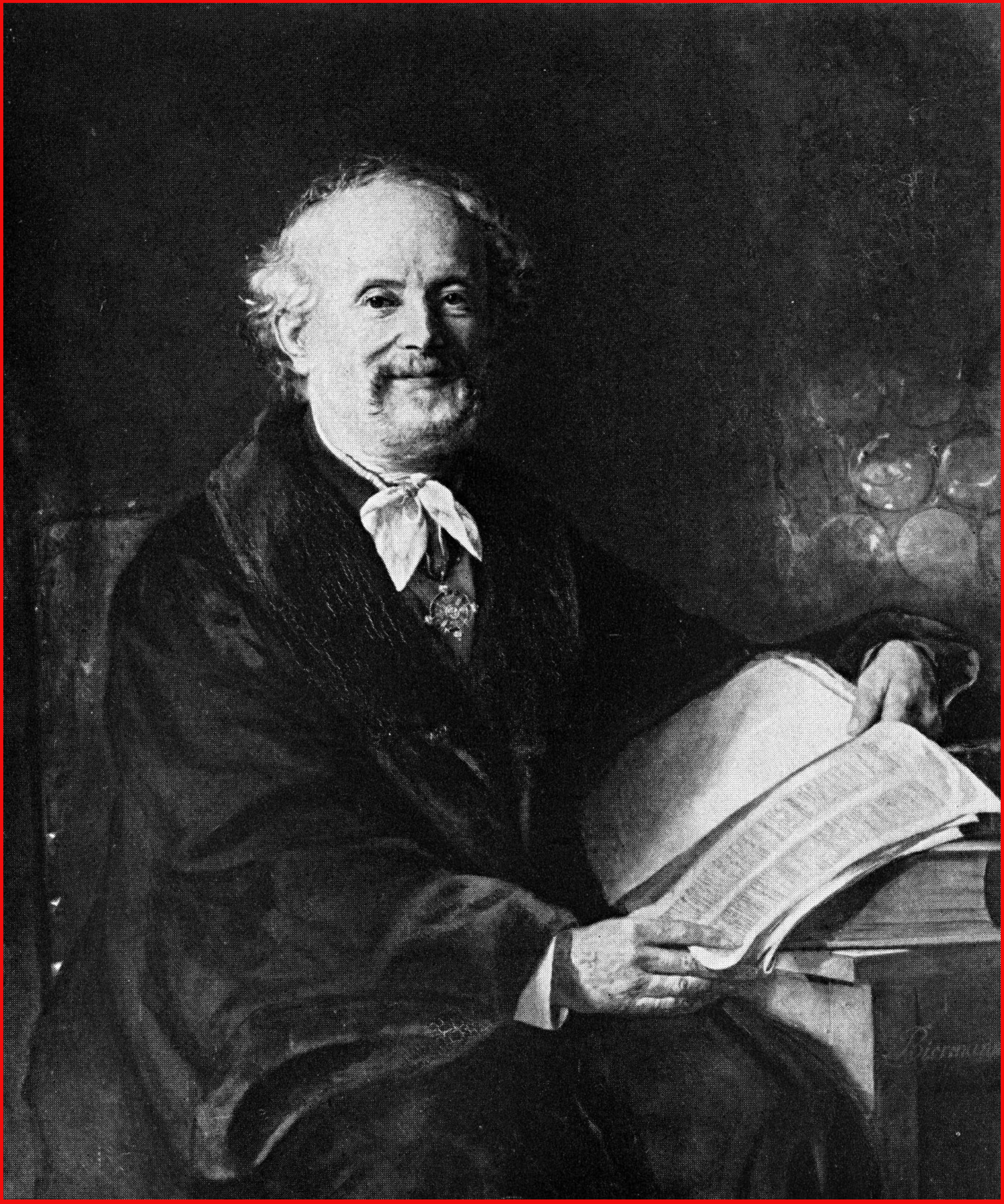
Wilhelm Weber's Main Works on Electrodynamics Translated into English

Volume IV: Conservation of Energy, Weber's Planetary Model of the Atom and the Unification of Electromagnetism and Gravitation



Edited by Andre Koch Torres Assis

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Front cover of Volume IV: The picture on the cover of Volume IV comes from a 1885 portrait of Weber made by the German painter Gottlieb Biermann (1824-1908). In 1864 Weber had received the Orden Pour le Mérite der Friedensklasse für die Verdienste um die Wissenschaften und die Künste (honor given for achievement in sciences and arts). Twenty years later the Prussian minister of culture proposed to the emperor that he should be painted as Ritter der Friedenklasse (Knight of the Peace Class) for the National Gallery. He appears in the painting in the gown of Göttingen University with the medal of the Order Pour le Mérite. The original colour painting belongs to the Alte Nationalgalerie of the Staatliche Museen zu Berlin. A replica of this picture, together with a portrait of Carl Friedrich Gauss (1777-1855), was donated by the Prussian state in 1887 for the jubilee of the University of Göttingen for its Auditorium.

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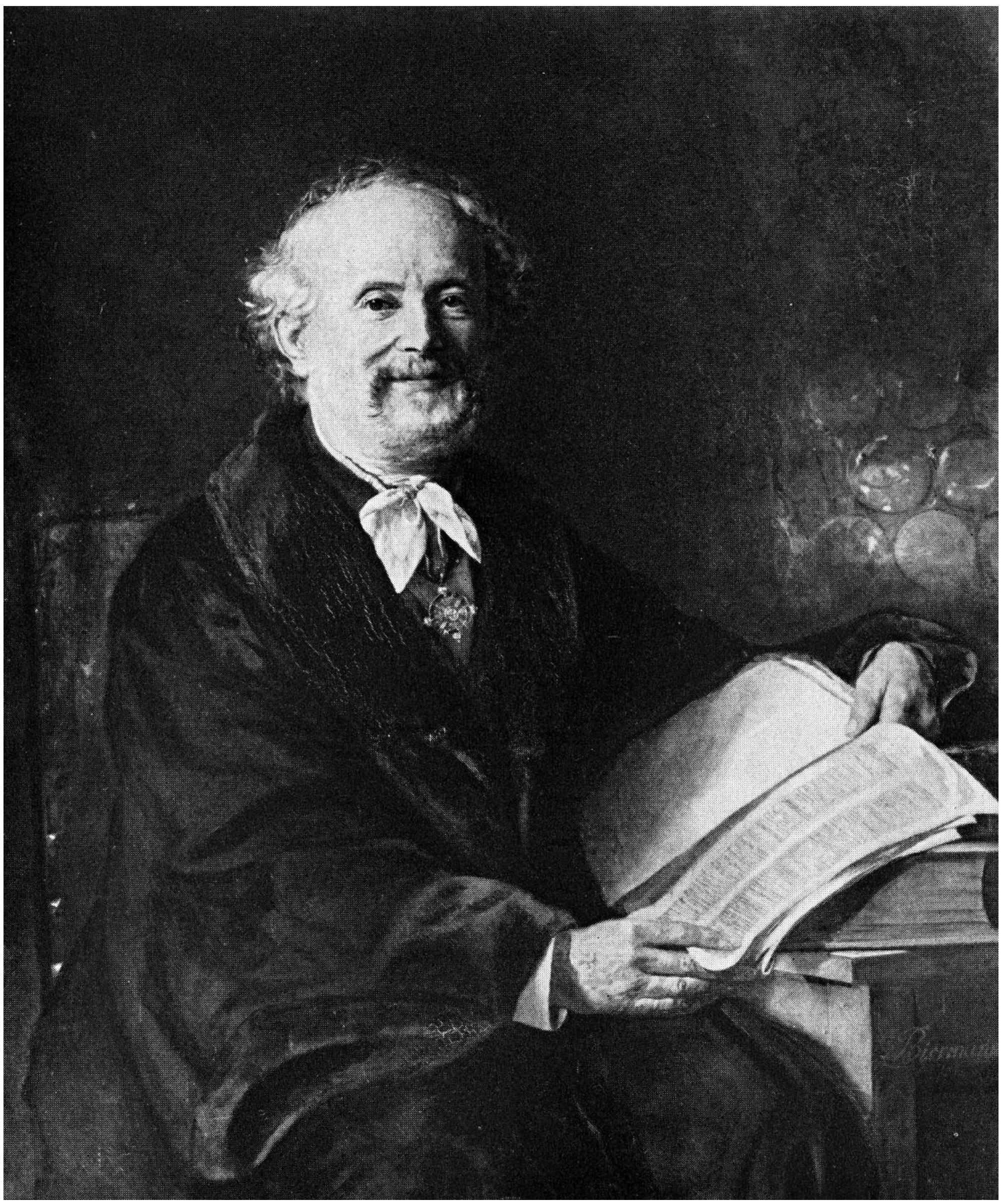

Edited by Andre Koch Torres Assis

Contents

Chapter 1

Introduction to Volume IV

A. K. T. Assis[1]

The picture on the cover of Volume 3 comes from a 1885 portrait of Weber made by the German painter Gottlieb Biermann (1824-1908).[2] Weber had received in 1864 the Orden Pour le Mérite der Friedensklasse für die Verdienste um die Wissenschaften und die Künste (honor given for achievement in sciences and arts). Twenty years later the Prussian minister of culture proposed to the emperor that he should be painted as Ritter der Friedensklasse (Knight of the Peace Class) for the National Gallery. He appears in the painting in the gown of Göttingen University with the medal of the Order Pour le Mérite. The original colour painting belongs to the Alte Nationalgalerie of the Staatliche Museen zu Berlin. A replica of this picture, together with a portrait of Carl Friedrich Gauss (1777-1855), was donated by the Prussian state in 1887 for the jubilee of the University of Göttingen for its Auditorium.

This fourth Volume begins with the English translation of Gauss' posthumous paper published in 1867. He arrived at a force law depending on the positions and velocities of the interacting electrified particles from which he could deduce not only electrostatics, but also the force between current elements. Then comes Carl Neumann's 1868 paper on the principles of electrodynamics. In this work Neumann introduced the Lagrangian and Hamiltonian formulations of Weber's electrodynamics. Moreover, he also showed that Weber's law was compatible with the principle of the conservation of energy. At that time he was not aware of Weber's 1848 potential energy, as Neumann acknowledged in 1880.

In the sequence I included Weber's 1871 Sixth major Memoir on Electrodynamic Measurements. He showed once more in detail that his force law was compatible with the principle of the conservation of energy. Moreover, he studied the two-body problem according to his electrodynamics. He showed that in some conditions two particles electrified with charges of the same sign might attract one another. This was at the origin of his planetary model of the atom.

This Volume contains the English translation of Tisserand's 1872 paper on the motion of planets around the Sun according to Weber's law. He calculated, for instance, the precession of the perihelion of the planets.

Weber's Seventh major Memoir on Electrodynamic Measurements was published in 1878. It is devoted to the energy of interaction. His Eight major Memoir, thought to be written in

[1]Homepage: www.ifi.unicamp.br/~assis

[2]It appears, for instance, in [Wie67, p. 5].

the 1880's, was published only posthumously in 1894 in his collected works. It is related to the connection of Weber's fundamental law of electrodynamics with the law of gravitation. Moreover, it contains his mature planetary model of the atom in which the nucleus is held together by purely electrodynamic forces according to Weber's law. Weber's aphorisms are also translated in this fourth Volume.

I close this work with an overview of Weber's law applied to electromagnetism and gravitation, together with some possible future developments of his theory. The main topics are the unifications in physics: (a) Ampère's unification of magnetism, electrodynamics and electromagnetism; (b) Weber's unification of the laws of Coulomb, Ampère and Faraday; (c) unification of optics with electrodynamics; (d) unification of nuclear physics with electrodynamics; (d) applications of Weber's law for gravitation; (e) unification of gravitation with electrodynamics; and (e) unification of inertia with gravitation. I discuss, in particular, the implementation of Mach's principle and the deduction of Newton's second law of motion from Weber's force.

Chapter 2

Editor's Introduction to Gauss' Work on the Fundamental Law for All Interactions of Galvanic Currents

A. K. T. Assis[3]

In 1835 Carl Friedrich Gauss (1777-1855) discovered a force between two point particles depending on their positions and velocities. This work was only published posthumously in 1867 in his collected works. This is the date when Gauss' force law became known to the scientific community.[4]

Gauss considered two electrified particles with charges e and e' located at points $(x,\ y,\ z)$ and $(x',\ y',\ z')$ relative to the origin of a rectangular coordinate system. These particles were separated by a distance $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. His force law was given by:

$$\frac{ee'}{r^2}\left\{1 + k\left[\left(\frac{d(x'-x)}{dt}\right)^2 + \left(\frac{d(y'-y)}{dt}\right)^2 + \left(\frac{d(z'-z)}{dt}\right)^2 - \frac{3}{2}\left(\frac{dr}{dt}\right)^2\right]\right\} . \qquad (2.1)$$

Gauss said that the constant $\sqrt{1/k}$ in this expression represents a certain velocity. However, he did not specify the value of this constant.

As the title of Gauss' work indicates, he was studying the interactions of galvanic currents. His force depends not only on the square of the *relative* velocity dr/dt between the elements of electricity e and e', but also on the magnitude $(d(x'-x)/dt)^2 + (d(y'-y)/dt)^2 + (d(z'-z)/dt)^2$. The difference of these magnitudes can be seen as follows:[5]

$$\frac{dr}{dt} = \frac{d}{dt}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$= \frac{(x-x')(dx/dt - dx'/dt) + (y-y')(dy/dt - dy'/dt) + (z-z')(dz/dt - dz'/dt)}{r} . \qquad (2.2)$$

[3] Homepage: www.ifi.unicamp.br/~assis
[4] [Gau67].
[5] [Ass94, Section 3.2].

This shows that, in general,

$$\frac{dr}{dt} \neq \sqrt{\left(\frac{d(x-x')}{dt}\right)^2 + \left(\frac{d(y-y')}{dt}\right)^2 + \left(\frac{d(z-z')}{dt}\right)^2} \, . \tag{2.3}$$

It is relevant to compare Gauss and Weber's force laws on this respect. Assuming that Weber's constant c can be written as $\sqrt{2} \cdot v_L$, where v_L is light velocity in vacuum, Weber's force between e and e' is given by:

$$\frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right] = \frac{ee'}{r^2}\left[1 - \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2 + \frac{r}{v_L^2}\frac{d^2r}{dt^2}\right] \, , \tag{2.4}$$

Weber's force depends on the distance r between e and e', on their *relative velocity* dr/dt, and on their *relative acceleration* d^2r/dt^2. These are intrinsic magnitudes of the system of interacting particles. Moreover, they have the same value in all frames of reference, even for non-inertial ones. I have called them *relational magnitudes*. Gauss' force, on the other hand, does not depend on the accelerations of e and e'. Moreover, Gauss' force depends not only on the relative velocity dr/dt, but also on $(d(x'-x)/dt)^2 + (d(y'-y)/dt)^2 + (d(z'-z)/dt)^2$. This last magnitude is not relational. In particular, its value in one inertial frame of reference may be different from the value of the same magnitude in another non-inertial frame of reference. A detailed discussion of this topic can be found in Appendix A of the book *Relational Mechanics and Implementation of Mach's Principle with Weber's Gravitational Force*.[6]

James Clerk Maxwell (1831-1879) criticized Gauss' force in the last Chapter of his book *Treatise on Electricity and Magnetism*.[7] Maxwell said that it leads to Ampère's force between current elements.[8] However, Maxwell mentioned that it was inconsistent with the principle of the conservation of energy, and must therefore be abandoned. He also pointed out that Gauss' force fails to explain Faraday's law of induction.[9]

In Appendix B (Alternative Formulations of Electrodynamics) of the book *Weber's Electrodynamics*, I discussed the forces of Gauss, Bernhard Riemann (1826-1866), Rudolf Clausius (1822-1888) and Walter Ritz (1878-1909).[10]

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[6] [Ass14].

[7] [Max73a] and [Max54a]. German translation in [Max83a]. Portuguese translation in [Ass92c].

[8] André-Marie Ampère (1775-1836). Ampère's masterpiece was published in 1826, [Amp26] and [Amp23]. There is a complete Portuguese translation of this work, [Cha09] and [AC11]. Partial English translations can be found at [Amp65] and [Amp69c]. Complete and commented English translations can be found in [Amp12] and [AC15].

A huge material on Ampère and his force law between current elements can be found in the homepage *Ampère et l'Histoire de l'Électricité*, [Blo05].

[9] Michael Faraday (1791-1867). See [Far32a]. German translation in [Far32b] and [Far89]. Portuguese translation in [Far11].

[10] In English: [Ass94]. In Portuguese: [Ass92a], [Ass95a] and [Ass15a].

Chapter 3

[Gauss, 1867] Fundamental Law for All Interactions of Galvanic Currents

Carl Friedrich Gauss[11,12]

(Discovered in July 1835.)

Two elements of electricity in relative motion attract or repel one another, but not in the same way as if they are in relative rest.

$$e, \ x, \ y, \ z \qquad \text{Element and coordinates}$$

$$e', \ x', \ y', \ z'$$

$$(x'-x)^2 + (y'-y)^2 + (z'-z)^2 = rr$$

Mutual action (repulsion)

$$= \frac{ee'}{rr}\left\{1 + k\left(\left(\frac{d(x'-x)}{dt}\right)^2 + \left(\frac{d(y'-y)}{dt}\right)^2 + \left(\frac{d(z'-z)}{dt}\right)^2 - \frac{3}{2}\left(\frac{dr}{dt}\right)^2\right)\right\}$$

where $\sqrt{\frac{1}{k}}$ represents a certain velocity.

[11][Gau67].

[12]Translated by A. K. T. Assis, www.ifi.unicamp.br/~assis

Chapter 4

Editor's Introduction to Carl Neumann's 1868 Paper

A. K. T. Assis[13]

This is the first English translation of Carl Neumann's 1868 paper "Die Principien der Elektrodynamik".[14] It was translated by Laurence Hecht and Urs Frauenfelder.[15]

One important aspect of Neumann's paper was the introduction of the Lagrangian and Hamiltonian formulations of Weber's electrodynamics.[16] Moreover, he also showed that Weber's theory was compatible with the principle of the conservation of energy.

In 1846 Weber presented his force F between two particles electrified with charges e and e' separated by a distance r and moving with relative radial velocity dr/dt and relative radial acceleration d^2r/dt^2.[17] In 1852 he expressed this force as:[18]

$$F = \frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right] . \tag{4.1}$$

Weber's constant c was only measured in 1854-55 by Weber and R. Kohlrausch. Their measured value published in 1857 was 4.39×10^8 m/s.[19]

Weber had also introduced in 1848 a potential energy from which he could deduce his force law.[20] Representing this potential energy by V, it can be written in terms of Weber's constant c as:

$$V = \frac{ee'}{r}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] . \tag{4.2}$$

In 1857 Weber and Kirchhoff, working independently from one another, showed that

[13] Homepage: www.ifi.unicamp.br/~assis

[14] [Neu68a].

[15] larryhecht33@gmail.com and urs.frauenfelder@math.uni-augsburg.de.

[16] [Hol70] with English translation in [Hol17], [Sch97], [Arc86], [Ass94, Section 3.5: Lagrangian and Hamiltonian Formulations of Weber's Electrodynamics] and [Sch04].

[17] [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

[18] [Web52b] with English translation in [Web21a].

[19] [KW57] with English translation in [KW21].

[20] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

Weber's electrodynamics led to the complete telegraph equation for a signal propagating along a conducting wire.[21] In modern terminology, they included not only the capacitance and resistance of the wire, but also its self-inductance. Both of them worked with Weber's force law. They showed, in particular, that when the resistance of the wire was negligible, the electric wave propagates along the wire with a velocity given by $c/\sqrt{2} = 3.1 \times 10^8$ m/s. That is, with the same magnitude as the known value of light velocity in vacuum v_L, namely, $c/\sqrt{2} = v_L = 3 \times 10^8$ m/s. Weber's force and potential energy can then be written in terms of light velocity in vacuum v_L as:

$$F = \frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right] = \frac{ee'}{r^2}\left[1 - \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2 + \frac{r}{v_L^2}\frac{d^2r}{dt^2}\right] , \tag{4.3}$$

and

$$V = \frac{ee'}{r}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] = \frac{ee'}{r}\left[1 - \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2\right] . \tag{4.4}$$

When Carl Neumann published the present paper of 1868, he was not aware of Weber's potential energy V introduced in 1848. This information was acknowledged by Carl Neumann in a supplementary remark of 1880 when his paper of 1868 was reprinted. In any event, in 1868 Carl Neumann arrived at an expression for the Lagrangian energy W. I will present it here as a function of Weber's constant c and also as a function of light velocity v_L:

$$W = \frac{ee'}{r}\left[1 + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] = \frac{ee'}{r}\left[1 + \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2\right] . \tag{4.5}$$

The functions V and W differ in the sign in front of the square of the relative velocity dr/dt. Neumann showed that for a system of particles interacting through Weber's force, the total energy E, given by the sum of V with the kinetic energy T of the particles, is a constant in time:

$$E = T + V = \text{constant in time.} \tag{4.6}$$

Calling L the Lagrangian of the system and H its Hamiltonian we then have:

$$L = T - W , \tag{4.7}$$

and

$$H = E = T + V = \text{constant in time.} \tag{4.8}$$

Before presenting the translation of Carl Neumann's paper, I would like to discuss briefly another topic. In his work Neumann presented two main conceptions of an electric current. What he called the two-fluid theory of electric current is the assumption that the positive and negative electric fluids or particles move relative to the matter of the conductor with opposite drift velocities. The one-fluid theory of electric current (also called the unitary point of view), on the other hand, assumes that only one of these fluids move relative to the conductor, while the other fluid is attached to the ponderable matter of the conductor.

[21] [Kir57b] with English translation in [Kir57a], [Pog57] with English translation in [Pog21], and [Web64] with English translation in [Web21b].

This is analogous to the modern theory of metallic conduction in which only free electrons move relative to the conductor, while the positive ions remain fixed in the lattice. Wilhelm Weber initially adopted the two-fluid theory of an electric current (or the double current hypothesis), but later on he changed his mind and adopted the one fluid conception of an electric current.[22]

[22]See Chapter 5 (The evolution of Weber's conception of an electric current: from a double current to a simple current) of the book *Weber's Planetary Model of the Atom*, available in English, [AWW11], Portuguese, [AWW14], and German, [AWW18].

Chapter 5

[Carl Neumann, 1868] Principles of Electrodynamics

Carl Neumann[23,24,25]

The individual areas of physical science could aptly be subdivided into *two* parts, according to the nature of the elementary forces which are assumed to explain the relevant phenomena. On one side stands *celestial mechanics, elasticity, capillarity*, in general those areas for which the direction and magnitude of the force is fully determined by the relative position of the material parts; on the other side are to be considered the investigations of *friction, electricity* and *magnetism*, and perhaps also *optics*, in general those areas of physics in which the known forces depend upon other conditions in addition to their relative positions — their velocities and accelerations, for example.

Now, if the *law (or principle) of vis viva*[26] rules completely over the natural phenomena

[23] [Neu68a] with English translation in [Neu20a].

[24] Translated by Laurence Hecht, larryhecht33@gmail.com, and Urs Frauenfelder, urs.frauenfelder@math.uni-augsburg.de. Edited by A. K. T. Assis.

[25] The Notes by Carl Neumann are represented by [Note by CN:]; the Notes by Laurence Hecht are represented by [Note by LH:]; the Notes by Urs Frauenfelder are represented by [Note by UF:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[26] [Note by AKTA:] The Latin expression *vis viva* (*living force* in English or *lebendige Kraft* in German) was coined by G. W. Leibniz (1646-1716).

Originally the *vis viva* of a body of mass m moving with velocity v relative to an inertial frame of reference was defined as mv^2, that is, twice the modern kinetic energy. However, during the XIXth century many authors like Weber and Helmholtz defined the *vis viva* as $mv^2/2$, that is, like the modern kinetic energy.

In 1847 Helmholtz expressed himself as follows, [Hel47, p. 9] with English translation in [Hel53, p. 119]:

> For the sake of better agreement with the customary manner of measuring the intensity of forces, I propose calling the quantity $\frac{1}{2}mv^2$ the quantity of *vis viva*, by which it is rendered identical with the quantity of work.

In 1872 he made an analogous definition, [Hel72a] with English translation in [Hel72b, p. 533]:

> If we, as has always hitherto been done, name *vis viva* or *actual energy* the sum of the moved inert masses multiplied each by half the square of its velocity, then, [...]

Weber also utilized the expression *vis viva* as $mv^2/2$. This can be seen, for instance, in [Web71, footnote 1, pp. 256-257 of Weber's *Werke*] with English translation in [Web72, p. 9], and footnote 140 on page 74 of Chapter 9.

(and all previous experience speaks for this), as appears to apply for the first subdivision as a direct consequence of the underlying ideas, for the second subdivision it seems to be a *matter of chance*. For the elementary forces of the first type subject themselves to the rule of that law, but those of the second type do not.

> "It seems" — says Fechner in his *Psychophysik* (1860),[27] Vol. I, page 34 — "that these (last) elementary forces work together in such a way that the law[28] remains applicable to all actions of nature. In the case of the magnetic forces (and therefore electric currents as well) this is self-evident, insofar as they actually can be represented as the effects of central forces, which are independent of velocity and acceleration. Moreover, Prof. W. Weber has responded orally to my questioning, that in all cases to which his investigation has led, even beyond the limit of the latter forces, the law is found to be valid, even if its full applicability to the region of these forces still requires strict proof."

But this is actually not a proof, but a discovery. Because that law represents a relation between the *vis viva* and the potential, and thus a relation between two magnitudes, the latter of which is known as an elementary force of the first type, *but is completely unknown for the second type*. Respecting the latter forces, it is therefore not the proof of the law, but the *discovery of its content*, and the determination of its magnitude, which would be regarded as the potential of those forces.

Three years ago, stimulated by the just cited words of Fechner, I began to interest myself in this question and thus directed my attention to *those* elementary forces of the second kind which Weber assumed between two electric particles, and I soon found that the potential of such a force could be viewed with certain authority by the following expression:

$$W = \frac{mm_1}{r} + G\frac{mm_1}{r}\left(\frac{dr}{dt}\right)^2,$$

where m, m_1 indicate the masses of the two particles, r their distance apart, t the moment of time under consideration, and G a constant.[29] Then it is seen that the force assumed by Weber can be derived from this expression by *variation* of the coordinates in exactly the same way in which an elementary force of the *first* kind is obtained from its potential though a *differentiation* of its coordinates.

And simultaneously it resulted that during the motion of both particles a very simple relationship prevails between the *vis viva* and the two parts of the expression W adopted as potential, namely:

$$(\text{vis viva}) + \frac{mm_1}{r} - G\frac{mm_1}{r}\left(\frac{dr}{dt}\right)^2 = \text{constant} .$$

It can scarcely be doubted that this relationship represents the law to be discovered for the force assumed by Weber.

I also had already back then, in accordance with the expression for W, formulated the potential for *two elements of electric current*, and found that from the potential so obtained,

[27] [Note by AKTA:] Gustav Theodor Fechner (1801-1887), see [Fec60, p. 34].

[28] [Note by AKTA:] Fechner is referring here to the law of the conservation of energy.

[29] [Note by AKTA:] By "electric masses" m and m_1 we should understand here the charges of the particles, [Arc86, p. 787].

both the repulsive and the inductive action of the two elements on one another could be derived in a very simple way, namely the former could be deduced by variation according to the distance, the latter by variation according to direction of an element.

Amazing as it may seem at first sight, and in some contrast to the hitherto prevailing view, variation must take the place of differentiation. However, as I want to remark right away, this contrast is to some extent tempered, when one observes that a similar treatment already applies in the area of elementary forces of the *first* type, for example in investigation into elasticity. Namely, let u, v, w be those functions of the coordinates through which the internal displacement of a given elastic body are represented, and Φ the potential which the particles of the body collectively exert on any *one* of them, then the force acting on the latter is found through *variation* of Φ according to u, v, w (as was developed in detail by me in an essay on elasticity, Borchardt's *Journal*, Vol. 57, page 304).[30]

Some time ago I was prompted to resume and continue my investigations into the subject in question by a posthumous essay of Riemann's, published in Poggendorff's *Annalen* (Vol. 131, page 237),[31] in which the attempt (which, however was not very successful, and perhaps, as a result of the too brief presentation should not be judged) is made to explain the repulsive action of two current elements on each other by elementary forces of the *first* kind, under the assumption that the potential of this force — similar to light — is propagated through space with a certain constant velocity. To my surprise I found that this assumption leads directly to my conjecture, namely by assuming such a progressive propagation, the ordinary potential $\frac{mm_1}{r}$ (corresponding to the Newtonian gravitational force),[32] transforms into a magnitude whose effective constituent is completely identical with the previously mentioned expression W.

Already in May of this year I made a short communication to the Göttingen Scientific Society about the starting point and results of the investigation in question (*Nachrichten der Gesellschaft*, June 16, 1868).[33] If I now intend to present these investigations, or at least a part of them, in detail and as carefully as possible, this is not because I consider these investigations to be completely thorough, but rather because of the *extraordinary importance of the subject at hand*, and because I am of the opinion that my researches may be necessary, or at least not without use, for a deeper penetration into this subject.

5.1 Overview

5.1.1 Basis of the Investigation

In the present investigation I will share the nomenclature of those authors who understand *vis viva (Lebendige Kraft)* as the sum of the masses multiplied by *one half* the square of their velocity, and who further understand the potential as that function of the coordinates whose *negative* differential coefficient represents the force.[34,35] By applying this nomencla-

[30] [Note by AKTA:] [Neu60, p. 304].

[31] [Note by AKTA:] [Rie67b] with English translation in [Rie67a] and [Rie77a].

[32] [Note by AKTA:] [New34], [New52] and [New99]. Portuguese translation in [New90], [New08] and [New10].

[33] [Note by AKTA:] [Neu68b].

[34] [Note by CN:] By this definition *vis viva* and *potential* are identical to the magnitudes the English call *actual* and *potential energy*. *Potential* is also identical to the magnitude Helmholtz calls *Spannkraft*.

[35] [Note by AKTA:] Helmholtz introduced the concept *Spannkraft* in 1847, [Hel47, p. 14]. It was translated as *tension*, [Hel66, p. 122]. According to Elkana, Helmholtz coined the phrase "Spannkraft" for the clearly

ture (which in hindsight seems especially appropriate to the mechanical theory of heat) the *Principle of Vis Viva* assumes the form

$$(\text{vis viva}) + (\text{potential}) = \text{constant} \ .$$

At the same time another general principle of mechanics, the *Hamiltonian Principle*, finds its expression in the formula

$$\delta \int [(\text{vis viva}) - (\text{potential})] dt = 0 \ ,$$

where the integration is carried out over any chosen time interval, and where δ designates the *internal variation*, that is a variation which does not affect the limits but only the inside of that time interval.

