2 N''},
\]
\[
\cos \rho = 1.
\]

Hence it is clear that already \( \mu a^2 w'/N'' \) being a very small fraction, \( \sin \rho = \mu a^2 w'/(4n^2 N'') \) all the more may be considered as vanishingly small the larger
the place number \( n \). Therefore also for very long and thin conductors and for very rapid rotation of the small magnet we may approximately assume

\[
\gamma = \rho_0 \quad \text{and} \quad k = g_0 \sin \rho_0 ,
\]

whence we find

\[
i = -\frac{g_0}{w'} \sin \rho_0 \cos (\mu t + \rho_0) .
\]

As \( g_0/w' \) and \( \rho_0 \) have values independent of \( s \), it follows that the electric oscillations have equal phase and oscillation amplitude in all parts of a circular conductor, even if the electromotive forces exerted by the rotating magnet are very unevenly distributed along the different parts of the conductor.

From the evenness of oscillation phases and amplitudes in all parts of the circular conductor, it follows that the current intensity at any point always equals the average current intensity in the whole conductor. But we had derived the law for the averages of the current intensities in closed conductors already in Section 9 where, denoting the average of the external electromotive force by \([1/2\pi a] \cdot S\) and putting

\[
\frac{8}{e^2} \int M''_0 ds + \frac{4\pi a}{r \mathcal{E}} = p ,
\]

\[
\frac{2\pi a}{\pi \alpha^2 k} = w = 2\pi aw' ,
\]

we had the result

\[
i = \frac{1}{p} e^{-\omega t/p} \cdot \int e^{\omega t/p} \cdot S dt .
\]

Now applying this law to our case where the oscillations in a conductor are produced by a rotating magnet and where the average of the electromotive forces exerted by the rotating magnet on the conductor was equal to

\[
\frac{1}{2\pi a} \cdot S = g_0 \sin \mu t ,
\]

we get

\[
i = \frac{2\pi ag_0}{p} e^{-\omega t/p} \cdot \int e^{\omega t/p} \cdot \sin \mu t \cdot dt = \frac{2\pi ag_0}{p} \cdot \frac{\omega}{\mu^2 + \mu^2} ,
\]

\[
= \frac{2\pi ag_0}{\mu p} \cdot \sqrt{\left(\frac{\omega}{\mu p}\right)^2 + 1} \cdot \cos \left( \mu t + \arctan \frac{w}{\mu p} \right) .
\]
As now
\[ p = \frac{8}{c^2} \cdot \int M''_0 ds + \frac{4\pi a}{r c} = \frac{8}{c^2} \cdot \int M''_0 (1 + \lambda) ds , \]
and
\[ w = 2\pi a w' , \]
one gets, putting \( M''_0 \) instead of \( M''_0 (1 + \lambda) \) for simplification as in Section 20,
\[ \frac{w}{\mu p} = \frac{\pi a c^2 w'}{4\mu \int M''_0 ds} = \tan \rho_0 , \]
\[ \frac{w}{\mu p \sqrt{\left(\frac{w}{\mu p}\right)^2 + 1}} = \frac{2\pi a w'}{\mu p \sqrt{\left(\frac{w}{\mu p}\right)^2 + 1}} = \sin \rho_0 , \]

hence agreeing with the above result found for \textit{circular} conductors
\[ i = -\frac{g_0 \sin \rho_0}{w'} \cdot \cos(\mu t + \rho_0) . \]

As the above law for the averages of the current intensities in closed conductors depending on the averages of the electromotive forces in Section 9 was found not to be restricted to \textit{circular} conductors only, but also to be independent of the consideration of the shape of the closed conductor, this yields that the law, derived for the case where the electromotive forces originating from a rotating magnet are given, equally holds for closed conductors of any shape.

The presented result that the \textit{phases} and \textit{amplitudes} of electric oscillations in circular conductors be equal everywhere, is based on the assumption that the ratios \( \mu a^2 w'/N'' \) and \( \mu a/c \) are very small fractions. As now these fractions increase with the length and the fineness of the conductor and with the rotational velocity of the magnet, it is interesting to calculate their actual values for some examples of long and thin wires at great rotational velocities. Choosing the three conducting wires already exemplified in Section 16, we get the values presented in the following Table.
The two bottom rows of this Table contain the values of the two ratios for the three exemplified wires if \( \mu = 100 \), that is at 15.965 turns of the magnet per second. We see that in all these cases the values of these ratios are very small fractions, while we also see that, as these values may be 10 times larger at 159.65 turns per second and 100 times larger at 1596.5 turns per second, indeed cases may occur where these ratios become considerably large quantities and where hence the law of equality of the \textit{phases} and \textit{amplitudes} in the conductor would not hold any more.

### Distribution of the Free Electricity in a Circular Conductor During the Electric Oscillation

The law of the distribution of the free electricity in a circular conductor during the electric oscillation is contained in the expression for the [linear] density, \( E \), found in Section 20, namely

\[
E = \frac{2}{\mu aw'} \cdot k' \sin(\mu t + \gamma'),
\]

where the coefficient \( k' \) was determined by the equation

\[
k'^2 = \left( \sum n \sin^2 \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right)^2 + \left( \sum n \sin \rho \cos \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right)^2.
\]

Hence we see that also the amount of the charge with free electricity in each point of the circular conductor changes in proportion to the sine of an
are increasing in proportion to \( t \), but that the maximum charge \( = 2k'/[\mu aw'] \)
which takes place when the sine \( = 1 \), is different in different points of the conductor
and that in fact the change from element to element is greatest approximately in those points where the electromotive force exerted by the
rotating magnet deviates most from its average; where this electromotive
force equals its average, also the charge is approximately constant, in fact it
equals zero. Hence in the whole conductor there would be no free electricity
anywhere if the rotating magnet acted equally on all points of it, where
it is assumed that the circular conductor would have no charge from free electricity independent of the rotating magnet.

Since \( \sin \rho \) and \( \cos \rho \) retain finite values for \( n = 0 \) according to the pre-
ceding Section, it is clear that the above value for \( k'^2 \) can be written as

\[
k'^2 = \left( \sum_{1}^{\infty} \sin^2 \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right) ^2
+ \left( \sum_{1}^{\infty} n \sin \rho \cos \rho \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right) \right) ^2.
\]

Furthermore, putting

\[
\sin \rho = \frac{\mu a^2 w'}{4n^2 N''},
\]

\[
\cos \rho = 1,
\]

under the assumptions made in the previous Section and when the value of
\( \sin \rho \) is very small, the first part of \( k'^2 \), containing the factor \( \sin^2 \rho \) in the
sum, may be neglected compared to the second term, whence we thus get

\[
k' = \frac{\mu a^2 w'}{4} \sum_{1}^{\infty} \frac{1}{nN''} \left( f_n \cos \frac{ns}{a} - g_n \sin \frac{ns}{a} \right).
\]

This now yields

\[
\frac{dk'}{ds} = -\frac{\mu aw'}{4} \sum_{1}^{\infty} \frac{1}{N''} \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right).
\]

If finally \( \log[8a/\alpha] \) is a very large number and if furthermore the series

\[
\sum \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right)
\]
converges so fast that all terms of the series may be neglected for \( n > \nu \), while

\[
2 \left[ 1 + \frac{1}{3} + \ldots + \frac{1}{2\nu - 1} \right] + \frac{\nu^2 \alpha}{8a}
\]

vanishes in comparison with \( \log[8a/\alpha] \), then we may put \( N'' = 2\log[8a/\alpha] \) and

\[
\frac{dk'}{ds} = -\frac{\mu aw'}{8 \log \frac{8a}{\alpha}} \left( \sum_{0}^{\nu} \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) - g_0 \right).
\]

Now the factor

\[
\sum_{0}^{\nu} \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right) - g_0
\]

equals the difference between the electromotive force exerted at the point \( s \) by the rotating magnet and its average along the whole length of the conductor; hence \( \frac{dk'}{ds} \) or the change of \( k' \) with respect to the change of \( s \) is proportional to this difference.

As is easily seen, the discharge of sparks\(^{53}\) and the necessary degree of insulation of the conductor depend on the amount of these charges, if the flash over is to be avoided, a topic to be treated in more detail only when the conductors in question are not just circular but constitute a system of closely spaced windings, a case that has been excluded here.

### 23 Guide to the Observations

It remains to use the results of the previous development as a guide to the observations by means of which those results shall be checked with experience. Such a guide is particularly necessary if there are no analogies of other motion phenomena which may be used for this purpose, and from what was previously said it follows that here such analogies are missing in many respects.

In the absence of analogies with other already known and investigated motion phenomena, above all the question is to know the determination of objects of observation, which are particularly important and suited for more detailed determination by observations. Furthermore the closer knowledge of the conditions is required under which the most exact determinations can

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\(^{53}\)Note by AKTA:] \( \text{Das Ueberspringen elektrischer Funken} \) in the original.
be obtained via these objects of observation. Now it is clear that the more detailed discussion of these conditions is best combined with the discussion of the tools for its effective representation and with the very execution of the observations, which will be the subject of the following Section of this treatise. Therefore, at the end of this Section the objects of observation shall be only briefly indicated which, according to the preceding development, seem to be particularly important and suited for a more detailed determination by observations.

The velocity of propagation which is so important for other motion phenomena, seemingly has not to be included here, as already mentioned in Section 18, but instead other different topics are available for the observation.

According to the developed laws, there are essentially three topics which prove to be particularly suited as topics of observation to test the formulated laws, namely first the comparison of the phases and the amplitudes of the oscillations of the electricity at various places of a long closed conductor on which a rotating magnet acts inductively; second the law of the dependence of the oscillation amplitude on the rotation velocity of the magnet; finally third another important topic suggests the observations of the dependence of the oscillation amplitude produced by a rotating magnet in a closed conducting wire on the shape of this wire.

The equality of the phase and of the amplitude of the oscillation which, according to the formulated laws, should occur in all parts even in a very long closed circuit and at a high velocity of rotation, is a topic all the more suited for an experimental test the more unexpected this result would seem. For without a more detailed definition of the conditions it would be expected in a very long circuit, where all motions start in one place and are subject to a very strong damping or absorption while propagating, that all motions would arrive only very faint at the most distant parts of the circuit even if the oscillations were continuously excited. As further the propagation from the place of excitation occurs in both directions, one would expect that the encounter of interchanging positive and negative oscillations from opposite sides would result in amplification at some places, and in cancellation at other places, as with interference phenomena. Finally, even if oscillations, due to such an encounter which are perfectly synchronous in all parts of the circuit, were possible, one would indeed expect that this possible case were tied to special conditions, for instance certain velocities of rotation, but not that such synchronous oscillations would arise in all parts of the circuit at any velocity of rotation. Hence the presented result is highly unexpected as regards all analogies offered by the propagation of motion in other known cases and therefore is especially suited for an experimental test of the results
of the theory built on our hitherto existing knowledge about electricity.

Further, the dependence of the oscillation amplitude on the velocity of rotation of the magnet is suitable from a different perspective, namely, the quantitative test of the formulated law by observations and measurements which are arranged in a sequence of increasing velocity of rotation.

Finally, if one succeeded in addition to gain more detailed determinations of the dependence of the oscillation amplitude on the shape of the circuit by exact observations and measurements, one would not only obtain a new test of the formulated laws, but also an essential supplement of our knowledge of the very electricity from which these laws were derived. From our hitherto existing knowledge, the electricity as a body must be attributed with a mass, and this mass exerts a force on another similar mass; still missing is the knowledge of the ratio of that mass to this force. Now the knowledge of this ratio was also not required as long as we dealt with equilibrium phenomena or with steady motions, where it was sufficient to know the forces; here the different amounts of electricity could be distinguished according to the strength of the forces they exert on the same amount of electricity at the unit distance, instead of their masses, and the latter amount of electricity could be determined by means of the force it exerts on an identical amount of electricity at the unit distance. Indeed such a specific quantity of electricity thus determined was the so called electrostatic unit of measure. If we do not deal with a mere equilibrium or a mere maintenance of an already existent motion but, instead, if an amount of electricity is to receive a new motion not existing before, then the pure knowledge of the forces is not sufficient but also the knowledge of the mass of the electricity to be set in motion is required, or of the ratio of this mass and of the force it exerts on the electrostatic unit of measure at the unit distance, that is the knowledge of the number of electrostatic units of measure, contained in the unit of mass (milligram) of electricity. Above,\(^{54}\) this number has been denoted by \(r\) and hence the mass of each specific amount of electricity, \(\mathcal{E}\), determined in electrostatic measure, is thus found equal to \([1/r] \cdot \mathcal{E}\). Now if any force, \(f\), acts on this mass, it is clear that the ratio of this force through the mass \([1/r] \cdot \mathcal{E}\) on which it acts, yields the velocity of the motion imparted by the force to this mass per unit time, \(= fr/\mathcal{E}\).\(^{55}\)

Now our knowledge of the existing amounts of electricity in electrostatic units of measure is indeed limited to the free amounts of electricity contained in the bodies obtained by the observations and does not cover the amounts.

\(^{54}\)[Note by AKTA:] See page 35 on Section 5.

\(^{55}\)[Note by AKTA:] That is, \(fr/\mathcal{E}\) represents the acceleration acquired by this mass \([1/r] \cdot \mathcal{E}\) through the action of the force \(f\).
of electricity contained in the neutral fluid. Likewise, our knowledge of the forces, \( f \), is limited to those acting on free amounts of electricity, while the observations yield only the knowledge of the coefficient, \( f' \), denoted by the name electromotive force, which has to be multiplied by the unknown number of electrostatic units of measure contained in the neutral fluid, \( \mathcal{E} \), in order to obtain \( f = f' \cdot \mathcal{E} \). On the other hand, in the whole electrodynamics we do not have to investigate the velocity itself, but only the current density and its changes, that is, the product of the number of electrostatic units of measure, \( \mathcal{E} \), contained in the flowing electricity, and that velocity \( rf/\mathcal{E} \), that is, \( rf = f' \cdot r \mathcal{E} \), where the electromotive force, \( f' \), is already known, hence where only the product \( r \mathcal{E} \) remains to be determined.

If hence, in agreement with the previous development of the determination of the current densities and of their changes, we do not need the very knowledge of the number of electrostatic units of measure, \( r \), contained in the unit of mass (milligram), but only the knowledge of the product \( r \mathcal{E} \), it is clear that, on the other hand, from the observation of the current densities and their changes, also only the knowledge of the product \( r \mathcal{E} \) can be obtained; but the importance of the knowledge of \( r \mathcal{E} \) is immediately clear, the tentative to know it by means of detailed and exact observations on the dependence of the oscillation amplitude on the shape of the circuit, according to the guide based on the derived laws, turns out to be the most appropriate.

To this end, the most exact knowledge of the conditions under which definite statements about this dependence can be obtained is necessary, the discussion of which, as already mentioned, shall be combined with the discussion of the tools for the practical presentation in the following Sections of this treatise.

### Part II

#### Observations of Oscillations

##### 24 Method of Observation

According to the guide for the observations presented in the previous Section mainly the differences of the phases and amplitudes of electric oscillations shall be observed and measured in closed conductors. But no galvanometers are suited for these observations and measurements, as for the observation and measurement of the intensities of steady currents. For if an electric oscillation is present in the multiplier of a galvanometer, instead of a steady
current, the needle of the galvanometer cannot stay at rest and in equilibrium, but must also perform oscillations that become the weaker the smaller the fraction of the period of electric oscillation compared with the period of oscillation of the magnetometer needle; if these oscillations were becoming vanishingly weak, the galvanometer needle would exactly behave as if there were no electric oscillation in the multiplier, it would stay fixed in the equilibrium position without any deflection, so that nothing could be determined. Therefore, the observation of electric oscillations and in particular the measurement of its differences of amplitudes and phases demand that the closed conductor in which the oscillations take place constitutes not only a multiplier as part of a galvanometer, but also a solenoid supported as a torsion balance which, together with the multiplier, makes an electrodynamometer, whose construction has already been described in the first treatise on Electrodynamic Measurements (Abhandlung bei Begründung der Königl. Sächs. Gesellschaft der Wissenschaften, Leipzig 1846)\textsuperscript{56,57} and the use of which for the observation of electric oscillations has been discussed in general and analyzed with an example at this very place.