If I now notice that we know the potential by the given forces, but also that, inversely, by specifying the potential the forces are determined, and if I accordingly allow myself to consider the potential as primary, as the actual *driver of impulse to motion*, and the forces as secondary, as the *form* in which the impulse manifests itself, this is not a real but at best a formal innovation. On the other hand, what is essentially new (albeit related to the conjecture already made by Riemann) is my assumption that *the motive impulse represented by the potential does not pass from one mass point to the other instantaneously, but progressively, that it propagates in space with a certain albeit extremely great velocity.* This velocity is considered constant and will be designated by c.[36]

The idea just mentioned, and the assumption that *Hamilton's Principle* is applicable without restriction, form the basis of my investigation; they form the source from which the laws of electric phenomena (discovered by Ampère, Weber, and my father)[37] come out on their own, without bringing in any further assumption.

It is scarcely necessary to remark that the *ordinary* conception of an instantaneous propagation of the potential is contained as a special case in the conception *put forth here* of a progressive propagation, namely that this conception goes over to the ordinary one as soon as one sets the constant $c = \infty$.

5.1.2 Weber's Law

First consider only *two* points m and m_1, which move under their mutual influence. Then, proceeding from the conception of a progressive propagation of the potential, for each given instant of time t, two different potentials appear, the emissive and the receptive.

The *emissive potential* is that which *is sent out at the time t* from each of the two points, and which therefore reaches the other point a little later. Let r represent the distance of

---

defined mechanical entity that we call "potential energy", [Elk70, p. 280]. Caneva translated it as "tensional force", [Can19].

[36][Note by LH:] Weber's constant c is not the speed of light, being equal to $\sqrt{2}$ times the speed of light.

[37][Note by AKTA:] Carl Neumann is referring to André-Marie Ampère (1775-1836), Wilhelm Eduard Weber (1804-1891) and Franz Ernst Neumann (1798-1895). Ampère's main works were mentioned on footnote 8 on page 10.

Weber's force between two electrified particles was published in 1846, [Web46], with partial French translation in [Web87] and a complete English translation in [Web07].

Franz Neumann's works on induction can be found in [Neu46] and [Neu47], with French translation in [Neu48a]; [Neu48b] and [Neu49].

the two points at time t, and $\tilde{\omega}$ the emissive potential corresponding to the same time, then according to Newton's law $\tilde{\omega} = \frac{mm_1}{r}$, or generally:[38]

$$\tilde{\omega} = mm_1\varphi \ , \tag{5.1}$$

where $\varphi = \varphi(r)$ represents any given function of r.

The *receptive potential* on the other hand, is that which *is received at time* t and which therefore was already sent out a little earlier from the other point. The receptive potential belonging to the *given* time is accordingly identical to the emissive potential of an *earlier* time. The distance at time t is again designated as r, and the receptive potential corresponding to that time is ω, so there results after some calculation

$$\omega = w + \frac{d\mathfrak{w}}{dt} \ , \tag{5.2}$$

where

$$\begin{aligned} w &= mm_1\left[\varphi + \left(\frac{d\psi}{dt}\right)^2\right] \ , \\ \mathfrak{w} &= mm_1\left[\chi + \frac{d\Phi}{dt}\right] \ . \end{aligned} \tag{5.3}$$

Here φ is the function contained in the emissive potential; and at the same time ψ, χ, Φ are certain other functions, also only depending upon r, which allow derivation out of the given function φ through fairly simple operations. So, for example

$$\psi = \frac{1}{c}\int\sqrt{-r\frac{d\varphi}{dr}}dr \ . \tag{5.4}$$

The function φ is, as emerges directly from its definition, independent of the propagation velocity c; φ, χ however are affected by the factor $\frac{1}{c}$ and Φ with the factor $\frac{1}{c^2}$.

$$\left.\begin{array}{c}\textit{Still to be noticed is that for the case of the} \\ \textit{Newtonian emission law, namely for } \varphi = \frac{1}{r}, \textit{ the function} \\ \psi \textit{ assumes the value: } \psi = \frac{2\sqrt{r}}{c} \ . \end{array}\right\} \tag{5.5}$$

Of the two parts of the receptive potential, we denote w as the *effective potential* and the other one $\frac{d\mathfrak{w}}{dt}$ the *ineffective potential*.

Since Hamilton's principle is considered to be valid without limitation, one has to be able to derive the dynamics of the points m and m_1 from the formula

$$\delta\int(\tau - \omega)dt = 0 \ ,$$

where τ is the vis viva of the two points and ω the already mentioned receptive potential.

Substituting for ω its value (5.2), the formula reduces to

[38][Note by AKTA:] Neumann numbered the equations in each Section of his paper beginning with (1). This creates a possible misunderstanding related to which specific equation he might be referring to in later portions of the work. In this English translation we numbered sequentially the equations of the whole paper.

$$\delta \int (\tau - w) dt = 0 \ .$$

If one carries out the variation from this expression, the six differential equations needed to determine the dynamics follow. These equations explain *how* the dynamics take place, i.e., they explain the *force* acting between the two points. The result obtained in this way is the following:

I. *A force, R, acts between the two points as they move, the force acting along the straight line r connecting the points at each point in time.*

II. *If one considers this force as a repulsive one and if w is the (already mentioned) effective potential of the two points, then R equals the negative variation coefficient of w with respect to r.*[39]

An immediate consequence of this is

$$R = mm_1 \left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} \right] \ . \tag{5.6}$$

In the special case mentioned in (5.5), namely $\varphi = \frac{1}{r}$ and $\psi = \frac{2\sqrt{r}}{c}$, this formula becomes

$$R = mm_1 \left[\frac{1}{r^2} + \frac{4}{c^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2} \right] \ . \tag{5.7}$$

Formula (5.6) precisely coincides with the law on which I based ten years ago my study dealing with the magnetic rotation of the plane of polarization of light. And formula (5.7) is literally *identical to Weber's law.*[40]

A closer examination leads to the following additional results

III. *If W is the effective potential of an arbitrary system of points and if x, y, z are the coordinates of the point having the mass m, the components of the force acting on m become equal to the negative variation coefficient of W with respect to x, y, z.*

[39][Note by UF:] In German: *dem negativen Variationscoefficienten von w nach r*. Neumann invented the German term "Variationskoefficient". This term does not seem to be used anymore in this context. It arises in statistics, but there it has a quite different meaning than the one Neumann assigned to it. The statistics term is translated into English as "coefficient of variation".

[40][Note by AKTA:] That is:

$$\begin{aligned} R &= mm_1 \left[\frac{1}{r^2} + \frac{4}{c^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2} \right] = mm_1 \left[\frac{1}{r^2} + \frac{2}{c^2\sqrt{r}}\frac{d(\dot{r}r^{-1/2})}{dt} \right] \\ &= mm_1 \left[\frac{1}{r^2} + \frac{2}{c^2\sqrt{r}} \left(\ddot{r}r^{-1/2} - \frac{1}{2}r^{-3/2}\dot{r}^2 \right) \right] = \frac{mm'}{r^2} \left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2} \right] \ . \end{aligned}$$

Remembering that Neumann's "electric masses" m and m_1 represent the charges of the particles which Weber represented by e and e', then this expression is equivalent to Weber's equation of 1852, [Web52b, p. 366 of Weber's *Werke*] with English translation in [Web21a], namely:

$$\frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) = \frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right] \ .$$

IV. *If P is the component of that force in an arbitrary given direction p, then P equals the negative variation coefficient of W with respect to p.*

The term variation coefficient used several times needs a short explanation. Suppose that u, v, ... w are *undetermined* functions of a base variable (for example the time), or *undetermined* functions of any number of base variables α_1, α_2, ... α_n, and G is a *given* expression from the variables α_1, α_2, ... α_n, from the functions u, v, ... w and from some derivatives of these functions with respect to these variables. Then it is well-known that the *internal variation* coming from a change of u, v, ... w

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \dots d\alpha_n$$

always can be put into the form

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \cdots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v + \cdots + c\delta w\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n \ ,$$

in which the coefficients a, b, ... c only depend on α_1, α_2, ... α_n, u, v, ... w, being independent of the variations δu, δv, ... δw. These coefficients $a, b, \dots c$ I call *variation coefficients with respect to* u, v, ... w.

5.1.3 The Laws of Electric Repulsion and Induction

Since the hypotheses employed led to Weber's universal law of electrical action, they obviously must also guide us to those special laws regarding the repulsion and induction of electric currents, which were discovered earlier and later unified under Weber's law. Nevertheless I examined this topic more closely and found,[41] that for the deduction of the known special laws it almost does not matter if one starts from the *two-fluid* or *one-fluid* theory of electrical current. A difference in this respect only shows up in the laws of induction and here as well in the (probably still not sufficiently examined) cases dealing with induction of *non-closed* currents.

Let ds denote an element of an electric current. Moreover, $+eds$ and $-eds$ are the quantities of positive and negative electric fluids contained in it. Finally $s' = \frac{\partial s}{\partial t}$ and S' are the velocities of these quantities with respect to *one and the same* direction s.

Putting $S' = -s'$, then both fluids move with the same speed in opposite directions. This is in full correspondence with the two-fluid theory one usually starts with.

However, if one puts $S' = 0$ the negative fluid is considered to be attached to the ponderable matter or even to be identical with this matter. In this case only *one* fluid is moving. This latter point of view I denoted before as the unitary one.

If one follows simultaneously *both* concepts and keeps the function φ in the emissive potential undetermined, one obtains the following results. Here ds, eds, $s' = \frac{\partial s}{\partial t}$ have the already mentioned meaning and $d\sigma$, $\eta d\sigma$, $\sigma' = \frac{\partial \sigma}{\partial t}$ have an analogous meaning with respect to a second current element.

I. *Assume that W is the effective potential of the two current elements and r their distance, then*

[41] [Note by CN:] What I state concerning electric repulsion and induction as a result of my studies will not be justified and carried further in the current article. I plan to do this in a later note.

$$W = \frac{(2n)^2 ds d\sigma \cdot es'\,\eta\sigma'}{2}\frac{\partial\psi}{\partial s}\frac{\partial\psi}{\partial\sigma}\ , \tag{5.8}$$

where ψ represents the function mentioned in (5.4) and n is an integer, which $= 2$ or $= 1$ depending on whether one assumes the two-fluid or one-fluid theory.

As mentioned in (5.5) for the special case $\varphi = \frac{1}{r}$ one has $\psi = \frac{2\sqrt{r}}{c}$. In this case the value of the potential becomes

$$W = \left(\frac{2n}{c}\right)^2 \frac{ds d\sigma \cdot es'\,\eta\sigma'}{2r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial\sigma}\ . \tag{5.9}$$

II. *The repulsive force $\mathfrak{R}$ by which the two current elements act on each other equals the negative variation coefficient of the potential W with respect to r.*

From that follows the formula

$$\mathfrak{R} = (2n)^2 ds d\sigma \cdot es'\,\eta\sigma'\,\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial s\partial\sigma}\ , \tag{5.10}$$

which in the case $\varphi = \frac{1}{r}$, $\psi = \frac{2\sqrt{r}}{c}$ becomes

$$\mathfrak{R} = \left(\frac{2n}{c}\right)^2 \frac{2 ds d\sigma \cdot es'\,\eta\sigma'}{\sqrt{r}}\frac{\partial^2\sqrt{r}}{\partial s\partial\sigma}\ . \tag{5.11}$$

However, this last formula is *identical to Ampère's law*, as follows easily.

III. *If $d\sigma$ and ds are two elements of closed currents, and $\mathfrak{E}$ denotes the electromotive force exerted by $d\sigma$ on ds along the direction s, then $\mathfrak{E}$ equals the negative variation coefficient of W with respect to s.*

This formula, which is valid in general, whether the induction is due to a change of relative position or a change of the intensity of the current, immediately leads to the formula

$$\mathfrak{E} = \frac{d\overline{W}}{dt}\ , \tag{5.12}$$

if one understands by $\overline{W}$ the value of the potential W for $s' = 1$. This formula precisely represents *the induction law as stated by my father.*

IV. Up to this point there is a complete correspondence between the results obtained from the two-fluid and one-fluid theories. However, I examined as well the case of induction between *non-closed* currents and found that in this case there is quite a *difference* between the results obtained from the two points of view.

5.1.4 The Principle of Vis Viva

Our assumption is that Hamilton's principle is valid without restriction. An immediate consequence of this assumption is that the principle of vis viva always holds as well. However, it might change its usual form.

If only *two* points m and m_1 are given and w is the effective potential of the two points, then according to (5.3) we have

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dt}\right)^2\right] , \tag{5.13}$$

or equivalently

$$w = u + v , \tag{5.14}$$

where u and v are given by

$$\begin{aligned} u &= mm_1\varphi , \\ v &= mm_1\left(\frac{d\psi}{dt}\right)^2 . \end{aligned} \tag{5.15}$$

From the meaning of φ and ψ (compare (5.1) and (5.4)) it follows that u is independent of the propagation velocity c, whereas v is affected by the factor $\frac{1}{c^2}$. On the other hand one immediately sees from (5.15), that v vanishes, as soon as the two points are at *rest*, and that in this case the potential w becomes u. For this reason I call u the *static* and v the *motive potential*.[42] It is worth noticing that the static potential coincides with the emissive potential, which follows not only from the formulas, but also directly from the definition of these potentials.

We consider now the dynamics of an arbitrary system of points and let W be its effective potential. We decompose W (as was done for w) into two terms

$$W = U + V . \tag{5.16}$$

The term U independent from c represents the static potential, while the term V affected by the factor $\frac{1}{c^2}$ represents the motive potential. Using these notions we will show the validity of the following theorem for the vis viva:

During the movement of an arbitrary system of points the vis viva, increased by the static and decreased by the motive potential, always has the same value. Mathematically one has

$$T + U - V = \text{constant} , \tag{5.17}$$

where T is the vis viva of the system. In the case of instantaneous propagation, i.e., for $c = \infty$, the expression V affected by the factor $\frac{1}{c^2}$ vanishes. In this case the formula (5.17) simplifies to the well-known formula $T + U = \text{constant}$.

As regards the expressions T, U, V, we remark that the first one only depends on the velocities of the points, the second one only on their relative position and the third one simultaneously on *both* the velocities and the relative position.

[42][Note by AKTA:] In German: *Das motorische Potential*.

5.2 The Variation Coefficients

5.2.1 Preliminary Remark

If f and φ are functions of the three variables α, β, γ the following equations hold

$$\begin{aligned} f\frac{\partial^3\varphi}{\partial\alpha\partial\beta\partial\gamma} &= \frac{\partial}{\partial\alpha}\left(f\frac{\partial^2\varphi}{\partial\beta\partial\gamma}\right) - \frac{\partial f}{\partial\alpha}\frac{\partial^2\varphi}{\partial\beta\partial\gamma} \ , \\ \frac{\partial f}{\partial\alpha}\frac{\partial^2\varphi}{\partial\beta\partial\gamma} &= \frac{\partial}{\partial\beta}\left(\frac{\partial f}{\partial\alpha}\frac{\partial\varphi}{\partial\gamma}\right) - \frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial\varphi}{\partial\gamma} \ , \\ \frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial\varphi}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\left(\frac{\partial^2 f}{\partial\alpha\partial\beta}\varphi\right) - \frac{\partial^3 f}{\partial\alpha\partial\beta\partial\gamma}\varphi \ . \end{aligned}$$

If these equations are multiplied by $(-1)^0$, $(-1)^1$, $(-1)^2$, respectively, and than added together one gets

$$\begin{aligned} \frac{\partial^3\varphi}{\partial\alpha\partial\beta\partial\gamma} &= \frac{\partial}{\partial\alpha}\left((-1)^0 f\frac{\partial^2\varphi}{\partial\beta\partial\gamma}\right) + \frac{\partial}{\partial\beta}\left((-1)^1\frac{\partial f}{\partial\alpha}\frac{\partial\varphi}{\partial\gamma}\right) \\ &\quad + \frac{\partial}{\partial\gamma}\left((-1)^2\frac{\partial^2 f}{\partial\alpha\partial\beta}\varphi\right) + (-1)^3\frac{\partial^3 f}{\partial\alpha\partial\beta\partial\gamma}\varphi \ . \end{aligned} \tag{5.18}$$

Analogously, if f and φ are functions of arbitrarily many (for instance p) variables α, β, $\ldots$ π, one obtains a formula of the following form

$$f\frac{\partial^p\varphi}{\partial\alpha\partial\beta\ldots\partial\pi} = \frac{\partial A}{\partial\alpha} + \frac{\partial B}{\partial\beta} + \cdots + \frac{\partial P}{\partial\pi} + (-1)^p\frac{\partial^p f}{\partial\alpha\partial\beta\ldots\partial\pi}\varphi \ . \tag{5.19}$$

Let there be in total n variables α_1, α_2, $\ldots$ α_n on which f, φ depend, and let α, β, $\ldots$ π represent any number of these n variables each one with arbitrary many repetitions, then the formula (5.19) is still valid. If one multiplies that formula by $d\alpha_1 d\alpha_2\ldots d\alpha_n$ and integrates over an arbitrary domain, it follows that

$$\begin{aligned} &\int^{(n)} f\frac{\partial^p\varphi}{\partial\alpha\partial\beta\ldots\partial\pi}d\alpha_1 d\alpha_2\ldots d\alpha_n \\ &= \Sigma + (-1)^p\int^{(n)}\frac{\partial^p f}{\partial\alpha\partial\beta\ldots\partial\pi}\varphi d\alpha_1 d\alpha_2\ldots d\alpha_n \ , \end{aligned} \tag{5.20}$$

where Σ is a sum of $(n-1)$-fold integrals on the *boundary* of the domain of integration. Moreover, it follows from the meaning of A, B, $\ldots$ P that *these integrals vanish when the function φ and its derivatives vanish at that boundary.*

5.2.2 Definition of the Variation Coefficients

Assume that u is an *undetermined* function in the variables α_1, α_2, $\ldots$ α_n. As before α, β, $\ldots$ π is an arbitrary selection of these variables each with arbitrary many repetitions. We abbreviate

$$\frac{\partial^p u}{\partial\alpha\partial\beta\ldots\partial\pi} = u' \ . \tag{5.21}$$

Moreover,

$$G = G(\alpha_1,\, \alpha_2,\, \ldots\, \alpha_n,\, u,\, u') \tag{5.22}$$

is a *given* expression from those variables as well as from u and u'. We have to examine the variation of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n \tag{5.23}$$

over an arbitrary given domain for a modification of u, under the simplifying assumption that the function u and all its derivatives are fixed at the boundary. In the future we refer to this as *internal variation* of this integral. For that we immediately obtain

$$\begin{aligned}\delta\int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n &= \int^{(n)} \delta G \cdot d\alpha_1 d\alpha_2 \ldots d\alpha_n \\ &= \int^{(n)} \left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u'\right) d\alpha_1 d\alpha_2 \ldots d\alpha_n .\end{aligned} \tag{5.24}$$

By (5.21) we have

$$\frac{\partial G}{\partial u'}\delta u' = \frac{\partial G}{\partial u'}\frac{\partial^p \delta u}{\partial\alpha\partial\beta\ldots\partial\pi} \ , \tag{5.25}$$

hence according to (5.20)

$$\begin{aligned}&\int^{(n)} \frac{\partial G}{\partial u'}\delta u' d\alpha_1 d\alpha_2 \ldots d\alpha_n \\ =\ & \Sigma + (-1)^p \int^{(n)} \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi}\frac{\partial G}{\partial u'} \cdot \delta u d\alpha_1 d\alpha_2 \ldots d\alpha_n \ .\end{aligned} \tag{5.26}$$

The previously mentioned case where Σ vanishes takes place here. In fact the function δu vanishes with all its derivatives at the *boundary* of the integration domain, since the variation is an *inner* one. Consequently, by substituting (5.26) into (5.24) one has

$$\delta\int^{(n)} G d\alpha_1 d\alpha_2 \cdots d\alpha_n = \int^{(n)} a\delta u d\alpha_1 d\alpha_2 \cdots d\alpha_n \ , \tag{5.27}$$

where a is given by

$$a = \frac{\partial G}{\partial u} + (-1)^p \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi}\frac{\partial G}{\partial u'} \ , \tag{5.28}$$

i.e.,

$$a = \frac{\partial G}{\partial u} + (-1)^p \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi}\frac{\partial G}{\partial\frac{\partial^p u}{\partial\alpha\partial\beta\ldots\partial\pi}} \ . \tag{5.29}$$

We abbreviate this quantity by

$$a = \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} \ , \tag{5.30}$$

where $D_{u'}$ indicates the differentiation with respect to all variables used to build the derivative u'. The symbol $\varepsilon_{u'}$ denotes a number, which is either $+1$ or -1 depending if u' is a derivative of even or odd order.

Analogously a more general task can be carried out. Assume that u is an *undetermined* function of the variables α_1, α_2, … α_n and u', u'', … are arbitrarily many derivatives of this function of arbitrarily high degree. Moreover,

$$G = G(\alpha_1,\, \alpha_2,\, \ldots\, \alpha_n,\, u,\, u',\, u''\, \ldots) \tag{5.31}$$

is a *given* expression of those variables, functions and derivatives. Then for the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots \alpha_n \tag{5.32}$$

one obtains the following value:

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} a \delta u d\alpha_1 d\alpha_2 \ldots d\alpha_n \ , \tag{5.33}$$

where using the notion introduced in (5.30) one can express a by

$$a = \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} + \varepsilon_{u''} D_{u''} \frac{\partial G}{\partial u''} + \cdots \ . \tag{5.34}$$

With the same ease an even more general task can be treated. Assume that u, v, … w are arbitrarily many *undetermined* functions of the variables α_1, α_2, … α_n. Moreover, let G be a *given* expression composed from the variables α, the functions u, v, … w and arbitrary many derivatives of these functions with respect to α. The task at hand is to determine the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n \tag{5.35}$$

by simultaneous perturbation of u, v, … w. It is easy to see that the result in this case is

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v + \ldots + c\delta w\big) d\alpha_1 d\alpha_2 \ldots d\alpha_n \ , \tag{5.36}$$

where a, b, … c are given by

$$\begin{aligned} a &= \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} + \varepsilon_{u''} D_{u''} \frac{\partial G}{\partial u''} + \cdots \ , \\ b &= \frac{\partial G}{\partial v} + \varepsilon_{v'} D_{v'} \frac{\partial G}{\partial v'} + \varepsilon_{v''} D_{v''} \frac{\partial G}{\partial v''} + \cdots \ , \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c &= \frac{\partial G}{\partial w} + \varepsilon_{w'} D_{w'} \frac{\partial G}{\partial w'} + \varepsilon_{w''} D_{w''} \frac{\partial G}{\partial w''} + \cdots \ . \end{aligned} \tag{5.37}$$

Here it is understood that

$$\begin{array}{c} u',\, u'',\, \ldots\ , \\ v',\, v'',\, \ldots\ , \\ \ldots\ldots\ldots \\ w',\, w'',\, \ldots \end{array}$$

are the derivatives of u, v, ... w, on which G depends.

It seems appropriate to call the quantities a, b, ... c used to represent the variation of the integral of G the *variation coefficients of* G *with respect to* u, v, ... w (cf. page 23). We denote them in an analogous way to the differential coefficients with the only difference that we use Δ instead of ∂.[43] With this convention we have

$$\begin{aligned} a &= \frac{\Delta G}{\Delta u}\ , \\ b &= \frac{\Delta G}{\Delta v}\ , \\ &\ldots\ldots \\ c &= \frac{\Delta G}{\Delta w}\ . \end{aligned} \tag{5.38}$$

The lowercase letter δ is reserved to denote the variation itself.

As follows from (5.37) the variation coefficients of G with respect to u, v, ... w transform to the differential coefficients $\frac{\partial G}{\partial u}$, $\frac{\partial G}{\partial v}$, ... $\frac{\partial G}{\partial w}$ as soon as the expression G only contains the functions u, v, ... w themselves, but not their derivatives.

5.2.3 A Theorem on Variation Coefficients

For the following discussion we need to derive a theorem which in many cases simplifies computations involving variation coefficients. I noted this result before in “Untersuchungen über Elasticität” which appeared in Crelle’s *Journal*, Vol. 57, p. 299.[44]

Apart from the variables α_1, α_2, ... α_n and the m undetermined functions u, v, ... w we might have an additional M new undetermined functions U, V, ... W, which also only depend on α_1, α_2, ... α_n, but are connected to the previous functions u, v, ... w by certain prescribed relations

$$\begin{aligned} U &= \varphi(\alpha_1,\, \alpha_2,\, \ldots \alpha_n,\, u,\, v,\, \ldots\, w)\ , \\ V &= \psi(\alpha_1,\, \alpha_2,\, \ldots \alpha_n,\, u,\, v,\, \ldots\, w)\ , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ W &= \chi(\alpha_1,\, \alpha_2,\, \ldots \alpha_n,\, u,\, v,\, \ldots\, w)\ . \end{aligned} \tag{5.39}$$

[43] [Note by CN:] According to my knowledge the term *differential coefficient* is seldom used, always in the same meaning as *derivative* or *differential quotient*. Analogously like the term *differential coefficient*, we refer here to *variation coefficient*. (In the original article from 1868 instead of Δ a reversed ρ was used).

[44] [Note by AKTA:] [Neu60, p. 299].

M might be bigger or smaller than m or the two integers might be equal.

We assume that G is a *given* expression of the variables α_1, α_2, ... α_n, of the functions U, V, ... W and of the derivatives of arbitrary high degree of these functions. We want to determine the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \dots d\alpha_n \tag{5.40}$$

subject to a perturbation of u, v, ... w. This task can be solved in two ways.

First way. As soon as u, v, ... w are varied by arbitrary given quantities δu, δv, ... δw, the functions U, V, ... W contained in G are varied by quantities δU, δV, ... δW which in view of the relations (5.39) can be expressed as

$$\begin{aligned}
\delta U &= \frac{\partial U}{\partial u}\delta u + \frac{\partial U}{\partial v}\delta v \dots + \frac{\partial U}{\partial w}\delta w \ , \\
\delta V &= \frac{\partial V}{\partial u}\delta u + \frac{\partial V}{\partial v}\delta v \dots + \frac{\partial V}{\partial w}\delta w \ , \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\delta W &= \frac{\partial W}{\partial u}\delta u + \frac{\partial W}{\partial v}\delta v \dots + \frac{\partial W}{\partial w}\delta w \ .
\end{aligned} \tag{5.41}$$

As a consequence of these perturbations δU, δV, ... δW the integral (5.40) will be subject to a variation described by

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \dots d\alpha_n = \int^{(n)} \big(A\delta U + B\delta V \dots + C\delta W\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n, \tag{5.42}$$

where A, B, ... C are the variation coefficients of G with respect to U, V, ... W.

Second way. One can *eliminate* the functions U, V, ... W contained in G and their derivatives by replacing them by the functions u, v, ... w and their derivatives using the relations (5.39). Doing this, the change in the integral (5.39) which arises on the basis of the given changes δu, δv ... δw, is represented by the formula

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \dots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v \dots + c\delta w\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n \ , \tag{5.43}$$

where a, b, ... c are the variation coefficients of G with respect to u, v, ... w.

Comparison of the results. The results obtained in (5.42) and (5.43) have to agree for arbitrary values of δu, δv, ... δw under the hypothesis that by δU, δV, ... δW one understands the expressions found in (5.41). For example the coefficient of δu in (5.43) has to be the same as in (5.42). Therefore

$$a = A\frac{\partial U}{\partial u} + B\frac{\partial V}{\partial u} \dots + C\frac{\partial W}{\partial u}.$$

Analogous formulas one gets by equating the coefficients of δv, ... δw.

Using the notion just introduced for the variation coefficients, then these formulas become

$$
\begin{aligned}
\frac{\Delta G}{\Delta u} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial u} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial u}\cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial u} \ , \\
\frac{\Delta G}{\Delta v} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial v} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial v}\cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial v} \ , \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\Delta G}{\Delta w} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial w} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial w}\cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial w} \ .
\end{aligned}
\tag{5.44}
$$

These formulas are the theorem we wanted to prove . It can be seen as a generalization of a known theorem in calculus. In case the expression G only depends on U, V, $\ldots W$, but not on their derivatives, then after the elimination by the relations (5.39) it depends as well only just on u, v, $\ldots w$, but not on the derivatives of these functions. In such a case the variation coefficients appearing in (5.44) become the corresponding differential coefficients and the formulas themselves turn into well-known formulas of calculus.

To make the general theorem contained in (5.44) clear, we remark that if there is just one function u, v, $\ldots w$ and just one function U, V, $\ldots W$ as well, then the assertion becomes the following.

If G depends on an undetermined function U and its derivative, and if the function U in turn depends on a different undetermined function u, then the variation coefficient of G with respect to u is obtained by building the variation coefficient of G with respect to U and multiplying it by the differential coefficient of U with respect to u. The formula

$$
\frac{\Delta G}{\Delta u} = \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial u}
\tag{5.45}
$$

holds. Here u and U are functions in arbitrary many variables α_1, α_2, $\ldots$ α_n and the derivatives of these functions are derivatives with respect to α_1, α_2 $\ldots$ α_n where the differentiation with respect to each of these variables can be repeated as many times as one likes.

5.3 The Emissive and Receptive Potential

We[45,46] consider two points m and m_1 moving under their mutual interaction. We denote their distance for a *given* moment of time t by r, and for a *previous* moment of time $t - \Delta t$ by $r - \Delta r$. Putting

$$
r = f(t) \ ,
\tag{5.46}
$$

the function f is to be understood as *unknown*, like the dynamics of the points. Anyway one has to put as well

$$
r - \Delta r = f(t - \Delta t) \ ,
\tag{5.47}
$$

or equivalently

$$
r - \Delta r = f(t) - \frac{\Delta t}{1} f'(t) + \frac{\Delta t^2}{1 \cdot 2} f''(t) - \ldots \ .
\tag{5.48}
$$

[45] [Note by CN:] *More details* about this (probably a bit too short) Section can be found in these *Annalen*, Vol. 1, pp 317–324.

[46] [Note by AKTA:] [Neu69a] with English translation in [Neu20b], see Chapter 6.

From (5.46) we have the equations

$$\frac{dr}{dt} = f'(t) \ , \quad \frac{d^2r}{dt^2} = f''(t) \ , \quad \ldots$$

so that the formula above becomes

$$r - \Delta r = r - \frac{\Delta t}{1}\frac{dr}{dt} + \frac{\Delta t^2}{1 \cdot 2}\frac{d^2r}{dt^2} - \ldots \ . \tag{5.49}$$

Using the notions introduced before (see page 20) we denote by $\tilde{\omega}$ the *emissive potential* of the two points at time t. We have

$$\tilde{\omega} = mm_1\varphi(r) \ , \tag{5.50}$$

where $\varphi(r)$ is any *given* function which in the case of Newton's law would be $\frac{1}{r}$.

On the other hand we denote the *receptive potential* of the two points at time t by ω. To fix the ideas we think of m as the absorber and m_1 as the emitter. Then ω is the potential which m receives at time t and which therefore at a *previous time* $t - \Delta t$ was emitted by m_1. In this case ω coincides with the emissive potential at this previous time and has therefore the value

$$\omega = mm_1\varphi(r - \Delta r) \ . \tag{5.51}$$

Using (5.49) this value becomes

$$\omega = mm_1\varphi\left(r - \frac{\Delta t}{1}\frac{dr}{dt} + \frac{\Delta t^2}{1 \cdot 2}\frac{d^2r}{dt^2} - \ldots\right) \ . \tag{5.52}$$

The expression Δt here represents that time, which the potential needs to pass through the path r. Since we denoted the propagation velocity of the potential by c (page 20), i.e., we understand by c the distance through which the potential propagates in time 1, we have $\Delta t : r = 1 : c$, implying

$$\Delta t = \frac{r}{c} \ . \tag{5.53}$$

In the following we assume that the velocity c is huge and therefore the fraction $\frac{r}{c}$ is tiny, so that we can ignore its *third* power. Substituting the value (5.53) into (5.52) we obtain

$$\omega = mm_1\varphi\left(r - \frac{r}{c}\frac{dr}{dt} + \frac{r^2}{2c^2}\frac{d^2r}{dt^2}\right) \ , \tag{5.54}$$

which leads to the expansion

$$\omega = mm_1\left[\varphi - \frac{r}{c}\frac{dr}{dt}\varphi' + \frac{r^2}{2c^2}\frac{d^2r}{dt^2}\varphi' + \frac{r^2}{2c^2}\left(\frac{dr}{dt}\right)^2\varphi''\right] \ , \tag{5.55}$$

or after rearrangement

$$\omega = mm_1\left[\varphi + \frac{r^2\varphi''}{2c^2}\left(\frac{dr}{dt}\right)^2 + \frac{r^2\varphi'}{2c^2}\frac{d^2r}{dt^2} - \frac{r\varphi'}{c}\frac{dr}{dt}\right] \ . \tag{5.56}$$

Here we abbreviated $\varphi(r)=\varphi$, $\frac{d\varphi(r)}{dr}=\varphi'$, $\frac{d^2\varphi(r)}{dr^2}=\varphi''$. If Φ is an arbitrary function of r, we have the following general formulas

$$\begin{aligned}\Phi\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(\Phi\frac{dr}{dt}\right)-\frac{d\Phi}{dr}\left(\frac{dr}{dt}\right)^2 ,\\ \Phi\frac{dr}{dt} &= \frac{d}{dt}\left(\int\Phi dr\right) .\end{aligned}$$

Applying these formulas to the last two terms in the expression (5.56) for ω one gets

$$\begin{aligned}\omega = mm_1\left[\varphi+\frac{r^2\varphi''}{2c^2}\left(\frac{dr}{dt}\right)^2-\frac{(r^2\varphi')'}{2c^2}\left(\frac{dr}{dt}\right)^2\right]\\ +mm_1\frac{d}{dt}\left[\frac{r^2\varphi'}{2c^2}\frac{dr}{dt}-\frac{\int r\varphi' dr}{c}\right] ,\end{aligned} \tag{5.57}$$

where we put $(r^2\varphi')'$ for $\frac{d(r^2\varphi')}{dr}$ which equals $r^2\varphi''+2r\varphi'$. Substituting this value and noting further that $\int r\varphi' dr=r\varphi-\int\varphi dr$, then the expression for ω takes the following form

$$\omega=mm_1\left[\varphi-\frac{r\varphi'}{c^2}\left(\frac{dr}{dt}\right)^2\right]+mm_1\frac{d}{dt}\left[\frac{(\int\varphi dr)-r\varphi}{c}+\frac{r^2\varphi'}{2c^2}\frac{dr}{dt}\right] . \tag{5.58}$$

We thought so far m_1 as emitter and m as absorber of the potential. As one easily sees, the *same* consideration involving the *same* formulas can be carried out in the opposite case where m is the emitter and m_1 the absorber of the potential.

It follows from this that the potential value ω found in (5.58) not only is the one which *reaches* m in the moment t *emitted from* m_1, but simultaneously the one which *reaches* in that instant m_1 *emitted from* m.

We obtained the following result:

If two points m and m_1 are moving under their common interaction, r denoting the distance at time t and moreover ω the receptive potential of the two points corresponding to the same time, then

$$\omega=w+\frac{d\mathfrak{w}}{dt} , \tag{5.59}$$

where w and $\mathfrak{w}$ represent the following expressions:

$$\begin{aligned}w &= mm_1\left[\varphi-\frac{r}{c^2}\frac{d\varphi}{dr}\left(\frac{dr}{dt}\right)^2\right] ,\\ \mathfrak{w} &= mm_1\left[\frac{(\int\varphi dr)-r\varphi}{c}+\frac{r^2}{2c^2}\frac{d\varphi}{dr}\frac{dr}{dt}\right] .\end{aligned} \tag{5.60}$$

Here φ abbreviates $\varphi(r)$ and moreover c is a huge constant speed by which the potential propagates through space.

We further remark that the value of the expression w can be represented more easily by

$$w = mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] , \tag{5.61}$$

where ψ is the function

$$\psi = \int \sqrt{-r \frac{d\varphi}{dr}} \cdot \frac{dr}{c} . \tag{5.62}$$

According to (5.59) the receptive potential consists of the two terms w and $\frac{d\mathfrak{w}}{dt}$. We refer to the first term, namely w, as the effective potential, and the other one, namely $\frac{d\mathfrak{w}}{dt}$, as the ineffective potential.

The notions introduced here seem quite necessary in order to avoid that the following discussion becomes cumbersome. *How* the notions are chosen should become clearer during the exposition.

For the case of Newton's law, namely $\varphi = \frac{1}{r}$, one obtains $\psi = \frac{2\sqrt{r}}{c}$. In this case the formulas (5.59), (5.60), and (5.61) become

$$\omega = w + \frac{d\mathfrak{w}}{dt} , \tag{5.63}$$

$$w = mm_1 \left[\frac{1}{r} + \frac{4}{c^2} \left(\frac{d\sqrt{r}}{dt} \right)^2 \right] = \frac{mm_1}{r} \left[1 + \frac{1}{c^2} \left(\frac{dr}{dt} \right)^2 \right] , \tag{5.64}$$

$$\mathfrak{w} = mm_1 \left[\frac{\log r}{c} - \frac{1}{2c^2} \frac{dr}{dt} \right] . \tag{5.65}$$

5.4 Weber's Law

5.4.1 Derivation of the Law

The task at hand is to determine the dynamics of two points m and m_1 under the hypothesis that the potential emitted by one point reaches the other point at a later time.

For a time t the coordinates of the points are denoted by x, y, z, x_1, y_1, z_1 and their distance to each other by r. Moreover, for that moment, ω is the receptive potential derived in (5.59) up to (5.61):

$$\omega = w + \frac{d\mathfrak{w}}{dt} , \tag{5.66}$$

and τ is their vis viva

$$\tau = \frac{m}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] + \frac{m_1}{2} \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dy_1}{dt} \right)^2 + \left(\frac{dz_1}{dt} \right)^2 \right] . \tag{5.67}$$

As mentioned on page 20, we consider Hamilton's principle applicable without restriction. Therefore the dynamics of the points m and m_1 is characterized by the formula

$$\delta \int (\tau - \omega) dt = 0 \ . \tag{5.68}$$

According to page 20, the integration is carried out over an arbitrary interval of time. By δ we understand the *internal variation*, i.e., the variation which is only concerned with the interior of the time interval, but not its boundaries.

Through the substitution of (5.66) the formula (5.68) takes the form

$$\delta \int \tau dt = \delta \int \left(w + \frac{d\mathfrak{w}}{dt} \right) dt = \delta \left(\mathfrak{w}_{''} - \mathfrak{w}_{'} + \int w dt \right) \ , \tag{5.69}$$

or, since δ is an *internal* variation and therefore $\delta \mathfrak{w}_{''} = \delta \mathfrak{w}_{'} = 0$, one obtains

$$\delta \int \tau dt = \delta \int w dt \ . \tag{5.70}$$

Taking into account that the undetermined functions contained in τ and w are represented by x, y, z and x_1, y_1, z_1, we obtain the following six equations using our previous notation introduced on page 29:

$$\begin{aligned} \frac{\Delta \tau}{\Delta x} &= \frac{\Delta w}{\Delta x} \ , & \frac{\Delta \tau}{\Delta x_1} &= \frac{\Delta w}{\Delta x_1} \ , \\ \frac{\Delta \tau}{\Delta y} &= \frac{\Delta w}{\Delta y} \ , & \frac{\Delta \tau}{\Delta y_1} &= \frac{\Delta w}{\Delta y_1} \ , \\ \frac{\Delta \tau}{\Delta z} &= \frac{\Delta w}{\Delta z} \ , & \frac{\Delta \tau}{\Delta z_1} &= \frac{\Delta w}{\Delta z_1} \ . \end{aligned} \tag{5.71}$$

If one computes the variation coefficients on the left hand side using the value of τ given in (5.67), then the six equations become

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\frac{\Delta w}{\Delta x} \ , & m_1\frac{d^2x_1}{dt^2} &= -\frac{\Delta w}{\Delta x_1} \ , \\ m\frac{d^2y}{dt^2} &= -\frac{\Delta w}{\Delta y} \ , & m_1\frac{d^2y_1}{dt^2} &= -\frac{\Delta w}{\Delta y_1} \ , \\ m\frac{d^2z}{dt^2} &= -\frac{\Delta w}{\Delta z} \ , & m_1\frac{d^2z_1}{dt^2} &= -\frac{\Delta w}{\Delta z_1} \ . \end{aligned} \tag{5.72}$$

These equations show, *that the negative variation coefficients of the righthand side represent the components of that forces, which act on the points during their movement*. To explicitly determine these variation coefficients we observe that by (5.61) the effective potential w has the value

$$w = mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] = mm_1 \left[\varphi + \left(\frac{d\psi}{dr} \frac{dr}{dt} \right)^2 \right] \ . \tag{5.73}$$

In particular, it depends on r and $\frac{dr}{dt}$, where r itself depends on the undetermined functions x, y, z, x_1, y_1, z_1 through the equation

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \ . \tag{5.74}$$

Therefore the variation coefficients can be computed by the theorem stated on page 31, namely using the formulas

$$\begin{aligned}
\frac{\Delta w}{\Delta x} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial x} \ , & \frac{\Delta w}{\Delta x_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial x_1} \ , \\
\frac{\Delta w}{\Delta y} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial y} \ , & \frac{\Delta w}{\Delta y_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial y_1} \ , \\
\frac{\Delta w}{\Delta z} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial z} \ , & \frac{\Delta w}{\Delta z_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial z_1} \ .
\end{aligned} \tag{5.75}$$

Substituting these expressions into (5.72) and using the values for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, ..., which follow from (5.74), one obtains the equations

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{x - x_1}{r} \ , & m_1\frac{d^2x_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{x_1 - x}{r} \ , \\
m\frac{d^2y}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{y - y_1}{r} \ , & m_1\frac{d^2y_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{y_1 - y}{r} \ , \\
m\frac{d^2z}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{z - z_1}{r} \ , & m_1\frac{d^2z_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{z_1 - z}{r} \ .
\end{aligned} \tag{5.76}$$

It remains to compute the variation coefficient $\frac{\Delta w}{\Delta r}$. Abbreviating $\frac{dr}{dt}$ by r' and $\frac{d^2r}{dt^2}$ by r'', it follows from (5.73) that

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dr}r'\right)^2\right] \ . \tag{5.77}$$

Therefore

$$\begin{aligned}
\frac{\partial w}{\partial r} &= mm_1\left[\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dr^2}(r')^2\right] \ , \\
\frac{\partial w}{\partial r'} &= mm_1 \cdot 2\left(\frac{d\psi}{dr}\right)^2 r' \ ,
\end{aligned}$$

or equivalently

$$\frac{\partial w}{\partial r} = mm_1\left[\frac{d\varphi}{dr} + 2\frac{d\psi}{dt}\frac{d\frac{d\psi}{dr}}{dt}\right] \ , \tag{5.78}$$

$$\frac{\partial w}{\partial r'} = mm_1 \cdot 2\frac{d\psi}{dr}\frac{d\psi}{dt} \ . \tag{5.79}$$

Differentiating the last formula we obtain

$$\frac{d\frac{\partial w}{\partial r'}}{dt} = mm_1\left[2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} + 2\frac{d\psi}{dt}\frac{d\frac{d\psi}{dr}}{dt}\right] \ . \tag{5.80}$$

Since w only depends on r and r' by (5.77), one has

$$\frac{\Delta w}{\Delta r} = \frac{\partial w}{\partial r} - \frac{d\frac{\partial w}{\partial r'}}{dt} \ . \tag{5.81}$$

Therefore by (5.78) and (5.80)

$$\frac{\Delta w}{\Delta r} = mm_1 \left[\frac{d\varphi}{dr} - 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} \right] . \tag{5.82}$$

From (5.76) and (5.82) the following theorems follow:

Between two points m and m_1 a force R is acting during their movement, which at each moment coincides with the connecting line r.

If one considers this force R as a repulsive one and if w is the effective potential of the two points with respect to each other, then R equals at each moment the negative variation coefficient of w with respect to r, so that it has the value

$$R = -\frac{\Delta w}{\Delta r} \ . \tag{5.83}$$

In case the emission law of the potential is arbitrary, i.e., the emissive potential equals $mm_1\varphi(r)$, where φ is an arbitrary function, using the abbreviation

$$\begin{aligned} \varphi(r) &= \varphi \ , \\ \frac{1}{c}\int \sqrt{-r\frac{d\varphi}{dr}}dr &= \psi(r) = \psi \ , \end{aligned} \tag{5.84}$$

the values of the effective potential w and the force R become

$$\begin{aligned} w &= mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] \ , \\ R = -\frac{\Delta w}{\Delta r} &= mm_1 \left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} \right] . \end{aligned} \tag{5.85}$$

In the special case of Newton's emission law one has

$$\begin{aligned} \varphi &= \frac{1}{r} \ , \\ \psi &= \frac{2\sqrt{r}}{c} \ , \end{aligned} \tag{5.86}$$

and therefore

$$\begin{aligned} w &= mm_1 \left[\frac{1}{r} + \frac{4}{c^2}\left(\frac{d\sqrt{r}}{dt} \right)^2 \right] \ , \\ R = -\frac{\Delta w}{\Delta r} &= mm_1 \left[\frac{1}{r^2} + \frac{4}{c^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2} \right] \ , \end{aligned} \tag{5.87}$$

i.e.,

$$R = \frac{mm_1}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right] .$$

Here c always represents the constant but huge speed through which the potential propagates in space.[47]

The general formula (5.85) coincides completely with the law I supposed in my PhD thesis "*Explicare tentatur quomodo fiat, ut lucis planum polarisationis per vires electricas vel magneticas declinetur. Halis Saxonum 1858*,"[48] which discussed the mutual interaction of an *electric* and an *aether particle*. In fact formula (5.85) can be written as follows

$$R = mm_1\left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dr^2}\left(\frac{dr}{dt}\right)^2 + 2\left(\frac{d\psi}{dr}\right)^2\frac{d^2r}{dt^2}\right] . \tag{5.88}$$

Putting

$$-\frac{d\varphi}{dr} = F , \qquad\qquad 2\left(\frac{d\psi}{dr}\right)^2 = \Phi , \tag{5.89}$$

it becomes

$$R = mm_1\left[F + \frac{1}{2}\frac{d\Phi}{dr}\left(\frac{dr}{dt}\right)^2 + \Phi\frac{d^2r}{dt^2}\right] . \tag{5.90}$$

[47][Note by CN:] The value of R in (5.85) can also be deduced from the formula

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dt}\right)^2\right]$$

by the following reasoning. According to the theorem on variation coefficients on page 31 we have

$$\frac{\Delta w}{\Delta r} = \frac{\Delta w}{\Delta\varphi}\frac{\partial\varphi}{\partial r} + \frac{\Delta w}{\Delta\psi}\frac{\partial\psi}{\partial r} = mm_1\frac{\partial\varphi}{\partial r} - mm_1\cdot 2\frac{d^2\psi}{dt^2}\frac{\partial\psi}{\partial r} ,$$

hence

$$R = mm_1\left[-\frac{\partial\varphi}{\partial r} + 2\frac{\partial\psi}{\partial r}\frac{d^2\psi}{dt^2}\right] .$$

[48][Note by AKTA:] [Neu58]. See also [Neu63].

However, this is the law supposed in that thesis on page 3.[49,50]

The most important point however is the fact that (5.87) coincides literally with the well-known *law of Weber*.

5.4.2 Addenda

We denote by R the force which acts on m during the movement of the two points m and m_1. Its components we abbreviate by X, Y and Z. According to (5.72) we have the equations

$$\begin{aligned} X &= -\frac{\Delta w}{\Delta x} \,, \\ Y &= -\frac{\Delta w}{\Delta y} \,, \\ Z &= -\frac{\Delta w}{\Delta z} \,. \end{aligned} \tag{5.94}$$

We think that there is a line through m whose direction is determined by the direction cosine α, β, γ. We denote the component of the force R in this direction by P. Then

$$P = X\alpha + Y\beta + Z\gamma = -\left[\frac{\Delta w}{\Delta x}\alpha + \frac{\Delta w}{\Delta y}\beta + \frac{\Delta w}{\Delta z}\gamma\right] . \tag{5.95}$$

We think for a moment that the motion of the point m or x, y, z is constrained to that line. We thus put

$$x = a + p\alpha \,, \qquad y = b + p\beta \,, \qquad z = c + p\gamma \,,$$

where a, b, c is a fixed point of the line and p is the distance between this point and the point x, y, z. We then have

[49][Note by CN:] According to (5.84), the formulas (5.89) can as well be written as

$$-\frac{d\varphi}{dr} = F \,, \qquad -\frac{2r}{c^2}\frac{d\varphi}{dr} = \Phi \,. \tag{5.91}$$

Therefore one has between F and Φ the relation

$$\frac{2F}{c^2} = \frac{\Phi}{r} \,. \tag{5.92}$$

In the mentioned thesis I kept the relation between F and Φ undetermined, so that there is not the slightest contradiction between that thesis and the theory developed in this paper. The mentioned optical phenomenon I have treated later in more depth in my note "Ueber die Magnetische Drehung der Polarisationsebene des Lichtes. Halle. 1863." Unfortunately I assumed there in order to make the exposition simpler a certain relation between F and G, namely

$$\frac{2F}{c^2} = -\frac{d\Phi}{dr} \,. \tag{5.93}$$

For the *special case* $\varphi = \frac{1}{r}$, i.e., $F = \frac{1}{r^2}$, this is identical to the relation (5.92) and leads as well to the value $\Phi = \frac{2}{c^2 r}$. *But in general* it contradicts (5.92). I remark that the assumption of the relation (5.93) in the above-mentioned paper was not motivated by internal reasons, but just to give the exterior form more simplicity. In fact the function F does not play a role at all in my investigation of the rotation of the plane of polarization. It drops out of the computations quite at the beginning. Therefore the results in that investigation are *the same* whatever relation between F and Φ we assume.

[50][Note by AKTA:] [Neu63].

$$\alpha = \frac{\partial x}{\partial p} , \qquad \beta = \frac{\partial y}{\partial p} , \qquad \gamma = \frac{\partial z}{\partial p} .$$

Consequently the formula (5.95) becomes

$$P = -\left[\frac{\Delta w}{\Delta x}\frac{\partial x}{\partial p} + \frac{\Delta w}{\Delta y}\frac{\partial y}{\partial p} + \frac{\Delta w}{\Delta z}\frac{\partial z}{\partial p}\right] . \tag{5.96}$$

As regards the dependence between w and p, we first remark that w depends on x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, while on the other hand x, y, z depend on p. The expression in square brackets in (5.96) is then nothing else than the variation coefficient of w with respect to p as follows from the theorem on page 23. It follows that

$$P = -\frac{\Delta w}{\Delta p} , \tag{5.97}$$

which is analogous to the formulas (5.94) and contains these as special cases.

In case we have arbitrary many points m, m_1, m_2, m_3, $\cdots$, and denote by w_1, w_2, w_3, ... the effective potentials for each pair of points (m, m_1), (m, m_2), (m, m_3), $\cdots$, then we obtain from (5.97) that the expression

$$-\left(\frac{\Delta w_1}{\Delta p} + \frac{\Delta w_2}{\Delta p} + \frac{\Delta w_3}{\Delta p} + \cdots\right) \tag{5.98}$$

represents that force through which the point m is driven along the direction p by all other points together. This expression can be written more compactly by using the effective potential of the whole system of points, W, in the form

$$-\frac{\Delta W}{\Delta p} . \tag{5.99}$$

Hence the theorem follows:

If W is the effective potential of an arbitrary system of points, the force by which any of these points is driven along a given direction, is always equal the negative variation coefficient of W in that direction.

5.5 The Principle of Vis Viva

5.5.1 Consideration of Two Points

We start with a rather easy case, namely the one where only *two* points m and m_1 exist. Moreover, we assume that m is *moveable*, while m_1 is *fixed*.

Let x, y, z and x_1, y_1, z_1 be the coordinates of the two points, r their distance and furthermore ω the *receptive potential* of the two points. Finally τ is their vis viva.

According to page 33 the receptive potential consists of two parts

$$\omega = w + \frac{d\mathfrak{w}}{dt} . \tag{5.100}$$

We refer to the first term as the *effective* and the last term as the *ineffective* potential. Moreover, according to page 33 the effective potential w has the value

$$w = mm_1\left[\varphi(r) + \left(\frac{d\psi(r)}{dt}\right)^2\right] , \tag{5.101}$$

where $\varphi(r)$ and $\psi(r)$ are given functions of r. In the case of Newton's emission law they are represented by $\frac{1}{r}$ and $\frac{2\sqrt{r}}{c}$, where c is the propagation velocity which was mentioned several times. We denote the two parts of w by u and v, namely

$$\begin{aligned} w &= u + v , \qquad (5.102) \\ u &= mm_1\varphi(r) = mm_1\varphi , \\ v &= mm_1\left(\frac{d\psi(r)}{dt}\right)^2 = mm_1\left(\frac{d\psi}{dt}\right)^2 . \end{aligned}$$

In the state of rest, i.e., when r is constant, v vanishes and w becomes u. We refer to the first part u of the *effective potential* w as the *static potential* and to the second part v as the *motive potential*.

Since m_1 is *fixed*, the vis viva τ is given by

$$\tau = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] . \tag{5.103}$$

We denote the derivatives with respect to the time by primes. Since x_1, y_1, z_1 are *constant*, we can also write formulas (5.102) and (5.103) as

$$\begin{aligned} w &= u + v , \qquad (5.104) \\ u &= mm_1\varphi , \\ v &= mm_1\left(\frac{\partial\psi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' + \frac{\partial\psi}{\partial z}z'\right)^2 , \\ \tau &= \frac{m}{2}\left((x')^2 + (y')^2 + (z')^2\right) . \qquad (5.105) \end{aligned}$$

According to Hamilton's principle for the dynamics of the points the formula

$$\delta\int(\tau - \omega)dt = 0 \tag{5.106}$$

holds, i.e., according to (5.100):

$$\delta\int\tau dt = \delta\int\left(w + \frac{d\mathfrak{w}}{dt}\right)dt = \delta\mathfrak{w}'' - \delta\mathfrak{w}' + \delta\int w dt , \tag{5.107}$$

or since the boundaries of integrals are considered as fixed with respect to position and velocity:

$$\delta\int\tau dt = \delta\int w dt . \tag{5.108}$$

Since x_1, y_1, z_1 are constant and only x, y, z variable, we obtain three equations after carrying out the variation δ. These are

$$
\begin{aligned}
-mx'' &= \frac{\partial w}{\partial x} - \frac{d}{dt}\frac{\partial w}{\partial x'} \ , \\
-my'' &= \frac{\partial w}{\partial y} - \frac{d}{dt}\frac{\partial w}{\partial y'} \ , \\
-mz'' &= \frac{\partial w}{\partial z} - \frac{d}{dt}\frac{\partial w}{\partial z'} \ ,
\end{aligned}
\tag{5.109}
$$

where the primes indicate differentiation with respect to time. Multiplying the equations of (5.109) by $-x'$, $-y'$, $-z'$ and adding them together, one obtains in view of (5.106):

$$
\frac{d\tau}{dt} = -\left(x'\frac{\partial w}{\partial x} + y'\frac{\partial w}{\partial y} + z'\frac{\partial w}{\partial z}\right) + \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + y'\frac{d}{dt}\frac{\partial w}{\partial y'} + z'\frac{d}{dt}\frac{\partial w}{\partial z'}\right) \ , \tag{5.110}
$$

or in abbreviated form

$$
\frac{d\tau}{dt} = -\left(x'\frac{\partial w}{\partial x} + \cdots\right) + \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + \cdots\right) \ . \tag{5.111}
$$

Differentiating the effective potential w (5.104) with respect to time and noting that this w not only depends on x, y, z, but as well on x', y', z', one gets the formula:

$$
\frac{dw}{dt} = \left(x'\frac{\partial w}{\partial x} + \cdots\right) + \left(x''\frac{\partial w}{\partial x'} + \cdots\right) \ , \tag{5.112}
$$

or equivalently

$$
\frac{dw}{dt} \;=\; \left(x'\frac{\partial w}{\partial x} + \cdots\right) + \frac{d}{dt}\left(x'\frac{\partial w}{\partial x'} + \cdots\right) - \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + \cdots\right) \ . \tag{5.113}
$$

Adding (5.111) and (5.113) it follows that:

$$
\frac{d(\tau + w)}{dt} = \frac{d}{dt}\left(x'\frac{\partial w}{\partial x'} + y'\frac{\partial w}{\partial y'} + z'\frac{\partial w}{\partial z'}\right) \ . \tag{5.114}
$$

By (5.104) we have $w = u + v$, moreover v is independent of x', y', z' and on the other hand v is a homogeneous expression of degree two in x', y', z', so that:

$$
x'\frac{\partial w}{\partial x'} + y'\frac{\partial w}{\partial y'} + z'\frac{\partial w}{\partial z'} = x'\frac{\partial v}{\partial x'} + y'\frac{\partial v}{\partial y'} + z'\frac{\partial v}{\partial z'} = 2v \ .
$$

Therefore equation (5.114) becomes

$$
\frac{d(\tau + w)}{dt} = \frac{d(2v)}{dt} \ . \tag{5.115}
$$

This implies

$$
\tau + w - 2v = \text{constant}, \tag{5.116}
$$

or using $w = u + v$:

$$
\tau + u - v = \text{constant}. \tag{5.117}
$$

This means that if one adds the static and subtracts the motive potential from the vis viva one gets a constant of motion.

5.5.2 Examination of an Arbitrary System of Points

The same discussion can be applied to a system of arbitrary many points, say n, not only in the case where the system is freely movable, but also in the case where there are some constraints. However, if there are constraints, we assume that they can be expressed by equations involving *only the coordinates* of the points, but not their velocities. These equations we denote by

$$B_1 = 0 \ , \qquad B_2 = 0 \ , \qquad B_3 = 0 \ , \qquad \cdots \ . \tag{5.118}$$

We denote the vis viva of the system by T and the receptive potential by Ω. In this case T is a sum of n terms each having the form

$$\tau = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \frac{m}{2}\left((x')^2 + (y')^2 + (z')^2\right) \ . \tag{5.119}$$

On the other hand Ω is a sum of $\frac{n(n-1)}{2}$ terms, each belonging to two points of the form

$$\omega = w + \frac{d\mathfrak{w}}{dt} = u + v + \frac{d\mathfrak{w}}{dt} \ . \tag{5.120}$$

In this case Ω itself has an analogous form, namely:

$$\Omega = W + \frac{d\mathfrak{W}}{dt} = U + V + \frac{d\mathfrak{W}}{dt} \ , \tag{5.121}$$

where W represents the effective and $\frac{d\mathfrak{W}}{dt}$ the ineffective potential of the system. As regards the two parts of W, we refer to U as the static and to V as the motive potential of the system.