Now the method how to determine the differences of phases and amplitudes of electric oscillations in closed conductors from observations by means of a dynamometer shall be considered more closely here, where it may be assumed for simplification that the multiplier formed by the conductor carrying the electric oscillations be formed by a vertical ring having a quite large diameter, similar to that of a tangent galvanometer,\textsuperscript{58} with the solenoid formed by the same conductor concentrated in a volume as small as possible supported in its center, replacing the pivoted needle of the galvanometer there. There is a fundamental difference between this needle and the solenoid formed by the conductor that carries the oscillation, [namely,] that the needle has a constant magnetic moment on which the steady current in the multiplier acts, while the solenoid has a galvanic moment which, according to Ampère’s law, indeed would be completely equivalent to the magnetic moment of the needle, which, however, would not be constant for an electric oscillation in the solenoid, but varies with the phase of the electric oscillation. Furthermore, it is not a steady current from the multiplier that acts on this variable galvanic moment of the solenoid, but the electric oscillation existing in the multiplier, whose action on the solenoid is also variable with the phase of the oscillation.

Let \( a \) and \( n \) be the average radius and the number of windings, respec-

\textsuperscript{56}[Note by HW:] Wilhelm Weber’s Werke, Vol. III, pp. 35 and 123.
\textsuperscript{57}[Note by AKTA:] [Web46, Sections 1 and 16, pp. 35 and 123 of Weber’s Werke] with partial French translation in [Web87] and a complete English translation in [Web07].
\textsuperscript{58}[Note by AKTA:] Tangenten-Boussole in the original. The tangent galvanometer was invented by J. J. Nervander (1805-1848), [Ner33] and [Sih21].
tively, of the multiplier and likewise $a$ and $n'$ for the solenoid, further $i$ and $i'$ the current intensities in multiplier and solenoid, respectively, expressed in *absolute magnetic measure* as used in galvanometry, whence $[c/\sqrt{8}] \cdot i dt$ and $[c/\sqrt{8}] \cdot i'dt$ represent the amount of positive electricity passing through the cross section of the conductors during the time element $dt$; then $n\pi a^2 i$ and $n'\pi a'^2 i'$ represent the galvanic moments of the multiplier and the solenoid. Twice the product of these two galvanic moments divided by the third power of the distance $a$ between the solenoid concentrated at the center and the ring of the multiplier yields the *directive force*\footnote{Note by AKTA: Direktionskraft in the original. This expression can also be translated as “force of direction” or “directional force”. The concept of “Direktionskraft” (directive force) was introduced by Gauss in 1838, [Gau38, p. 4] with English translation in [Gau41, p. 254].} exerted on the solenoid by the multiplier which, multiplied by the sine of the angle formed between the solenoid axis and the multiplier axis, or, equivalently, multiplied by the cosine of the deflection angle $\varphi$ between the solenoid axis and the plane of the multiplier ring, yields the *torque*\footnote{Note by AKTA: Drehungsmoment in the original. This expression can also be translated as “moment of force”, “moment” or “rotational force”.} exerted by the multiplier on the solenoid, namely

$$= 2 \frac{n\pi a^2 i \cdot n'\pi a'^2 i'}{a^3} \cdot \cos \varphi$$

With this complete analogy between the theory of the *electrodynamometer* and that of the *galvanometer*, no further explication is needed, but instead, considering how to use the instrument, we can immediately pass over to the case when the conducting wire which includes multiplier and solenoid carries electric oscillations, where thus the current intensities $i$ and $i'$ vary with the sine of an angle, which increase in proportion with time $t$. If in this case $i$ and $i'$ represent the maximum current intensities corresponding to the maximum values of the sine, then the current intensities can be represented by $i \sin(\mu t + \gamma)$ and $i' \sin(\mu t + \gamma')$ for any moment at the end of the time $t$. If $E$ denotes the amount of positive electricity contained in the unit length of the conductor, the distance between an oscillating particle in the multiplier or solenoid from its equilibrium position during this oscillation for this moment will be represented by $[i/\mu E] \cos(\mu t + \gamma)$ and $[i'/\mu E] \cos(\mu t + \gamma')$, where $i/\mu E$ and $i'/\mu E$ is the amplitude of oscillation to be determined here. — As, however, with currents one renounces the knowledge of the current velocity itself, being satisfied with the product of this velocity and the unknown factor $E$, similarly one contents oneself here with the determination of the product of this oscillation amplitude and this
very factor \( \mathcal{E} \), because the observations will only permit us to express this product in absolute measure.

In this case, with these new specifications of the current intensities, one now obtains the torque exerted by the multiplier on the solenoid:

\[
\tau = 2 \frac{n \pi a^2 i \cdot n' \pi a'^2 i'}{a^3} \sin(\mu t + \gamma) \sin(\mu t' + \gamma') \cdot \cos \varphi ,
\]

wherein \( i \) and \( i' \) have constant values independent of the time \( t \).

Under the influence of this torque whose magnitude changes incessantly with time \( t \), the mobile solenoid clearly cannot get to rest at all; therefore the question arises what kind of observations can be performed with this incessant motion of the solenoid and what can be determined from these observations. In order to answer this question we have to develop the laws of motion of the solenoid under the influence of such a variable torque.

To simplify this development we may at first stick to the case where the current intensities in the multiplier and the solenoid are always the same, where hence we may put

\[
i = i', \quad \text{and} \quad \gamma = \gamma' = 0.
\]

This case yields the variable torque acting on the solenoid equal to

\[
\tau = 2 \frac{\pi^2 n n' a^2}{a} \cdot i^2 (\sin \mu t)^2 \cos \varphi ,
\]

which may be written as

\[
= \frac{\pi^2 n n' a'^2}{a} \cdot i^2 (1 - \cos 2\mu t) \cos \varphi .
\]

But from the construction of the electrodynamometer we know that the solenoid is supported in a bifilar fashion, whence with the given length and distance of the two supporting wires there is a static directive force which can easily be determined and shall be denoted by \( S \). If now this bifilar suspension of the solenoid is normally regulated, so that the torque resulting from the static directive force equals zero, when the axis of the solenoid is parallel to the plane of the ring of the multiplier, or when the angle of deflection \( \varphi = 0 \), then we get the static torque acting on the solenoid for any value of the angle \( \varphi \) equal to

\[
= -S \sin \varphi .
\]

Adding this static torque to the above electrodynamic torque, the sum of both torques acting on the solenoid divided by the moment of inertia \( K \) of
the solenoid, yields the rotational acceleration of the solenoid, \(d^2\varphi/dt^2\), at the end of the time \(t\), whence follows the equation of motion of the solenoid, namely

\[
\frac{\pi^2 \nu \nu' a^2}{a} \cdot i^2 (1 - \cos 2\mu t) \cos \varphi - S \sin \varphi = K \frac{d^2 \varphi}{dt^2}.
\]

Putting herein

\[
\varphi = v + \alpha,
\]

in which one assumes the constant value for \(v\) determined by the following equation

\[
\tan v = \frac{\pi^2 \nu \nu' a^2}{aS} \cdot i^2,
\]

hence \(d^2 \varphi/dt^2 = d^2 \alpha/dt^2\), so that one gets

\[
\frac{d^2 \alpha}{dt^2} + \frac{S}{K} \left[ (1 + (1 - \cos 2\mu t) \tan^2 v) \cos v \sin \alpha + \cos 2\mu t \cdot \sin v \cos \alpha \right] = 0.
\]

Under the assumption that \(v\) and \(\alpha\) have small values (which, as a rule, is possible because the solenoid, equipped with a mirror, is to be observed quite like a magnetic needle, where the deflection of the solenoid shall always stay within narrow limits given by the length of the scale), we can write

\[
\frac{d^2 \alpha}{dt^2} + \frac{S}{K} (\alpha \sec v + \cos 2\mu t \cdot \sin v) = 0,
\]

whence we get by integration:

\[
\alpha = \frac{\sin v}{4\mu^2 S - \sec v} \cdot \cos 2\mu t + A \sin(t - B) \sqrt{\frac{S \sec v}{K}},
\]

with \(A\) and \(B\) the two constants of integration. Now denoting by \(\tau\) the period of oscillation of the solenoid,\(^{61}\) which corresponds to the directive force \(S\) and the moment of inertia \(K\), and by \(\vartheta\) the period of oscillation of the electricity in the conducting wire, we get

\(^{61}\) [Note by AKTA:] *Die Schwingungsdauer des Solenoids* in the original.

Gauss and Weber utilized the old French definition of the period of oscillation \(\tau\) which is half of the English definition of the period of oscillation \(T\), that is, \(\tau = T/2\). [Gil71, pp. 154 and 180]. For instance, the period of oscillation for small oscillations of a simple pendulum of length \(\ell\) is \(T = 2\pi \sqrt{\ell/g}\), where \(g\) is the local free fall acceleration due to the gravity of the Earth, while \(\tau = T/2 = \pi \sqrt{\ell/g}\).
\( \frac{K}{S} = \frac{\pi^2}{\tau^2} \quad \text{and} \quad \mu = \frac{\pi}{\vartheta} ; \)

consequently

\[ \alpha = \frac{\sin v}{4 \tau^2 \sec v} \cdot \cos \frac{2\pi}{\vartheta} t + A \sin \frac{\pi}{\tau} (t - B) \sqrt{\sec v} , \]

or, for the assumed small value of \( v \) and putting \( A = 0 \), that means apart from that small oscillation which would be performed by the solenoid if solely under the influence of the static directive force \( S \) and the electrodynamic \([\text{directive force}] \) \( nn'\pi^2 a^2 t^2 / a \) (as this oscillation is easily suppressed during the observation by standard damping devices), it follows

\[ \alpha = \frac{\vartheta^2}{4 \tau^2 - \vartheta^2} \cdot v \cos \frac{2\pi}{\vartheta} t . \]

As an example we choose the case which will occur in the following observations where we had, expressed in seconds

\[ \tau = 15 , \quad \vartheta = \frac{1}{520} , \]

which hence yields

\[ \alpha = \frac{1}{243 \cdot 10^6} \cdot v \cos \frac{2\pi}{\vartheta} t , \]

that means where \( \alpha \) vanishes completely compared to \( v \). The same is valid for all observations to be treated here.

When \( \alpha \) vanishes, the constant deflection of the solenoid \( v \) can now be observed directly with utmost precision and one finds

\[ i = \frac{1}{\pi a'} \sqrt{aS \tan v} \]

whence the electric oscillation in the closed conductor is determined completely if the period of oscillation, \( \vartheta \), were known from counting the revolutions of the rotating magnet, namely

\[ i \sin \frac{\pi}{\vartheta} t = \sin \frac{\pi}{\vartheta} t \cdot \sqrt{aS \tan v} \]

If, instead of the electric oscillation, there were a constant current having intensity \( i \sqrt{\frac{1}{2}} \), then the torque exerted by the multiplier on the solenoid would be
\[ = \frac{\pi^2 mn'a'^2}{a} \cdot \cos \varphi , \]

and this torque, together with the static torque, \(-S \sin \varphi\), would amount to zero for equilibrium, whence the deflection \(\varphi\) of the solenoid for equilibrium would be given by

\[ \varphi = v . \]

Therefore the result from the above consideration can be expressed as follows:

*If the period of oscillation of the electricity in the closed conductor makes a very small fraction of the static period of oscillation of the solenoid, the solenoid behaves as if there were a constant current in the conductor whose intensity is to the maximum intensity, \(i\), of the currents due to the electric oscillation as \(1 : \sqrt{2}\).*

There is then a deflection of the solenoid which can be observed similarly as if there were a constant current in the closed conductor and if from this observed deflection (according to the same law as with galvanometers) the intensity of the constant current is calculated which could cause it, then this intensity has only to be multiplied by \(\sqrt{2}\) in order to obtain the maximum intensity, \(i\), of the currents occurring in the electric oscillation, or multiplied by \(c/[2\mu E]\) in order to obtain the amplitude of the electric oscillation in the closed conductor, whereby, however, as already remarked, \(E\) must be left indefinite as an unknown coefficient and where only \(ci/[2\mu]\) can be expressed in absolute units of measure. Hereby the problem is solved how to observe and to determine with an electrodynamometer the electric oscillation caused in a closed conductor by a magnet rotating at a known speed.

The solution of the problem, however, has here been limited to the case where the multiplier and solenoid belong to adjacent parts of the closed conductor where there is no noticeable difference of the oscillation amplitude and the oscillation phase of the electricity. If the multiplier and solenoid belonged to two parts of the closed conductor where the period of oscillation of the electricity were indeed the same, but where the current maxima \(i\) and \(i'\), as well as the phases of the oscillation \(\lambda\) and \(\lambda'\), would have to be distinguished; then the starting point of the time \(t\) may always be chosen so that the arithmetic mean \((\lambda + \lambda')/2\) of both phases of the oscillation is equal to zero. Then the current intensities related to the electric oscillation can be represented in these two parts of the closed conductor by

\[ i \sin \frac{\pi}{\vartheta} (t + \lambda) \quad \text{and} \quad i' \sin \frac{\pi}{\vartheta} (t - \lambda) . \]
One then observes first the deflection of the solenoid, \( v \), when both multiplier and solenoid belong to the first part of the closed conductor. According to the established rules, the maximum intensity, \( i \), of the currents of the electric oscillation in this part can be determined from this observation, namely

\[
  i = \frac{1}{\pi a'} \sqrt{\frac{aS \tan v}{nn'}}.
\]

Second, one observes the deflection of the solenoid, \( v' \), when both multiplier and solenoid belong to the other part of the closed conductor and finds the maximum intensity, \( i' \), of the currents of the existing electric oscillation in this part

\[
  i' = \frac{1}{\pi a'} \sqrt{\frac{aS \tan v'}{nn'}}.
\]

Third finally, one observes the deflection of the solenoid, \( v'' \), when the multiplier belongs to the first part and the solenoid to the latter part of the closed conductor. Then from the third observation also the difference of the phases of the oscillation, \( 2\lambda \), in both parts of the closed conductor can be determined. According to the previous details we then have the equation of motion of the solenoid, namely:

\[
2\pi^2 nn' a'^2 \cdot ii' \sin(\mu t + \lambda) \sin(\mu t - \lambda) \cdot \cos \varphi - S \sin \varphi = \frac{K}{dt^2} d^2 \varphi,
\]

for which, as

\[
\sin(\mu t + \lambda) \sin(\mu t - \lambda) = \sin^2 \mu t - \sin^2 \lambda = \frac{1}{2} (1 - \cos 2\mu t - 2 \sin^2 \lambda)
\]

holds, we can write:

\[
\frac{\pi^2 nn' a'^2}{a} \cdot ii'(1 - \cos 2\mu t - 2 \sin^2 \lambda) \cos \varphi - S \sin \varphi = \frac{K}{dt^2} d^2 \varphi.
\]