The effective potential $W = U + V$ of the system consists of $\frac{n(n-1)}{2}$ terms of the form $w = u + v$. If m and m_1 are any two points of the system, r their distance and, moreover, x, y, z and x_1, y_1, z_1 their coordinates, the term $w = u + v$ belonging to these two points has the value [cf. formula (5.102)]

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi(r) = mm_1\varphi \ , \\ v &= mm_1\left(\frac{d\psi(r)}{dt}\right)^2 = mm_1\left(\frac{d\psi}{dt}\right)^2 \ , \end{aligned} \tag{5.122}$$

or written in more detail

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi \ , \\ v &= mm_1\left(\frac{\partial\psi(r)}{\partial x}(x' - x_1') + \frac{\partial\psi}{\partial y}(y' - y_1') + \frac{\partial\psi}{\partial z}(z' - z_1')\right)^2 \ . \end{aligned} \tag{5.123}$$

The following formula holds for an unconstrained dynamical system

$$\delta \int T dt = \delta \int \Omega dt \ .$$

However, if the system is constrained by the conditions (5.118), the above formula has to be replaced by

$$\delta \int T dt = \delta \int (\Omega + \lambda_1 B_1 + \lambda_2 B_2 + \cdots) dt \ , \tag{5.124}$$

where λ_1, λ_2, $\cdots$ are unknown functions of time. Putting $\Omega = W + \frac{d\mathfrak{M}}{dt}$ this formula becomes:

$$\delta \int T dt = \delta \int (W + \lambda_1 B_1 + \lambda_2 B_2 + \cdots) dt \ . \tag{5.125}$$

If one carries out the variation δ one obtains $3n$ differential equations, namely the same number of equations as one has of variables x, y, z. The equations which belong to the point m with coordinates x, y, z read:

$$\begin{aligned} -mx'' &= \frac{\partial W}{\partial x} - \frac{d}{dt}\frac{\partial W}{\partial x'} + \lambda_1 \frac{\partial B_1}{\partial x} + \lambda_2 \frac{\partial B_2}{\partial x} + \cdots \ , \\ -my'' &= \frac{\partial W}{\partial y} - \frac{d}{dt}\frac{\partial W}{\partial y'} + \lambda_1 \frac{\partial B_1}{\partial y} + \lambda_2 \frac{\partial B_2}{\partial y} + \cdots \ , \\ -mz'' &= \frac{\partial W}{\partial z} - \frac{d}{dt}\frac{\partial W}{\partial z'} + \lambda_1 \frac{\partial B_1}{\partial z} + \lambda_2 \frac{\partial B_2}{\partial z} + \cdots \ . \end{aligned} \tag{5.126}$$

After multiplication by $-x'$, $-y'$, $-z'$ and addition, one obtains in view of (5.119) the equation

$$\begin{aligned} \frac{d\tau}{dt} \ &= \ -\left(x'\frac{\partial W}{\partial x} + y'\frac{\partial W}{\partial y} + z'\frac{\partial W}{\partial z}\right) \\ &+ \left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + y'\frac{d}{dt}\frac{\partial W}{\partial y'} + z'\frac{d}{dt}\frac{\partial W}{\partial z'}\right) \\ &- \lambda_1\left(x'\frac{\partial B_1}{\partial x} + \cdots\right) - \lambda_2\left(x'\frac{\partial B_2}{\partial x} + \cdots\right) - \cdots \ . \end{aligned} \tag{5.127}$$

There are as many of these equations as there are points. Adding all these equations together, one obtains in view of (5.118) the formula:

$$\frac{\partial T}{\partial t} = -\sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) + \sum\left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + \cdots\right) \ . \tag{5.128}$$

The effective potential W depends on the coordinates and the velocities. If one differentiates it with respect to time one gets

$$\frac{dW}{dt} = \sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) + \sum\left(x''\frac{\partial W}{\partial x'} + \cdots\right) ,$$

or equivalently

$$\begin{aligned} \frac{dW}{dt} = \sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) &+ \frac{d}{dt}\sum\left(x'\frac{\partial W}{\partial x'} + \cdots\right) \\ &- \sum\left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + \cdots\right) \ . \end{aligned} \tag{5.129}$$

Adding (5.128) and (5.129) one obtains

$$\frac{d(T+W)}{dt} = \frac{d}{dt}\sum\left(x'\frac{\partial W}{\partial x'} + y'\frac{\partial W}{\partial y'} + z'\frac{\partial W}{\partial z'}\right) . \tag{5.130}$$

With the help of (5.123) one sees that U is independent of the $3n$ magnitudes x', y', z' and, on the other hand, V is a homogeneous expression of degree two in these $3n$ magnitudes. With $W = U + V$ it follows that:

$$\sum\left(x'\frac{\partial W}{\partial x'} + y'\frac{\partial W}{\partial y'} + z'\frac{\partial W}{\partial z'}\right) = 2V .$$

Equation (5.130) therefore becomes

$$\frac{d(T+W)}{dt} = \frac{d(2V)}{dt} . \tag{5.131}$$

This implies

$$T + W - 2V = \text{constant}, \tag{5.132}$$

or, since $W = U + V$:

$$T + U - V = \text{constant}. \tag{5.133}$$

This formula reduces in the case of an *instantaneous* propagation of the potential, i.e., $c = \infty$, to the well-known formula $T + U = \text{constant}$ (cf. page 20). The general formula (5.133) contains the following theorem:

The vis viva increased by the static and decreased by the motive potential is a constant of motion for an arbitrary system of points. This holds not only in the unconstrained case, but also in the case where the coordinates of the points are constrained by some conditions.

This theorem was derived under the assumption that in the system of points only *internal* forces are acting. In the case where a system consisting of points m_1, m_2, $\cdots$ m_n is subject not only to its *internal* forces, but also to *external* forces, one can always find *fixed points* M_1, M_2, $\cdots$ M_p which are the centers of these latter forces. The system consisting of all these $n + p$ points is then subject only to *internal* forces and therefore the theorem above applies to it. The fact that among the $n + p$ points some are fixed is not a problem for the utilization of the theorem.

5.5.3 Afterword

If one assumes (as almost always happens since Newton) that *spatially* separate objects act directly on one another, it should be just as permissible to assume a direct mutual action between two objects which are *temporally* separated from one another; provided naturally that such an assumption leads to equally happy consequences as the first. Accordingly Professor Weber, to whom I am indebted for his gracious communication, remarks that the hypothesis put forth by me (for the case $\varphi = \frac{1}{r}$) can be formulated in this way:

> "*The potential values stemming from a particle of matter are inversely proportional to the distances, and are valid for later moments of time in proportion to the distance*. The reason why they are valid for later moments of time, *may* lie in a

propagation, of which it is only possible to speak under the assumption of a *higher mechanics* (as for example, the propagation of waves in air can only be treated with knowledge of fluid mechanics), from which it would follow that the propagation can be disturbed and interrupted at every point of the medium."

If the question raised here, whether the presumed effect between *temporally* separated objects should be regarded as primary (not further explicable) or as something secondary (derivable from simpler processes), would have to be decided *right away*, I would safely give preference to the first conception. But even in this case, the mode of expression I have chosen should be legitimate at least as a *figurative* and not inappropriate one.

Tübingen, in May 1868.

5.6 Supplementary Remarks of Carl Neumann in the Year 1880

The *last words* of this article *considered by itself* make it already quite clear that the criticism of *Clausius* in the year 1869 (in Poggendorff's *Annalen*, Vol. 135, page 606)[51] against it is not applicable. Concerning this point one should also compare it to my note in *Math. Ann.*, Vol. 1, page 317–324.[52]

A brief look at the *first few pages* of this article (pages 400-402)[53] show that in the year 1868, when I wrote it, I was not aware of two important considerations of *Weber* and *Riemann*.

An *argument by Weber* (which appeared as a short note in Poggendorff's *Annalen*, Vol. 73, page 229 in the year 1848)[54] *shows in an elementary way, that the principle of vis viva continues to be valid for Weber's fundamental law.* — I regret, that at the time of writing I did not know this note. In my later publications (like for example in the *Abhandlungen der Kgl. Sächs. Ges. d. Wiss.*, Vol. 11, 1874, page 115)[55] I made an effort to bring to light the argument by Weber.

On the other hand the considerations by Riemann (compare the work of Hattendorff about weight, electricity and magnetism, Hannover, Rümpler, 1876, pag. 316–336)[56] *already contain the idea to introduce an electrodynamic potential and to deduce from it the electric forces by variation.* This idea is crucial in this article and is developed in great detail. There is no need to apologize that *I did not know* the considerations by Riemann when I wrote the article in the year 1868. Although part of it were already contained in a lecture by Riemann in the year 1861 as Hattendorff mentions, they appeared in print only in 1876 (in the work by Hattendorff referred to above).

[51][Note by AKTA:] [Cla68] with English translation in [Cla69].
[52][Note by AKTA:] [Neu69a] with English translation in [Neu20b], see Chapter 6.
[53][Note by AKTA:] Pp. 400-402 of the 1880 reprint of Neumann's 1868 paper, [Neu68a].
[54][Note by AKTA:] [Web48] with English translation in [Web52c], [Web66] and [Web19a].
[55][Note by AKTA:] [Neu74].
[56][Note by AKTA:] [Rie76] with partial English translation in [Rie77b]. See also [Rie67b] with English translation in [Rie67a] and [Rie77a].

Other people might decide if under these circumstances one should just call *Riemann* the author of these ideas, or if it is not more appropriate to give credit as well to *the person* who, independently of Riemann, had this same idea and published it *first* (and in greater detail). On the other hand *Clausius* in a recent note used the idea of introducing an electrodynamic potential and deriving the forces from it by variation, *without mentioning my work*. I think that those who do not know the literature well could easily get quite a wrong impression from this.

Leipzig, in November 1880.

Chapter 6

[Carl Neumann, 1869] Notes on a Recently Published Essay on the Principles of Electrodynamics

Carl Neumann in Leipzig[57,58,59]

My paper written for the Jubilee of the University of Bonn (*The Principles of Electrodynamics*, Tübingen, 1868),[60] a preliminary report of which had already appeared in the *Nachrichten der Göttinger Societät der Wissenschaften* (June 1868),[61] has been subjected by Clausius in the latest volume of *Poggendorff's Annalen* (Vol. 135, p. 606)[62] to a judgment with which I cannot agree, and which prompts me to make these brief comments.

Looking back now on that paper, it can be considered as composed of two different parts, one of which is preceded by the other both in respect to the nature of its content and the strength of its reasoning. Accordingly, it seems to me appropriate first to address the more important and then the subordinate part, such that no separation in the content of my paper be allowed to occur.

§ 1.
The First Part of the Cited Paper

As starting point of this part, two ideas are to be considered: One is the idea that for every electrical force there must exist a potential, which, however, depends not only upon the relative position of the electrical masses, but at the same time on their velocity. The other idea is that the well-known Hamiltonian Principle[63] (ruling over the whole of Mechanics) is just as applicable to potentials of this kind, as to the common potential dependent only upon relative position. My research proceeds accordingly from a certain hypothetical formula for

[57][Neu69a] with English translation in [Neu20b].

[58]Translated by Laurence Hecht, larryhecht33@gmail.com. Edited by A. K. T. Assis.

[59]The Notes by Carl Neumann are represented by [Note by CN:], the Notes by Laurence Hecht are represented by [Note by LH:], while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[60][Note by AKTA:] [Neu68a] with English translation in [Neu20a], see Chapter 5.

[61][Note by AKTA:] [Neu68b].

[62][Note by AKTA:] [Cla68] with English translation in [Cla69].

[63][Note by AKTA:] Due to William Rowan Hamilton (1805-1865).

the potential of electrical masses:

$$w = \frac{mm_1}{r}\left[1 + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] . \qquad (1)$$

Here m and m_1 signify the two masses, and r their distance apart at time t; and by c is understood the constant contained in Weber's law.[64,65]

My research now shows how, by application of Hamilton's Principle, one is led directly from hypothesis (1) to the known laws of electrical repulsion and induction, and simultaneously also to a comprehensive electrodynamic form of the principle of *vis viva*. — The latter result lends itself directly to an investigation recently carried out by me, on the oscillating charge of a Franklin pane[66] (cf. *Nachrichten der Göttinger Gesellschaft der Wissenschaften*, 13 January 1869),[67] in which it is shown that in addition to Kirchhoff's differential equation for this phenomenon,[68] one can achieve success in a completely different way already indicated by W. Thomson,[69] namely by direct application of the principle of *vis viva*. — It should be noted at this occasion, that in judging the magnitude of a given mass, no distinction is made in my paper between the effect the mass produces and its inertial mass. However, such a distinction must be made when one is dealing with the masses of different matter (for example electrical and ponderable masses) simultaneously. If one therefore considers the magnitude of the mass, as measured by its effect, as equal to m, as is usually the case with electrical masses, then the measured value of the inertial mass of the same is in no way equal to m, but is rather designated as fm, where f represents a constant factor whose value depends only on the nature of the matter under consideration. This constant factor, included in some places in my paper, has been neglected. The oversight is easily corrected; the results obtained remain completely intact.

Also in my paper, the mutual potential of two masses is brought under consideration using Formula (1) at the same time as the general formula

$$w = mm_1\left[\varphi(r) - \frac{r}{c^2}\frac{d\varphi(r)}{dr}\left(\frac{dr}{dt}\right)^2\right] , \qquad (1a)$$

where $\varphi(r)$ represents a certain function of r. Just as one arrives, on the basis of hypothesis (1), at the law of electrical repulsion and induction, one may also arrive, on the basis of hypothesis (1a), as my paper shows, at that law which I had supposed (in 1858) from my investigation of the magnetic rotation of the plane of polarization of light for the force obtaining between electricity and ether.[70]

[64][Note by CN:] This expression is identified right at the beginning of my paper (page 2) as the proper starting point for my observations; the constant $1/c^2$ is there called G. Regarding that, permit me to remark that due to a disturbing printing error in another place in my paper (page 24) this same expression appears as $1/ccr$ instead of $1/cc$.

[65][Note by AKTA:] Wilhelm Eduard Weber (1804-1891). Weber presented his force law in several publications. For instance, [Web46] with partial French translation in [Web87] and a complete English translation in [Web07]; [Web52b] with English translation in [Web21a]; [Web52a] with English translation in [Web21c]; [KW57] with English translation in [KW21]; and [Web64] with English translation in [Web21b].

[66][Note by AKTA:] Named after Benjamin Franklin (1705-1790).

[67][Note by AKTA:] [Neu69b].

[68][Note by AKTA:] [Kir64].

[69][Note by AKTA:] [Tho53b].

[70][Note by AKTA:] [Neu58] and [Neu63].

Clausius has raised no concerns against the part of my paper discussed up to this point. His concerns are directed only against that portion which I have here designated as standing second in line. I will therefore address these latter in detail.

§ 2.
The Second Part of the Cited Paper

Clausius is concerned with the already identified hypothetical formula (1) assumed for the potential. He seeks to give this formula a further basis; he seeks to replace the formula with concepts.

I view the Newtonian potential (the product of the masses divided by the distance) of two bodies or masses, m and m_1, as an impulse for motion,[71] or (better expressed) an order or command,[72] which is given and emitted by one body, and received and followed by the other. At the same time it is assumed that the order requires a certain time to get from the place of emission to the place of receipt, in other words, that it requires a certain time to traverse the space between the two bodies.

The distance of the two bodies from one another at the time t_0 may be designated as r_0. In the instant t_0, a certain order is issued by one body, and that with reference to the instantaneous relationship, that is the instantaneous distance r_0; the command rings out accordingly: mm_1/r_0. Given and emitted at time t_0, the order traverses the space between the two bodies without suffering any change en route. Because the passage of that space requires a certain time, the order is therefore not received and obeyed by the obedient body at time t_0, but at a later time t, a time when the distance between the two bodies is no longer r_0, but another magnitude r.

The command (the value of the potential mm_1/r_0) can accordingly be designated on the one hand as the emissive potential corresponding to the time t_0, and on the other hand it can also be designated as the receptive potential corresponding to the time t. In the time t_0 the order is given; in the time t it goes into force.[73]

The interval $t - t_0$ is that which the command requires to traverse the space between

[71][Note by LH:] In German: *Bewegungsantrieb*.

[72][Note by LH:] In German: *Befehl*.

[73][Note by CN:] This is stated in fuller detail, for example on pages 6 and 7 of my paper:

> "If we consider only two points m and m_1, there are, proceeding from the concept of a progressive propagation of the potential for each instant of time t, we must distinguish between two different potentials, the emissive and the receptive."
>
> "The emissive potential is that which is sent out at the time t by each of the two points, and which thus only reaches the other point a little later..."
>
> "The receptive potential on the other hand is that which is received in time t, by each of the two points in time t, and which thus has already been sent out a little earlier by the other point. The receptive potential belonging to a given time is accordingly identical with the emissive potential of an earlier time..."

On the basis of the just cited definition for emissive and receptive potential, the last theorem could also be stated as:

> "The value of the potential received by a point in a given time is identical with that which is emitted at an earlier time by the other point."

Because, in the cited pages of my paper, the potential is understood to be completely determined by the value expressed in the formulas.

the two bodies. In my paper, the concept of this traversal is based on what seems to me the **simplest** representation, namely, to assume that the command proceeds forward, at the constant velocity c, along the radius vector which originates at the body giving the order and ends at the obeying body. The velocity designated as c is thus a relative motion, because the radius vector along which the order proceeds is itself in motion, carried away along the same path as the moving body.[74]

The order mm_1/r_0 received by the obeying body in time t, must now obviously traverse that length of radius vector which is present at the instant of its reception; thus it must traverse the length of radius vector r which exists at time t. The time required is r/c; therefore $t - t_0 = r/c$.

Let the receptive potential mm_1/r_0 at time t be designated by ω, and also let $t_0 = t - \Delta t$, $r_0 = r - \Delta t$, then it results that:

$$\omega = \frac{mm_1}{r_0} = \frac{mm_1}{r - \Delta r} \; ; \qquad (2)$$

and at the same time for the interval Δt corresponding to length Δr, we get the formula:

$$\Delta t = t - t_0 = \frac{r}{c} \; . \qquad (3)$$

However, by further manipulation, and neglecting the third power of $1/c$, Formulas (2) and (3) give for the value of the receptive potential ω:

$$\omega = w + \frac{d\mathfrak{w}}{dt} \; ,$$

where w represents the expression (1), while on the other hand, $\mathfrak{w}$ is a rational combination of $\log r$ and dr/dt; and further, by application of Hamilton's Principle, the just named value is equivalent to the simpler value: $\omega = w$; — to which there are no objections, and in fact no one is in doubt.

Consequently it is shown that the given concepts really lead to the hypothetically assumed potential formula (1). Whether this substitution of an odd formula by means of a no less odd concept implies progress, is very difficult to judge at present. In writing my paper I also did not attach the same weight to this second line of thought. This explains the striking brevity with which I treated it, taking up only 3 pages of my 38-page paper. So it happens that this part of the underlying concept in my paper is not explained in detail, but implied only briefly and in passing.[75] It is left to the reader to some extent, first to extract these

It is clear from this, however, how little is said in my writing about a direct analogy between the propagation of the potential and that of light. It would be completely absurd to say that the value of the light (either in quantity or intensity) received by a point at a given instant would be identical to that emitted at an earlier time by the other point.

Overall, as one sees, my suppositions about the potential bear such an extraordinarily glaring difference to the laws of light, that it could scarcely have occurred to me to instead propose a similarity.

[74][Note by CN:] This is the first time I have made use of these words to explain this concept. Earlier when writing my paper, I had clothed the concept differently, more pictorially. I thought then of the body giving the order as surrounded by an infinitely extended atmosphere, which was to a certain extent rigidly bound to the body and took part in all its motions; thus I thought of the order from the emitting body as proceeding in this pure ideal atmosphere with constant velocity and without suffering any change in its original constitution. I thus made use of the word "*propagation*", which might better have been replaced by "*transmission*".

[75][Note by CN:] I thought it permissible earlier, as I considered that paper only provisional, to be followed by a more extensive publication on the subject, and such was also my intention.

concepts from the given formulae; and that, I gladly concede, were no easy task and certainly a thankless one. For, as one sees, these ideas, at least in their present form, are very varied in comparison with those usually employed in the explanation of physical processes.

Still it seems remarkable, to me[76,77] at least, that the same concepts also lead to the law of the mutual interaction of electricity and ether. Thus if one replaces the function $1/r$ in the Newtonian potential by any function $\varphi(r)$, then, based on those ideas, instead of formulas (2) and (3), there result

$$\omega = mm_1\varphi(r_0) = mm_1\varphi(r - \Delta r) \ , \qquad (2a)$$

$$\Delta t = t - t_0 = \frac{r}{c} \ . \qquad (3a)$$

However, on further treatment this result leads immediately to the potential formula (1a), consequently to the law of the mutual interaction of electricity and ether.

Overall, I would like to view this second part of my paper as by no means completely superfluous. Rather I am inclined to consider it as a preparatory work, allowing a deeper insight into any obstacles, which, if not eliminated, can at least be analyzed and illuminated. Clausius' objections are directed against this second part of my work. In the following paragraphs I take the opportunity to respond to them.

§ 3.
Potential and Light

My suppositions concerning potential show a very great difference with the laws of light. So, for example, the following differences between emission, transmission and reception come into view: The light emitted by a luminous body is independent of the illuminated body, while the potential emitted in any instant by an attracting body is in the strictest sense dependent on the instantaneous position of the attracted body, (they are the same, namely $= mm_1/r$, or $= mm_1\varphi(r)$, where r signifies the instantaneous distance). Further: The light emitted by the luminous body in a given instant diminishes in intensity the further away it is from the body; while the emitted potential travels without any change in its original value up to the attracted body. Finally: the light received (i.e. absorbed) by the illuminated body is in general a fraction of the outgoing light; while the potential received by the attracted body is identical (i.e. equal) to the arriving potential.

The laws by which the potential of a body is transmitted to another are thus (according to my suppositions) so extraordinarily different from the corresponding laws of light, that one can scarcely speak of an analogy. At least there would be only one circumstance in

The disproportionately large attention paid to some of its parts is not in keeping with the provisional character of the paper; since in some parts cited by others in the greatest detail, only the end results were given. This is also why I did not publish my paper more widely, and intentionally withheld it from the book sellers.

[76][Note by CN:] I cannot insist that the argument made here be recognized as generally applicable. For the law of the mutual interaction of electricity and the ether, on which my case here rests, could be put in doubt by the experimental investigations of Verdet on the dispersion arising from the rotation of the plane of polarization light by a magnet (*Ann. d. chim.* (3) Vol. 69, p. 415). As for me, I believe for obvious reasons that the provisional results of the dispersion observations should be accorded no significant weight as to the correctness or incorrectness of that law.

[77][Note by AKTA:] [Ver63].

which a kind of analogy could be asserted. This consists in the fact that light, like potential, propagates with a very large constant velocity; and even this analogy is not perfect, for the constant velocity for light and potential possess different values, and it applies in the case of light to an absolute motion, while for potential it is a relative motion.

Coincidentally, the just mentioned similarity in a certain passage in the introduction to my paper (page 3) has been pointed out. There it is noted that Riemann had assumed that the "*Potential — similar to light — propagates through space with a certain constant velocity*"; at the same time it is added that this assumption is also the basis of my research.[78]

That this comparison with light was completely accidental and unintended should be clear not only from the text of my paper, but from the fact that one finds such a comparison in only **one** place in my whole paper, and it is even clearer when it is recognized that such a comparison with light is **not found anywhere** in my communication to the Göttingen Society. (The word "*light*" is not contained anywhere in that communication.)

In any case I must also add that the referenced passage from the introduction to my paper, which gave only a casual historical note, namely that Riemann should be designated as the originator of the idea of a progressive motion of the potential, possesses less exactness. While writing my paper, it seemed petty to wish to emphasize in the introduction that my idea on this progressive motion differs fundamentally from that of Riemann (so far as the latter is understood by me). Thus I omitted mention of this difference; which could more easily lead to misunderstanding as the disputes in question are treated extremely briefly in the text of my paper.

Thus, it is easily understandable that Clausius, in judging my paper, proceeded from the opinion that I had supposed that the potential propagates from one body to another in a way similar to that of light; while in reality my suppositions concerning the progressive movement of the potential exhibit the greatest difference with the laws of light. — I can only welcome the fact that I have been made aware of the danger of such a misunderstanding, and begun to eliminate it.

Leipzig, January 19, 1869.

[78][Note by AKTA:] Carl Neumann is referring to Riemann's posthumous 1867 work, [Rie67b] with English translation in [Rie67a] and [Rie77a].

Chapter 7

[Weber, 1869] On a Simple Formulation of the General Fundamental Law of Electric Action

Wilhelm Weber[79,80,81]

In these *Annalen* 1848, Vol. 73, p. 193 and following,[82] where I have given an excerpt from my first treatise on "Electrodynamic Measurements,"[83] I have added on p. 229[84,85] that the expression given in that treatise for the general fundamental law of electrical action could be simplified by specifying the expression of the *potential* instead of the expression of the *force*, that is, the function of the coordinates x, y, z, whose negative partial differential coefficients with respect to x, y, z, correspond to the *components of the force* parallel to these coordinates. If we denote by e, e' two electric particles, by r their distance from one another, and by c a certain constant, then the expression of the *force* was[86]

$$\frac{ee'}{r^2}\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}+\frac{2r}{c^2}\frac{d^2r}{dt^2}\right) ,$$

while the expression of the *potential* was[87,88]

[79] [Web69] with English translation in [Web21d].

[80] Translated by H. Härtel, haertel@astrophysik.uni-kiel.de and http://www.astrophysik.uni-kiel.de/~hhaertel/index_e.htm. Edited by A. K. T. Assis.

[81] The Notes by H. Weber, the editor of the fourth volume of Weber's *Werke*, are represented by [Note by HW:]; the Notes by Wilhelm Weber are represented by [Note by WW:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[82] [Note by AKTA:] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

[83] [Note by AKTA:] [Web46] with a partial French translation in [Web87] and a complete English translation in [Web07].

[84] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 245.

[85] [Note by AKTA:] [Web48, p. 229 of Weber's original 1848 paper and p. 245 of Vol. III of Weber's *Werke*]. See also [Web52c, p. 520], [Web66, p. 520] and [Web19a, p. 42].

[86] [Note by AKTA:] The expression dr^2/dt^2 should be understood as $(dr/dt)^2$.

[87] [Note by WW:] Independently of this, Neumann in his investigations on the principles of electrodynamics started from the hypothetical formula for the potential of electrical masses as $(ee'/r)\left(1+\frac{1}{c^2}\frac{dr^2}{dt^2}\right)$.

[88] [Note by AKTA:] Let V represent Weber's potential energy:

$$\frac{ee'}{r}\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}\right) .$$

The latter statement of the law can now be put into words in the following way, whereby the *physical meaning* of the law and the dependence of the *stimulation of motion*[89] on the *existing motion* emerge more clearly, namely:

> Between every two electrical particles there is partly mutual motion, partly stimulation to mutual motion. If one calls the following values, namely that of mutual motion when no stimulation takes place, and that of mutual stimulation when no motion takes place, *limit values*, then the fraction missing from one limit value is always represented by an equal fraction of the other limit value.

The *latter* of the two limit values is the well-known *electrostatic potential* ee'/r, while the *former* limit value is always the same, namely the *value of a mutual motion with the velocity* c, which can be represented by ac^2. — If there is now a mutual motion between e and e' with the velocity $dr/dt < c$, the value of which is $= a\frac{dr^2}{dt^2}$, and therefore the following fraction is missing at the first limit value

$$\frac{ac^2 - a\frac{dr^2}{dt^2}}{ac^2} = 1 - \frac{1}{c^2}\frac{dr^2}{dt^2} ,$$

then this missing fraction is represented by an equal fraction of the other limit value ee'/r, i.e. by $(ee'/r)(1-[1/c^2][dr^2/dt^2])$, which is the general expression of the *potential*, as given above.

If e and e' indicate the *masses*[90] of the electric particles and α, β the *velocities of* e in the direction r and perpendicular to it, α', β' the same *velocities for* e', after which $\alpha-\alpha' = dr/dt$ is the relative velocity of both particles, then we have

---

$$V = \frac{ee'}{r}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] .$$

Neumann's Lagrangian energy W, on the other hand, is expressed nowadays as:

$$W = \frac{ee'}{r}\left[1+\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] .$$

Neumann presented this expression in 1868 when he introduced the Lagrangian and Hamiltonian formulations of Weber's electrodynamics, [Neu68a], with English translation in [Neu20a], see Chapter 5.

Weber's potential energy $V = (ee'/r)[1-(1/c^2)(dr/dt)^2]$ differs from Neumann's Lagrangian energy $W = (ee'/r)[1+(1/c^2)(dr/dt)^2]$ in the sign in front of the square of the relative velocity dr/dt.

For a system of two particles interacting through Weber's force, the sum of V with the kinetic energy T of the particles is a constant in time, namely, $E = T + V =$ constant in time. The Lagrangian L and the Hamiltonian H of this system are given by, respectively, $L = T - W$ and $H = E = T + V$. I discussed this topic in [Ass94, Section 3.5: Lagrangian and Hamiltonian Formulations of Weber's Electrodynamics]. In Portuguese: [Ass92a], [Ass95a] and [Ass15a].

[89][Note by AKTA:] In German: *Bewegungsanregung*. This expression can also be translated as "excitation of motion".

[90][Note by AKTA:] Weber is utilizing the symbols e and e' to indicate not only the values of the electric charges of the two particles, but also the values of their inertial masses. In 1871 he will represent the charges by e and e', while the inertial masses will be represented by ε and ε', [Web71] with English translation in [Web72], see Chapter 9.

$$\frac{1}{2}e\left(\alpha^2+\beta^2\right)+\frac{1}{2}e'\left(\alpha'^2+\beta'^2\right)$$

as the living force[91] or work belonging to the two particles, which expresses their motion, according to size, proportional to the moving masses and to the squares of their velocities. If we now set

$$\begin{array}{ll} \text{for } \alpha, & \dfrac{e\alpha+e'\alpha'}{e+e'}+\dfrac{e'(\alpha-\alpha')}{e+e'}\ , \\ \text{for } \alpha', & \dfrac{e\alpha+e'\alpha'}{e+e'}+\dfrac{e(\alpha'-\alpha)}{e+e'}\ , \end{array}$$

and note that $\alpha-\alpha'=dr/dt$, then one can represent this living force or work of the two masses e and e' in the following *two parts*, namely

$$=\frac{1}{2}\frac{ee'}{e+e'}\cdot\frac{dr^2}{dt^2}+\frac{1}{2}\left(\frac{(e\alpha+e'\alpha')^2}{e+e'}+e\beta^2+e'\beta'^2\right)\ .$$

The *former* may be called the *internal work*, the *latter* the *external work*, because for the *former* the knowledge of the particles e and e' and the increase or decrease of their distance from one another is sufficient, while for the *latter apart from the particles* e and e', a fixed coordinate system must be given in order to be able to observe and measure the velocities $(e\alpha+e'\alpha')/(e+e')$, β and β'.

It is now evident that this *internal work* $(1/2)[ee'/(e+e')]\cdot[dr^2/dt^2]$ is the exact value of the *mutual motion* of both particles, which was denoted above with $a[dr^2/dt^2]$, so that $a=(1/2)[ee'/(e+e')]$.

This *internal work* and the *potential* of the two particles e, e' at the distance r can have very different values, but if one value increases, the other decreases, and the increase and decrease are always in the same proportion. If the *potential* has decreased by ee'/r, the *internal work* has increased by $(1/2)[ee'/(e+e')]c^2=ac^2$. If this *internal work*, which has taken the place of the vanished potential, is called the *work equivalent*[92] of that potential, the *work equivalent of an arbitrary potential* V results from the same relationship $=[rc^2/2(e+e')]\cdot V$.

The *existing internal work* and the *work equivalent of the existing potential* form together the *sum of the existing internal work values*. Understood in this way, the following simple formulation of our law results, namely:

> For two electrical particles e and e', at any distance from each other, the sum of the existing internal work values is always the same, equal to $(1/2)[ee'/(e+e')]\cdot c^2$.

Since the existing *internal work* is $(1/2)[ee'/(e+e')]\cdot[dr^2/t^2]$, the existing potential is V and *its work equivalent* is $[rc^2/2(e+e')]\cdot V$. Consequently, the *sum of the existing internal work values* is equal to

$$\frac{1}{2}\frac{ee'}{e+e'}\cdot\frac{dr^2}{dt^2}+\frac{rc^2}{2(e+e')}\cdot V=\frac{1}{2}\frac{ee'}{e+e'}\cdot c^2\ ,$$

[91][Note by AKTA:] See footnote 26 on page 17.
[92][Note by AKTA:] In German: *Arbeitsäquivalent*.

or, divided by the last term,

$$\frac{1}{c^2} \cdot \frac{dr^2}{dt^2} + \frac{r}{ee'} \cdot V = 1 \ ,$$

from which the *potential* $V = (ee'/r)(1 - [1/c^2][dr^2/dt^2])$ is obtained, as above.

The formulation of the law discussed here is only intended to show in the simplest way the dependence of two particles on each other in their motions, especially the dependence of their mutual stimulation on their existing motion. Quite different needs emerge when it comes to the task to find the complete mathematical development of all the consequences of this law in connection with the general principles of mechanics in the case of larger electrical masses connected in various ways with other bodies. For this task the principles of electrodynamics are to be brought into other forms, which was not the objective of this work.

Chapter 8

Editor's Introduction to Weber's Sixth Memoir on Electrodynamic Measurements

A. K. T. Assis[93]

I would like to discuss some aspects related to Weber's Sixth major Memoir on Electrodynamic Measurements.[94] In particular, I wish to comment upon Maxwell's reaction to Weber's electrodynamics and its connection with the principle of the conservation of energy.[95] This may help to contextualize Weber's work.

The principle of the conservation of energy by taking into account thermal energy had been established by Julius Robert von Mayer (1814-1878) in 1842 and also by James Prescott Joule (1818 - 1889) in 1843.[96]

In 1846 Weber published his First major Memoir on Electrodynamic Measurements introducing his force law which depends on the distance r, relative velocity dr/dt and relative acceleration d^2r/dt^2 between the interacting electrified particles.[97] In this work he did not discuss the conservation of energy.

In 1847 Hermann von Helmholtz (1821-1894) published his famous and very influential work on the conservation of energy.[98] In this work he put this principle in a solid theoretical foundation developing the mathematical consequences of central forces. He utilized the expression "conservation of force" for what we would nowadays call "conservation of energy". At that time the common name for the magnitude mv^2 was *vis viva*,[99] but in this paper of 1847 Helmholtz explicitly stated that he would call $mv^2/2$ (our kinetic energy) by *vis viva*, as this latter quantity appeared more frequently in mechanics and seemed more useful:[100]

> For the sake of better agreement with the customary manner of measuring the inten-

[93]Homepage: www.ifi.unicamp.br/~assis

[94][Web71] with English translation in [Web72], see Chapter 9.

[95][Ass94, Section 3.6: Maxwell and the Electrodynamics of Weber].

[96][May42] with English translation in [May62] and Portuguese translation in [May84]; [Jou43]. See also [Mar84].

[97][Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

[98][Hel47] with English translation in [Hel53] and [Hel66].

[99]See footnote 26 on page 17.

[100]See [Hel47, p. 9] with English translation in [Hel53, p. 119].

sity of forces, I propose calling the quantity $\frac{1}{2}mv^2$ the quantity of *vis viva*, by which it is rendered identical with the quantity of work.

Helmholtz called "tension" what we call nowadays by the name potential energy (like the gravitational potential energy mgh of a body of mass m at the height h above the ground, with g being the free fall acceleration). The main results of his paper were stated as follows:[101]

> The preceding propositions may be collected together as follows:
>
> 1. Whenever natural bodies act upon each other by attractive or repulsive forces, which are independent of time and velocity, the sum of their *vires vivae* and tensions must be constant; the maximum quantity of work which can be obtained is therefore a limited quantity.
>
> 2. If, on the contrary, natural bodies are possessed of forces which depend upon time and velocity, or which act in other directions than the lines which unite each two separate material points, for example, rotatory forces, then combinations of such bodies would be possible in which force might be either lost or gained *ad infinitum*.

In 1848 Weber presented his potential energy from which he could deduced his force law.[102] The conservation of energy was implicit in his deduction, although he did not emphasize this topic.

In the Introductory Chapter to Weber's First major Memoir on Electrodynamic Measurements presented in Volume 1 of these English translations of his main works on electrodynamics, I presented several quotes from Maxwell in which he highly praised Weber's electrodynamics. But if Maxwell knew so well Weber's electrodynamics and appreciated it so much, why did he not work with it and develop its properties and applications? The main reason was that Maxwell believed that Weber's electrodynamics did not comply with the principle of the conservation of energy. He was directly influenced by Helmholtz' 1847 work. Helmholtz' results were understood by Maxwell, among others, as implying that Weber's electrodynamics did not comply with the principle of conservation of energy. The reason was that although Weber's force was a central one (directed along the straight line connecting the particles), it depended on the velocity of the electrified particles. This can be seen in his paper of 1855 (published in 1858) where Maxwell pointed out only this problem in Weber's electrodynamics:[103]

> There exists however a professedly physical theory of electro-dynamics, which is so elegant, so mathematical, and so entirely different from anything in this paper, that I must state its axioms, at the risk of repeating what ought to be well known. It is contained in M. W. Weber *Electro-dynamic Measurements*, and may be found in the Transactions of the Leibnitz-Society, and of the Royal Society of Sciences in Saxony.[104,105] The assumptions are [...]. From these axioms are deducible Ampère's laws

[101] [Hel53, p. 126] and [Hel66, p. 126].

[102] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

[103] [Max58, pp. 207-208 of Niven's book].

[104] [Note by Maxwell:] When this was written, I was not aware that part of M. Weber's Memoir is translated in Taylor's *Scientific Memoirs*, Vol. V. Art. XIV. The value of his researches, both experimental and theoretical, renders the study of his theory necessary to every electrician.

[105] Maxwell was referring in his footnote to the 1848 excerpt of Weber's First major Memoir on Electrodynamic Measurements, [Web48] with English translation in [Web52c], [Web66] and [Web19a].

of the attraction of conductors, and those of Neumann and others, for the induction of currents. Here then is a really physical theory, satisfying the required conditions better perhaps than any yet invented, and put forth by a philosopher whose experimental researches form an ample foundation for his mathematical investigations.

There are also objections to making any ultimate forces in nature depend on the velocity of the bodies between which they act. If the forces in nature are to be reduced to forces acting between particles, the principle of the Conservation of Force requires that these forces should be in the line joining the particles and functions of the distance only. The experiments of M. Weber on the reverse polarity of diamagnetics, which have been recently repeated by Professor Tyndall, establish a fact which is equally a consequence of M. Weber's theory of electricity and of the theory of lines of force.

What Maxwell called here the "principle of the Conservation of Force" should be understood as the principle of the conservation of energy. This expression utilized by Maxwell indicates the direct influence of Helmholtz on his thinking, as the title of Helmholtz' work of 1847 was *Über die Erhaltung der Kraft*, which had been translated in 1853 as *On the Conservation of Force*.

In his paper of 1864 Maxwell stated even more explicitly the reason why he rejected Weber's electrodynamics and decided to follow another approach (our italics):[106]

(1) The most obvious mechanical phenomenon in electrical and magnetical experiments is the mutual action by which bodies in certain states set each other in motion while still at sensible distance from each other. The first step, therefore, in reducing these phenomena into scientific form, is to ascertain the magnitude and direction of the force acting between the bodies, and when it is found that this force depends in a certain way upon the relative position of the bodies and on their electric and magnetic condition, it seems at first sight natural to explain the facts by assuming the existence of something either at rest or in motion in each body, constituting its electric and magnetic state, and capable of acting at a distance according to mathematical laws.

In this way mathematical theories of statical electricity, of magnetism, of the mechanical action between conductors carrying currents, and of the induction of currents have been formed. In these theories the force acting between the two bodies is treated with reference only to the condition of the bodies and their relative position, and without any express consideration of the surrounding medium.

These theories assume, more or less explicitly, the existence of substances the particles of which have the property of acting on one another at a distance by attraction or repulsion. The most complete development of a theory of this kind is that of M. W. Weber,[107,108] who has made the same theory include electrostatic and electromagnetic phenomena.

In doing so, however, he has found it necessary to assume that the force between two electric particles depends on their relative velocity, as well as on their distance.

[106] [Max65, pp. 526-527 of Niven's book].

[107] [Note by Maxwell:] Electrodynamische Maassbestimmungen. Leipzic Trans. vol. i. 1849, and Taylor's Scientific Memoirs, vol. v. art. xiv.

[108] Maxwell was referring to [Web46] with English translation in [Web07]; and [Web48] with English translation in [Web52c], [Web66] and [Web19a].

> This theory, as developed by MM. W. Weber and C. Neumann,[109,110] is exceedingly ingenious, and wonderfully comprehensive in its application to the phenomena of statical electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement, and by actually determining electrical quantities with an accuracy hitherto unknown.
>
> (2) *The mechanical difficulties*, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such at to *prevent me from considering this theory as an ultimate one*, though it may have been, and may yet be useful in leading to the coordination of phenomena.
>
> *I have therefore preferred to seek an explanation of the fact in another direction*, by supposing them to be produced by actions which go on in the surrounding medium as well as in the excited bodies, and endeavouring to explain the action between distant bodies without assuming the existence of forces capable of acting directly at sensible distances.

William Thomson (1824-1907), usually known as Lord Kelvin, and Peter Guthrie Tait (1831-1901), were some of the main opponents of Weber's electrodynamics in Great Britain. In 1867 they published the book *Treatise on Natural Philosophy*. There is an authorized translation into German made by Helmholtz and Gustav Wertheim (1843-1902). On paragraphs 381-385, pages 310-312, they classified the mathematical theories of natural philosophy. They considered Weber's electrodynamics as pernicious and dangerous (my emphasis in italics):[111]

> 381. It may perhaps be advisable to say a few words here about the use of hypotheses, and especially those of very different gradations of value which are promulgated in the form of Mathematical Theories of different branches of Natural Philosophy.
>
> [...] *And this leads us to a fourth class, which, however ingenious, must be regarded as in reality pernicious rather than useful.*
>
> 385. A good type of such a theory is that of Weber, which professes to supply a physical basis for Ampère's Theory of Electro-dynamics, just mentioned as one of the admirable and really useful third class. Ampère contents himself with experimental data as to the action of closed currents on each other, and from these he deduces mathematically the action which an element of one current ought to exert on an element of another — if such a case could be submitted to experiment. This cannot possibly lead to confusion. But Weber goes further, he assumes that an electric current consists in the motion of particles of two kinds of electricity moving in opposite directions through the conducting wire; and that these particles exert forces on other such particles of electricity, when in relative motion, different from those they would exert if at relative rest. *In the present state of science this is wholly unwarrantable, because it is impossible to conceive that the hypothesis of two electric fluids can be*

[109] [Note by Maxwell:] "Explicare tentatur quomodo fiat ut lucis planum polarizationis per vires electricas vel magneticas declinetur." — Halis Saxonum, 1858.

[110] [Neu58]. See also [Neu63].

[111] [TT67, pp. 310-312] with German translation in [TT71, pp. 349-351].

> *true, and besides, because the conclusions are inconsistent with the Conservation of Energy, which we have numberless experimental reasons for receiving as a general principle of nature. It only adds to the danger of such theories, when they happen to explain further phenomena, as those of induced currents are explained by that of Weber.* Another of this class is the Corpuscular Theory of Light, which for a time did great mischief, and which could scarcely have been justifiable unless a luminous corpuscle had been actually seen and examined. *As such speculations, though dangerous, are interesting, and often beautiful (as, for instance, that of Weber), we will refer to them again under the proper heads.*

However, Maxwell, Kelvin and Tait were wrong as regards the conservation of energy in Weber's electrodynamics. Weber had already presented in 1848 the potential energy from which he could deduce his force law and Maxwell was aware of this paper which had been translated into English in 1852.

In 1868 Carl Neumann (1832-1925) showed more explicitly the conservation of energy in Weber's electrodynamics. At that time Neumann was not aware of Weber's potential energy of 1848, as he emphasized in 1880 in the supplementary remarks when this paper was reprinted:[112]

> An *argument by Weber* (which appeared as a short note in Poggendorff's *Annalen*, Vol. 73, page 229 in the year 1848)[113] *shows in an elementary way, that the principle of vis viva continues to be valid for Weber's fundamental law.* — I regret, that at the time of writing I did not know this note. In my later publications (like for example in the *Abhandlungen der Kgl. Sächs. Ges. d. Wiss.*, Vol. 11, 1874, page 115)[114] I made an effort to bring to light the argument by Weber.

In any event, this lack of knowledge of Weber's potential energy of 1848 had a positive effect for Neumann, as it led him to introduce in this paper of 1868 the Lagrangian and Hamiltonian formulations for Weber's electrodynamics, as I discussed in the Introduction to the English translation of his paper, Chapter 4.[115]

In 1869 and 1871 Weber proved in detail that his force law satisfied the principle of the conservation of energy.[116]

Maxwell changed his mind only in 1871, after Weber's proof. Harman reproduced a postcard from Maxwell to P. G. Tait, from 7 November 1871, where he informed Tait that Weber had reason. His force has a potential. Hence in any cyclic operation no work is spent or gained. There is then conservation of energy in Weber's electrodynamics.[117]

Helmholtz' proof does not apply to Weber's electrodynamics. The reason is that Weber's force depends not only on the distance and velocity of the interacting particles, but also on their accelerations. And this general case had not been considered by Helmholtz in 1847.

Weber said the following when he discussed the conservation of energy with his force law in the Sixth major Memoir on Electrodynamic Measurements:[118]

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[112] [Neu68a] with English translation in [Neu20a], see Section 5.6 of Chapter 5.

[113] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

[114] [Neu74].

[115] See also Section 3.5 (Lagrangian and Hamiltonian Formulations of Weber's Electrodynamics) of [Ass94].

[116] [Web69] with English translation in [Web21d]; and [Web71] with English translation in [Web72]. See Chapters 7 and 9.

[117] [Har82, pp. 96-97] and [Max95, pp. 686-688].

[118] [Web72, pp. 1-2], see also page 67 of Chapter 9.

> The law of electrical action announced in the First Memoir on Electrodynamics Measurements (*Elektrodynamische Maasbestimmungen*, Leipzig, 1846) has been tested on various sides and been modified in many ways. It has also been made the subject of observations and speculations on a more general kind; these, however, cannot by any means be regarded as having as yet led to definitive conclusions. The First Part of the following Memoir is limited to a discussion of the relation which this law bears to the *Principle of the Conservation of Energy*, the great importance and high significance of which have been brought specially into prominence in connexion with the Mechanical Theory of Heat. In consequence of its having been asserted that the law referred to is in contradiction with this principle, an endeavour is here made to show that no such contradiction exists. On the contrary, the law enables us to make an addition to the Principle of Conservation of Energy, and to alter it so that its application to each pair of particles is no longer limited solely to the time during which the pair does not undergo either increase or diminution of *vis viva* through the action of other bodies, but always holds good independently of the manifold relations to other bodies into which the two particles can enter.
>
> Besides this, in the Second Part the law is applied to the development of *the equations of motion of two electrical particles subjected only to their mutual action.* Albeit this development does not lead directly to any comparisons or exact control by reference to existing experience (on which account it has hitherto received little attention), it neverthless leads to various results which appear to be of importance as furnishing clues for the investigation of the molecular conditions and motions of bodies which have acquired such special significance in relation to Chemistry and the theory of Heat, and to offer to further investigation interesting relations in these still obscure regions.

When he wrote the *Treatise* in 1873, Maxwell presented his new point of view that Weber's force was consistent with the principle of energy conservation:[119]

> 852.] The two expressions[120] lead to precisely the same result when they are applied to the determination of the mechanical force between two electric currents, and this result is identical with that of Ampère. But when they are considered as expressions of the physical law of the action between two electrical particles, we are led to enquire whether they are consistent with other known facts of nature.
>
> Both of these expressions involve the relative velocity of the particles. Now, in establishing by mathematical reasoning the well-known principle of the conservation of energy, it is generally assumed that the force acting between two particles is a function of the distance only, and it is commonly stated that if it is a function of anything else, such as the time, or the velocity of the particles, the proof would not hold.
>
> Hence a law of electrical action, involving the velocity of the particles, has sometimes been supposed to be inconsistent with the principle of the conservation of energy.

[119][Max54a, Vol. 2, Articles 852-853, pp. 483-484] with Portuguese translation in [Ass92c].

[120]Maxwell was referring here to the forces between electrified particles due to C. F. Gauss (1777-1855), see [Gau67] and Chapter 3, and Wilhelm Weber, [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

853.] The formula of Gauss is inconsistent with this principle, and must therefore be abandoned, as it leads to the conclusion that energy might be indefinitely generated in a finite system by physical means. This objection does not apply to the formula of Weber, for he has shewn[121,122] that if we assume as the potential energy of a system consisting of two electric particles,

$$\psi = \frac{ee'}{r}\left[1 - \frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right] ,$$

the repulsion between them, which is found by differentiating this quantity with respect to r, and changing the sign, is that given by the formula

$$\frac{ee'}{r^2}\left[1 + \frac{1}{c^2}\left(r\frac{\partial^2 r}{\partial t^2} - \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right] .$$

Hence the work done on a moving particle by the repulsion of a fixed particle is $\psi_o - \psi_1$, where ψ_o and ψ_1 are the values of ψ at the beginning and at the end of its path. Now ψ depends only on the distance, r, and on the velocity resolved in the direction of r. If, therefore, the particle describes any closed path, so that its position, velocity, and direction of motion are the same at the end as at the beginning, ψ_1 will be equal to ψ_o, and no work will be done on the whole during the cycle of operations.

Hence an indefinite amount of work cannot be generated by a particle moving in a periodic manner under the action of the force assumed by Weber.

I now present the translation of Weber's Sixth major Memoir on Electrodynamic Measurements.

[121] [Note by Maxwell:] *Pogg. Ann.* lxxiii. p. 229 (1848).

[122] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

Chapter 9

[Weber, 1871, EM6] Electrodynamic Measurements, Sixth Memoir, relating specially to the Principle of the Conservation of Energy

Wilhelm Weber[123,124,125]

I - Introduction

The law of electrical action announced in the First Memoir on Electrodynamic Measurements (*Elektrodynamische Maassbestimmungen*, Leipzig, 1846)[126,127] has been tested on various sides and been modified in many ways. It has also been made the subject of observations and speculations of a more general kind; these, however, cannot by any means be regarded as having as yet led to definite conclusions. The First Part of the following Memoir is limited to a discussion of the relation which this law bears to the *Principle of the Conservation of Energy*, the great importance and high significance of which have been brought specially into prominence in connexion with the Mechanical Theory of Heat. In consequence of its having been asserted that the law referred to is in contradiction with this principle, an endeavour is here made to show that no such contradiction exists. On the contrary, the law enables us to make an addition to the Principle of the Conservation of Energy, and to alter it so that its application to each pair of particles is no longer limited solely to the time during which the pair does not undergo either increase or diminution of *vis viva*[128] through the action of

[123] [Web71] with English translation by George Carey Foster (1835-1919) in [Web72].

[124] Wilhelm Weber's Notes are represented by [Note by WW:]; the Notes by H. Weber, the editor of the fourth volume of Weber's *Werke*, are represented by [Note by HW:]; the Notes by the Editors of the Philosophical Magazine which published the English translation of this work are represented by [Note by EPM:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[125] [Note by EPM:] Translated by Professor G. C. Foster, F.R.S., from the *Abhandlungen der mathem.-phys. Classe der Königl. Sächsischen Gesellschaft der Wissenschaften*, vol. x. (January 1871).

[126] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 25.

[127] [Note by AKTA:] See [Web46, p. 25 of Weber's *Werke*] with partial French translation in [Web87] and a complete English translation in [Web07].

[128] [Note by AKTA:] See footnote 26 on page 17. As will become evident from footnote 140 on page 74, what Weber calls the *vis viva*, living force or *lebendige Kraft* of a particle should be understood as the modern

other bodies, but always holds good independently of the manifold relations to other bodies into which the two particles can enter.

Besides this, in the Second Part the law is applied to the development of *the equations of motion of two electrical particles subjected only to their mutual action.* Albeit this development does not lead directly to any comparisons or exact control by reference to existing experience (on which account it has hitherto received little attention), it nevertheless leads to various results which appear to be of importance as furnishing clues for the investigation of the molecular conditions and motions of bodies which have acquired such special significance in relation to Chemistry and the theory of Heat, and to offer to further investigation interesting relations in these still obscure regions.

kinetic energy, namely, $mv^2/2$.

II - On the Relation between the Laws of Electricity and the Principle of the Conservation of Energy

9.1 Electrical Particles and Electrical Masses

Particles of positive and of negative electricity are denoted by the same letters, for instance by e or e' etc., but a positive or a negative value is assigned to e or e'... according to whether it represents a particle of the positive or of the negative fluid.

If the measurable force of repulsion exerted by the first particle e upon another exactly equal particle e at the constant measurable distance r be denoted by f, and also the measurable force of repulsion exerted by the second particle e' upon another exactly equal particle e', at the same distance r, be denoted by f', then $\pm r\sqrt{f}$ is taken as the measure of e, and $\pm r\sqrt{f'}$ as the measure of e', where the upper or the lower sign is to be taken according to whether the particle is a particle of positive or of negative fluid. The unit of force which is here adopted for the measurement of f and f' is the unit recognized in Mechanics, namely the force which, when it acts upon the unit of mass recognized in Mechanics (1 milligramme), imparts to this mass one unit of velocity in one unit of time. The repulsive force of the two particles e and e', so long as their distance r remains unchanged, is, in accordance with the electrostatical law,

$$= \frac{ee'}{r^2} .$$

A negative value of this expression denotes attractive force.

In this mode of denoting particles of the electric fluids, however, e and e' have not the signification of *masses* in the mechanical sense, as appears from the simple consideration that e and e' may have at one time positive and at another time negative values; but nevertheless the values of e and e' are closely related to the masses of the particles. For if we denote the *masses* of the particles e and e' (in the mechanical sense, according to which the unit of mass [1 milligramme] is determined by the mass of *one* ponderable body, and different masses are compared with each other in proportion to the reciprocals of the accelerations produced in them by the same force) by ε and ε', of which the values are always positive, we get for *positive* values of e and e',

$$\frac{e}{\varepsilon} = \frac{e'}{\varepsilon'} = a ,$$

and for negative values of e and e',

$$\frac{e}{\varepsilon} = \frac{e'}{\varepsilon'} = b ,$$

where a has a definite *positive* and b a definite *negative value.* Whether or not we have here $a^2 = b^2$, or what ratio a^2 bears to b^2, has not as yet been made out, any more than the numerical value of a or b. In many cases the electrical mass ε is connected with a ponderable mass m, so that it is impossible for it to be moved independently of it; in such cases, only the combined mass $m + \varepsilon$ comes into account, and in general ε may be regarded as vanishingly small in comparison with m. Consequently it is only seldom that the masses ε and ε' have to be considered.

The distinction here indicated between the particles e and e', and their masses ε and ε', is not always made; on the contrary, the symbols of the particles e and e' are also used to denote the corresponding masses. It is, however, to be observed that, when this is done, no regard can be had to the signs of e and e'. The omission of the unknown factors a and b is always allowable when we are dealing only with the *relative values* of masses of positive or of negative electricity.

9.2 The Law of Electrical Force

The Law of Electrical Force is thus stated in, "Electrodynamic Measurements" (Leipzig, 1846, p. 119):[129,130]

> If e and e' denote two electrical particles, the repulsive force exerted by the two particles on each other at the distance r is represented by[131]

$$\frac{ee'}{r^2}\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}+\frac{2r}{c^2}\frac{d^2r}{dt^2}\right) ,$$

where c is the constant denoted at the place quoted by $4/a$.

But this expression for the *force* which the particles e and e' mutually exert upon each other, it is easy to see, is dependent on a magnitude which contains as a factor the very force that is to be determined. This is readily seen when the relative acceleration of the two particles, namely d^2r/dt^2, is broken up into two parts, thus,

$$\frac{d^2r}{dt^2}=\frac{d^2r'}{dt^2}+\frac{d^2r''}{dt^2} ,$$

where the first part, d^2r'/dt^2, is the relative acceleration due to the mutual action of the two particles, and the second part, d^2r''/dt^2, is the acceleration due to other causes (namely to the acquired velocity of the particles perpendicular to r, and to the mutual action between them and other bodies). The first part, however, or that due to the mutual action of the two particles, is proportional to the force arising from this mutual action, and is represented by the quotient of this force by the mass upon which it acts.

Hence there easily follows, as was shown in the memoir already quoted (page 168),[132,133] another expression for the force which the particles e and e' mutually exert upon each other, containing only terms which are independent of the force to be determined, namely the expression

$$\frac{ee'}{r^2-\frac{2r}{c^2}(e+e')}\left(1-\frac{1}{c^2}\frac{dr^2}{dt^2}+\frac{2rf}{c^2}\right) ,$$

[129] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 157

[130] [Note by AKTA:] See [Web46, p. 157 of Weber's *Werke*] and [Web07, p. 98].

[131] [Note by AKTA:] Nowadays this expression would be written as:

$$\frac{ee'}{r^2}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2+\frac{2r}{c^2}\frac{d^2r}{dt^2}\right] .$$

[132] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 212.

[133] [Note by AKTA:] [Web46, p. 212 of Weber's *Werke*].

(in which f is put for d^2r''/dt^2), or, if the electrical particles e and e' are distinguished from their masses ε and ε' in accordance with the previous Section (a distinction which was not made in the memoir quoted above), the expression

$$\frac{ee'}{r^2 - \frac{2r}{c^2} \cdot \frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'} ee'} \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2rf}{c^2}\right) .$$

From this it results that the law of electrical force is by no means so simple as we expect a fundamental law to be; on the contrary, it appears in two respects to be particularly complex.

In the first place, it is a consequence of this expression for the force, that, as was pointed out in the memoir referred to, the force which two electrical particles exert upon each other does not depend exclusively upon these particles themselves, but also upon the portion of their relative acceleration denoted by f, *which is in part due to the action of other bodies.* It was also pointed out that, inasmuch as the forces exerted by two bodies upon each other have been called by Berzelius *catalytic forces* when they depend upon the presence of a third body,[134] *electrical forces* considered generally are, in this sense, *catalytic forces.*

In the second place, another noteworthy result follows from this expression for the force — namely, that when the particles e and e' are *of the same kind*,[135] *they do not by any means always repel each other*; thus when $dr^2/dt^2 < c^2 + 2rf$, they repel only so long as

$$r > \frac{2}{c^2}\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}ee' ,$$

and, on the contrary, they attract when

$$r < \frac{2}{c^2}\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}ee' .$$

An exception to this rule occurs only in the case in which

$$\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\frac{ee'}{c^2}\right) ,$$

which is always a factor of the denominator, becomes likewise a factor of the numerator. This case occurs when the two electrical particles are at *permanent relative rest*, so that $dr/dt = 0$ and $d^2r/dt^2 = 0$.

The general expression for the force given above becomes in fact

$$\frac{ee'}{r\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{c^2}\right)} \cdot \left(1 + \frac{2r}{c^2}f\right)$$

when $dr/dt = 0$; and by dividing this by the mass $\varepsilon\varepsilon'/(\varepsilon+\varepsilon')$ we find the part of the acceleration which is due to the forces exerted upon each other by the two electrical particles, namely

$$\frac{(\varepsilon+\varepsilon')ee'}{\varepsilon\varepsilon' r\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{c^2}\right)} \cdot \left(1 + \frac{2r}{c^2}f\right) .$$

By adding to this the other part of the acceleration, namely f, which is due to the acquired motion of the particles at right angles to r and to the action of other bodies, we obtain the *total* acceleration, namely

[134] [Note by AKTA:] Jöns Jacob Berzelius (1779-1848). See [Ber36c], [Ber36a] and [Ber36b].

[135] [Note by AKTA:] That is, both of them positive, or both of them negative, such that $ee' > 0$.

$$\frac{d^2r}{dt^2} = f + \frac{(\varepsilon+\varepsilon')ee'}{\varepsilon\varepsilon' r\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{c^2}\right)}\cdot\left(1+\frac{2r}{c^2}f\right) \ ,$$

which, when the particles are *at permanent relative rest*, $=0$. Hence *for permanent relative rest* we have

$$f = -\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{r^2} \ .$$

If this value of f be substituted in the expression for the force

$$\frac{ee'}{r\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{c^2}\right)}\cdot\left(1+\frac{2r}{c^2}f\right) \ ,$$

the latter becomes

$$\frac{ee'}{r\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{c^2}\right)}\cdot\frac{1}{r}\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{c^2}\right) \ .$$

Hence it appears that, in the case of permanent relative rest, the factor

$$\left(r - 2\frac{\varepsilon+\varepsilon'}{\varepsilon\varepsilon'}\cdot\frac{ee'}{c^2}\right)$$

is common to numerator and denominator. The value of the quotient, which is thus independent of this factor, namely ee'/r^2, consequently gives the expression for the force, in the case of permanent relative rest, in complete agreement with the fundamental laws of electrostatics, according to which this force has a *positive* value for particles of the *same kind at all distances*.

9.3 The Law of Electrical Potential

In the previous Section the law of electrical force is shown to be, in two respects, of a very complicated character, namely: — in the first place, in that the repulsive force between two electrical particles is dependent on things that do not appertain either to the nature of the particles which exert the force upon each other, or to their relative positions in space, or their existing relative motion, but *depends upon other bodies*; and secondly, in that *repulsion* may be exerted upon each other at certain distances by the same particles, and *attraction* at other distances.

Compared with this complicated law of *electrical force*, the law of *electrical potential* is very simple.

The value of the potential V of two electrical particles e and e', in fact, as I pointed out as long ago as the year 1848 in Poggendorff's *Annalen* (vol. lxxiii, p. 229),[136,137] is determined by the following law,

$$V = \frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right) \ .$$

[136][Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 245.

[137][Note by AKTA:] [Web48, p. 245 of Weber's *Werke*] with English translations in [Web52c], [Web66] and [Web19a].