Let

\[
\varphi = u + \alpha,
\]

where we take

\[
\tan u = \frac{\pi^2 nn' a'^2}{aS} \cdot ii'.
\]
As after that we have $d^2 \varphi / dt^2 = d^2 \alpha / dt^2$, we obtain

$$\frac{d^2 \alpha}{dt^2} + \frac{S}{K} \left[(1 + (1 - \cos 2\mu t - 2 \sin^2 \lambda) \tan^2 u) \cos u \sin \alpha + (\cos 2\mu t + 2 \sin^2 \lambda) \sin u \cos \alpha\right] = 0.$$ 

Under the assumption that $u$ and $\alpha$ have small values we get

$$\frac{d^2 \alpha}{dt^2} + \frac{S}{K} \left[(1 - 2 \sin^2 \lambda \sin^2 u) \frac{\alpha}{\cos u} + (\cos 2\mu t + 2 \sin^2 \lambda) \sin u\right] = 0,$$

or, when

$$\beta = (1 - 2 \sin^2 \lambda \sin^2 u) \alpha,$$

and

$$S' = (1 - 2 \sin^2 \lambda \sin^2 u) S,$$

[we get]

$$\frac{d^2 \beta}{dt^2} + \frac{S'}{K} \left[\sec u \cdot \beta + (\cos 2\mu t + 2 \sin^2 \lambda) \sin u\right] = 0.$$ 

By integration we get from this

$$\beta = \frac{\sin u}{4\mu^2 K - \sec u} \cdot \cos 2\mu t - \sin 2u \sin^2 \lambda + A \sin(t - B) \sqrt{\frac{S' \sec u}{K}},$$

$$\alpha = \frac{S \sin u}{4\mu^2 K - S' \sec u} \cdot \cos 2\mu t - \frac{\sin 2u \cdot \sin^2 \lambda}{1 - 2 \sin^2 u \sin^2 \lambda} + A' \sin(t - B) \sqrt{\frac{S' \sec u}{K}}.$$

If now, with fast oscillations of the electricity and after the solenoid has settled to rest, the first and second parts of $\alpha$ vanish, one obtains the constant value of the deflection $\varphi$, denominated by $v''$, namely

$$v'' = u - \frac{\sin 2u \cdot \sin^2 \lambda}{1 - 2 \sin^2 u \sin^2 \lambda},$$

whence we get
\[
\sin^2 \lambda = \frac{u - v''}{2 \sin u (\cos u + [u - v'' \sin u])}.
\]

As \(u\) is already known from the values of \(i\) and \(i'\) determined by the previous observations by means of the equation
\[
\tan u = \frac{\pi^2 n n' a' \cdot ii'}{a},
\]
we solved the problem to determine the phase difference \(2\lambda\) of the electric oscillations at two different places of the closed circuit from the observed deflection \(v''\).

### 25 The Commutators

To meet the aim of an exact comparison of the amplitudes and phases of oscillations at two places of a closed conductor, if they were only slightly different, the execution of the described observations after the method described in the preceding Section would demand a very detailed fineness and exactness, hardly attainable if they had to be performed separately and independently. But the achievement of this aim can be extremely simplified if these observations can be combined pairwise and performed simultaneously with the same closed conductor and the same rotation of the magnet. For this aim, two as equally as possible constructed electrodynamometers are required with their multipliers and solenoids making part of the same closed circuit. If a system of exactly corresponding observations is to be performed by means of two such electrodynamometers belonging to the same circuit, the most essential condition to be fulfilled is that the period of oscillation of the solenoids of the two electrodynamometers, suspended in a bifilar manner, be identical, which is very easily realized if the construction of the electrodynamometer provides the possibility to control at will the distance of the suspending wires of one solenoid or of both, whereby the period of oscillation of one solenoid can be exactly matched with that of the other solenoid. If the two solenoids are at complete rest before an observation series is started, then a more expanded observation series may be performed in such a way that all of the observed elongations of both solenoids set in motion by electric oscillations in the circuit are pairwise valid for equal moments.

The complete correspondence of both electrodynamometers in other respects is considerably much less taken into consideration as with the period of oscillation. For it is easily seen that if both electrodynamometers in the closed circuit are closely spaced in series, so that both belong to the same
part of the circuit where there are no noticeable differences of the oscillation amplitudes and phases, then a very exact comparison of both instruments is possible by means of simultaneous observations with both instruments in correspondence which can be continued over a longer duration at equal period of oscillation of the solenoids, whence all observations performed with one of the instruments can be reduced exactly so as to yield the same results that would be obtained with the other instrument if this instrument were identical.

Under this assumption both electrodynamometers, exactly adjusted with each other, can be applied at two different very distant places of one and the same conductor and then, by simultaneously observing both instruments with one electric oscillation in the conductor, a much finer comparison of the oscillation amplitudes at both places of the circuit can be gained than would be possible if one and the same electrodynamometer would be applied and observed at both places and different times, whereby it would have to be assumed that the rotation of the magnet were identical at both times, an assumption that never can be fulfilled in reality and which can be completely economized with these synchronized corresponding observations.

Furthermore, after mutually comparing them exactly, both electrodynamometers can also serve to place the solenoid of one electrodynamometer at another distant part of the closed conductor, while the multiplier of the same instrument stays at its former place, and then to perform simultaneous corresponding observations by means of this electrodynamometer and the other one, fixed at its place, by means of which any ever so minute phase difference of the electric oscillation is recognized at the two very distant places of the closed conductor without the necessity to assume an identical rotation of the magnet at different times.

Finally, it is now of great importance for the exactness and the reliability of the results derived from these observations that the different series of observation, namely first those for the comparison of the instruments and second those for the comparison of the oscillation amplitudes and phases, be performed alternately in direct succession and repeated while the magnet is continuously rotated in an utmost uniform fashion, where it is required to be able to replace either the whole electrodynamometer or one of its parts, for example the solenoid, momentarily between two observations, which is easily realized by means of suitably designed commutators.

These commutators, as they will be applied in the following experiments, consist of a number of twin cells, that is, cells pairwise connected by a conductor and connected to the ends of the various parts of the conducting wire. These twin cells then can be connected again pairwise with each other, namely, combined together in two different ways, by distinguishing the front
cell from the rear cell. One of these methods of pairwise combination of the twin cells can indeed be obtained by means of a fixed system of connective wires which are simultaneously immersed in all front cells; the other method of pairwise combination of the twin cells can be obtained by another fixed system of connective wires which are simultaneously immersed in all rear cells. And these two different systems of connective wires can behave like the two arms of a lever, so that immersing one of the systems causes the other to emerge and vice versa. It is immediately clear that by means of such a commutator with 6 twin cells one part of the conducting wire may be disconnected from its connection with two other parts of the conducting wire, thus connecting the two latter parts among themselves, and finally the hitherto connected two parts may be disconnected and the previously disconnected part may be inserted between them. All this is performed by means of a simultaneously operated interchange, namely by turning a lever whereby the system of connecting wires is immersed into the front cells, while the other system emerges from the rear cells, or vice versa. — Moreover, one needs commutators with 4 twin cells, if not during the observations, but beforehand. In fact, before the observations the solenoids of the two electro-dynamometers are to be damped, wherefore first a current is needed which passes through the solenoid and the dynamometer to be damped, second a commutator with 4 twin cells, two of which are connected to the ends of the multiplier wire and the other two of which to the ends of the solenoid wire. By means of this commutator the multiplier can be connected at will now in parallel, now crosswise. With one kind of connection the current through the multiplier exerts a positive torque on the solenoid carrying the same current, with the other kind a negative torque, and the solenoid is damped if the former kind of connection is established during the backward oscillation, the latter kind during the forward oscillation. As the effect of a current simultaneously present in multiplier and solenoid is completely independent of the direction of the current, the alternating current induced in the circuit by the rotating magnet can be used, instead of a steady current, whereby it is possible to let the damping of the dynamometer, after the rotation of the magnet has started, immediately precede the observations.

If now all these operations, namely, the damping of the dynamometer and then all observations required at different places of the circuit for the comparison of the oscillation amplitudes and phases, while the magnet is continuously rotated, are executed successively without interruption, this requires five commutators in total which have to be connected with the various parts of the circuit in a systematic way to be explained in more detail.

For an easier overview first the 5 commutators, second the various parts of the circuit to be connected with the commutators are now to be denoted
exactly and distinguished. Then Figure 1 will serve to give the overview of
the set up as a whole and of all connections in detail.

The first commutator, denoted by \( A \), is required either in order to put one
of the electrodynamometers (multiplier and solenoid in one) alternately into
two different places of the conducting wire, or in order to put the solenoid
of this electrodynamometer into two different places of the conducting wire
while the respective multiplier stays in its place. This requires a commutator
with 6 twin cells, two of which are required for both ends of the electrodyna-

The second commutator, denoted by \( B \), is required as auxiliary commuta-
tor, the setting of which determines whether the multiplier including solenoid
or the solenoid of one of the electrodynamometers alone is put into two dif-
ferent places of the conducting wire by alternative operation of commutator
\( A \), which equally requires a commutator with 6 twin cells.

The third and fourth commutator, namely \( C \) and \( C' \), respectively, are
used to dampen the solenoids before the observations start. This requires
commutators with 4 twin cells, two of which for the two ends of the leads to
the solenoid and two for the leads to the respective multiplier including the
remaining conducting wire from which, however, the part belonging to the
other electrodynamometer has to be excluded, so that the solenoid at rest is not disturbed while the other one is damped.

Finally, the fifth commutator, denoted by $D$, is required in order to establish the connection between the dynamometers with the commutators $C$ and $C'$, respectively, now with one, now with the other solenoid. To this purpose, a commutator with 4 twin cells is needed, two of which for the ends of the conducting wire at the switch-on location of the commutator, the other two for two connecting wires through which the current can be led to the conducting wire by-passing one or the other electrodynamometer.

The following parts of the closed conducting wire have to be distinguished which can be connected by means of the commutators in various ways.

The first wire, denoted by $a$, is the multiplier wire of the first dynamometer whose ends lead to two twin cells of the commutator $C$, besides two short connecting wires of the two other twin cells of this commutator [leading] to two twin cells of commutator $B$. These various parts of wire $a$, which always stay connected in the same way during the observations, shall be distinguished by symbols $a^I, a^{II}, a^{III}$.

The second wire, denoted by $b$, is the solenoid wire of the first dynamometer, whose ends are connected to a twin cell of commutator $B$ and to a twin cell of commutator $A$.

The third and fourth wires, denoted by $c$ and $d$, are two short connecting wires of two twin cells of commutator $B$ [leading] to two twin cells of commutator $A$, whose resistance may be considered as vanishingly small.

The fifth wire, denoted by $e$, is one of the two very long pieces of wire which are needed during the observations in order to either remove both dynamometers from each other or to remove the solenoid of the first dynamometer from its multiplier by connecting either the two wire endings of one dynamometer with those of the other one, or the two wire endings of the multiplier and the wire endings of the solenoids by a long piece of wire. Both ends of the long piece of wire, $e$, are connected to two twin cells of commutator $A$.

The sixth wire, denoted by $f$, is the whole rest of the conducting circuit and comprises the inductor ring of the rotating magnet, further the second long piece of wire just mentioned before, then the wire of the second dynamometer, of the solenoid as well as the multiplier, and finally a connecting wire leading back to the first wire. These various parts of the wire $f$, which stay connected in the same way during the observations, are to be distinguished by the symbols $f^I, f^{II}, f^{III}, f^{IV}, f^V$. The commutator $C'$ is plugged in between the solenoid wire $f^{III}$ and the multiplier wire $f^{IV}$ of the second dynamometer, the latter, however, will not be used during the observations. Likewise, there is a plug in the connecting wire $f^V$ for the
commutator $D$ which, however, stays closed because also this commutator is not needed during the observations.

Hence now Figure 1 has been sketched for a better illustration where the various twin cells of the commutators $A$, $B$, $C$, $C'$, $D$ are represented by the symbol

and one of the two ways of connection is sketched by the upper dashed arcs and the other way is sketched by the lower dashed arcs.

The commutators $C$ and $C'$, keeping the top setting, and $D$, being excluded from the circuit by means of a wire connecting its first and last cell after the solenoid is damped, are not used to displace the first dynamometer, consisting of the multiplier $a^I$ and the solenoid $b$, from the first switching place to the other one during the observations, but the displacement is performed simply by a change of the setting of the commutator $A$ after the connections, sketched by dashed arcs on top, are established by means of commutator $B$. Because the setting of commutator $A$, sketched by means of dashed top arcs, completes a closed circuit with the shown setting of commutator $B$, where the wires denoted follow in the following sequence:

$$abefdca;$$

the setting of commutator $A$, sketched by means of the bottom dashed arcs, completes a closed circuit with the following sequence of wires:

$$abfdeca.$$

Breaking up $f$ into its parts $f^I$, $f^{II}$, $f^{III}$, $f^{IV}$, $f^V$, and representing the whole circuit by means of the four sides of a rectangle with its long sides symbolizing the long connecting wires, denoted by $e$ and $f^{II}$, then Figure 2 sketches the former case and Figure 3 the latter case.
In addition, the place of the inductor, $f^I$, with the rotating magnet has been marked with +, the two places of the multipliers, $a$ and $f^{IV}$, with larger circles, the two places of the solenoids $b$ and $f^{III}$ by smaller circles. The inductor, $f^I$, with the rotating magnet is always at the top side of the rectangle, the dynamometer, $f^{III} f^{IV}$, is always at the opposite bottom side. The place of the other dynamometer, $ab$, is alternated and, in the former case, is situated at the bottom side besides the first dynamometer, $f^{III} f^{IV}$, opposite to the inductor, $f^I$, in the latter case at the top side besides the inductor, $f^I$, opposite to the dynamometer, $f^{III} f^{IV}$. Hence by operating the commutator $A$, the dynamometer $ab$ is switched now at a place of the circuit very far from the inductor, $f^I$, now at a place very close to it as was demanded for the first series of observations.

Likewise, the commutators $C$, $C'$ and $D$ are not used to displace the solenoid $b$ of the first dynamometer from one switching place to the other one, but the displacement is performed by a simple change of the setting of commutator $A$ after the connections, symbolized by the lower dashed arcs, have been established by means of commutator $B$. Because the setting of commutator $A$, sketched by means of dashed top arcs, completes a closed circuit where the wires denoted follow in the following sequence:

\[ adebefa; \]

the setting of commutator $A$, sketched by means of the bottom dashed arcs, completes a closed circuit with the following sequence of wires:

\[ adecbfa. \]

The former case is represented by Figure 4, the latter case by Figure 5.
We see that the places of the inductor $f^I$ of the dynamometer, $f^{III} f^{IV}$, and also of the multiplier, $a$, always stay unchanged and that simply the place of the solenoid $b$ alternates, which in the former case, is situated at the bottom side of the rectangle besides the multiplier, $a$, opposite to the inductor, $f^I$, and in the latter case is situated at the top side of the rectangle besides the inductor, $f^I$, opposite to the multiplier, $a$. Hence by operating the commutator $A$, the solenoid $b$ is switched now at a place of the circuit very far from the multiplier, $a$, now at a place very close to it as was demanded for the second series of observations.

In order to dampen the solenoids before starting the observations, the piece of the wire, $f^V$, connecting the first and fourth twin cell of the commutator, $D$, is taken out. Then, in order to dampen the solenoid of the first dynamometer, $b$, the commutator, $D$, is given the setting symbolized by the upper dashed arc, whereby the wire, $f^I$, together with the second dynamometer, $f^{III} f^{IV}$, are connected and whereby, depending on the setting of commutator $C$, a circuit is completed with the following sequence of wires:

\[
\begin{align*}
\text{with top setting of } C: & \ a^I a^{III} b e f^I f^V a^{III} a^I, \\
\text{with bottom setting of } C: & \ a^I a^{III} c d f^V f^I e b a^{III} a^I,
\end{align*}
\]

where the top settings have been assumed for the commutators $A$ and $B$. Hence we see that, with the given direction of the current through the multiplier $a^I$, the direction of the current through the solenoid $b$ with the top setting of $C$ is given by $a^{III} b e$, while it is given by $e b a^{III}$ with the bottom setting, being hence opposite to the former, whereby in both cases opposite equal torques are exerted on the solenoid $b$, one of which can always be used to dampen the motion of the solenoid.