Observing that both r and dr/dt have different values at different times for both the particles e and e', and that consequently both are functions of the time, it follows that dr/dt may also be regarded as a function of r, which may be denoted by fr. We thus obtain

$$V = \frac{ee'}{r}\left(\frac{1}{c^2}\cdot(fr)^2 - 1\right) \ ,$$

and from this, by differentiation, the expression for the force

$$\frac{dV}{dr} = -\frac{ee'}{r^2}\left(\frac{1}{c^2}\cdot(fr)^2 - 1\right) + 2\frac{ee'}{rc^2}\cdot fr\cdot\frac{dfr}{dr} \ ,$$

or, if we again put dr/dt for fr,

$$\frac{dV}{dr} = \frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2} + \frac{2r}{c^2}\cdot\frac{dr}{dt}\cdot\frac{d\frac{dr}{dt}}{dr}\right) \ ,$$

for which we may write

$$\frac{dV}{dr} = \frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2} + \frac{2r}{c^2}\cdot\frac{d^2r}{dt^2}\right) \ .$$

From this it appears that

$$\frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right)$$

is a function whose differential coefficient with respect to r represents the repulsive force between the two particles e and e', where r and dr/dt denote respectively their distance and relative velocity regarded as functions of the time. But since

$$\frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right)$$

becomes equal to nothing when e and e' are separated infinitely far from each other,

$$\frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right)$$

is the *potential* of the electrical particles e and e', that is to say, the *work* which is expended in causing the particles to approach each other from an infinite distance while under the action of their mutual repulsion, and to arrive at the distance r with the relative velocity dr/dt.[138,139]

It likewise results from the foregoing that the work, which is expended when a given relative arrangement and state of motion of a system of particles e and e' are changed to

[138][Note by WW:] This law of electrical *potential* has also been taken as his starting-point by Beer in his 'Introduction to Electrodynamics' (see *Einleitung in die Elektrostatik, die Lehre vom Magnetismus und die Elektrodynamik*, von August Beer. Nach dem Tode des Verfassers herausgegeben von Julius Plücker: Braunschweig, 1865. p. 250). The placing of the law of *potential* in the foreground as the fundamental law, and deriving the law of force from it, ought not to give rise to any misgiving. We have in many respects a better justification for speaking of the *physical existence of the work expressed by the potential* than for speaking of the *physical existence of a force*, as to which all we can say is that it *tends to change the physical relations of bodies*.

[139][Note by AKTA:] [Bee65, p. 250].

another arrangement and another state of motion, depends only on the initial and final arrangements and movements of the particles, and is independent of the way by which the transition has been effected, and also independent of states of motion which may have existed during the transition.

9.4 Fundamental Electrical Laws

The law of *electrical potential* certainly appears to stand, in view of its simplicity, in a much closer relation to the true fundamental laws of electricity than the far more complex law of *electrical force*; but the expression of the former law may still be resolved into two simpler laws, which may be stated in the following manner:

First Law. — If two particles e and e' are at relative rest or possess the same relative motion at two different distances r and ρ, the quantities of work V and U which are expended in separating the particles, while mutually acting on each other, from these distances to an infinite distance, are to each other inversely as these two distances, that is,

$$V : U = \rho : r \ . \qquad (1)$$

Second Law. — The work U, which is expended in separating the particles e and e' while subject to the force exerted by them on each other from *a given distance* $\rho = ee'/a$ *proportional to the quantity* ee' to an infinite distance, makes *together with the vis viva* x, which belonged to the particles in consequence of their relative motion at the distance ρ, *a constant sum*, namely a, that is,

$$U + x = a \ . \qquad (2)$$

For from equation (1) it follows that

$$U = \frac{r}{\rho} V \ ,$$

and hence, by equation (2),

$$\frac{r}{\rho} V + x = a \ ,$$

or, since $\rho = ee'/a$,

$$V = \frac{ee'}{r}\left(1 - \frac{x}{a}\right) \ .$$

But the relative *vis viva* x is proportional to the square of the relative velocity dr/dt, so that we may substitute for a a new constant c^2 such that[140]

[140][Note by WW:] If ε and ε' denote the masses of the particles e and e', and α and β the velocities of ε in the direction of r and at right angles thereto, and α' and β' the same velocities for ε', so that $\alpha - \alpha' = dr/dt$ is the relative velocity of the two particles, then

$$\frac{1}{2}\varepsilon\left(\alpha^2 + \beta^2\right) + \frac{1}{2}\varepsilon'\left(\alpha'^2 + \beta'^2\right)$$

is the total *vis viva* of the two particles. If we now put for α

$$\frac{\varepsilon\alpha + \varepsilon'\alpha'}{\varepsilon + \varepsilon'} + \frac{\varepsilon'(\alpha - \alpha')}{\varepsilon + \varepsilon'} \ ,$$

$$\frac{x}{a} = \frac{1}{c^2} \cdot \frac{dr^2}{dt^2} \ .$$

We thus obtain

---

and for α'

$$\frac{\varepsilon\alpha + \varepsilon'\alpha'}{\varepsilon + \varepsilon'} - \frac{\varepsilon(\alpha - \alpha')}{\varepsilon + \varepsilon'} \ ,$$

we get the total *vis viva* of the two particles represented as the sum of two parts in the following manner — namely,

$$= \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} + \frac{1}{2}\left[\frac{(\varepsilon\alpha + \varepsilon'\alpha')^2}{\varepsilon + \varepsilon'} + \varepsilon\beta^2 + \varepsilon'\beta'^2\right] \ ,$$

the *first* part of which, or $\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot \frac{dr^2}{dt^2}$, is the *relative* vis viva of the particles which was denoted above by x. But a is also a relative *vis viva* of the same particles, namely that which corresponds to a definite relative velocity c, so that $a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot c^2$. Hence we get $\frac{x}{a} = \frac{1}{c^2} \cdot \frac{dr^2}{dt^2}$, as was given above.

It may be further observed that the *second* part of the above sum, namely

$$\frac{1}{2}\left[\frac{(\varepsilon\alpha + \varepsilon'\alpha')^2}{\varepsilon + \varepsilon'} + \varepsilon\beta^2 + \varepsilon'\beta'^2\right] \ ,$$

may be again represented, after another subdivision, as the sum of two parts, thus

$$= \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{ds^2}{dt^2} + \frac{1}{2}\left[\frac{(\varepsilon\alpha + \varepsilon'\alpha')^2}{\varepsilon + \varepsilon'} + (\varepsilon + \varepsilon')\gamma^2\right] \ ,$$

where ds/dt represents the velocity with which the two particles move relatively to each other in space perpendicularly to r, while γ represents the velocity, perpendicular to r, of the centre of gravity of the two particles. We thus get the total *vis viva* of the two particles divided into three parts, namely,

$$i. \qquad \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \ ,$$

$$ii. \qquad \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{ds^2}{dt^2} \ ,$$

$$iii. \qquad \frac{1}{2}\left[\frac{(\varepsilon\alpha + \varepsilon'\alpha')^2}{\varepsilon + \varepsilon'} + (\varepsilon + \varepsilon')\gamma^2\right] \ ;$$

the *first* of which, namely

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \ ,$$

represents the *relative* vis viva of the two particles; while the *first* two parts taken together, namely

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'}\left(\frac{dr^2}{dt^2} + \frac{ds^2}{dt^2}\right) \ ,$$

represent the total *internal* vis viva, or the total *internal kinetic energy of the system*; and the *third* part, namely

$$\frac{1}{2}\left[\frac{(\varepsilon\alpha + \varepsilon'\alpha')^2}{\varepsilon + \varepsilon'} + (\varepsilon + \varepsilon')\gamma^2\right] \ ,$$

represent the *external* vis viva, or the *external kinetic energy of the system* (that is, the *vis viva* of the centre of gravity of the two particles).

$$V = \frac{ee'}{r}\left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2}\right) \ .$$

Here V denotes the work expended in separating the two particles from the distance r to an infinite distance. If V is to denote the work done in bringing the particles from an infinite distance to the distance r, as it is usually understood to do, so that positive values of dV/dr may indicate *repulsion*, we obtain

$$V = \frac{ee'}{r}\left(\frac{1}{c^2} \cdot \frac{dr^2}{dt^2} - 1\right) \ ,$$

that is to say, the *law of electrical potential*.

9.5 Principle of the Conservation of Energy for Two Particles which Form an Isolated System

The two fundamental laws laid down in the foregoing Section, which may be called

- The Law of the dependence of the Potential on the distance for *a constant relative motion*, and
- The Law of the dependence of the Potential on the relative motion for *a constant distance*,

require to be further discussed in relation to their bearing upon the principle of the Conservation of Energy.

In accordance with the principle of the conservation of energy, three forms of energy are to be distinguished from each other, namely, *energy of motion* (kinetic energy),[141] *potential energy*, and *energy of heat* (thermal energy).[142]

The *energy of motion* is that part of the energy which depends upon the existing movements; and a special determination is given of the way in which it depends upon movement, namely, partly upon the magnitude of the moving mass, and partly upon the velocity with which this mass moves.

The same determination also apples to *thermal energy*, if this is regarded, in accordance with the mechanical theory of heat, as an *internal motion of the particles of bodies*. But if we are dealing with a system of two *elementary particles* (that is to say, particles such that there can be no motion *within* them), it is obvious that in the case of such a system thermal energy has no existence, and *energy of motion* and *potential energy* alone remain.

Lastly, the *potential energy* is that part of the energy which depends on the existing potential; and a special determination is needed of the way in which potential energy *depends upon the potential*, exactly as, in the case of the energy of motion, it is needful to determine the special way in which it depends on movement.

[141][Note by AKTA:] In German: *Bewegungsenergie*.
[142][Note by AKTA:] In German: *Wärmeenergie*.

Now this special determination has been made by *equating potential energy* (without regard to the sign) *and potential.*[143]

The justification for this proceeding has been found in the fact that the potential is a magnitude which is homogeneous with kinetic energy, which, when taken with the negative sign and added to the kinetic energy, gives always the same sum, so long as the two particles constitute an *isolated* system[144] which does not undergo either gain or loss of energy from without.

For instance, if we have a system of two ponderable particles m and m', its *potential* is

$$V = \frac{mm'}{r} \ ,$$

and the internal *vis viva*, or the *internal kinetic energy of the system*, is

$$W = \frac{1}{2}\frac{mm'}{m+m'}\left(u^2 + \alpha^2\right) \ ,$$

where $u = dr/dt$ is the relative velocity of the two particles, and α the difference of the velocities in space perpendicularly to r. But, for such an isolated system, if we put $r = r_0$ and $\alpha = \alpha_0$ when $u = 0$, the following value is easily got, namely[145]

[143][Note by WW:] The sign of the *potential*, V, is so determined that positive values of dV/dr indicate repelling forces; the sign of the *potential energy* is fixed by the sign of the work which is done, in consequence of the mutual action of the particles, when the two particles are separated from the distance r to an infinite distance. Consequently, for two ponderable particles m and m', the potential is $V = mm'/r$, and the potential energy $= -mm'/r$. For two electrical particles e and e' the potential is

$$= \frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right) \ ,$$

and the potential energy

$$= \frac{ee'}{r}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2}\right) \ .$$

[144][Note by AKTA:] In German: *ein abgesondertes System*. This expression was always translated by G. C. Foster as "a detached system", [Web72]. Here I am utilizing everywhere the analogous and more common expression "an isolated system".

[145][Note by WW:] The force with which the two particles mutually act on each other, namely dV/dr, divided by m, gives the acceleration of the particle m, that is, $\frac{1}{m}\cdot\frac{dV}{dr}$; divided by m' it gives the acceleration of the particle m', namely $\frac{1}{m'}\cdot\frac{dV}{dr}$. Consequently that part of the relative acceleration of the two particles which arises from their mutual action is

$$= \left(\frac{1}{m} + \frac{1}{m'}\right)\frac{dV}{dr} \ ,$$

while that part of the relative acceleration of the two particles which arises from their rotation about one another is represented by α^2/r. If now this last portion be subtracted from the total acceleration du/dt, the following equation results:

$$\frac{du}{dt} - \frac{\alpha^2}{r} = \left(\frac{1}{m} + \frac{1}{m'}\right)\frac{dV}{dr} \ .$$

Putting $r = r_0$ and $\alpha = \alpha_0$ for the instant at which $u = 0$, we obtain the expression

$$\alpha r = \alpha_0 r_0 \ ;$$

as applicable for the case in which the only forces acting on the two particles are those due to their mutual action. Accordingly we get, by integrating the above differential equation after it has been multiplied by

$$u^2 = \frac{r_0 - r}{r_0} \left[\frac{2(m+m')}{r} - \frac{r_0 + r}{r_0} \alpha^2 \right] ,$$

and consequently the sum

$$W - V = -\frac{mm'}{r_0} + \frac{1}{2} \frac{mm'}{m+m'} \cdot \alpha_0^2 .$$

This sum always retains the same value as long as the values of r_0 and α_0 remain unchanged, that is, so long as the system of the two particles undergoes neither loss nor gain of energy from without. The *external kinetic energy* of such an isolated system amounts *separately to a constant sum*.

Now the same thing holds good also for two *electrical* particles e and e'; for their potential, taken with the negative sign and added to their kinetic energy, gives in like manner always the same sum so long as the particles constitute an *isolated* system.

We have, for the *potential* of such a system of two electrical particles,

$$V = \frac{ee'}{r} \left(\frac{u^2}{c^2} - 1 \right) ,$$

and, for the *internal kinetic energy of the system*,

$$W = \frac{1}{2} \frac{\varepsilon \varepsilon'}{\varepsilon + \varepsilon'} \left(u^2 + \alpha^2 \right) = \frac{ee'}{\rho c^2} \left(u^2 + \alpha^2 \right) ,$$

if $u = dr/dt$ denotes the relative velocity of the two particles, and α the difference of their velocities in space at right angles to r. But, for such an *isolated* system, when we put $r = r_0$ and $\alpha = \alpha_0$ for $u = 0$, it is easy to obtain[146]

$$\alpha = \frac{r_0}{r} \alpha_0 ,$$

$$u^2 = \frac{r - r_0}{r - \rho} \left(\frac{\rho}{r_0} c^2 + \frac{r_0 + r}{r} \alpha_0^2 \right) ,$$

and consequently the sum

$$W - V = \frac{ee'}{r_0} + \frac{ee'}{\rho} \cdot \frac{\alpha_0^2}{c^2} = \frac{ee'}{r_0} + \frac{1}{2} \frac{\varepsilon \varepsilon'}{\varepsilon + \varepsilon'} \cdot \alpha_0^2 .$$

This sum likewise retains the same value so long as the values of r_0 and α_0 remain unchanged, that is, so long as the system of two particles undergoes neither loss nor gain of

$2dr = 2udt$,

$$u^2 + \alpha_0^2 r_0^2 \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) = 2 \left(\frac{1}{m} + \frac{1}{m'} \right) \left(\frac{mm'}{r} - \frac{mm'}{r_0} \right) ,$$

and hence

$$u^2 = \frac{r_0 - r}{r} \left(\frac{2(m+m')}{r_0} - \frac{r_0 + r}{r} \alpha_0^2 \right) = \frac{r_0 - r}{r_0} \left(\frac{2(m+m')}{r} - \frac{r_0 + r}{r_0} \alpha^2 \right) .$$

[146] [Note by WW:] See Section 9.11.

energy from without.[147,148] *The same principle holds good in relation to the external kinetic energy of an isolated system of two electrical particles and to that of two ponderable particles.*

9.6 Extension of the Principle of the Conservation of Energy to Two Electrical Particles which do Not Form an Isolated System

If potential energy is taken, as is done in the previous Section, as equal and opposite to potential, the principle of the conservation of energy holds good for two particles only so long as these two particles constitute an *isolated* system, that is, so long as the system formed of the two particles undergoes neither gain nor loss of energy from without.

If the *total* energy of such an isolated system of two particles were at first $= A$, but, the system ceasing to be isolated, it received from without a quantity of kinetic energy $= a$, it seems to follow that, if the system were now again to become isolated, the *total* energy would again become and remain constant so long as it remained isolated, but that the total energy of the system in its final isolated state would have the value $A + a$ (that is, a value exceeding that corresponding to its previous isolated state by a). This, however, does not by

[147] [Note by WW:] In Professor Tait's very instructive work, "A Sketch of Thermodynamics" (Edinburgh, 1868), the following passage occurs at page 76, in reference to the investigations of Riemann and Lorenz which appeared in Poggendorff's *Annalen* for 1867 [Phil. Mag. S. 4. vol. xxxiv. pp. 368 and 287]:

> "But the investigations of these authors are entirely based on Weber's inadmissible theory of the forces exerted on each other by *moving electric particles*, for which the conservation of energy is not true, while Maxwell's result is in perfect consistence with that great principle."

This assertion of Professor Tait's seems to be in contradiction with the above. At page 56 of the same work Mr. Tait mentions that Helmholtz has based the doctrine of energy on Newton's principle and on the following postulate:

> "Matter consists of ultimate particles which exert upon each other forces whose directions are those of the lines joining each pair of particles, and whose magnitudes depend solely on the distances between the particles."

It is evident the contradiction of the fundamental law of electricity *with this postulate*, but not at all *with the principle of the conservation of energy*, — a distinction which Professor Tait seems to have confused.

[148] [Note by AKTA:] See [Tai68, pp. 56 and 76]; [Rie67b] with English translation in [Rie67a] and [Rie77a]; [Lor67b] with English translation in [Lor67a]. Tait was referring to Helmholtz's work of 1847, [Hel47] with English translation in [Hel66].

The last sentence of footnote 147 runs as follows in German:

> Es leuchtet der Widerspruch des elektrischen Grundgesetzes *mit diesem Postulate* wohl ein, aber keineswegs *mit dem Principe der Erhaltung der Energie*, was Herr Tait verwechselt zu haben scheint.

The "elektrischen Grundgesetzes" here refers to Weber's 1846 fundamental law of electricity. I preferred to translate this sentence as presented in footnote 147, instead of G. C. Foster's translation which runs as follows, [Web72, p. 13]:

> The contradiction between the fundamental law of electricity and *this postulate* is evident; but the contradiction between it and the *principle of the conservation of energy* is by no means evident, — a distinction which Professor Tait seems to have overlooked.

any means conclusively prove the impossibility of extending the principle of the conservation of energy to two electrical particles which do not constitute an isolated system.

For, strictly speaking, this has only been proved on the assumption that the *potential energy* of the system depends solely on the distance between the two particles; while if, on the other hand, the potential energy does not depend simply on the distance of the two particles, but also on their relative *motion*, it is evident that while the system receives from without an amount of *kinetic energy* $= a$, a change in its *potential energy* may be indirectly produced thereby. It is thus possible that the change of *potential energy*, so caused indirectly from without, might be $= -a$, so that the *total* energy (kinetic energy and potential energy together) of the two particles, even if they did not constitute an isolated system, would retain always the same value.

This, however, certainly does not occur in reality for a system of two electrical particles, if the *potential energy* is taken as *equal and opposite to the potential*; but this assumption, which would thus make the extension of the principle impossible, has by no means been proved to be a necessary one. In general, all that is required is a *special determination of the way in which the potential energy depends upon the potential*; and here all that is self-evident is, that inasmuch as potential and potential energy are homogeneous magnitudes, a purely numerical relation must exist between them. But whether this numerical relation is always that of $+1$ to -1, or whether it is to be fixed otherwise, must still be regarded as in general doubtful; so that the possibility of the extension of the principle still remains.

We understand, in fact, by the *potential* of two particles, the amount of *work* which, in consequence of the mutual action of the two particles, is done when they are transferred in any way whatever from an infinite distance to the actually existing distance r with the actually existing relative velocity dr/dt.

It is, however, evident that *work* is done, in consequence of the mutual action of the two particles, not only during their transference from a *greater* distance to the distance r, but also during their transference from a *smaller* distance to the distance r. And there is no obvious reason why the *energy ascribed to the system* should be made to depend on the work done in the *former* case, and not on that done in the *latter* case also.

For example, if the *first* quantity of work were denoted, according to Section 9.4, by V, and the *second* by $\frac{\rho-r}{\rho}V$, the potential energy ascribed to the system might be taken as the *difference of these two amounts of work*, namely

$$= \frac{\rho - r}{\rho}V - V = -\frac{r}{\rho}V \ .$$

This difference of the two amounts of work is evidently the quantity of work which is done, in consequence of the mutual action of the two particles, during their transference from the limiting value of *small* distances to the limiting value of *great* distances — that is to say, the value which

$$-V = \frac{ee'}{r}\left(1 - \frac{u^2}{c^2}\right)$$

assumes when r is taken therein as equal to the limiting value of *small* distances, or when we put $r = \rho$, where ρ denotes the limiting value of small distances. According to this, therefore, this *difference of the two quantities of work*

$$= \frac{ee'}{\rho}\left(1 - \frac{u^2}{c^2}\right) = -\frac{r}{\rho}V \ .$$

In order to determine in this way the potential energy of a system of two electrical particles when the *first* quantity of work above referred to is

$$V = \frac{ee'}{r}\left(\frac{u^2}{c^2} - 1\right) ,$$

it is only necessary further, for the determination of the *second* quantity of work, to determine the value of ρ, that is, of the *smaller distance* which is to be taken account of in that portion of the work.

Now this *smaller distance*, equally with the *greater distance*, must be determined *on its own account, independently of the actually existing conditions* of the two particles. This was done in the case of the *greater* distance by assigning to it an infinitely great value; in the case of the *smaller* distance the same thing is accomplished if we assign to it the value

$$2\frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{c^2} ,$$

a distance which is given by the particles e and e', by their masses ε and ε', and by the known electrical constant c.

If we now put the smaller distance equal to the value of ρ, we get, in virtue of the equations

$$V = \frac{ee'}{r}\left(\frac{u^2}{c^2} - 1\right) ,$$

$$\frac{\rho - r}{\rho} V = \frac{\rho - r}{\rho} \cdot \frac{ee'}{r}\left(\frac{u^2}{c^2} - 1\right) ,$$

the required value of the *potential energy*, namely

$$-\frac{r}{\rho} V = -\frac{ee'}{\rho}\left(\frac{u^2}{c^2} - 1\right) = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'}\left(c^2 - u^2\right) .$$

In accordance with the distinction which is here drawn between the *potential* and the *potential energy* of two electrical particles and with the corresponding determination of their relation to each other, an analogous distinction may also be made between the *vis viva* and the *kinetic energy* of two particles. For there is no necessity that the *kinetic energy* of two particles should be taken as being equal to *the total* vis viva *of the two particles*; all that is generally essential is a *definite determination of the relation subsisting between the kinetic energy of two particles and the total vis viva belonging to them both.*

Now the total *vis viva* possessed by the two particles was represented in the footnote to Section 9.4 as the sum of two parts, of which the *first* part, namely

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} ,$$

was called the *relative* vis viva. The *second* part was that which the two particles possessed in virtue of their revolution about each other in space, and in virtue of the motion of their centre of gravity in space.

If now, in order to establish the conception of the *energy* of two particles, we take it as our starting-point that the *principle of the conservation of energy* of two particles must be based upon the essential characters of the two particles, and in fact upon what is *essential* *to them when regarded as constituting an isolated system*, it is obvious that for this purpose the conception of the *energy* of two particles must be made to depend only on the relations presented by the system of the two particles as such, quite irrespectively of the relations in which these particles may stand to all other bodies in space.

Applying this fundamental principle to the *kinetic energy* of two particles in the same way as it has just been done in respect of the *potential energy*, we see that the *kinetic energy* must be taken as dependent upon the *first* part of the total *vis viva* belonging to the two particles — that is to say, upon their *relative* vis viva — and not upon the *second* part of the total *vis viva*, or that which the two particles possess in virtue of their revolution about one another in space or of the motion of their centre of gravity in space; for this latter part depends upon relations which the two particles do not of themselves directly present. For the two particles taken by themselves do not directly present any relation to space except their distance apart, from which no knowledge can be had of their rotation or of the motion of their centre of gravity in space.

Consequently, in what follows, by the *kinetic energy* of two particles is to be understood, not the total *vis viva* possessed by the two particles, but only their *relative* vis viva.

But it is easy to see that, in accordance with this, while a system of two electrical particles e and e' receives from without an amount of kinetic energy $= a$, it really undergoes an alteration of its *potential energy* $= -a$; so that the *whole* energy of the system must always retain the same value not only when the two particles constitute an isolated system, but also when they do not do so. For if we represent the *kinetic energy* communicated from without by

$$a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}v^2 \ ,$$

while the kinetic energy of the particles *before* the communication of this portion was

$$= \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot u_0^2 \ ,$$

the *kinetic energy* existing *after* the communication is

$$\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot u^2 = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\left(u_0^2 + v^2\right) \ .$$

Consequently the *potential energy before the communication* is

$$-\frac{r}{\rho}V_0 = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\left(c^2 - u_0^2\right) \ ,$$

whereas the *potential energy after the communication* is

$$-\frac{r}{\rho}V = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\left(c^2 - u^2\right) = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\left(c^2 - u_0^2\right) - \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}v^2 \ ;$$

so that, in consequence of the communication from without of *kinetic energy* equal to $+a$, a change of *potential energy* has occurred which is represented by

$$-\frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}v^2 = -a \ .$$

9.7 Application to Other Bodies

If we distinguish, in accordance with the last Section, between the potential and the potential energy of two particles — that is to say, if we define

> *Potential* as the amount of *work* which, in consequence of the mutual action of the two particles, is done during the transference of the particles from an infinite distance to the actual distance r with the existing relative velocity dr/dt; and
>
> *Potential energy* as that amount of *work*, taken *negatively*, which, in consequence of the mutual action of the two particles, is done during the transference of the particles *from the greater distance* $r = \infty$ *to the smaller distance* $r = \rho$ determined by the particles e and e', their masses ε and ε' and by the constant c, with the existing relative velocity dr/dt,

the latter (that is to say, the *potential energy in the sense that has been indicated*) may be resolved into two parts, one of them equal and opposite to the *potential*, and therefore identical with the magnitude which has *hitherto* been *alone called potential energy*, but which, regarded henceforward as only a part of the potential energy, we may call the *free potential energy*; the remainder is the *second* part, which may be called the *latent potential energy*.

Hence the principle of the conservation of energy may be enunciated in the first place in the *earlier* wider sense as follows:

> For an *isolated* system of two particles the sum of the *kinetic energy* and of the *free potential energy* is always the same.

For so long as no kinetic energy is either lost or communicated from without, every change in the free potential energy will be compensated by an equal and opposite change in the kinetic energy.

But the principle of the conservation of energy may also be enunciated, secondly, in the *narrower sense* as follows (potential energy and kinetic energy being understood in the sense that has just been defined):

> The *relative kinetic energy* of two particles, and the *total potential energy* which they possess along with this kinetic energy, together give always the same sum.

Upon this the following remarks may be made:

(1) One particle regarded by itself can only possess *kinetic energy*.

(2) Two particles likewise possess in the first place kinetic energy, which is the sum of those which they possess when considered separately.

(3) This sum consists of a part A, which may be ascribed partly to the motion of their centre of gravity, and partly to their rotation about one another in space — and of another part B, which the particles possess relatively to each other when considered by themselves. This latter part, B, is called the *relative kinetic energy*, or *that belonging to the system formed by the two particles*.

(4) But in the *system of two particles* there is a something, in addition to its kinetic energy, which does not belong to the two particles taken separately, namely a greater or less *capacity for doing work* in virtue of the mutual action of the two particles upon each other. The *measure* of this capacity for doing work is termed the *potential energy of the system*,

or the *relative potential energy of the two particles*; and that quantity of work serves as the *measure of this working-power* which is done in consequence of the mutual action of the two particles during their transference *from the smaller distance* $r = \rho$ *to the greater distance* $r = \infty$, where ρ is determined by the particles themselves e and e', by their masses ε and ε', and by the constant c.

(5) The principle of the conservation of energy, however, when specially defined as above, is only applicable to two particles when their *potential* is of the same form as that of two electrical particles, namely

$$V = \frac{ee'}{r}\left(\frac{1}{c^2} \cdot \frac{dr^2}{dt^2} - 1\right) .$$

The potential of two ponderable masses m and m', on the contrary, is

$$V = \frac{mm'}{r} ,$$

which (neglecting the sign) can be included under the above general form only if the value of the constant c for ponderable masses is infinitely great. It is evident, however, that it would in reality suffice for the constant c to have only a very great value instead of an infinite value, in order that there might not be any thing perceptibly inconsistent with the results of experiment. And, considering the extraordinarily high value which must be ascribed to the constant c in the case of electrical particles, it does not seem at all necessary, for the avoidance of all sensible contradictions, to adopt any other value for ponderable bodies; consequently it must be permissible to represent the *potential* of two ponderable particles m and m' by

$$V = \frac{mm'}{r}\left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2}\right) ,$$

where the constant c retains *the same value as in the potential of two electrical particles.*

But even if it should hereafter result from more accurate experimental results that it is not permissible thus to ascribe the same value to the constant c in the case of ponderable particles, the possibility would always remain of assigning to the constant c a still greater value for ponderable particles; and this could easily be taken so great that any sensible disagreement with experiment would completely vanish.

III - On the Movement of Two Electrical Particles in Consequence of Their Action on Each Other

9.8 On the Validity of the Laws for Molecular Motions

The fundamental electrical law determines the action exerted by any given particle upon another under any circumstances. The simplest and most obvious application that can be made of this law, would seem to be to develop the laws of the motion of two particles which act mutually upon each other. Greater practical interest, however, attached to the determination, in the first place, of the laws of the distribution of electricity at rest upon conductors, and of the laws of the forces exerted by a current of electricity in a closed conductor, by reason of the current existing in another conductor, upon this latter conductor itself — as well as to the development of the laws of the (electromotive) forces exerted by closed currents (or by magnets) on the electricity in closed conductors — inasmuch as the results of these developments admitted of being directly tested and confirmed by experiment. But although this important practical interest is wanting to the development of the laws of motion of two particles subject only to their mutual action, many of its results cannot fail to merit attention in other respects.

The interest which belongs to these results relates indeed specially to the *molecular movements* of two particles, movements which are shut out from all direct experimental investigation, so that there is no authority for the application to them of the law that has been established, so far as it is regarded as an experimental law. Consequently the development of the laws of the *molecular movements* of two particles in accordance with the law that has been established must be considered only as an attempt to find a clue to the theory (which as yet we are entirely without) of these movements — a clue which by itself is certainly not sufficient, but is still in need of being supplemented in essential respects. For so long as the *molecular forces acting only at molecular distances*, which doubtless cooperate in the molecular movements, are not known and taken exact account of, the results that may be acquired cannot have any exact *quantitative* application, but only a *qualitative* value within certain limits, and can be of consequence only for a first *reconnaissance* of the territory.