In order to dampen the solenoid of the second dynamometer, $f^{III}$, the
commutator $D$ is given the bottom setting symbolized by a dashed arc whereby the wires $eba^Ia^Ia^III$, including the first dynamometer, are connected and whereby, depending on the setting of commutator $C^I$, a circuit with the following sequence of wires is completed:

- with top setting of $C^I$: $f^IVf^Vf^If^IIIf^I$,  
- with bottom setting of $C^I$: $f^IVf^If^IIIf^I f^IV f^IV$.

We see that, given the current direction through multiplier $f^IV$, the current direction through the solenoid $f^III$ with the top setting of $C^I$ is given by $f^If^III f^IV$, while by the bottom setting it is given by $f^IV f^If^III f^IV$, hence opposite, whereby in both cases opposite torques are exerted on the solenoid $f^III$, one of which can always be used to dampen the motion of the solenoid.

After damping both solenoids, the commutator $D$ is opened and the removed piece of wire $f^V$ is again inserted to connect the first and last cell.

### 26 The Long Conducting Wires

For the displacement of the solenoid of an electrodynamometer or for the displacement of the whole electrodynamometer (solenoid and multiplier) from one switching place of the closed circuit to the other one, it is a matter of great importance for the observations that the two conducting wires connecting the two switching places be of almost equal and great length. Therefore two parts of the closed circuit, namely the wires $e$ and $f^II$, have been explicitly mentioned for serving this purpose in the previous Section. In the circuit used in the following experiments each of these two wires had a length of 36 600 meters or almost 5 miles.

Considering the great length of the whole circuit containing these two long wires, it is immediately clear that it is practically impossible to give them the exact shape of a circle as has been assumed for simplification in the previous Section when developing the laws. But even apart from this great length to be associated with the closed conductor, the simple shape of a circle could not be applied in a circuit that must contain an inductor ring for the rotating magnet and two dynamometers for the purpose of the observations, because pieces of the conducting wire have to be employed whose shape and position are determined by the rules valid for the construction of these instruments.

Obviously this in practice unavoidable deviation of the shape of the closed conductor from a circle has an influence on the electric oscillations caused by the rotating magnet in the conductor, and thereby the law of the dependence of the amplitude of the electric oscillations on the rotation velocity of the magnet is essentially changed. If, however, it is not a question of observations
by which the amplitude is exactly determined and measured, but only of those
to compare the amplitudes at two different places of the conductor (or to
determine only the phase difference at both places), then the deviation from
the circular shape is of minor importance. Because if, according to the laws
developed in the previous Section, there were really no noticeable difference
of oscillation amplitude and phase at two very distant places of a circular
conductor even at high rotation velocities, there would be no reason to assume
that such a difference would be brought forth simply by a deviation of the
conductor from the circular shape; still more so, if the observations show that
in a closed conductor with an arbitrary shape entirely different from a circle
that there are no noticeable differences of oscillation amplitude and phase,
it might be allowed conversely to consider this result as generally valid, also
for differences of oscillation amplitudes and phases being unnoticeable for a
perfect circular shape.

It is of special importance for these two 5 mile wires making part of the
closed conductor that, if no telegraph wires are used but if these long wires
shall instead be housed in the closed laboratory where the observations are
made as is necessary to completely control all essential external conditions of
the observations, these long wires have to be wound on spools to save space.
Now it is clear, however, that in the case of electric currents now abruptly
arising and now dying off again as happens due to the electric oscillations
caused by the fast rotation of the magnet, all windings of the conducting
wire on the spool must mutually exert electromotive forces according to the
laws of Volta-induction\[^{62}\] which sum up to a strong damping force, thereby
essentially reducing the amplitude of the electric oscillations so that the latter
could not any more be observed even with the most sensitive dynamometers
at faster rotations of the magnet. In order to perform the observations it is
therefore utterly important to find a method to wind the long wires on spools
so that such a mutual induction between the wire windings is avoided.

To achieve this aim, provided the wires are well braid,\[^{63}\] the simplest and
most perfect way is to combine the two halves of each piece to a twin wire
before winding it on one coil. This combination is best obtained by braiding
the two halves, each of which is already braid and thus kept isolated from
each other by means of this double braiding, wound once again together with
cotton or silk. Then, connected at one of the ends, the two halves form
a conductor through which a current entering by the other open end and
passing through one of the wire halves is led back to the open end passing

\[^{62}\]See footnote 10 on page 9.

\[^{63}\]Wenn die Drähte gut umsponnen sind in the original. A braided
wire here means silk or cotton woven around the conductor for insulating purposes.
through the other half wire on almost the same path. Then the end where the two wire halves are connected is fixed at the spool intended for the braiding and subsequently the whole twin wire is braided on this spool, so that the end which leaves the two wire halves non-isolated are freely exposed on top and the whole twin wire can be connected to the remainder of the circuit by means of these two non-isolated ends of the two wire halves.

In this way all current elements pertaining to such a twin wire are ordered pairwise, so that only oppositely equal current elements are close neighbors. It is clear that, even at the fastest changes of intensity, such pairs of current elements cannot exert an electromotive force on any other more distant conductor element and that hence this twin wire, in whatever way it may be wound by braiding it on the spool, is not subject to any damping force as a consequence of these windings which otherwise the electric oscillations would have experienced by the rotating magnet, as would have been the case had the wire simply been wound in a unidirectional fashion alongside its whole length.

Without the above method fast electric oscillations in such a long conducting circuit would become vanishingly weak, though not in consequence of the great resistance of the circuit, but in consequence of the mutual induction between all the windings, making their observation practically impossible. Here it is barely necessary to remark that the same method may find fruitful applications also in other cases where similar conditions are met under which this very method will perform in a similar way.

This is valid, in particular, for long distance electric telegraphs, where for the purpose of [sending] telegraph signals, electric currents in a very long circuit arise and vanish very rapidly. We have indeed already mentioned that the damping forces, exerted hereby by the electricity in the various wire elements (and suffered especially under the influence of an enclosing conductor), cause great impediments due to delays in signaling which threaten to thwart the further expansion especially of undersea telegraph lines. For example from Europe to America, quite independent of the technical difficulties in connection with the laying and maintenance. Hereby, not only the forces mutually exerted between the electricities in the various wire elements come into play, but also those forces exerted by the electricity on the neighboring conductors and sustained by them, and even the forces exerted by the magnetism of the earth and its variability on the electricity of the various wire elements. All impediments arising from this for fast signal processing and great extension of the circuit can be totally or almost totally avoided by applying the above method, always placing two wire elements closely together in which the electric current and charge are practically equal but opposite. Hence it is very clear that, as a rule for further extension of the telegraph
line, a cable must be designed so as to have a wire with current passing in one direction very close to a second wire leading the current back, where hence the return of the current via the earth must be given up. That the insulation between closely spaced wires does not cause difficulties seems to be shown by the example of our circuit where the two wires packed closely together by means of a common braiding are insulated from each other only by covering each individual wire with silk before joining them. The thickness of the insulator coating here was less than 1/10 of a millimeter and yet the insulation was to be considered perfect for currents so strong that the scale range barely sufficed for the dynamometer deviations caused by them, as will be demonstrated by the corresponding observations to be described below.

27 Observations to Compare the Amplitude of Electric Oscillations at Two Different Places of a Long Closed Circuit

According to the arrangement discussed in the previous Sections now four series of observations have been performed, all on one day, September 28, 1860, alternately to compare the amplitude and to determine the phase difference at two distant places of the above long closed circuit while electric oscillations were excited therein by a fast rotating magnet. However, while not performed in immediate succession, the two series of observations to compare the amplitude shall both be considered together in the present Section, likewise both series of observations to determine the phase difference in the following Section.

The corresponding observations at both electrodynamometers were made by Mr. Schering and myself, while Mr. Klinkerfues and Mr. H. Weber performed the uniform rotation of the magnet in the inductor coil and determined its velocity. This velocity was kept as close as possible to 260 turns per second whereby only slight deviations occurred which were noticeable as small variations of the deviations of the solenoids of both dynamometers.

The duration of the periods of the solenoids was regulated in such a way that it was equal in size and lasted almost exactly 15 seconds. In doing so, the sensitivity of both instruments, however, was very different as a consequence of using indeed equal spools for both solenoids, but having different wire thickness and hence different numbers of windings. The more sensitive dynamometer, the one with the solenoid having a greater number of windings,

64[Note by AKTA:] Ernst Christian Julius Schering (1824-1897), Ernst Friedrich Wilhelm Klinkerfues (1827-1884) and Heinrich Weber (1839-1928).
was used for those observations by means of which the oscillation amplitude
was to be determined alternately at two different places of the conducting
circuit, while the less sensitive dynamometer served for the corresponding
observations, in order to account for the influence of small variations of the
rotation velocity, to which purpose the same had to stay fixed at a certain
place of the conducting wire.

First Series.

According to the setup prescribed in Section 24, the first series of ob-
servations was performed in order to compare the intensity or oscillation
amplitude of the electric oscillations at two different places of the long closed
circuit.

All observations are expressed in parts of a millimeter scale, the image
of which, 2 100 divisions away from it and attached to the solenoid, plane
mirror on the magnetometer, was observed in the usual way with a telescope.
In order to exploit the whole expansion of the scale, the telescopes including
their scales were placed before the mirrors in such a way that the solenoid
position at rest with the magnet at rest or with the open circuit did not
correspond, as usual, to the center of the scale above the telescope, but to
a point near the beginning of the scale, because the solenoid was always
deflected towards one and the same side from its position at rest.

During the whole series of observations the magnet was kept in continuous
uniform rotation. Between the various sets of observation distinguished by
number the commutator A described in Section 25 was operated, namely the
first time, when it had previously been open, it was closed, and afterward the
top and bottom settings were simply exchanged. During this the commutator
B was kept closed in the top setting, likewise the two commutators C and C'
with 4 cells each which had been used for the damping of the solenoids before
the beginning of the observations, while the commutator D with 4 cells was
opened and completely taken out of the circuit by re-inserting the piece of
wire connecting the first and fourth cell which had been removed while the
two solenoids were damped.

Before starting the observations the solenoids of both dynamometers, as
explained in Section 25, were damped as much as possible. — As the setting
of the three commutators B, C, C' was kept fixed during the whole series
of observations, it sufficed to remark in the headlines of the various sets
whether the circuit was open or closed and whether, in the latter case, the
top or bottom setting of commutator A was employed according to the scheme
in Section 25. — With the closed circuit, where the solenoids moved more
vividly, the second averages have been chosen from the subsequently observed
elongations to determine the state of rest.

Comparing the corresponding deflections of both simultaneously observed dynamometers which we obtain by subtracting the state of rest observed with open circuit (set number 1) from that observed with closed circuit and hereby restricting ourselves at first to those cases, sets numbers 2, 4, 6, where both dynamometers have been positioned symmetrically and close together and separated from either side from the inductor of the rotating magnet by means
of the long conducting wires, where hence the oscillation amplitude and phase should always be equal in both dynamometers; we get the ratio of their sensitivities from the ratio of the observed deflections of both dynamometers. Hence we obtain the sensitivity of the first dynamometer expressed in parts of that of the second dynamometer:

from sets number 1 and number 2: \[ \frac{844.94}{623.86} = 1.3544 \, , \]
from sets number 1 and number 4: \[ \frac{848.59}{626.67} = 1.3541 \, , \]
from sets number 1 and number 6: \[ \frac{844.05}{623.84} = 1.3530 \, , \]

hence the average ratio of the sensitivity of dynamometer number 1 and of dynamometer number 2 behaves as

\[ 1.3538 : 1 \, . \]

After this mutual comparison of the sensitivities of both dynamometers the number of readings of one dynamometer alternatively switched onto two different places of the circuit may be reduced by making use of the observed deflections of the other dynamometer, always kept fixed at the same place of the circuit, as if the deflections were observed simultaneously at the two places of the circuit by means of identical dynamometers. Namely from the corresponding deflections of the auxiliary dynamometer it is now always possible to calculate the deflections of the main dynamometer as would have been observed if the main dynamometer had stayed in its original place, for which the comparison of its sensitivity with that of the other dynamometer is valid, and this comparison calculated for the first position of the main dynamometer in the circuit may then be compared with the deflection really observed for the second position of the main dynamometer.

That is, if the obtained ratio of 1.3538 is multiplied by the deflections of the auxiliary dynamometer observed in the 3rd, 5th, and 7th set, after subtracting the state of rest found in set number 1,

\[ 619.55, \quad 626.89, \quad 625.58 \, , \]

one gets the deflections which would have been observed at the main dynamometer if the latter had kept its place in the circuit like it has been in the 2nd, 4th, and 6th set.

The following Table contains the values of the calculated deflections in the second column; the third column contains the actually observed deflections at position II of the main dynamometer for which these calculated deflections of the main dynamometer at position I were valid; the fourth column finally lists the differences between both.
<table>
<thead>
<tr>
<th>Set number</th>
<th>Calculated deflection for position I</th>
<th>Observed deflection for position II</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>838.75</td>
<td>839.41</td>
<td>+0.66</td>
</tr>
<tr>
<td>5</td>
<td>848.69</td>
<td>848.88</td>
<td>+0.19</td>
</tr>
<tr>
<td>7</td>
<td>846.92</td>
<td>846.85</td>
<td>−0.07</td>
</tr>
<tr>
<td>Average</td>
<td>844.78</td>
<td>845.05</td>
<td>+0.26</td>
</tr>
</tbody>
</table>

These deflections, observed in scale units, divided by the distance, 2100 scale units, of the mirror from the scale, now yield the tangents of twice the angles designated by $v$ and $v'$ in Section 24. Hence we have for position I:

$$\tan v = \tan \frac{1}{2} \arctan \frac{844.78}{2100} = 0.193601,$$

and for position II:

$$\tan v' = \tan \frac{1}{2} \arctan \frac{845.05}{2100} = 0.193656.$$

But now, according to Section 24, the ratio of the squares of the intensities $i$ and $i'$, or the squares of the amplitudes of the electric oscillations at the two positions I and II in comparison where the main dynamometer has been placed by means of the top and bottom settings of commutator $A$, behaves as

$$i^2 : i'^2 = \tan v : \tan v';$$

hence one gets

$$i' = 1.000142 \cdot i.$$

Position I in the circuit, however, is almost 5 miles away from the inductor which houses the rotating magnet, while position II is very close to the inductor. It seems that this indeed means that the amplitude of the electric oscillations, produced by the rotating magnet in the whole circuit are somewhat weaker at the great distance from the inductor where the excitation started, namely at the position denoted by I, than very close to the inductor, at position II; the difference found, however, is exceedingly small, so that it cannot be safely established even by means of the most exact observations, it is in fact less than $1/7000$ of the total oscillation amplitude corresponding to the full deflection of the dynamometer. In fact these observations hence show that at two positions of the circuit at a mutual distance of almost 5 miles practically no difference of the amplitudes of the electric oscillations can be detected even by the most exact observations.
Concerning the *exactness of the observations* it is indeed clear that its closer determination cannot yet be gained from just so few repetitions as in this first series of observations; yet, as no deviation from the average surpasses 0.40 scale units, one may consider this average obtained from all 3 observations as reliable, which corresponds to one part in 845 of the total oscillation amplitude. — Such an accuracy of the intensity measurements of electric oscillations surpasses the precision that could be obtained hitherto by means of intensity measurements of almost any other kind of oscillation. In *acoustics* and *optics* the intensity of sound and light depends on the oscillation amplitude and it is known how far the intensity measurements of sound and light stay behind this precision. Only the observations of the oscillation amplitude of a magnetic needle or generally by means of a torsion balance in unifilar or bifilar suspension after Gauss’ method allow equal or, under favorable conditions, a still somewhat higher precision.\(^{65}\) — It is worthwhile to remark that the same precision by means of the same inductor and the same dynamometers, which served for the production and observation of electric oscillations, 520 of which took place every second, would have been obtained equally easily in a circuit almost 10 miles long even if the frequency of the electric oscillations were increased to more than 1000 per second and the length of the circuit to more than 30 miles without having to reinforce the wire of the prolonged circuit; for the electric oscillation and its effect were deliberately weakened during the above experiments; namely *first* by excluding one half of the inductor on which the rotating magnet acted; *second* by increasing the static directive force of the solenoids of both dynamometers; otherwise the length of the scale would not have sufficed to perform the observations. The observed effects would have been equally strong in a much longer circuit making use of the whole inductor and decreasing the static directive force of the solenoids, whereby the duration of their oscillation would have been increased from 15 to 20 seconds.

In order to eliminate any doubt that this precision be just an apparent one and that the coincidence of the observations repeated only 3 times in the above series of observations be just accidental, finally a *second series* of observations were performed by means of the identical set up and on this very day, the results of which are compiled in the same way for comparison with the preceding series in the following Table.

Second Series.

The remarks preceding the first series are equally valid for the second

---

\(^{65}\) [Note by AKTA:] See [Web38] with English translation in [Web41a] and [Web66]; and [Web94a].
The ratios of the sensitivity of the first dynamometer, expressed in parts of [the sensitivity] of the second [dynamometer], from the observations listed in the second column yield the following values:

from sets number 1 and number 2: $\frac{846.53}{625.42} = 1.3535$ ,
from sets number 1 and number 4: $\frac{849.27}{626.79} = 1.3549$ ,
from sets number 1 and number 6: $\frac{841.42}{621.92} = 1.3529$ ,

<table>
<thead>
<tr>
<th>Top Setting of Commutator B.</th>
<th>Set number 1, open circuit</th>
<th>Set number 2, closed circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dynamoeter no. 1</td>
<td>Dynamoeter no. 2</td>
</tr>
<tr>
<td>Observed elongations</td>
<td>State of rest</td>
<td>Observed elongations</td>
</tr>
<tr>
<td>30.3</td>
<td>32.35</td>
<td>$-8.5$</td>
</tr>
<tr>
<td>34.4</td>
<td>32.35</td>
<td>+5.2</td>
</tr>
<tr>
<td>30.3</td>
<td>32.20</td>
<td>$-8.3$</td>
</tr>
<tr>
<td>34.1</td>
<td>32.20</td>
<td>+5.0</td>
</tr>
<tr>
<td>30.3</td>
<td>32.25</td>
<td>$-8.0$</td>
</tr>
<tr>
<td>34.2</td>
<td>32.45</td>
<td>+4.9</td>
</tr>
<tr>
<td>Average 32.27</td>
<td>Average $-1.58$</td>
<td>Average 878.80</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set number 3, closed circuit</th>
<th>Set number 4, closed circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom setting of commutator A</td>
<td>Top setting of commutator A</td>
</tr>
<tr>
<td>837.0</td>
<td>596.0</td>
</tr>
<tr>
<td>918.8</td>
<td>880.60</td>
</tr>
<tr>
<td>847.8</td>
<td>885.58</td>
</tr>
<tr>
<td>927.9</td>
<td>881.70</td>
</tr>
<tr>
<td>823.2</td>
<td>880.00</td>
</tr>
<tr>
<td>945.7</td>
<td>669.4</td>
</tr>
<tr>
<td>Average 881.97</td>
<td>Average 625.87</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set number 5, closed circuit</th>
<th>Set number 6, closed circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom setting of commutator A</td>
<td>Top setting of commutator A</td>
</tr>
<tr>
<td>794.0</td>
<td>566.4</td>
</tr>
<tr>
<td>962.4</td>
<td>875.90</td>
</tr>
<tr>
<td>784.8</td>
<td>876.73</td>
</tr>
<tr>
<td>974.9</td>
<td>879.92</td>
</tr>
<tr>
<td>785.1</td>
<td>874.35</td>
</tr>
<tr>
<td>952.3</td>
<td>672.1</td>
</tr>
<tr>
<td>Average 876.73</td>
<td>Average 621.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Set number 7, closed circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom setting of commutator A</td>
</tr>
<tr>
<td>Dynamoeter no. 1</td>
</tr>
<tr>
<td>Oberved elongations</td>
</tr>
<tr>
<td>783.5</td>
</tr>
<tr>
<td>969.6</td>
</tr>
<tr>
<td>790.3</td>
</tr>
<tr>
<td>976.7</td>
</tr>
<tr>
<td>785.4</td>
</tr>
<tr>
<td>981.0</td>
</tr>
<tr>
<td>Average 881.09</td>
</tr>
</tbody>
</table>
hence the average ratio of the sensitivity of the first and the second dynamometer behaves as

\[ 1.3538 : 1 \, . \]

Now multiplying this ratio 1.3538 with the observed deflections of the second dynamometer which was always fixed at its position in the circuit during all observations, namely the deflections resulting from the difference between the positions at rest with open circuit in set number 1 and with closed circuit in sets numbers 3, 5 and 7, yield:

\[ 627.45, \quad 623.33, \quad 626.79 \, , \]

so the products

\[ 849.45, \quad 843.86, \quad 848.55 \, , \]

yield the values of the deflections which would have been observed at the first dynamometer if the latter had kept its position as in sets numbers 2, 4 and 6, while the deflections of the positions at rest in sets numbers 3, 5 and 7 correspond to the changed position of the dynamometer. The following Table lists the deflections from the initial position together with the corresponding deflections of the first dynamometer in the changed position.

<table>
<thead>
<tr>
<th>Set number</th>
<th>Calculated deflection for position I</th>
<th>Observed deflection for position II</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>849.45</td>
<td>849.70</td>
<td>+0.25</td>
</tr>
<tr>
<td>5</td>
<td>843.86</td>
<td>844.46</td>
<td>+0.60</td>
</tr>
<tr>
<td>7</td>
<td>848.55</td>
<td>848.82</td>
<td>+0.27</td>
</tr>
<tr>
<td>Average</td>
<td>847.29</td>
<td>847.66</td>
<td>+0.37</td>
</tr>
</tbody>
</table>

Herewith, like with the previous series of observations, we get the comparison of the oscillation amplitude or of the current intensity at position I and II, [respectively]. For position I:

\[ \tan v = \tan \left( \frac{1}{2} \arctan \frac{847.29}{2100} \right) = 0.194134 \, , \]

and for position II:

\[ \tan v' = \tan \left( \frac{1}{2} \arctan \frac{847.66}{2100} \right) = 0.194212 \, , \]

hence, as according to Section 24 [one has]
\[
\frac{i^2}{i'^2} = \tan v : \tan v',
\]
it follows that
\[
i' = 1.000201 \cdot i.
\]
Thus the difference between the oscillation amplitude at both positions, one of which was almost 5 miles away from the inductor of the rotating magnet while the other one was close to the inductor, makes barely 1/5000 of the total oscillation amplitude corresponding to the total deflection of the dynamometer. It is clear that also this difference is too small in order to be safely declared even for the most exact observations and therefore also this second series of observations will confirm that there is practically no difference of the amplitudes of the electric oscillations that can be safely established even for two positions almost 5 miles apart.

28 Observations to Determine the Difference of the Phase of Electric Oscillations at Two Different Places of a Long Closed Circuit

On the basis of an equal setup as described for the two foregoing series of observations, a third series of observations was performed, however, not for the comparison of the oscillation amplitudes, but for the determination of the phase difference of the electric oscillations at two different places of a long circuit. To this aim, like with the previous series of observations, the commutator \( A \) described in Section 25, was closed, if previously open, between the various sets of observation distinguished by number, or it was opened if previously closed, that means, the top and bottom setting were exchanged. Commutator \( B \), on the other hand, was kept closed, but in the bottom setting (instead of in the top setting as before). Finally, the two commutators \( C \) and \( C' \), equipped with 4 cells each, used to dampen the solenoids before starting the observations, were again kept closed and set exactly as before during the observations. After damping the solenoids, the commutator \( D \) was opened before starting the observations and completely excluded from the circuit by means of a wire connecting its first and last cell.

Third Series.
For the ratio of the sensitivity of the first dynamometer, expressed in parts of the second one, the observations of this third series yield the following values:

from sets number 1 and number 2: $\frac{848.45}{630.14} = 1.3464$,
from sets number 1 and number 4: $\frac{840.08}{622.92} = 1.3486$,
from sets number 1 and number 6: $\frac{846.66}{629.03} = 1.3460$,

hence the average ratio of the respective first and second dynamometer sen-
Now multiplying this ratio $1.3740$ by the observed deflections of the second dynamometer which has kept its position in the circuit during all observations, namely the deflections resulting from the differences of the positions of rest with open circuit in set number 1 and with closed circuit in sets numbers 3, 5 and 7, one obtains:

$$625, 20, \quad 625.29, \quad 621.70,$$

so that one gets the products

$$842.15, \quad 842.26, \quad 837.44,$$

which yield the values of the deflections which would have been observed at the first dynamometer if the solenoid had kept its positions during the observations of sets numbers 2, 4 and 6, while the deflections taken from the positions of rest of sets numbers 3, 5 and 7 correspond to the changed position of the solenoid in the circuit. The following Table lists the deflections of the solenoid of the original position and the corresponding deflections of the changed position.

<table>
<thead>
<tr>
<th>Set number</th>
<th>Calculated deflection for position I of the solenoid</th>
<th>Observed deflection for position II of the solenoid</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>842.15</td>
<td>841.56</td>
<td>−0.59</td>
</tr>
<tr>
<td>5</td>
<td>842.26</td>
<td>842.41</td>
<td>+0.15</td>
</tr>
<tr>
<td>7</td>
<td>837.44</td>
<td>835.80</td>
<td>−1.64</td>
</tr>
<tr>
<td>Average</td>
<td>840.62</td>
<td>839.92</td>
<td>−0.70</td>
</tr>
</tbody>
</table>

Also this series of observations has been repeated again in order to test the exactness attributed to these observations and we let this *fourth series of observations* follow immediately.

Fourth Series.
### Bottom Setting of Commutator $B$

<table>
<thead>
<tr>
<th>Set number 1, open circuit</th>
<th>Set number 2, closed circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Top setting of commutator</strong> $A$</td>
<td><strong>Top setting of commutator</strong> $A$</td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td>44.8</td>
<td>36.50</td>
</tr>
<tr>
<td>28.2</td>
<td>36.35</td>
</tr>
<tr>
<td>44.5</td>
<td>36.20</td>
</tr>
<tr>
<td>27.9</td>
<td>36.25</td>
</tr>
<tr>
<td>44.6</td>
<td>36.35</td>
</tr>
<tr>
<td>28.1</td>
<td>36.50</td>
</tr>
</tbody>
</table>

**Average** 36.33 | **Average** 36.33 | **Average** 881.72 | **Average** 623.04

<table>
<thead>
<tr>
<th>Bottom setting of commutator $A$</th>
<th><strong>Set number 4, closed circuit</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Set number 3, closed circuit</strong></td>
<td><strong>Set number 4, closed circuit</strong></td>
</tr>
<tr>
<td><strong>Bottom setting of commutator $A$</strong></td>
<td><strong>Bottom setting of commutator $A$</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td>794.5</td>
<td>559.9</td>
</tr>
<tr>
<td>970.1</td>
<td>882.67</td>
</tr>
<tr>
<td>976.0</td>
<td>882.95</td>
</tr>
<tr>
<td>969.7</td>
<td>855.48</td>
</tr>
<tr>
<td>980.5</td>
<td>887.97</td>
</tr>
<tr>
<td>969.2</td>
<td>686.4</td>
</tr>
</tbody>
</table>

**Average** 884.77 | **Average** 625.36 | **Average** 883.67 | **Average** 623.98

<table>
<thead>
<tr>
<th>Set number 5, closed circuit</th>
<th><strong>Set number 6, closed circuit</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bottom setting of commutator $A$</strong></td>
<td><strong>Bottom setting of commutator $A$</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td>741.1</td>
<td>324.2</td>
</tr>
<tr>
<td>1023.3</td>
<td>883.92</td>
</tr>
<tr>
<td>745.0</td>
<td>883.75</td>
</tr>
<tr>
<td>1021.7</td>
<td>885.80</td>
</tr>
<tr>
<td>754.8</td>
<td>883.37</td>
</tr>
<tr>
<td>1002.2</td>
<td>879.68</td>
</tr>
<tr>
<td>795.5</td>
<td>353.8</td>
</tr>
</tbody>
</table>

**Average** 883.30 | **Average** 624.06 | **Average** 881.39 | **Average** 622.61

<table>
<thead>
<tr>
<th>Set number 7, closed circuit</th>
<th><strong>Set number 7, closed circuit</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bottom setting of commutator $A$</strong></td>
<td><strong>Bottom setting of commutator $A$</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Dynamometer no. 1</strong></td>
<td><strong>Dynamometer no. 2</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td><strong>Observed elongations</strong></td>
<td><strong>Observed elongations</strong></td>
</tr>
<tr>
<td><strong>State of rest</strong></td>
<td><strong>State of rest</strong></td>
</tr>
<tr>
<td>960.0</td>
<td></td>
</tr>
<tr>
<td>805.4</td>
<td>884.25</td>
</tr>
<tr>
<td>996.2</td>
<td>882.20</td>
</tr>
<tr>
<td>791.0</td>
<td>876.15</td>
</tr>
<tr>
<td>956.4</td>
<td>875.35</td>
</tr>
<tr>
<td>797.6</td>
<td></td>
</tr>
</tbody>
</table>

**Average** 879.49 | **Average** 621.84

From the observations of this fourth series we obtain the following values for the sensitivity of the first dynamometer in parts of that of the second dynamometer:

from sets number 1 and number 2: $\frac{845.39}{624.27} = 1.3542$ ,
from sets number 1 and number 4: $\frac{847.34}{625.21} = 1.3553$ ,
from sets number 1 and number 6: $\frac{845.06}{623.84} = 1.3546$ ,

hence the average ratio of the respective first and second dynamometer sen-
sitivities equals

$$1.3547 : 1.$$  

Now multiplying this ratio $1.3547$ by the observed deflections of the second dynamometer which has kept its position in the circuit during all observations, namely the deflections resulting from the differences of the positions of rest with open circuit in set number 1 and with closed circuit in sets numbers 3, 5 and 7, one obtains:

$$626.59, \quad 625.29, \quad 623.07,$$

so we get the products

$$848.84, \quad 847.10, \quad 844.10,$$

which yield the values of the deflections which would have been observed at the first dynamometer if the solenoid had kept its positions during the observations of sets numbers 2, 4 and 6, while the deflections taken from the positions of rest of sets numbers 3, 5 and 7 correspond to the changed position of the solenoid in the circuit. The following Table lists the deflections of the solenoid of the original position and the corresponding deflections of the changed position.