9.9 Motion of Two Electrical Particles in the Direction of the Straight Line which Joins Them

For two particles, e and e', moving simply in consequence of their mutual action, we have, according to the fundamental laws of Section 4, by putting

$$\rho = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{ee'}{c^2} \ ,$$

$$x = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \ ,$$

$$a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot c^2 \ ,$$

and also giving a negative sign to U and V, so as to denote thereby the *potentials*,

$$V : U = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{ee'}{c^2} : r \ ,$$

$$-U + \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot c^2 \ ;$$

and therefore

$$V = \frac{2}{r}\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{ee'}{c^2} \cdot U = \frac{ee'}{r}\left(\frac{1}{c^2} \cdot \frac{dr^2}{dt^2} - 1\right) \ .$$

If there is no *motion of rotation of the particles about each other* in space, $[1/\varepsilon][dV/dr]$ is the acceleration of the particle e in the direction of r, and $[1/\varepsilon'][dV/dr]$ is the acceleration of the particle e' in the opposite direction. Hence the *relative acceleration* of the two particles becomes

$$\frac{d^2r}{dt^2} = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{dV}{dr} \ ,$$

and from this, by integrating between the limits $r = r_0$ and $r = r$ (r_0 denoting the value of r for the moment when $dr/dt = u = 0$), since ρ was made $= 2\,(1/\varepsilon + 1/\varepsilon')\,[ee'/c^2]$ we obtain

$$\frac{dr^2}{dt^2} = u^2 = \frac{r - r_0}{r - \rho} \cdot \frac{\rho}{r_0} \cdot c^2 \ .$$

ρ/r_0 has always a positive or negative value differing from nothing; for

$$\rho = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \cdot \frac{ee'}{c^2}$$

has a given finite although very small value, which is positive or negative according as ee' is positive or negative; and

$$r_0 = \frac{r}{1 + \frac{u^2}{c^2} \cdot \frac{r-\rho}{\rho}}$$

has also a positive or negative value differing from nothing, since the initial values of r and u^2, by which r_0 is to be determined, must be considered as positive measurable quantities to be determined by experiment.

When ρ/r_0 is positive because both numerator and denominator are positive, all the movements are confined to the distances outside the interval ρr_0, and are divisible into *movements at a distance* and *molecular movements* which are separated from each other by the interval ρr_0.

But if ρ/r_0 is positive because numerator and denominator are both negative, the movements extend to all possible distances, since the interval ρr_0 then lies outside all possible distances.

When ρ/r_0 is negative, in which case the interval ρr_0 lies partly outside and partly within the possible distances, all the movements are confined to the part of the interval ρr_0 lying within possible distances; and if ρ is positive and r_0 negative, they are *molecular movements.*

From this it follows, when ρ and r_0 are positive, that, in the first place, no transition from *movements at a distance* to *molecular movements* takes place; secondly, that u^2 always remains less than c^2, if it was smaller at first; and thirdly, that when u^2 is less than c^2, r and r_0 are (both at once) either greater or less than ρ.

If we keep merely to experience, some of these relative movements of the two particles may be left entirely out of account, for it is evident that infinitely great relative velocities are never met with in reality; on the contrary, $[1/c^2][dr^2/dt^2]$ is almost always to be considered a very small fraction.

This limitation, derived from the nature of things, is also tacitly assumed when

$$V = \frac{ee'}{r}\left(\frac{1}{c^2}\cdot\frac{dr^2}{dt^2} - 1\right)$$

is taken as the *potential*, since this must be $= 0$ for an infinitely great value of r. For if dr^2/dt^2 were infinitely great, the expression $[ee'/r]([1/c^2][dr^2/dt^2] - 1)$ might have a value differing from nothing even for infinitely great values of r.

But if the value of dr^2/dt^2 is never infinitely great, there must be a finite value which dr^2/dt^2 never exceeds. We may assume c^2 as such a value.

Presupposing this limitation of the relative velocities, r_0 is always positive; and for every value of r_0 there exists only a single, always continuous series of corresponding values of r and dr^2/dt^2; and

when ρ is positive and r_0 is $> \rho$,

the corresponding values of r and dr^2/dt^2 extend from $r = r_0$ to $r = \infty$ and from $dr^2/dt^2 = 0$ to $dr^2/dt^2 = (\rho/r_0)c^2$.[149] The movements are in this case it movements at a distance.

If ρ is positive and $r_0 < \rho$, or if ρ is negative,

the corresponding values extend from $r = r_0$ to $r = 0$, and from $dr^2/dt^2 = 0$ to $dr^2/dt^2 = c^2$. In the first case, when ρ is positive and $r_0 < \rho$, and likewise in the second case, when ρ is negative and $r_0 < \rho$, the movements are *molecular movements*; but if, in the second case, r_0 is $> \rho$, the movements are partly *movements at a distance* and partly *molecular movements*.

Hence, with the above limitation of the movements, we obtain for two particles e and e', moving solely in consequence of their reciprocal action, if there is *no motion of rotation of the particles about each other* in space, the following equation of motion, namely,

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho}\cdot\frac{\rho}{r_0} \ ,$$

in which u is put $= dr/dt$, and where ρ has a value that is given by the particles e and e', their masses ε and ε', and the constant c, and r_0 denotes a constant to be determined, according to this very equation, by the initial value of r (which must be positive and not equal to ρ, but otherwise may be any thing whatever) and the initial value of u^2 (which must be positive and less than c^2, but otherwise may be any thing whatever).

[149][Note by AKTA:] Due to a misprint, the original text had here $dr^2/dt^2 = \rho/r_0$ instead of $dr^2/dt^2 = (\rho/r_0)c^2$.

9.10 Two States of Aggregation of a System of Two Particles of the Same Kind

For two like particles the value of ρ is positive. And since, moreover, for every value of r the relative velocity u may have two equal but opposite values, the value of r may, in accordance with the above equation

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho} \cdot \frac{\rho}{r_0} \ ,$$

> either *at first* decrease from $r = \infty$ to $r = u_0$, u at the same time increasing from $u = -c\sqrt{\rho/r_0}$ to $u = O$, and *afterwards* r may increase again from $r = r_0$ to $r = \infty$, u at the same time increasing from $u = 0$ to $u = +c\sqrt{\rho/r_0}$;
>
> or r may *at first* decrease from $r = r_0$ to $r = 0$, u at the same time decreasing from $u = 0$ to $u = -c$, and then *afterwards* r may increase from $r = 0$ to $r = r_0$, u at the same time decreasing from $u = +c$ to $u = 0$.

It is easily seen that in the *first* case the motion is *not a reverting one*; for, after the distance r has diminished from any given value to r_0, it increases again without limit; that is, it never decreases again. In the *latter* case, on the other hand, the motion is *reverting*, for the distance r alternately diminishes from r_0 to 0 and increases again from 0 to r_0.

There seems indeed to be a sudden change in the value of the velocity u from $-c$ to $+c$ at the moment when $r = 0$; but no sudden change occurs in reality; for, when r vanishes, $-c$ denotes the same velocity as $+c$ does when r is increasing again from zero.

These two cases of motion are moreover distinguished from each other by the fact that *no transition* takes place from one to the other; for, according to the above equation, such a transition, in the case of the interval ρr_0 or $r_0\rho$ could only occur by u taking imaginary values.

Now upon this separateness of the two kinds of motion a distinction may be founded between *two states of aggregation of a system of two similar particles*[150] — that is, between a state of aggregation in which the particles can only move at a distance from each other, and a state of aggregation in which they can take part only in molecular movements. A transition from the one state of aggregation to the other cannot take place so long as both particles move in consequence of their reciprocal action only.

It only remains to be noted further, that it has been here presupposed that the two particles, considered in space, possessed no motion except in the direction of r; but in the next Section the opposite case will be considered.

9.11 Motion of Two Electrical Particles which Move in Space with Different Velocities, in Directions at Right Angles to the Straight Line Joining Them

Let α denote the difference of the two velocities which two electrical particles e and e', at a distance r from each other, possess in space in a direction perpendicular to the straight

[150][Note by AKTA:] That is, two positive particles, or two negative particles.

line r which joins them; then α^2/r denotes the part of the relative acceleration du/dt which depends upon α.

If we deduct this part α^2/r from the total acceleration du/dt, the difference $(du/dt - \alpha^2/r)$ expresses that part of the relative acceleration of the two particles which results from the forces exerted by them upon each other. According to Section 9.9 this latter part was

$$= \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \frac{dV}{dr} \ ,$$

and hence we obtain the following equation,

$$\frac{du}{dt} - \frac{\alpha^2}{r} = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \frac{dV}{dr} \ .$$

Multiplying this equation by $udt = dr$, we get

$$udu - \alpha^2 \frac{dr}{r} = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \cdot \frac{dV}{dr} dr \ ,$$

and hence, by integrating from the instant at which $u = 0$, the value of r corresponding to this instant being denoted by r_0,

$$\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) (V - V_0) = \frac{1}{2} u^2 - \int_{r_0}^{r} \frac{\alpha^2}{r} dr \ ,$$

in which $V = (ee'/r)([u^2/c^2] - 1)$ and $V_0 = -ee'/r_0$, but where, in order to perform the last integration, α^2 must be represented as a function of r.

Now $r \cdot \alpha dt$ is the element of surface described by the line connecting the two repelling or attracting particles while they move about each other for the element of time dt; and for equal elements of time dt this superficial element retains always the same value, whence $r\alpha dt = r_0 \alpha_0 dt$. Introducing the resulting value

$$\alpha^2 = r_0^2 \alpha_0^2 \cdot \frac{1}{r^2}$$

in the last member of the above equation, and carrying out the integration, we obtain the following equation,

$$2 \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \frac{ee'}{c^2} \left(\frac{r - r_0}{r r_0} + \frac{1}{r} \cdot \frac{u^2}{c^2}\right) = \frac{u^2}{c^2} + \frac{\alpha_0^2}{c^2} \cdot \frac{r_0^2 - r^2}{r^2} \ ,$$

from which, by putting

$$2 \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \frac{ee'}{c^2} = \rho \ ,$$

the equation of motion

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho} \left(\frac{\rho}{r_0} + \frac{r + r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right)$$

is obtained. Putting this value of u^2/c^2 into the equation

$$V = \frac{ee'}{r} \left(\frac{u^2}{c^2} - 1\right) \ ,$$

we get

$$V = \frac{ee'}{r}\left(\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\cdot\frac{\alpha_0^2}{c^2}\right) - 1\right) ,$$

$$\frac{dV}{dr} = \frac{ee'}{r_0}\cdot\frac{r_0-\rho}{(r-\rho)^2} - \frac{ee'}{(r-\rho)^2}\left(1-\left(3-2\frac{\rho}{r}\right)\frac{r_0^2}{r^2}\right)\frac{\alpha_0^2}{c^2} .$$

9.12 States of Aggregation of These Two Particles

According to the last Section, there exists an equation between the relative velocity u and the relative distance r of two particles moving anyhow *in space* under the action of their reciprocal forces, namely the equation

$$\frac{u^2}{c^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\frac{\alpha_0^2}{c^2}\right) ,$$

in which ρ denotes a constant that is *positive* for two *similar* particles, and *negative* for two *dissimilar* particles.[151]

Now from this there follow results relative to the free motions of two particles *in space*, which move, under the influence of their own reciprocal action, with unequal velocities in a direction perpendicular to the straight line joining them, quite similar to those arrived at in relation to the motions considered in Section 9.10 *in the direction of the straight line* r. There results, in fact, in this case also, a distinction between two states of aggregation for two *similar* particles — namely, a state of aggregation in which the two particles move in such a way as to return periodically into the same position relatively to each other, and a state of aggregation in which the two particles move so as to become always more and more distant from each other and never return to the same position. No transition from one state of aggregation to the other takes place so long as the two particles move only under the influence of their own reciprocal forces.

9.13 No Circular Motion of These Two Particles Around Each Other

A rotation of the two particles about each other implies the existence of a certain *attracting force* if the two particles are to remain at a constant distance from each other during this rotation; and this attracting force required for the rotation increases, for the same distance, according to the square of velocity of rotation. According to this, one would expect that, for two *similar* electrical particles at a distance $r_0 < \rho$ (at which they attract each other), there would be always a *certain velocity of rotation* α_0 for which the attracting force required by the rotation should be equal to the attracting force resulting from the reciprocal action of the two particles, so that the two particles rotating about each other would remain, for this velocity of rotation, at the same distance r_0. This, however, is not the case, since the attracting force resulting from the reciprocal action of the two particles depends not only

[151][Note by AKTA:] Two similar particles have charges of the same sign, that is, charges of the same kind. Two dissimilar particles have charges of opposite sign, that is, charges of opposite kind. See also footnote 135 on page 71.

upon the distance r_0, but also upon the velocity of rotation α_0, and increases with the latter in such a manner that it always remains greater than the attracting force required by the rotation, so that with any such rotation there is always involved a mutual approach of the two particles.

It follows indeed easily that, in the case of two *similar* particles e and e', when ρ has a *positive* value and $r = r_0$, and consequently $u = 0$, there is no value of α_0 for which $du/dt = 0$, as must be the case if the two particles are to remain at an invariable distance r_0. For when $r = r_0$, it results from the equation at the end of Section 9.11 that

$$\frac{dV}{dr} = \frac{ee'}{r_0 \left(r_0 - \rho\right)} \left(1 + 2\frac{\alpha_0^2}{c^2}\right) ,$$

and from this it further follows, since

$$\frac{du}{dt} - \frac{\alpha^2}{r} = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \frac{dV}{dr} = \frac{\rho}{2} \frac{c^2}{ee'} \cdot \frac{dV}{dr} ,$$

that

$$\frac{du}{dt} = \frac{1}{2} \frac{c^2}{r_0 - \rho} \left(\frac{\rho}{r_0} + 2\frac{\alpha_0^2}{c^2}\right) ,$$

whence du/dt can be equal to nothing only when

$$\alpha_0^2 = -\frac{1}{2} \frac{\rho}{r_0} c^2 ,$$

which for a *positive* value of ρ (that is, when e and e' are of the *same kind*) is impossible.

It follows further that, in the case of two *similar* particles, if $r = r_0$, du/dt is either *positive* or *negative*, according as $r_0 > \rho$ or $r_0 < \rho$. Consequently the two particles separate always to a greater and greater distance from each other when $r = r_0 > \rho$, and approach always nearer to each other when $r = r_0 < \rho$ whatever value α_0 may have.

9.14 On the Period of Oscillation of an Electrical Atomic Pair

Two similar electrical particles at a distance $r_0 < \rho$ from each other (at which their relative velocity $= 0$) do not remain at this distance, but approach each other from $r = r_0$ to $r = 0$ with a velocity which increases from $u = 0$ to

$$u = \sqrt{c^2 + \frac{r_0^2 \alpha_0^2}{\rho} \cdot \frac{1}{r}} ,$$

that is to say, becomes infinite, if the velocity of rotation α_0 differed from nothing for the instant at which $r = r_0$. From this it follows that the interval of time ϑ in which the two particles approach each other from the distance $r = r_0$ to $r = 0$ has a finite value. The fact that for the instant at which r becomes equal to 0 the value of the relative velocity of the two particles becomes

$$\sqrt{c^2 + \frac{r_0^2\alpha_0^2}{\rho} \cdot \frac{1}{r}} = \pm\infty \ ,$$

signifies here only that this relative velocity is to be henceforward taken as a velocity of separation $= +\infty$, whereas it was, up to this point, a velocity of approach $= -\infty$. This being premised, it easily follows that, in a second equal interval of time ϑ, the two particles will separate from each other again from the distance $r = 0$ to the distance $r = r_0$. The interval of time 2ϑ, in which the two particles approach each other with increasing velocity from the distance $r = r_0$ to $r = 0$ and then separate again from the distance $r = 0$ to $r = r_0$, may be called the *period of oscillation*[152] of the *atomic pair* formed of the two electrical particles.

There still remains the problem of *determining the period of oscillation* 2ϑ *of such an atomic pair.*

This period of oscillation can be readily deduced from the equation

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho} \left(\frac{\rho}{r_0} + \frac{r_0 + r}{r} \cdot \frac{\alpha_0^2}{c^2} \right) \ ,$$

if it be assumed that therein r_0 is not greater than ρ.

For if we *first* consider the limiting case in which $r_0 = \rho$, it follows from the above equation that

$$u^2 = c^2 + \alpha_0^2 + \rho\alpha_0^2 \cdot \frac{1}{r} \ ,$$

and hence, putting $u = dr/dt$,

$$dt = -dr\sqrt{\frac{r}{\rho\alpha_0^2 + (c^2 + \alpha_0^2)\, r}} \ .$$

From this we obtain, by integration,

$$\vartheta = -\int_\rho^0 dr\sqrt{\frac{r}{\rho\alpha_0^2 + (c^2 + \alpha_0^2)\, r}} \ .$$

Accordingly we get:

$$\vartheta = \frac{\rho}{c^2 + \alpha_0^2}\sqrt{c^2 + 2\alpha_0^2} - \frac{\rho\alpha_0^2}{(c^2 + \alpha_0^2)^{3/2}} \log\left(\sqrt{1 + \frac{c^2}{\alpha_0^2}} + \sqrt{2 + \frac{c^2}{\alpha_0^2}} \right) \ ,$$

or, for small values of α_0/c,

$$\vartheta = \frac{\rho}{c}\left(1 - \frac{\alpha_0^2}{c^2} \log\frac{2c}{\alpha_0} \right) \ .$$

[152][Note by AKTA:] In German: *Schwingungsdauer*. This expression was translated by G. C. Foster as "time of oscillation". We are utilizing here the more usual expression "period of oscillation". It should be observed that Gauss and Weber utilized the French definition of the period of oscillation t which is half of the English definition of the period of oscillation T, that is, $t = T/2$, [Gil71a, pp. 154 and 180]. For instance, the period of oscillation for small oscillations of a simple pendulum of length ℓ is $T = 2\pi\sqrt{\ell/g}$, where g is the local free fall acceleration due to the gravity of the Earth, while $t = T/2 = \pi\sqrt{\ell/g}$.

If we *next* confine ourselves to the consideration of *small oscillations* (that is to say, those for which r_0/ρ is very small), it results from the above equation, when r_0 and r are taken as vanishingly small compared with ρ, that

$$u^2 = \frac{r_0^2 \alpha_0^2}{\rho} \cdot \frac{1}{r} + c^2 - \left(\frac{c^2}{r_0} + \frac{\alpha_0^2}{\rho} \right) r \ ,$$

whence, putting $u = dr/dt$,

$$cdt = -dr \sqrt{\frac{r}{\frac{r_0^2 \alpha_0^2}{\rho c^2} + r - \left(\frac{1}{r_0} + \frac{\alpha_0^2}{\rho c^2} \right) r^2}} \ ,$$

which leads to an elliptic integral. For vanishing values of α_0/c, we obtain

$$cdt = -dr \sqrt{\frac{1}{1 - r/r_0}} \ ,$$

whence there comes, by integration,

$$\vartheta = -\frac{1}{c} \int_{r_0}^{0} \frac{dr}{\sqrt{1 - r/r_0}} = \frac{2r_0}{c} \ .$$

When, as has been assumed, r is $< \rho$, r_0 may be called the amplitude of oscillation; and it follows that, for small values of α_0/c and for small amplitudes of oscillation, the period of oscillation 2ϑ of an electrical atomic pair is proportional to the amplitude of oscillation r_0. But the factor with which r_0 must be multiplied in order to give 2ϑ, though a constant $= 4/c$ for small amplitudes, diminishes for greater amplitudes, and becomes $= 2/c$ for the amplitude $r = \rho$.

If we put $c = 439450 \cdot 10^6$ millimetre/second, it follows from this last determination that the value of ρ must lie approximately between 1/4000 and 1/8000 of a millimetre in order that these oscillations may be equal in rapidity to those of light.

The difference of the electrical particles e and e' and of their masses ε and ε' in the case of small values of α_0/c and small amplitudes, does not affect the oscillations at all; and in the case of greater amplitudes it affects them only so far as the value of ρ depends upon it.

9.15 Applicability to Chemical Atomic Groups

The distinction between two or more states of aggregation of bodies, according as they consist of simple atoms, or of atomic pairs, or of groups of more than two atoms, has acquired great importance in relation to *chemistry*. Now one, and now another state of aggregation occurs; and in many chemical processes a transition takes place from one to another; but the intermediate states which occur in the case of such transition cannot exist permanently, and those states of aggregation are consequently completely separate from each other as *permanent states*.

Now it is obvious that the *permanence* of some atomic conditions, which are distinguished as special states of aggregation, and the *want of permanence* in all other atomic conditions, may have its cause in the laws of the reciprocal action of atoms — that is, in the difference between the forces exerted upon each other by atoms according to the different relations in which they may stand towards each other. The cause of the permanence of some atomic states and of the want of this permanence in others has not hitherto been recognized in the laws of the reciprocal action of atoms; and it would doubtless be difficult to succeed in discovering this cause in such laws of reciprocal action as it has hitherto been attempted to establish and to assume for ponderable atoms.

The question consequently presents itself, whether the cause of the permanence of certain atomic states may not perhaps be found in such laws of mutual action as have here been established and assumed for electrical particles. Hence the movements of two electrical particles under the influence of the reciprocal action assigned to them, which have been followed out in the preceding Sections, are of interest in connexion with this point also, since in them a cause has been really discovered upon which the existence of such permanent states of aggregation may be founded. And in relation to this it is to be specially observed that the same forces as those which determine the *states of aggregation of electricity* formed by simple atoms and by atomic pairs, may possibly also determine similar states of *aggregation of ponderable bodies*. For in the general distribution of electricity it must be assumed that an atom of electricity adheres to each ponderable atom. But if atoms of electricity adhere firmly to ponderable atoms, nothing will be altered in the relations of the electrical atoms except the *masses* which have to be moved by the forces acting on the electrical atoms. But in the preceding developments the *masses* are left undetermined, and are simply denoted by ε and ε'; while the electrical particles themselves, to which the masses ε and ε' belong, are determined, without a knowledge of the values ε and ε', by the measurable quantities e and e'. If now we take the values of ε and ε' so great as to include the masses of the ponderable atoms adhering to the electrical atoms, all the results that have been arrived at in reference first of all to *electrical atoms* merely, may also be applied to the *ponderable atoms* combined with the electrical atoms.

9.16 On the State of Aggregation and Oscillation of Two Dissimilar Electrical Particles

In the case of two *dissimilar* electrical particles, the same equations hold good as in the case of two similar particles, namely those of Section 9.11; that is to say,

$$\frac{u^2}{c^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\cdot\frac{\alpha_0^2}{c^2}\right) ,$$

$$V = \frac{ee'}{r}\left[\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\cdot\frac{\alpha_0^2}{c^2}\right) - 1\right] ,$$

$$\frac{dV}{dr} = \frac{ee'}{(r-\rho)^2}\left[\frac{r_0-\rho}{r_0} - \left(1 - \frac{3r-2\rho}{r^3}\cdot r_0^2\right)\frac{\alpha_0^2}{c^2}\right] ,$$

where

$$\rho = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{ee'}{c^2} ;$$

the only difference is, that when the particles are *dissimilar* ρ has a *negative* value, because the product ee' is *negative*. Besides these equations we have also $\alpha r = \alpha_0 r_0$ (since only such motions are considered as are made by two electrical particles under the action of their own reciprocal action), whence there follows, lastly, the equation

$$\frac{du}{dt} = \frac{1}{2}\frac{\rho c^2}{ee'} \cdot \frac{dV}{dr} + \frac{r_0^2\alpha_0^2}{r^3} \ .$$

Hence it results that, as in the case of two similar electrical particles, when $r = r_0$,

$$\frac{dV}{dr} = \frac{ee'}{r_0(r_0 - \rho)}\left(1 + 2\frac{\alpha_0^2}{c^2}\right) \ ,$$

$$\frac{du}{dt} = \frac{1}{2}\frac{c^2}{r_0 - \rho}\left(\frac{\rho}{r_0} + 2\frac{\alpha_0^2}{c^2}\right) \ ,$$

and that, when also $\alpha_0 = \sqrt{-\rho c^2/2r_0}$ (which has now a real value, since

$$-\rho = -2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right)\frac{ee'}{c^2}$$

is positive for dissimilar particles), $du/dt = 0$; according to which, when $r = r_0$ and $\alpha_0 = \sqrt{-\rho c^2/2r_0}$, the two particles in their rotation about each other *remain always at the same distance* $(= r_0)$ *apart*, a case which with *two similar* particles cannot occur at all.

It follows, however, further from the equation

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho}\left(\frac{\rho}{r_0} + \frac{r + r_0}{r}\frac{\alpha_0^2}{c^2}\right) \ ,$$

or, when we put n for the constant value $-r_0^2\alpha_0^2/\rho c^2$, from the following equation,

$$-\frac{r - \rho}{\rho} \cdot \frac{u^2}{c^2} = \left(\frac{r}{r_0} - 1\right) \cdot \left[n\left(\frac{1}{r_0} + \frac{1}{r}\right) - 1\right] \ ,$$

that besides the value $r = r_0$, for which $u = 0$ is given, there is in general also another value of r, namely $nr_0/(r_0 - n)$, for which likewise $u = 0$.

These two values of r, however, for which $u = 0$, differ from each other sometimes to a greater and sometimes to a smaller extent, according to the value of n; and when $n = r_0/2$ (that is to say, when $\alpha_0 = \sqrt{-\rho c^2/2r_0}$), they coincide completely; and it is only when the two values of r for which $u = 0$ coincide thus that the previously mentioned case occurs, for which we have at the same time $u = 0$ and $du/dt = 0$; and consequently the two particles, while revolving round each other, remain at the same distance.

In all other cases in which the velocity $u = 0$ (as, for example, when $r = 2n - x$, where $x < n$) there is also a second value of r — in this case $2n + nx/(n - x)$, — for which also the velocity $u = 0$. du/dt has then a positive value for $r = 2n - x$, but diminishes and becomes equal to nothing between $r = 2n - x$ and $r = 2n + nx/(n - x)$; so that, for $r = 2n + nx/(n - x)$, du/dt has a negative value. It is evident from this that repulsion of the two particles takes place from $r = 2n - x$ as far as the value of r for which $du/dt = 0$, and attraction from this point as far as $r = 2n + nx/(n - x)$, and consequently that the two particles must always remain in *oscillatory motion relatively to each other within the indicated limits.*

9.17 On Ampère's Molecular Currents

The molecular state of aggregation of two dissimilar electrical particles that has just been described, namely that in which the distance of the two particles alternately increases and diminishes between exactly defined limits and the path in which one particle moves about the other becomes a circular orbit at the two limits, is deserving of closer consideration, especially in those cases in which it is admissible to regard one of the particles as being at rest and the other particle as moving in a circle about the first. The relation between the particles in respect of their participation in the motion depends upon the ratio of their masses ε and ε'; and, according to Section 9.15, the values of ε and ε' must include the masses of the ponderable atoms adhering to the electrical atoms. Let e be the positive electrical particle, and let the negative particle be equal and opposite to it, and let it therefore be denoted by $-e$ (instead of by e'). Now let a ponderable atom adhere to the latter only, whereby its mass is so much increased that the mass of the positive particle becomes negligible in comparison. The particle $-e$ may then be regarded as being at rest, and the particle $+e$ alone as being in motion around the particle $-e$.

The two dissimilar particles, when in the molecular state of aggregation that has been described, consequently represent an *Ampèrian molecular current*; for it can be shown that they correspond completely to the assumptions which Ampère made in relation to the *molecular currents*.[153]

In order to show this, let us develop the expression for the force which *the moving particle* e exerts upon any given element of a current. Let ds' denote the length of the given element of current, $+e'ds'$ the positive, and $-e'ds'$ the negative electricity which it contains; and, lastly, let u' denote the velocity of the positive particle $+e'ds'$, and $-u'$ the velocity of the negative particle $-e'ds'$. Also, let r denote the distance of the element of current from the particle e, u the velocity of the particle e, x, y, z the coordinates of the particle e, x', y', z' the coordinates of the element of current, ϑ and ϑ' the angles which the directions of u and u' make with r, and ε the angle between the directions of u and u'.

Next, let the general expression for the repelling force of two electrical particles e and e' at the distance r, namely

$$\frac{ee'}{r^2}\left(1-\frac{1}{c^2}\cdot\frac{dr^2}{dt^2}+\frac{2r}{c^2}\frac{d^2r}{dt^2}\right) ,$$

be transformed as follows (see Beer, *Einleitung in die Elektrostatik, die Lehre vom Magnetismus und die Electrodynamik*, p. 251).[154] First, let the equation

$$r^2=(x-x')^2+(y-y')^2+(z-z')^2$$

be differentiated with respect to the time t; we then get

$$r\frac{dr}{dt}=(x-x')\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)+(y-y')\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)+(z-z')\left(\frac{dz}{dt}-\frac{dz'}{dt}\right) ,$$

or also

$$r\frac{dr}{dt}=r(u\cos\vartheta-u'\cos\vartheta') .$$

By a second differentiation we get

[153] [Note by AKTA:] See footnote 8 on page 10 for references on Ampère's main works in French, Portuguese and English. Ampère's molecular currents have been discussed on Chapter 5 (Ampère's Conception of Magnetism) of [AC15].