<table>
<thead>
<tr>
<th>Set number</th>
<th>Calculated deflection for position I of the solenoid</th>
<th>Observed deflection for position II of the solenoid</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>848.84</td>
<td>848.44</td>
<td>−0.40</td>
</tr>
<tr>
<td>5</td>
<td>847.10</td>
<td>846.97</td>
<td>−0.13</td>
</tr>
<tr>
<td>7</td>
<td>844.10</td>
<td>843.16</td>
<td>−0.94</td>
</tr>
<tr>
<td>Average</td>
<td>846.68</td>
<td>846.19</td>
<td>−0.49</td>
</tr>
</tbody>
</table>

Comparing these averages found in the *fourth* series of observations with those obtained from the *third* series of observations, we find the ratio of the deflections at both positions in such an agreement, that it obviously suffices for all further considerations to take into account the averages of both series, namely for the deflections of the solenoid at position I:

$$843.65,$$

and at position II:

$$843.055.$$
Now this yields the following values for the determination of the difference of the oscillation phases at position I and II according to Section 24:

\[ u = \frac{1}{2} \arctan \frac{843.65}{2100} = 10^\circ56'37.0'' , \]

\[ v'' = \frac{1}{2} \arctan \frac{843.055}{2100} = 10^\circ56'11.7'' . \]

The latter task would need a minor correction if, according to the result of the previous Section, one were to take into account not the oscillation amplitudes at positions I and II considered as equal, but instead the little difference, although it cannot be guaranteed in any way. In that case, we would have to put \( v'' = 10^\circ56'13.2'' . \) Here however we will keep to the first task because there is no reason to assume such an inequality of the oscillation amplitude because in practice it cannot be safely demonstrated at all.

This finally yields the determination of the phase difference \( 2\lambda \) at both positions I and II according to Section 24

\[ \sin^2 \lambda = \frac{u - v''}{2 \sin u (\cos u + |u - v''| \sin u)} = 0.000329 , \]

hence

\[ 2\lambda = 2^\circ4'43'' , \]

which corresponds to about \( 1/87 \) of the period of oscillation.

Also this determination of the phase difference is based on such a small difference of the observed deflections, being equal to only \( 3/5 \) of a scale unit, that it can be just as little considered as safely established by experience, as the small difference of the oscillation amplitude in the previous Section.

29 Result of the Test

The observations described in Sections 27 and 28 serve to test the laws developed on the previous Section in what concerns the behavior of the oscillation amplitudes and oscillation phases at various positions of a closed conductor and thereby the identity of amplitudes and phases has been established even for very fast oscillations in a very long closed conducting wire. The method used for these observations of simultaneous corresponding observations by means of two dynamometers with exactly corresponding periods of oscillation allowed thereby a very high exactness, and this test could be carried out much further if the means for a still faster rotation of the magnet and for the
production of still longer circuits were available. Such a further expansion of
the test, however, would not suffice, as it seems, to justify the effort and the
expenditure to be invested, and their expansion up to 520 oscillations per sec-
ond and up to a length of 10 miles of the circuit would already suffice. Even
if amplitudes and phases of electric oscillations in closed conductors were not
generally equal everywhere, it is indeed clear that yet their differences should
have to become the smaller, the longer the period of oscillation depending on
the velocity of rotation, and the shorter the conducting wire would be, so that
finally these differences should have to become unnoticeable with increasing
period of oscillation and decreasing length of the conducting wire. Hence
the intended test can only arrive at its aim if it were considerably extended
beyond the limits within which such a compensation would have to occur
in any case, and the question therefore arises whether a length of conductor
of 10 miles and a period of oscillation of 1/520 of a second would serve this
purpose.

Let a simple wave train start from a point and consider the same during
the first cycle. Let the wave period be 1/520 of a second and the propagation
velocity be equal to the usual \( c \sqrt{\frac{1}{2}} = 41950 \) miles [per second]. In this case,
if it were realizable, a decrease of the amplitude of the electric oscillation
would take place as the distance increases from the starting position of the
wave train together with an increase of the phase difference with the same
distance for any moment, both of which are easy to determine.

In fact, for such a simple wave train the displacement of an electric par-
ticle, \( \sigma \), can be represented by the following equation:

\[
\sigma = Ae^{-ct} \cdot \sin 520\pi \left( t - \frac{s}{c} \sqrt{2} \right),
\]

where, according to Section 15, we may approximately put

\[
\varepsilon = \frac{c^2}{16\pi \alpha^2 k M''},
\]

and

\[
M'' = 2 \log \frac{8a}{\alpha}.
\]

As now further, according to Section 16,

\[
\frac{1}{\pi \alpha^2 k} = w' = \frac{16 \cdot 10^6}{\pi \alpha^2 \cdot c^2},
\]

hence
\[ \varepsilon = \frac{10^6}{2\pi \alpha^2 \log \frac{8a}{\alpha}}, \]
and with \(2\pi a = 76 \cdot 10^6\) millimeter and \(\alpha = 1/8\) millimeter for our circuit, we obtain

\[ \varepsilon = 477000. \]

The distance of \(s = 30 \cdot 10^6\) millimeter (about 5 miles) now corresponds to a duration of

\[ t = \frac{s}{c} \cdot \sqrt{2} = \frac{1}{8177} \text{ second}, \]

hence the ratio of the amplitudes at the starting point and at a distance of 5 miles (equals):

\[ 1 : e^{-\varepsilon t} = 1 : e^{-54.7} = 573 \cdot 10^{21} : 1, \]

whence the amplitude has become so small at a distance of 5 miles that it completely vanishes compared to that of the wave train at its starting point.

At a given moment the phase difference at the starting point of the wave train and at a distance \(= s\) from there is represented by

\[ 520 \frac{\pi s}{c} \cdot \sqrt{2}, \]

thus equals \(0.0636 \cdot \pi = 10° 27'\) for \(s = 38 \cdot 10^6\), a phase difference which can by no means be considered unnoticeable in view of the exactness allowed by the observations according to the previous Section.

According to these rough figures taken from the consideration of the elementary waves, the experiments described in the previous Sections may be considered as sufficient for the test of the laws established in the preceding Section for the ratios of the amplitudes and phases in closed conductors.

Finally let us remark that the identity of the oscillation amplitude at different positions at a large mutual distance of the closed conducting wire serves as proof, too, that the wire braided with silk may be considered as sufficiently insulated for electric currents like those produced by the rotating magnet; for with an insufficient insulation the currents would have been weaker far away from the inductor than closer to it.
30 Observations of the Dependence of the Oscillation Amplitude on the Rotation Velocity of the Magnet

After having confirmed the identity of the oscillation amplitudes and oscillation phases for the longest circuit and the fastest rotation velocity of the magnet possible with the present means, whence it is clear that this identity holds all the more for shorter circuits and lower rotational velocities, essentially only the test of the law of the dependence of the oscillation amplitude on the rotation velocity of the magnet for the circuit in question remains for the quantitative test of the laws developed in the preceding Section.

From the identity of the oscillation amplitudes and oscillation phases in all parts of a closed conductor, it follows all by itself that the intensity of the current in any point always equals the average of the current intensity in the whole conductor. Now the law for the averages of the current intensities in closed conductors depending on the average values of the electromotive forces has been developed in Section 9, independent of the consideration of the shape of the closed conductor, whence in Section 21 the law of this dependence has been determined more closely for the case when this average value of the electromotive forces changes in proportion to the sine which increases in proportion to time, which happens when the electromotive forces are produced by rotation of a small magnet. Hence putting namely the average electromotive force equal to $g_0 \sin \mu t$, this yielded the following law for the average current intensity

$$i = -\frac{g_0}{w'} \cdot \sin \rho_0 \cos(\mu t + \rho_0) ,$$

where $w'$ denoted the resistance of the unit length of the conductor and where we had

$$\rho_0 = \frac{\pi ac^2 w'}{4\mu \int M''_0 ds} .$$

But, according to this law, with a magnet of given strength and position and a given circuit, for which the resistance $w'$ and the coefficient $\int M''_0 ds$ depending on the shape of the circuit, as well as the factor $g_0$ depending on the strength of the magnet and on its position in the closed circuit, have definite values, the current intensity $i$ depends still only on the faster or slower rotation velocity to be determined by $\mu$, with $\mu/[2\pi]$ designating the number of revolutions per unit time.
However, besides the law of the dependence of the current intensity on the rotation velocity of the magnet, also the dependence of the absolute value of the current intensity \( i \) on the absolute values of the constants \( w', \int M''_0 ds \) and \( g_0 \) in addition could be subjected to a further test; but in what concerns the dependence on \( w' \) and \( g_0 \), the same has already been tested for vanishing values of \( \mu \), which yields \( \rho_0 = \pi/2 \) and hence

\[
i = \frac{g_0}{w'},
\]

which constitutes the well known Ohm’s law firmly based on experience; but in what concerns the dependence on \( \int M''_0 ds \), this test would be easy to perform as soon as there were analytical methods at hand to determine the value of the constant \( \int M''_0 ds \) from the shape of the closed conductor. The knowledge of this value for a circular conductor is not sufficient, because the observations requiring one inductor and two dynamometers cannot be carried out using a circular conductor.

The performance of the demanded quantitative test, however, requires an exact knowledge of the instruments by means of which the observations are made, in particular an exact knowledge of the dynamometers in use. Provided a practical device in order to regulate the mutual position of the multiplier and the solenoid of each dynamometer and the period of oscillation of the solenoid is at hand, this above all is a matter of practical positioning and then of the test of the instrument. The positioning of the instrument is to be performed in such a way that the axis of the solenoid is horizontal and parallel to the magnetic meridian; the axis of the multiplier housing the solenoid shall be equally horizontal and perpendicular to the axis of the solenoid. The center of the multiplier is to coincide with the center of the solenoid. If this is approximately done guided by external markings, it remains to be discussed how one could test by observations performed with the same instrument, whether the conditions are realized exactly, or how large the remaining deviations still are, as well as which experiments are needed to determine also those elements of the instrument, the knowledge of which is required for the exact quantitative determinations by means of the observations performed with them.

### 31 Test of the Dynamometer

Aiming at such a test of the dynamometer, we let the current of a constant voltaic pile equally pass through the multiplier of a tangent galvanometer, used to determine the current intensity, now forward, now backward, through
the solenoid and through the multiplier, while the latter is connected with the solenoid, now in parallel, now crosswise, which is easily performed by means of a commutator whose twin cells are connected to the ends of the solenoid and multiplier wire. In all these 4 cases the deflection of the solenoid from the original equilibrium position is observed in the familiar way. These observations serve to determine

1. the deviation, $\mu$, of the solenoid axis from the magnetic meridian at the original position of equilibrium,

2. the deviation, $\delta$, of the angle between the solenoid axis and the multiplier axis from a right angle,

3. the ratio, $\varepsilon$, of the directive force, exerted by terrestrial magnetism on the solenoid with given current intensity, and the static directive force of the solenoid,

4. the ratio, $\kappa$, of the directive force, exerted by the multiplier on the solenoid with given current intensity in both of them, and the static directive force of the solenoid.

The observation yields the deflection of the solenoid from the original equilibrium position in scale units, and dividing this number of scale units by double the distance, $R$, between the mirror and the scale in scale units, we get for smaller deflections the same expressed in arc values, which we denote by $a$, $b$, $c$, $d$ for the 4 cases discussed. Then we get

$$\kappa = \frac{1}{2} \left( \frac{c + b}{c - b} + \frac{d + a}{d - a} \right),$$

$$\varepsilon = \frac{da - cb - (db - ca)\kappa}{d + c - b - a},$$

$$\delta = \frac{1}{2\varepsilon} \left( \frac{c + b}{c - b} - \frac{d + a}{d - a} \right),$$

$$\mu = \frac{1}{\kappa} \left( \varepsilon - (1 + \kappa - \delta\varepsilon)a \right).$$

As a proof we only have to determine the static, the geomagnetic, and the electrodynamic torques acting on the solenoid, whose sum is to be put equal to zero for the equilibrium at the observed deflection.
Let $s$ be the *static* directive force, then the *static* torque, at the deflection, $\varphi$, from the static equilibrium that had existed before a current passed through the circuit, is given by

$$= -s \sin \varphi .$$

Let further $i$ be the *current intensity*, positive if the current carrying solenoid is equivalent to a magnet pointing northward with its north pole, let $mi$ be the *geomagnetic* directive force, with $m$ being the product of the horizontal part of terrestrial magnetism and the area encircled by the solenoid wire; finally let $\mu$, as already mentioned, be the angle between the solenoid axis and the magnetic meridian at static equilibrium; then the *geomagnetic* torque equals

$$= -mi \sin(\varphi + \mu) ,$$

or, when $\mu$ is very small,

$$= -mi(\sin \varphi + \mu \cos \varphi) .$$

Let finally $(\pi/2 + \delta)$ be the angle between the multiplier axis pointing eastward and the solenoid axis pointing northward; let further $e_i^2$ be the electrodynamic directive force exerted by the multiplier on the solenoid, with positive $e$ when multiplier and solenoid are connected in such a way that the multiplier acting at a distance with $i$ positive is equivalent to a magnet with its south pole pointing eastward; then the *electrodynamic* torque equals

$$= e_i^2 \cos(\varphi - \delta) ,$$

or, if $\delta$ is very small,

$$= e_i^2(\cos \varphi + \delta \sin \varphi) .$$

Now the equilibrium condition for the solenoid with the observed deflection, $\varphi$, demands that the sum of the three torques equals to zero, that is

$$-s \sin \varphi - mi(\sin \varphi + \mu \cos \varphi) + e_i^2(\cos \varphi + \delta \sin \varphi) = 0 .$$

Dividing this equation by $-s \cos \varphi$ and noting that $\tan \varphi$ may be replaced by the *arc value of the observed deflection*, that is by $a$ in the first of the four cases considered, we obtain the following equation

$$a + \frac{mi}{s}(a + \mu) - \frac{e_i^2}{s}(1 + \delta a) = 0 .$$
In this first case, the current passed the solenoid wire in the forward direction and the solenoid was connected in parallel with the multiplier. In the second case, with the current also passing the solenoid wire in the forward direction but with the solenoid connected crosswise with the multiplier, the current intensity $i$ stays positive, but $e$ changes sign while the arc value of the deflection $b$ observed in this case has to replace $\tan \varphi$, whence

$$b + \frac{mi}{s}(b + \mu) + \frac{ei^2}{s}(1 + \delta b) = 0.$$ 

In the third case, with the current passing the solenoid wire backwards, but with the solenoid connected in parallel with the multiplier as in the first case, $i$ changes sign and $e$ is positive as in the first case, while the arc value of the deflection, $c$, observed in this case has to replace $\tan \varphi$, whence

$$c - \frac{mi}{s}(c + \mu) - \frac{ei^2}{s}(1 + \delta c) = 0.$$ 

Finally in the fourth case with the current passing the solenoid wire backwards as in the third case, and with the solenoid connected crosswise with the multiplier, as in the second case, $i$ is negative as in the third case and $e$ is negative as in the second case, while the arc value of the deflection, $d$, observed in this case, has to replace $\tan \varphi$, whence

$$d - \frac{mi}{s}(d + \mu) + \frac{ei^2}{s}(1 + \delta d) = 0.$$ 

Replacing $\kappa$ by $mi/s$ and $\varepsilon$ by $ei^2/s$ we get the given values of $\kappa$, $\varepsilon$, $\delta$ and $\mu$ from these four equations.

Let us take as example the dynamometer used for the following experiments for which the observations yielded in terms of scale units:

$$2Ra = +440.01,$$
$$2Rb = -443.81,$$
$$2Rc = +448.26,$$
$$2Rd = -450.68.$$ 

Here we had $2R = 5075$ scale units. It now follows from this that

$$\kappa = 0.008484,$$
$$\varepsilon = 0.0880,$$
$$\delta = -0.0397,$$
$$\mu = +0.0323.$$
The values of $\kappa$ and $\varepsilon$, which are easy to refer to the normal values valid for the unit of the current intensity after $i$ has been measured by means of a tangent galvanometer, yield the figures for the strength of the solenoid and for the sensitivity of the dynamometer. The other two values, $\delta$ and $\mu$, on the other hand, refer to the setup and indicate the deviations of this setup under the conditions defined for them. In fact, this yields that, instead of a right angle, the solenoid axis makes the angle

$$\frac{\pi}{2} + \delta = 87^\circ 43' 31''$$

with the multiplier axis and that the solenoid axis, instead of coinciding with the magnetic meridian at the static equilibrium, deviates eastward from it by the angle

$$\mu = 1^\circ 51'$$.

One can see from this that, with the instrument being equipped with fine gradings, the errors of the setup are very easy to correct. — Even if these small errors remain uncorrected, the observations made with this instrument can still be refined and those values can be calculated that would have been obtained for an exact setup.

For the purpose of the following oscillation experiments the latter deviation, designated by $\mu$, does not come into question because of the very rapidly alternating sign of $i$, but only the deviation designated by $\delta$, and for a deviation $x'$ observed in terms of scale units, we easily arrive at the corrected value $x$, namely

$$x = x' - \frac{\delta x'^2}{2R} = x' + \frac{x'^2}{127\,780}.$$  

The observations presented in the following Sections were the beginning of a common work performed by myself and R. Kohlrausch which has been interrupted by the illness and the passing away of my dear friend.\[66\] The devices used for the uniform fast rotation of the magnet and for the measurement of this velocity including the respective observations have been performed by him.

\[66\text{[Note by AKTA:]}\] Rudolf Hermann Arndt Kohlrausch (1809-1858) had collaborated with Weber on the measurement of his fundamental constant $c$, [Web55] with English translation in [Web21c]; [WK56] with English translation in [WK03] and Portuguese translation in [WK08]; [KW57] with English translation in [KW21]; see also [WK68].
32 First Series

The following series of observations has been performed jointly by R. Kohlrausch and myself on April 12, 1857. It concerns the dependence of the oscillation amplitude on the rotation velocity of the magnet and was made using four different circuits but always using the same dynamometer and the same inductor coil, the latter consisting of one piece of a very fine copper wire, about 1500 meters long and having a resistance of $23 \cdot 10^{12}$ in absolute units, which was the constant part of the four circuits. In addition we used

- in the first circuit $A$ a piece 2800 meters long with resistance $= 8.79 \cdot 10^{12}$,
- in the second circuit $B$ a piece 5600 meters long with resistance $= 18 \cdot 10^{12}$,
- in the third circuit $C$ a piece 8000 meters long with resistance $= 27 \cdot 10^{12}$,
- in the fourth circuit $D$ a piece 10800 meters long with resistance $= 35 \cdot 10^{12}$.

The scale was in a fixed position at a distance of 2537.5 scale units from the little plane mirror fixed at the solenoid and made a right angle with the perpendicular of the mirror at the static equilibrium of the solenoid, and the vertical plane of the perpendicular of the mirror cut it at the 800th scale mark. The telescope mounted behind the scale could be displaced in such a way that, at static equilibrium of the solenoid, now the 800th scale marking, now a higher or lower scale marking could be observed, in order to enable observing the deflection of the solenoid also when it surpassed half of the scale width.
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The values of \( x' \) in the penultimate column are calculated using the deflections \( y \) listed in the foregoing column by means of the formula \( x' = y - y^2/5075^2 \), whence we obtain \( x' = 2R \tan \varphi \) with sufficient precision when the observed deflection \( y = R \tan 2\varphi \). Only for the observations on circuit \( A \) where the telescope was displaced, the deflections listed in the third column from the right do not directly give the values of \( R \tan 2\varphi \), but the latter must be calculated as follows. Separating the value of the deflection, \( y' \), given in this column into two parts, namely into part \( y'' \) which reaches from the scale mark observed for the static equilibrium to the 800th scale mark, and into
part \( y''' \) reaching from the 800th scale mark to the scale mark observed for the deflection, we get

\[
y = R \tan 2\varphi = \frac{y'}{1 - \frac{y''y'''}{R^2}},
\]

from which the value of \( x' = 2R \tan \varphi \) is easily calculated for \( R = 2537.5 \).

Now the values of \( x' \) calculated in this way have finally to be corrected as indicated at the end of the previous Section, whence the values of \( x = x' + x'^2/127780 \) listed in the last column are calculated.

### 33 Calculations of the Observations

The described observations serve to test the law developed in the previous Section concerning the dependence of the oscillation amplitude on the rotation velocity of the magnet. According to Section 24 the oscillation amplitude is expressed by \( i/|\mu \mathcal{E}| \), where \( i \) denotes the maximum intensity of the electric currents accompanying the electric oscillations. As now further according to Section 30 the current intensity in each moment of the oscillation is given by the value

\[
-\frac{g_0}{w'} \sin \rho_0 \cos(\mu t + \rho_0),
\]

hence the maximum intensity of the currents accompanying the electric oscillations is given by

\[
i = \frac{g_0}{w'} \sin \rho_0,
\]

where we had

\[
\tan \rho_0 = \frac{\pi ac^2 w'}{4\mu \int M_0'' ds},
\]

then, according to Section 24, we get the following expression for the oscillation amplitude:

\[
\frac{i}{\mu \mathcal{E}} = \frac{\pi ac^2 g_0}{\mu \mathcal{E} \sqrt{16\mu^2 (\int M_0'' ds)^2 + \pi^2 a^2 c^4 w'^2}}.
\]

Further the observations yield according to Section 24

\[
i = \frac{1}{\pi a'} \sqrt{\frac{aS \tan v}{nn'}}.
\]
where \( v \) denotes the observed deflection of the solenoid, hence the oscillation amplitude equals

\[
\frac{i}{\mu \mathcal{E}} = \frac{1}{\pi a' \mu \mathcal{E}} \sqrt{\frac{aS \tan v}{nn'}} = \frac{\pi ac^2 g_0}{\mu \mathcal{E} \sqrt{16 \mu^2 (\int M''_0 ds)^2 + \pi^2 a^2 c^4 w'^2}}.
\]

For the test of the obtained laws by means of the observations this yields the value denoted by \( \pm x = 2R \tan v \) in the last column of the observation Table in the preceding Section as given by

\[
\pm x = \frac{\pi^4 a a'^2 nn' c^4 g_0'^2}{16 \mu^2 (\int M''_0 ds)^2 + \pi^2 a^2 c^4 w'^2} \cdot \frac{2R}{S}.
\]

But now \( \mu/\pi \) is the oscillation number\(^{67}\) denoted by \( m \) in the Table and \( g_0 \) is proportional to the rotation velocity according to the laws of magnetic induction, or \( g_0 = mg_0' \), where \( g_0' \) denotes the value of \( g_0 \) for the oscillation number \( m = 1 \); hence we have

\[
\pm x = \frac{\pi^2 a a'^2 nn' c^4 g_0'^2 m^2}{16 (\int M''_0 ds)^2 \cdot m^2 + a^2 c^4 w'^2} \cdot \frac{2R}{S},
\]

or, putting

\[
C = \frac{\pi^2 a a'^2 nn' g_0'^2}{a w'^2} \cdot \frac{2R}{S},
\]

[and]

\[
P = \frac{16 (\int M''_0 ds)^2}{a^2 c^4 w'^2},
\]

[one gets]

\[
C - P x - \frac{x}{m^2} = 0,
\]

where \( C \) and \( P \) have constant values for all observations which have been made by means of the same circuit, the same rotating magnet, and the same dynamometer. Here the value of \( x \), ignoring its sign, is always to be taken positive.

Hence the observations listed in the Table of the preceding Section yield the following equations for the determination of the constants \( C \) and \( P \) for the circuit \( A \):

\(^{67}\) [Note by AKTA:] Schwingunszahl in the original. On page 128 Weber had designated by \( \mu/(2\pi) \) the number of revolutions per unit time.
from which we obtain the most probable values of $C$ and $P$, namely:

\[
C = 0.037\,978, \quad P = 0.000\,021\,865.
\]

Calculating the values of $C$ and $P$ for the circuits $B$, $C$, and $D$ in a similar way, one gets the results listed in the following Table.

<table>
<thead>
<tr>
<th>Circuit</th>
<th>$C$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
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<td>0.000,021,865</td>
</tr>
<tr>
<td>$B$</td>
<td>0.022,795</td>
<td>0.000,047,050</td>
</tr>
<tr>
<td>$C$</td>
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<td>0.000,068,093</td>
</tr>
<tr>
<td>$D$</td>
<td>0.011,869</td>
<td>0.000,087,274</td>
</tr>
</tbody>
</table>

Now finally calculating the values of $x$ from the given values of $m$ one gets, in progressive order of the values of $m$, the following comparison of the calculated values of $x$ and those found by observation:
<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Observed value of x</td>
<td>Calculated value of x</td>
<td>Difference</td>
<td></td>
<td>Observed value of x</td>
<td>Calculated value of x</td>
<td>Difference</td>
</tr>
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<td>---</td>
<td>----------------------</td>
<td>----------------------</td>
<td>------------</td>
<td>---</td>
<td>----------------------</td>
<td>----------------------</td>
<td>------------</td>
</tr>
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<td>87.79</td>
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<td>170.92</td>
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<td>217.12</td>
<td>111.06</td>
<td>109.41</td>
</tr>
<tr>
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<td>+0.35</td>
<td>217.68</td>
<td>111.25</td>
<td>109.52</td>
</tr>
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<td>119.38</td>
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<td>189.64</td>
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<td>120.86</td>
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<td></td>
<td></td>
<td>430.80</td>
<td>125.21</td>
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</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>433.06</td>
<td>125.01</td>
<td>128.17</td>
</tr>
</tbody>
</table>

Now this first test can be tied to another test. Indeed, according to the established law the two constants $C$ and $P$ should be inversely proportional to the square of the average resistance of the unit length of the conducting wire, $w'$; but the other quantities on which their values depend according to the presented formulas:

$$C = \frac{\pi^2 a^2 c^2 M_0^2 g_0^2}{aw'^2} \cdot \frac{2R}{S},$$

[and]

$$P = \frac{16(\int M''_0 ds)^2}{a^2 c^4 w'^2},$$

are not equal for all four circuits to which the observations described in the previous Section refer, but the magnitude $g_0'$ on which the constant $C$ depends, and the magnitude $\int M''_0 ds$ on which the constant $P$ depends, have a special value for each circuit. But now designating the total length of the conducting wire by $l$ which also has different values for the four circuits and minding that $g_0' \sin \mu t$ at the time $t$ denotes the average electromotive force exerted by the rotating magnet at the velocity for which $m = 1$ on any unit length of the total conducting wire; then $lg_0' \sin \mu t$ at the time $t$ denotes the electromotive force exerted by the rotating magnet at the velocity for which $m = 1$ on the total conducting wire. As now the induction by the rotating magnet acted only on the inductor wire common to all four circuits, it follows that $lg_0'$ has the same value for all four circuits; hence the formula.
\[ C = \frac{\pi^2 a'^2 nn' g_0'^2}{aw'^2} \cdot \frac{2R}{S} = \frac{\pi^2 a'^2 nn'(lg_0')^2}{a} \cdot \frac{2R}{S} \cdot \frac{1}{(lw')^2} , \]

yields that the values of the constant \( C \) must be inversely proportional to the squares of the resistances of these four circuits; for \( lw' \) designates the resistance of the whole circuit, as \( w' \) was the average resistance of the unit length.

In order to test the established law also with this respect, the values of the resistances of these four circuits presented at the beginning of the previous Section must be added to the observations considered, we note, however, that their determination was not considered the main purpose of the previous considerations, but, without claiming special exactness (they were based partly on the mere comparison in terms of wire lengths) was to serve only as a short description to distinguish the four circuits. However, we use also these determinations for a test of the established law because they, like all other observations presented, had been determined several years ago without respect to the laws developed here.

But, as mentioned at the beginning of the previous Section, according to this determination the resistances of the four circuits are roughly in proportion to

\[ 31.79 : 41 : 50 : 58 , \]

while, according to the established law, dividing the number 6.2158 by \( \sqrt{C} \), the same proportions are obtained as

\[ 31.89 : 41.17 : 50.46 : 57.06 , \]

which agrees quite well with the above experimental results.

### 34 Second Series

The following series of observations, also jointly performed by Kohlrausch and myself on April 18 and 22, 1857, is basically a repetition of the preceding one, but other than the whole conducting wire consisting of simply wound wire spools in the former series, this was valid in the following [series] only for circuit \( A \), while another circuit was formed by switching a twin wire \( E \) into circuit \( A \). This twin wire \( E \) was composed of two very fine copper wires braided by silk and hence insulated, but fixed together by another common braiding, in order to reduce the damping of electric oscillations caused by mutual induction in accordance with the prescription of Section 26, which
should thereby be tested. — This damping was in a way additionally reduced by the fact that the twin wire $E$ was mounted on a special support keeping all windings at least 20 millimeters away from each other instead of being wound up on a spool.

According to Section 32 the circuit $A$ had a length of some 4300 meters with a resistance, in absolute measure, equal to $3179 \cdot 10^{10}$. The twin wire $E$ had a length of 1412 meters (hence the single wire was 2824 meters long) and a resistance equal to $4292 \cdot 10^{10}$.

The setup of the scale and the reading telescope was identical with what was mentioned in Section 32 for the previous series of observations. — For each circuit, two series of observations have been performed, the first one on April 18, the second one on April 22.
<table>
<thead>
<tr>
<th>Circuit $A + E$</th>
<th>Oscillation number $m$</th>
<th>Position of static equilibrium</th>
<th>Deflected position $y$</th>
<th>Deflection in scale units $x'$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>107.22</td>
<td>801.98</td>
<td>728.42</td>
<td>$-73.56$</td>
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<td>$-73.50$</td>
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<tr>
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<td>801.98</td>
<td>681.37</td>
<td>$-120.61$</td>
<td>$-120.54$</td>
<td>$-120.40$</td>
</tr>
<tr>
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<td>801.98</td>
<td>542.95</td>
<td>$-258.03$</td>
<td>$-258.36$</td>
<td>$-257.84$</td>
</tr>
<tr>
<td>279.44</td>
<td>801.98</td>
<td>402.21</td>
<td>$-399.77$</td>
<td>$-397.29$</td>
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</tr>
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<td>$+124.13$</td>
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<tr>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Circuit $A$</th>
<th>Oscillation number $m$</th>
<th>Position of static equilibrium</th>
<th>Deflected position $y$</th>
<th>Deflection in scale units $x'$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
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<td>$+1075.80$</td>
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</tr>
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</table>

<table>
<thead>
<tr>
<th>Circuit $A + E$</th>
<th>Oscillation number $m$</th>
<th>Position of static equilibrium</th>
<th>Deflected position $y$</th>
<th>Deflection in scale units $x'$</th>
<th>$x$</th>
</tr>
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<th>Deflected position $y$</th>
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The calculation of the $x$ values in this Table is the same as has been discussed for the preceding Table.

Now from the corresponding values of $m$ and $x$ in this Table we can calculate the most probable values of the constants $C$ and $P$ for the circuits $A$ and $A + E$ in the same way, as in Section 33 from the values of Section 32 for the circuits $A$, $B$, $C$, $D$. In this way, the results listed in the following Table have been obtained.
Finally, calculating the $x$ values from the given values of $m$, we obtain, in successive order of the $m$ values, the following comparison of the values calculated for $x$ and the values found by observation.

<table>
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<tr>
<th>Circuit</th>
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<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A + E$</td>
<td>0.006672</td>
<td>0.000004041</td>
</tr>
<tr>
<td>$A$</td>
<td>0.03763</td>
<td>0.00002196</td>
</tr>
<tr>
<td>$A + E$</td>
<td>0.006610</td>
<td>0.000004083</td>
</tr>
<tr>
<td>$A$</td>
<td>0.03714</td>
<td>0.00002163</td>
</tr>
</tbody>
</table>

Finally, as indicated at the beginning of this Section, the observed resistances of the circuits $A + E$ and $A$ to each other are roughly in the ratio of

$$7471 : 3179,$$

while this ratio, according to the law presented in the preceding Section, applied here in the same way, as the values of the constant $C$ for both circuits should be inversely proportional to the squares of the resistances of these circuits, and this proportion is obtained dividing 611.75 by $\sqrt{C}$ as

$$7507 : 3164,$$

which agrees with the proportions obtained from the observations as far as the modest exactness of the resistance measurement justifies to expect.
35 Proportion of the Electrostatic Force of Two Equal Amounts of Electricity to their Mass

Finally, the third subject for more detailed observations according to Section 23 remains to be considered, namely the dependence of the amplitudes of the oscillations produced by a rotating magnet in a closed conductor on the shape of the conducting wire. Exact observations concerning this subject can not only serve to test the formulated laws but can moreover, as already discussed in Section 23, be used to expand our knowledge about electricity, namely, in order to determine the still unknown ratio of the force due to the electrostatic interaction between two equal amounts of electricity to their mass.

Let $\mathcal{E}$ be the amount of positive electricity in the unit length of the conducting wire expressed in electrostatic measure, then it exerts the electrostatic force equal to $\mathcal{E}^2$ at unit distance on an equal amount of electricity, while its mass has been expressed by $[1/r]\mathcal{E}$, which yields the unknown proportion of that force to this mass,

$$\frac{\mathcal{E}^2}{r\mathcal{E}} = \frac{r\mathcal{E}}{1}.$$  

If now this proportion, or the unknown quantity $r\mathcal{E}$ has a value comparable with other determined quantities, then it can be easily shown that this value could be determined most exactly from observations of the dependence of the oscillation amplitude on the shape of the circuit.

From the differential equations of electric motions in closed conductors established in Sections 8 and 10, it is clear that they contain the magnitude $r\mathcal{E}$ only in the expression

$$\frac{4M''(1 + \lambda)}{c^2} = \frac{4M''}{c^2} + \frac{1}{r\mathcal{E}}.$$  

However, by no means not all effects determinable from the differential equations depend on this expression contained in the differential equations; because for the determination of some effects the differential equations can be simplified so that this expression does not occur at all any more. As shown in Sections 11 and 12, this holds for all equilibrium effects or for the conservation of already existing motions, whence it follows vice versa that observations of equilibrium effects or of effects due to steady currents cannot serve to determine the quantity $r\mathcal{E}$.

[Note by AKTA:] See page 35 on Section 5.
The other effects, on the contrary, determined when the expression containing \( r \mathcal{E} \) does not vanish from the differential equations, comprise the electric oscillations produced in a closed conducting wire by induction due to a rotating magnet whose laws have been developed in Section 20 from these differential equations, according to which the current intensity of such an oscillation in the conducting wire was obtained as given by

\[
i = -\frac{1}{w'} \sqrt{f'^2 + g'^2} \cdot \cos \left( \mu t + \arctan \frac{f}{g} \right),
\]

if we had

\[
f = \sum \sin^2 \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right),
\]

\[
g = \sum \sin \rho \cos \rho \left( f_n \sin \frac{ns}{a} + g_n \cos \frac{ns}{a} \right),
\]

\[
\tan \rho = \frac{\mu a^2 c^2 w'}{4(2\mu^2 a^2 M''(1 + \lambda) - n^2 c^2 N'')} = \frac{\mu a^2 w'}{2\mu^2 a^2 \left( \frac{4M''}{c^2} + \frac{1}{rE} \right) - 4n^2 N''}.
\]

According to Section 21 this determination of the current intensity \( i \) is simplified in most cases, namely those cases in which all remaining terms may be considered as vanishingly small compared with the one corresponding to the place number \( n = 0 \). Indeed for these cases we obtain

\[
i = -\frac{g_0}{\sqrt{w'^2 + 4\mu^2 \left( \frac{4M''}{c^2} + \frac{1}{rE} \right)^2}} \cdot \cos \left( \mu t + \arctan \frac{w'}{2\mu \left( \frac{4M''}{c^2} + \frac{1}{rE} \right)} \right),
\]

where, according to Section 10, we have to put

\[
M''_0 = 2 \log \frac{8a}{\alpha}.
\]

Hence it follows that the magnitude \( r \mathcal{E} \) could be found out by measuring the current intensity \( i \) when we have the rotation velocity of the magnet, namely \( \mu/[2\pi] \) turns per second, the length = \( 2\pi a \), the thickness = \( 2\alpha \), and the resistance = \( 2\pi aw' \) of the conducting wire, because also the induction coefficient \( g_0 \), proportional to the rotation velocity, can be determined from the strength of the inducing magnet and from its distance and position with respect to the induced conducting wire.
Hence it would be possible to directly determine directly the magnitude $r\mathcal{E}$ without considering the dependence of the amplitude of electric oscillations on the shape of the conducting wire; it is easily seen, however, that this direct way cannot practically lead to any exact result if the magnitude $1/[r\mathcal{E}]$ is a very small fraction of the magnitude $4M_{0}''/c^{2}$, especially when considering that the determination of $M_{0}''$ and $c$ cannot be performed very precisely. Considering in addition, however, the dependence of the amplitude of the electric oscillations on the shape of the conducting wire, the following indirect way allows to determine the magnitude $r\mathcal{E}$ with much more precision.

In order to use the observation of the dependence of the amplitude of electric oscillations on the shape of the conducting wire for a more exact determination of the magnitude $r\mathcal{E}$, it is essential to find a method to give a closed conducting wire two different shapes, which allow to exactly determine either the two values of $M_{0}''$, or at least their ratio $\nu : 1$.

Here it comes into question that in the first part of this treatise the development of the laws of motion of electricity had to be restricted to circular conductors, for which an essential simplification could be achieved because the values of the integrals denoted by $M, N, M''$ and $N''$ were equal for all points of the conductor shape. But the latter also holds for a system consisting of two equal and parallel circles, which could not be taken into consideration only because it forms two separate conductors. For practical purposes when performing the observations, however, such a system may be substituted by a closed double winding conductor with respect to almost all considerations, and hence, according to this substitution, the values of the definite integrals $M, N, M''$ and $N''$ may be put equal for all points of a closed conductor which forms two equal very close circular windings, whereby it is possible to expand the laws of motion of electricity, at first formulated for a circular conductor only, to a closed conductor consisting of two equal very close circular windings.

Such a conductor, however, as we see easily, presents a very essential alternative concerning the variety of the connection between both of its windings, which may either provide the same current passing through both windings in series in the same direction, or passing through the second winding in opposite direction as the first one. These two cases correspond to completely different values of $M, N, M''$ and $N''$ whose ratio can be determined exactly. In fact the same holds also for the value of $M''$ for $n = 0$, denoted by $M_{0}'$', and we may put the ratio of both values of $M_{0}'$ for the two above cases equal to $\nu : 1$.

Let $A$ be the greatest current intensity of an electric oscillation in this conductor in the first case, and $B$ in the second case, then we have, according to Section 23,
\[ A = \frac{g_0}{\sqrt{w' + 4\mu^2 \left( \frac{4M''_{0}}{c^2} + \frac{1}{r\mathcal{E}} \right)^2}}, \]

\[ B = \frac{g_0}{\sqrt{w' + \mu^2 \left( \frac{4M''_{0}}{\nu c^2} + \frac{1}{r\mathcal{E}} \right)^2}}, \]

and both values, \( A \) and \( B \), can be determined by measurement; but here we assumed that the induction due to the rotating magnet does not extend to both windings of the closed conductor, but that the induction be limited to one of both windings or only to one part of it. In reality, this element is represented by the small inductor which houses the rotating magnet.

But now these values of \( A \) and \( B \) can be determined by measurement for different rotation velocities, that is, for different values of \( \mu \), and it is clear that their difference must vanish completely before long with decreasing values of \( \mu \). Now let \( \mu_0 \) be the small value of \( \mu \) for which this difference is not noticeable. Let then \( C \) denote the values of \( A \) and \( B \) when considered equal; then we get

\[ C = \frac{\mu_0}{\mu} \cdot \frac{g_0}{w'}, \]

because the coefficient of induction, denoted by \( g_0 \), is proportional to the rotation velocity. Also this third value, \( C \), can be determined by measurement.

Eliminating now \( g_0 \) from the three obtained equations where \( A, B, C \) are known by measurement, we obtain the following two equations:

\[ \frac{4M''_{0}}{c^2} + \frac{1}{r\mathcal{E}} = \frac{w'}{2\mu} \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{A^2} - 1, \]

\[ \frac{4M''_{0}}{\nu c^2} + \frac{1}{r\mathcal{E}} = \frac{w'}{2\mu} \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{B^2} - 1, \]

and hence follows:

\[ \frac{4M''_{0}}{c^2} = \frac{\nu}{\nu - 1} \cdot \frac{w'}{2\mu} \left\{ \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{A^2} - 1 \right\} - \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{B^2} - 1, \]

\[ \frac{1}{r\mathcal{E}} = \frac{1}{\nu - 1} \cdot \frac{w'}{2\mu} \left\{ \nu \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{B^2} - 1 \right\} - \sqrt{\frac{\mu^2}{\mu_0^2}} \cdot \frac{C^2}{A^2} - 1. \]
The first of these equations where all quantities are known may either serve to test the theory or the observations, while the unknown quantity \( rE \) is found from the second equation, where not even the closer determination of the quantities \( M''_0, g_0, \) and \( C \) is required but only that of the ratio \( \nu : 1 \).

The latter of both equations thus found can finally be given a somewhat simpler form noting that the resistance of the unit length of the conducting wire equals \( w' = 1/[\pi \alpha^2 \kappa] \), with \( \alpha \) the radius of the wire and \( \kappa \) the specific conductivity of the metal, and that further the amount of positive electricity expressed in electrostatic units of measure contained in the unit length of the conducting wire equals \( E = \pi \alpha^2 \cdot E_0 \), where \( E_0 \) denotes the positive amount of electricity in the unit volume of the conducting wire. Substituting these values we get

\[
\frac{1}{rE_0} = \frac{1}{2(\nu - 1) \mu \kappa} \left\{ \nu \sqrt{\frac{\mu^2}{\mu_0^2} \cdot \frac{C^2}{B^2} - 1} - \sqrt{\frac{\mu^2}{\mu_0^2} \cdot \frac{C^2}{A^2} - 1} \right\}.
\]

Hence it remains to be considered how to determine the ratio denoted by \( \nu : 1 \).

Let \( 2\pi a \) be the length of the total closed conducting wire as before, each half of which, equal to \( \pi a \), constitutes one winding, and consider both of these windings as parallel circles the centers of which lie at a distance, \( \delta \), perpendicular to the plane of the circle; then the value of \( M''_0 \) may be split into two parts at any point of that entire conducting wire, namely the part due to the circle containing the point under consideration and the part due to the other circle at a distance \( \delta \) from the first circle. The former part is immediately found equal to the value of \( M''_0 \) for a circle of radius \( \frac{1}{2} a \), namely, according to Section 16, equal to double the logarithm of the ratio of the 8-fold circle radius to the wire radius, \( = 2 \log(4a/\alpha) \). As is easily shown, the latter part is obtained from the former part by merely substituting the radius, \( \alpha \), by the distance between the two circles, \( \delta \), namely \( = 2 \log(4a/\delta) \).

— If now the two windings are connected in such a way that the current passes them in the same direction, then the value of \( M''_0 \) of the entire closed conductor at any point of its first or second winding equals the sum of both parts,

\[
= 2 \log \frac{4a}{\alpha} + 2 \log \frac{4a}{\delta} ;
\]

if, on the other hand, both windings are connected so that the second winding is passed in the counter sense as the first one, then the value of \( M''_0 \) equals

\[\text{Note by AKTA and PM:} \text{ In the original we have here } c \text{ instead of } C. \text{ We are replacing the lowercase } c \text{ by the uppercase } C.\]
the difference of both parts,
\[= 2 \log \frac{4a}{\alpha} - 2 \log \frac{4a}{\delta};\]
Hence we get the desired ratio as given by
\[
\nu : 1 = \left(2 \log \frac{4a}{\alpha} + 2 \log \frac{4a}{\delta}\right) : \left(2 \log \frac{4a}{\alpha} - 2 \log \frac{4a}{\delta}\right) = \left(2 \log \frac{4a}{\alpha} - 1\right) : 1.
\]

36 Conclusion

The discussions concerning the determination of the magnitude \(rE\) in the previous Section serve mainly to specifically exemplify that the dependence of the amplitude of the oscillation produced by a rotating magnet in a closed conductor on the shape of the conducting wire, as presented in Section 23, constitutes a third topic important and particularly suitable for more exact observations, deserving a more careful and more extensive treatment because of its multiple meaning. If the performance of exact observations on this topic is to be of practical use, it becomes clear that this has to be done in connection first with the extensive discussion of the dependence of the values of the integrals, denoted by \(N, N'', M\) and \(M''\) in Section 8, on the shape of the conducting wire, restricted to the single case of a circular wire in Section 10, and second with a specific discussion of the value of the magnitude \(rE\), the determination of which is of great interest all by itself, as has been treated in the preceding Section. Now it has indeed been shown in the preceding Section how the determination of this magnitude \(rE\) would be possible from observations of a special case of the dependence of the oscillation amplitude on the shape of the conducting wire without entering into an extensive discussion of this dependence in general; but this, however, still demands a lot of work and observations even for the solution of the hereby essentially simplified and limited task, for which the measures described and used in this treatise are not sufficient.

As now furthermore also the determination of the magnitude \(rE\) concerned less the test of the laws developed in the first part of this treatise, but rather a novel application of the theory with a particular and separate aim — appropriate as a topic for a special treatise —; it seems adequate to restrict beforehand the performance of observations, intended in the second part of this treatise and concerning a test of the laws, to the observations, presented in the first section, concerning the first two topics presented in Section 23 — [namely, (1),] comparison of amplitudes and phases of electric oscillations at
various places of a long conducting wire — [and (2), the] law of the dependence of the oscillation amplitude on the rotation velocity of the magnet —; and to reserve the performance of all the more exact observations concerning the third topic — namely, the law of the dependence of the oscillation amplitude on the shape of the conducting wire, including the respective special questions and tasks according to the instruction of the preceding Section — for a future treatise for which, finally, the results of the present treatise were intended here as the preparatory foundations only.
References


[Cou88b] C. A. Coulomb. Quatrième mémoire sur l’électricité. Où l’on démontre deux principales propriétés du fluide électrique: La première, que ce fluidé ne se répand dans aucun corps par une affinité chimique ou par une attraction élective, mais qu’il se partage entre differens corps mis en contact uniquement par son action répulsive; La seconde, que dans les corps conducteurs le fluide parvenu à l’état de stabilité, est répandu sur la surface du corps, & ne pénètre pas dans l’intérieur. *Mémoires de l’Académie