[154] [Note by AKTA:] [Bee65, p. 251].

$$\frac{dr^2}{dt^2} + r\frac{d^2r}{dt^2} = \left(\frac{dx}{dt} - \frac{dx'}{dt}\right)^2 + \left(\frac{dy}{dt} - \frac{dy'}{dt}\right)^2 + \left(\frac{dz}{dt} - \frac{dz'}{dt}\right)^2$$

$$+ (x - x')\left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right) + (y - y')\left(\frac{d^2y}{dt^2} - \frac{d^2y'}{dt^2}\right) + (z - z')\left(\frac{d^2z}{dt^2} - \frac{d^2z'}{dt^2}\right) ,$$

wherein

$$\left(\frac{dx}{dt} - \frac{dx'}{dt}\right)^2 + \left(\frac{dy}{dt} - \frac{dy'}{dt}\right)^2 + \left(\frac{dz}{dt} - \frac{dz'}{dt}\right)^2 = u^2 + u'^2 - 2uu'\cos\varepsilon .$$

If now the acceleration of the one particle, whose components are d^2x/dt^2, d^2y/dt^2, d^2z/dt^2, be denoted by N, and the angle which its direction makes with r by ν, and in like manner the acceleration of the other particle, whose components are d^2x'/dt^2, d^2y'/dt^2, d^2z'/dt^2, by N', and the angle which its direction makes with r by ν', we obtain

$$\frac{x - x'}{r}\left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right) + \frac{y - y'}{r}\left(\frac{d^2y}{dt^2} - \frac{d^2y'}{dt^2}\right) + \frac{z - z'}{r}\left(\frac{d^2z}{dt^2} - \frac{d^2z'}{dt^2}\right)$$

$$= N\cos\nu - N'\cos\nu' .$$

The substitution of these values gives

$$2\frac{dr^2}{dt^2} + 2r\frac{d^2r}{dt^2} = 2\left(u^2 + u'^2 - 2uu'\cos\varepsilon\right) + 2r\left(N\cos\nu - N'\cos\nu'\right) ,$$

$$3\frac{dr^2}{dt^2} = 3\left(u\cos\vartheta - u'\cos\vartheta'\right)^2 .$$

The second equation subtracted from the first gives

$$-\frac{dr^2}{dt^2} + 2r\frac{d^2r}{dt^2} = 2\left(u^2 + u'^2 - 2uu'\cos\varepsilon\right) - 3\left(u\cos\vartheta - u'\cos\vartheta'\right)^2$$

$$+ 2r\left(N\cos\nu - N'\cos\nu'\right) ,$$

whence the general expression for the repelling force of two electrical particles e and e' at the distance r, namely

$$\frac{ee'}{r^2}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) ,$$

is obtained in the following transformed shape,

$$= \frac{ee'}{c^2r^2}\Big[c^2 + 2\left(u^2 + u'^2 - 2uu'\cos\varepsilon\right) - 3\left(u\cos\vartheta - u'\cos\vartheta'\right)^2$$

$$+ 2r\left(N\cos\nu - N'\cos\nu'\right)\Big] .$$

By substituting for the particle e' the positive electricity in the given element of current, namely $+e'ds'$, this expression gives the repelling force

$$\frac{ee'ds'}{c^2r^2}\Big[c^2 + 2\left(u^2 + u'^2 - 2uu'\cos\varepsilon\right) - 3\left(u\cos\vartheta - u'\cos\vartheta'\right)^2$$

$$+\ 2r\left(N\cos\nu - N'\cos\nu'\right)\Big]\ ;$$

but by putting for the particle e' the negative electricity in the given element of current, namely, $-e'ds'$, we obtain the repelling force

$$\frac{ee'ds'}{c^2r^2}\Big[-c^2 - 2\left(u^2 + u'^2 + 2uu'\cos\varepsilon\right) + 3\left(u\cos\vartheta + u'\cos\vartheta'\right)^2$$

$$-\ 2r\left(N\cos\nu + N'\cos\nu'\right)\Big]\ ,$$

since in this case $\varepsilon + \pi$, $\vartheta' + \pi$, and $\nu' + \pi$ take the place of ε, ϑ', and ν'; and these therefore give together the total repelling force between the moving particle e and the whole element of current, namely

$$\frac{4ee'ds'}{c^2r^2}\left(3uu'\cos\vartheta\cos\vartheta' - 2uu'\cos\varepsilon - rN'\cos\nu'\right)\ .$$

The repelling force between the stationary particle $-e$ and the whole element of current, on the other hand, if r denotes the distance of the stationary particle $-e$ from the given element of current, is

$$+\frac{4ee'ds'}{c^2r^2}\cdot rN'\cos\nu'\ ,$$

since in this case $u = 0$. But the difference between the value given to r here and that assigned to it previously (namely the distance from the particle $+e$, in motion about the particle $-e$, to the given element of current), may be regarded as a negligible fraction of r, so that we get, for the repelling force exerted by the moving particle $+e$ and stationary particle $-e$ together upon the element of current, the expression

$$\frac{4ee'ds'}{c^2r^2}\left(3\cos\vartheta\cos\vartheta' - 2\cos\varepsilon\right)\cdot uu'\ .$$

If we were to put in place of the moving electrical particle $+e$ a second element of current, the positive electricity of which, moving with the velocity $+\frac{1}{2}u$, was denoted by $+eds$, and whose negative electricity, moving with the velocity $-\frac{1}{2}u$, was denoted by $-eds$, we should obtain for the mutual repelling force of the two elements of current the value

$$= \frac{4eds\cdot e'ds'}{c^2r^2}\left(3\cos\vartheta\cos\vartheta' - 2\cos\varepsilon'\right)\cdot uu'\ ,$$

that is to say, the same expression as before, if the electrical particle previously denoted by $+e$ (and moving with the velocity u) were taken as equal to the positive electricity contained in the second element of current, namely $+eds$ (moving with the velocity $\frac{1}{2}u$).

It follows from this that the rotatory motion of the electrical particle $+e$ about the stationary particle $-e$ replaces a circular double current, if the positive electricity contained in the latter is equal to $+e$ and moves in its circular orbit with half the velocity of the aforesaid electrical particle $+e$, and if also the negative electricity contained in the current is equal to $-e$ and moves with the same velocity as the positive electricity but in the opposite direction.

Hence it appears that an electrical particle $+e$ moving in a circle about the electrical particle $-e$ exerts upon all galvanic currents the same effects as those assumed by Ampère in the case of his molecular currents.

The molecular currents assumed by Ampère, however, differ essentially from all other galvanic currents in this respect, that, according to Ampère's assumption, they *continue* without electromotive force; whereas all other galvanic currents, in accordance with Ohm's law,[155] are proportional to the electromotive force, and *cease* when the electromotive force vanishes. But it is evident that the electrical particle $+e$, spoken of above, must of itself, without electromotive force, continue indefinitely its rotatory motion about the particle $-e$, and therefore must correspond entirely with the molecular currents assumed by Ampère in this respect also.

We accordingly obtain in this way, as a deduction from the *laws of the molecular state of aggregation of two dissimilar electrical particles*, developed in the preceding Section, a simple construction for the molecular currents assumed by Ampère without proof that their existence was possible.

[155][Note by AKTA:] Georg Simon Ohm (1789-1854). Ohm's law is from 1826: [Ohm26a], [Ohm26c], [Ohm26d], [Ohm26b] and [Ohm27] with French translation in [Ohm60] and English translation in [Ohm66].

9.18 Movements of Two Dissimilar Particles in Space under the Action of an Electrical Segregating Force (*Scheidungskraft*)

If $\pi + v$ denotes the angle which the direction of the electrical segregating force[156] makes with r, and a denotes the magnitude of the relative acceleration of the two particles depending upon the segregating force, $-a \cos v$ and $a \sin v$ are the components of a, — the *former* expressing the part of the relative acceleration du/dt which is dependent on the segregating force, and the latter the part of $d\alpha/dt$ which depends on the same force, where α is the difference of the velocities of the two particles in a direction perpendicular to r. It is presupposed that the direction of the segregating force lies in the plane in which the two particles rotate about each other.

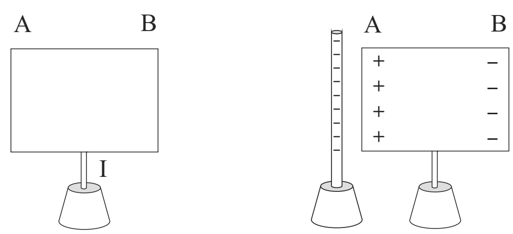
If now the *first* component, namely $-a \cos v$, as the part of du/dt which depends upon the *segregating force*, and also α^2/r, as the part of du/dt which depends upon the velocity α, be deducted from the total acceleration du/dt, the difference

$$\left(\frac{du}{dt} + a \cos v - \frac{\alpha^2}{r}\right)$$

denotes the part of the relative acceleration which results from the force which the two particles e and e' exert upon each other, namely

[156][Note by AKTA:] In German: *elektrischen Scheidungskraft*. This expression can also be translated as "electrical force of separation" or "electrical separating force".

I present here a simple example of a separating force. Consider a metal plate AB insulated from the ground by a dielectric support I as in Figure (a) of this footnote:



(a) (b)

If a negatively charged straw is placed close to side A of the plate, the charges on the plate become separated as illustrated in Figure (b). Side A of the plate becomes positively electrified, while side B becomes negatively electrified. This polarization of the plate is caused by the electric force of the negatively electrified straw acting on the free electrons of the plate. I presented several interesting experiments on this topic made with simple material, together with many quotes from original sources, in the 2 volumes of the book *The Experimental and Historical Foundations of Electricity* which is available in English, Portuguese, Italian and Russian: [Ass10a], [Ass10b], [Ass15b], [Ass17], [Ass18a], [Ass18b] and [Ass19].

Another effect of a separating force takes place in electrolysis. The electric forces in general are proportional to the charge q of the test particle on which they are acting. A positively electrified particle with $q > 0$ experiences a force in one direction, while a negatively electrified particle with $q < 0$ will be forced in the opposite direction. If these particles are free to move as in electrolysis, a double current will be produced due to this separating electric force. That is, the positive particles will move in one direction and the negative particles will move in the opposite direction.

$$\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\right)\frac{dV}{dr}=\frac{\rho}{2}\frac{c^2}{ee'}\cdot\frac{dV}{dr}\ ;$$

and hence the following equation is obtained:

$$\frac{du}{dt}+a\cos v-\frac{\alpha^2}{r}=\frac{\rho}{2}\frac{c^2}{ee'}\cdot\frac{dV}{dr}\ .$$

If we deduct the *last* component, namely $a\sin v$, as the part of the acceleration $d\alpha/dt$ which depends upon the *segregating force*, from the total value $d\alpha/dt$, the difference

$$\left(\frac{d\alpha}{dt}-a\sin v\right)$$

gives that part of the total acceleration $d\alpha/dt$ which results from *the existing motion under the sole influence of the forces exerted upon each other by the two particles*. But, under the sole influence of the attracting or repelling forces exerted upon each other by the two particles, the element of surface $\alpha r dt$, described in a given element of time dt, would have a constant value, or we should have

$$\alpha\frac{dr}{dt}+r\frac{d\alpha}{dt}=0\ ;$$

hence the resulting part of the acceleration $d\alpha/dt$ becomes

$$-\frac{\alpha}{r}\frac{dr}{dt}\ .$$

By equating this part with the above difference, we get the equation

$$\frac{d\alpha}{dt}-a\sin v=-\frac{\alpha}{r}\frac{dr}{dt}\ .$$

Besides these, we have, as is self-evident, a third equation,

$$dv=\frac{\alpha dt}{r}\ .$$

Accordingly, for the four variable magnitudes r, u, α, v, there are the following three equations:

$$a\cos v-\frac{\alpha^2}{r}=\frac{\rho c^2}{2ee'}\cdot\frac{dV}{dr}-\frac{du}{dt}\ , \qquad (1)$$

$$a\sin v-\frac{\alpha dr}{r dt}=\frac{d\alpha}{dt}\ , \qquad (2)$$

$$dv=\frac{\alpha dt}{r}\ . \qquad (3)$$

Multiplying equation (1) by $dr=udt$, and equation (2) by $rdv=\alpha dt$, we obtain

$$a\cos v\cdot dr-\frac{\alpha^2 dr}{r}=\frac{\rho c^2}{2ee'}\cdot\frac{dV}{dr}dr-udu\ , \qquad (4)$$

$$ar\sin v\cdot dv-\frac{\alpha^2 dr}{r}=\alpha d\alpha\ . \qquad (5)$$

The difference of these two equations gives

$$a \cdot d(r\cos v) = \frac{\rho c^2}{2ee'} \cdot \frac{dV}{dr} dr - \alpha d\alpha - u du \ . \qquad (6)$$

We also get from (2) and (3),

$$-2ar^3 \cdot d(\cos v) = d(\alpha^2 r^2) \ . \qquad (7)$$

The integration of the differential equation (6) gives, after multiplying by 2 and putting $V = (ee'/r)([u^2/c^2] - 1)$,

$$2ar\cos v = \frac{\rho c^2}{r}\left(\frac{u^2}{c^2} - 1\right) - \alpha^2 - u^2 + \text{constant} \ ; \qquad (8)$$

and from this, since $r = r_0$, $\alpha = \alpha_0$ and $\cos v = -1$ when $u = 0$, comes

$$-2ar_0 = -\frac{\rho c^2}{r_0} - \alpha_0^2 + \text{constant} \ . \qquad (9)$$

Equation (9), subtracted from equation (8), gives

$$2ar\cos v + 2ar_0 = \left(\frac{\rho}{r} - 1\right) u^2 + \rho c^2 \left(\frac{1}{r_0} - \frac{1}{r}\right) - \alpha^2 + \alpha_0^2 \ . \qquad (10)$$

By integrating the differential equation (7) we obtain, after dividing by r^3,

$$-2a\cos v = \frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2} \ ,$$

or, multiplying by r,

$$-2ar\cos v = \alpha^2 + 3r\int \frac{\alpha^2 dr}{r^2} \ , \qquad (11)$$

and hence, for the sum of (10) and (11),

$$2ar_0 = \left(\frac{\rho}{r} - 1\right) u^2 + \rho c^2 \left(\frac{1}{r_0} - \frac{1}{r}\right) + \alpha_0^2 + 3r\int \frac{\alpha^2 dr}{r^2} \ ,$$

and therefore

$$u^2 = \frac{1}{r-\rho}\left(\rho c^2 \left(\frac{r}{r_0} - 1\right) + r\alpha_0^2 + 3r^2 \int \frac{\alpha^2 dr}{r^2} - 2ar_0 r\right) \ . \qquad (12)$$

From equation (3) there follows further, since $dr = u dt$,

$$dv = \frac{\alpha}{u}\frac{dr}{r} \ , \qquad (13)$$

and since, by equation (7),

$$d(\cos v) = -\frac{d\left(\alpha^2 r^2\right)}{2ar^3} \ ,$$

and by equation (11),

$$\cos v = -\frac{1}{2\alpha}\left(\frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2}\right) \ ,$$

we get, by substituting these values in the identical equation[157]

$$dv = -\frac{d(\cos v)}{\sqrt{1 - \cos v^2}} \ ,$$

according to equation (13),

$$\frac{\alpha}{u}\frac{dr}{r} = \frac{\frac{d\left(\alpha^2 r^2\right)}{2ar^3}}{\sqrt{1 - \frac{1}{4a^2}\left(\frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2}\right)^2}} \ ;$$

and from this and equation (12),

$$\begin{aligned} u^2 &= \left(\frac{\alpha r^2 dr}{d\left(\alpha^2 r^2\right)}\right)^2 \cdot \left(4a^2 - \left(\frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2}\right)^2\right) \\ &= \frac{1}{r-\rho}\left(\frac{r - r_0}{r_0}\rho c^2 + r\left(\alpha_0^2 - 2ar_0\right) + \rho r^2 \int \frac{\alpha^2 dr}{r^2}\right) \ , \qquad (14) \end{aligned}$$

or the following equation for the two variables r and α:

$$4a^2 = \left(\frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2}\right)^2$$

$$+ \frac{4}{r-\rho}\left(\frac{d(\alpha r)}{dr}\right)^2 \cdot \left(\frac{r - r_0}{r_0} \cdot \frac{\rho c^2}{r^2} + \frac{\alpha_0^2 - 2ar_0}{r} + 3\int \frac{\alpha^2 dr}{r^2}\right) \ . \qquad (15)$$

[158]

If we now confine ourselves to small values of a, for which αr is not, indeed, constant, as it is for $a = 0$, according to Section 9.11, but for which it differs only little from a constant value $\alpha_0 r_0 = n$, we may put

$$\alpha r = n(1 + \varepsilon) \ , \qquad (16)$$

[157][Note by AKTA:] Nowadays the next equation would be written as

$$dv = -\frac{d(\cos v)}{\sqrt{1 - \cos^2 v}} \ .$$

[158][Note by WW:] If the segregating force a vanish, αr must, according to Section 9.11, assume a constant value. But for a constant value of αr and for $a = 0$, equation (15) reduces itself to

$$0 = \frac{\alpha^2}{r} + 3\int \frac{\alpha^2 dr}{r^2} \ ,$$

and this, divided by the constant value $\alpha^2 r^2$, gives the identical equation

$$0 = \frac{1}{r^3} + 3\int \frac{dr}{r^4} \ ,$$

in accordance with Section 9.11.

where ε has always a very small value. It then follows from this that

$$\frac{\alpha^2}{r} = (1+2\varepsilon)\frac{n^2}{r^3} \ , \qquad (17)$$

$$\frac{d(\alpha r)}{dr} = n\frac{d\varepsilon}{dr} \ . \qquad (18)$$

Further, by (11) and (17),

$$\int \frac{d\varepsilon}{r^3} = -\frac{a}{n^2}\cos v \ ,$$

or

$$d\varepsilon = \frac{a}{n^2}r^3 \sin v dv \ ; \qquad (19)$$

from (18) and (19),

$$\frac{d(\alpha r)}{dr} = \frac{a}{n}r^3 \sin v \cdot \frac{dv}{dr} \ ; \qquad (20)$$

and from (17) and (19),

$$\frac{\alpha^2}{r} = \frac{n^2}{r^3} + \frac{2a}{r^3}\int r^3 \sin v dv \ . \qquad (21)$$

If we now substitute the values of $d(\alpha r)/dr$ and α^2/r given by (20) and (21) in the following equation resulting from (ll) and (15), namely[159]

$$a^2 \sin v^2 = \frac{1}{r-\rho} \cdot \left(\frac{d(\alpha r)}{dr}\right)^2 \cdot \left(\frac{r-r_0}{r_0} \cdot \frac{\rho c^2}{r^2} + \frac{\alpha_0^2 - 2ar_0}{r} - \frac{\alpha^2}{r} - 2a\cos v\right) \ , \qquad (22)$$

we obtain, by again putting for n its value $\alpha_0 r_0$, the following equation between r and v, namely

$$\frac{\alpha_0^2 r_0^2}{r^4 c^2} \cdot \frac{dr^2}{dv^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right)$$
$$- \frac{2a}{(r-\rho)c^2}\left(r_0 r + \frac{3}{r}\int r^2 \cos v dr\right) \ .[160] \qquad (23)$$

159[Note by AKTA:] The left hand side of the next equation would be written nowadays as

$$a^2 \sin^2 v \ .$$

160[Note by WW:] From the above equation, since $[r/\alpha]u$ may be substituted for dr/dv we obtain

$$\frac{\alpha_0^2 r_0^2}{\alpha^2 r^2} \cdot \frac{u^2}{c^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right)$$
$$- \frac{2a}{(r-\rho)c^2}\left(r_0 r + \frac{3}{r}\int r^2 \cos v dr\right) \ ,$$

which, when the segregating force a vanishes, and therefore, according to Section 9.11, $\alpha r = \alpha_0 r_0$, passes

By differentiating this equation, after multiplying it by $r(r-\rho)$, we obtain

$$\frac{d}{dr}\left((r-\rho)\frac{\alpha_0^2 r_0^2}{r^3 c^2}\cdot\frac{dr^2}{dv^2}\right) = \frac{\rho r}{r_0} + (r+r_0)\frac{\alpha_0^2}{c^2}$$

$$+ (r-r_0)\left(\frac{\rho}{r_0}+\frac{\alpha_0^2}{c^2}\right) - \frac{2a}{c^2}\left(2r_0 r + 3r^2\cos v\right) .$$

If we here put, to consider a special case,

$$\rho = -\frac{2r_0}{c^2}\left(\alpha_0^2 + ar_0\right) ,$$

(that is to say, the case in which, for $a=0$, the two particles remain, according to Section 9.16, at the same distance during their rotation), we obtain

$$\frac{d}{dr}\left((r-\rho)\frac{\alpha_0^2 r_0^2}{r^3 c^2}\cdot\frac{dr^2}{dv^2}\right) = -\frac{2(r-r_0)}{c^2}\left(\alpha_0^2+ar_0\right) - \frac{6ar}{c^2}(r_0 + r\cos v) ,$$

which becomes $=0$, *first*, when $u=0$ and consequently $r=r_0$, $\alpha=\alpha_0$, and $\cos v = -1$, and, *secondly*, when

$$r_0 - r = \frac{3ar(r_0 + r\cos v)}{\alpha_0^2 + ar_0} ,$$

a case which occurs for small values of a, if $\cos v = +1$ and so $r = r_0 - 6ar_0^2/\alpha_0^2$ approximately.

Hence it follows that, just as, according to Section 9.16, one of two dissimilar electrical particles, for which $\rho = -2r_0\alpha_0^2/c^2$, could move round the other in a circular orbit when *not acted on by segregating force*, so also when two dissimilar electrical particles, for which

$$\rho = -2r_0\left(\frac{\alpha_0^2}{c^2} + ar_0\right) ,$$

are acted on by a segregating force ($=a$), one of them can revolve about the other in a closed orbit, though the orbit is not circular. The distance between the particles varies, in fact, according as the moving particle lies before or behind the central particle considered relatively to the direction of the segregating force, being in the latter case $=r_0$, and in the former case $=r_0 - 6[ar_0^2/\alpha_0^2]$.

Such an eccentrical position of the one particle in the plane of the orbit described (under the influence of a segregating force) by the other particle about this one, may be compared to the separation of electric fluids at rest under the influence of a similar segregating force; but the remarkable difference presents itself that the separation takes place in opposite directions in the two cases.

It follows from this, that in all conductors that have been charged in the usual way under the influence of a force of electrical segregation, the electricity cannot be contained only in the state of aggregation corresponding to Ampère's molecular currents, since in that case the resulting segregation would take place in the opposite direction to that which

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over into the equation

$$\frac{u^2}{c^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\cdot\frac{\alpha_0^2}{c^2}\right) ,$$

that is to say, into the same equation that was arrived at already for this case in Section 9.11.

actually does occur. But even if all the electricity in such a conductor existed in the form of Ampèrian molecular currents before the action of the segregating force began, there must have been amongst these molecular currents some which could not persist under the action of the segregating force (one particle continuing to revolve in a closed orbit round the other), and were accordingly broken up, the two particles separating more and more from each other until they arrived at the boundary of the conductor. Under the influence of the force of segregation, the positive and negative particles of the broken molecular currents could remain at rest only when distributed in a particular way on the surface of the conductor; but when the force of segregation ceased to act, they would enter into motion again until they had again united themselves two by two into Ampèrian molecular currents.

9.19 Electrical Currents in Conductors

If all the electricity in conductors were contained in them (before a segregating force began to act) in the state of aggregation corresponding to Ampèrian molecular currents, which, however, were incapable of persisting under the action of a segregating force, but were broken up, so that the two dissimilar electrical particles, which were revolving about each other, separated further and further from each other, until their paths finally approached asymptotically the direction of the segregating force, dissimilar electrical particles derived from different molecular currents would encounter each other before they could reach the boundaries of the conductor, and would form with each other new molecular currents. These newly formed molecular currents would then in their turn be broken up, and the particles constituting them would again separate further and further from each other in paths asymptotically approaching the direction of the segregating force, and so on.

Thus there would arise a current of electricity in the conductor in the direction of the segregating force. If the conductor had the shape of a uniform ring, and if the segregating force had the same intensity in every separate element of length of the ring and acted in the direction of the element, a constant circular current would be produced in the ring, and the laws of motion of electrical particles under the action of a force of electrical segregation, developed in the previous Section, would form the basis of the theory of these constant electrical currents in closed conductors.

Here it is evident that, during the existence of this current, *work* would be done by each particle, since it moves forward under the action of the segregating force in the direction of this force. And since all the other forces which act upon such a particle in a conductor must together balance each other, this work will make its appearance as an equivalent increase of the *vis viva* of the particle; whence it follows that the *vis viva* of all the Ampèrian molecular currents contained in the conductor must, while the current traverses the conductor, increase; that is to say, the square of the velocity with which the particles in the Ampèrian molecular current revolve about one another must increase proportionally to the force of segregation (*electromotive force*), and proportionally to the distance through which this force acts in its own direction (or to *the strength of the current*). If the ratio of the *electromotive force* to the *strength of the current* be called *resistance*, we may say instead of the above that the *vis viva* of all the molecular currents contained in the conductor increases, during the passage of the current, proportionally to the *resistance*, and proportionally to the *square of the strength of the current.*

This increase of *kinetic energy* of the electrical particles contained in a conductor while a current traverses it, follows therefore as a necessary consequence of the action of the electromotive force upon the particles, while these particles, as the result of the current, move onward in the direction of this force.

This theoretical conclusion receives, not indeed a direct, but an indirect confirmation from experiment, inasmuch as an increase of *thermal energy* is *observed* in the conductor while a current traverses it. And this *observed* increase of the *thermal energy* in the conductor is equal to the *calculated* increase of the *kinetic energy* of the electrical particles in the Ampèrian molecular currents of the conductor.

Now the *thermal energy* of a body is a *kinetic energy* resulting from movements in the *interior of the body*, which are therefore inaccessible to direct observation. In like manner, the *kinetic energy* belonging to the electrical particles in the Ampèrian molecular currents in a conductor is a kinetic energy which results from movements taking place in the *interior of the conductor*, and therefore inaccessible to direct observation.

But notwithstanding this agreement, the *thermal energy* of a body and this kinetic energy of the electrical particles in the Ampèrian currents contained in the same body might possibly be altogether different as to their essential nature. For it is possible that the *thermal energy* might be energy resulting from the motion of quite other particles than those of electricity, and the motion of these other particles might be of quite a different kind from those of the particles in Ampèrian currents.

In order to explain the identity of the increase of the energy of the Ampèrian molecular currents, as determined above, with the increase of thermal energy found by observation, it would then he absolutely necessary, *according to the principle of the conservation of energy*, that a *transference* should take place of the kinetic energy of the electrical particles in the Ampèrian currents to the other particles whose motion constitutes heat. And indeed it would be needful that *all* the kinetic energy produced by the current in the electrical particles of the Ampèrian currents should be *completely* transferred to these other particles at each instant.

But apart from the consideration that it is impossible to conceive how such a *complete* transference could take place, it is self-evident that any even partial transference of the kinetic energy of Ampèrian molecular currents to other particles is contradictory of the *permanence* which belongs to the essential nature of Ampèrian currents. If such a transference of kinetic energy from electrical particles in molecular currents to other particles were really to occur, it would simply prove that the molecular currents formed by these particles were not *Ampèrian molecular currents*, since they would not possess the permanence wherein the essence of Ampèrian molecular currents consists.

Hence it follows as a consequence that, if in conductors all the electrical particles exist in the state of aggregation corresponding to Ampèrian molecular currents, the observed increase in the *thermal energy* of a conductor, during the passage of a current through it, must result *immediately* from the increase of the *kinetic energy* of the electrical particles constituting the Ampèrian currents; that is to say, the *thermal energy* imparted to the conductor by the current must be *kinetic energy* due to motions in the interior of the conductor, and must in fact consist in *an increase in the strength of the Ampèrian currents formed by the electrical particles in the conductor*.

Reference may also be made, in connexion with the *identity of thermal energy and the kinetic energy of Ampèrian molecular currents*, to what is said respecting "the Transformation of the work of the current into Heat," in the 10th volume of the *Abhandlungen der K. Ges. d. Wiss. zu Göttingen* (1862), in the 33rd Section of the memoir entitled "Zur

Galvanometrie."[161,162]

9.20 On Thermomagnetism

The following remark readily connects itself with the hypothesis of the previous Section, that the electricity in conductors exists in the state of aggregation corresponding to Ampèrian molecular currents — and with the consequent identity of the *thermal energy* of the conductor and the kinetic energy of the Ampèrian currents in the conductor — namely, that *equality of temperature* in two conductors must depend upon certain relations between the strength and character of the Ampèrian currents in the two conductors, but that, along with the relation needed for this equality of temperature, the following difference may exist between the currents of the two conductors, namely: — that *greater masses* of electricity may move with *smaller velocity* in the Ampèrian currents of the one conductor, and *smaller masses* of electricity with *greater velocity* in those of the other conductor.

Let now a ring be conceived, formed of two such dissimilar conductors, through which a constant current passes, so that in the same time an equal quantity of electricity passes through every section of the ring; then it is evident that equal quantities of electricity must also traverse the two sections which bound the *first layer of the second conductor*. But the electricity which traverses the first section comes from the *first conductor*, in the molecular currents of which large masses of electricity move with small velocity. Hence, in consequence of this smaller velocity, this electricity which penetrates into the *first layer of the second conductor* possesses less *vis viva*. The electricity which passes through the second section comes from the above-mentioned first layer of the second conductor itself, where a smaller mass of electricity moves in the Ampèrian currents with a greater velocity, and therefore it possesses, in consequence of this greater velocity, a greater *vis viva*. It follows from this, that, as a consequence of the current, this *first layer of the second conductor* gives up more *vis viva* to the following layer of the second conductor than it receives from the last layer of the first conductor. Consequently a diminution takes place in the kinetic energy of the Ampèrian currents of this layer, or, in other words, a *diminution of the thermal energy or temperature*.

The opposite condition is found on considering the two sections which bound the *first layer of the first conductor*. The electricity which passes through the first section into this layer comes out of the end of the *second* conductor with a greater velocity; and that which passes out of this layer through the second section, leaves this section with a smaller velocity; whence it follows that, as a consequence of the current, the *first layer of the first conductor* gives up less *vis viva* to the following layer of the same conductor than it receives from the last layer of the second conductor; and thus an increase takes place in the kinetic energy of the Ampèrian currents of this layer, or, in other words, an *increase of the thermal energy or temperature*.

It will be seen that a foundation is here presented for the doctrine of *thermomagnetism*, and in particular for Peltier's fundamental experiment,[163] although it would lead us too far to pursue it further here.

It may suffice merely to add here a similar remark in relation to Seebeck's fundamental

[161] [Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 91.
[162] [Note by AKTA:] [Web62, p. 91 of Weber's *Werke*].
[163] [Note by AKTA:] Jean Charles Athanase Peltier (1785-1845). See [Pel34].

thermomagnetic experiment.[164] In a body which possesses the same temperature in all its parts, the heat is supposed to be in a state of *mobile equilibrium*; or we speak, with Fourier,[165] of a *reciprocal radiation* of the particles of the body, by virtue of which each particle parts with just as much heat to the surrounding particles as it receives from them. Now, if heat consists in Ampèrian molecular currents, which, however, are broken up by the positive and negative particles separating from each other until they encounter other particles, with which they form new molecular currents, equilibrium of temperature must consist in this, that the *vis viva* of the electrical particles which leave any part of the body is equal to the *vis viva* of the electrical particles which enter this part of the body.

Let us now consider the surface of contact of two conductors which differ from each other only by greater masses of electricity moving with smaller velocity in the Ampèrian currents of one, and smaller masses moving with greater velocity in those of the other. Then, when both the conductors are at the same temperature, the *vis viva* of the electrical particles which pass from the first conductor into the second must be equal to the *vis viva* of the electrical particles that pass from the second conductor into the first; but the *mass* of the electrical particles which pass from the first conductor into the second would be greater than the *mass* of the electrical particles which pass out of the second conductor into the first. But from this (if the electricity which passes over is always positive, while the negative electricity remains behind in the conductor, to the particles of which it adheres) there would result a *difference of electrical charge on the two sides of the surface of contact*; that is to say, there would result *an electromotive force* at this surface of contact; for the electromotive force of a surface of contact is a force whereby a difference of electrical charge is produced at the two sides of the surface of contact.

If now the two conductors are of such a nature that this difference of charge at the two sides of their surface of contact is not always the same, but is *greater or less according to variations of temperature*, there would follow the production of a current in a ring formed of these two conductors, if different temperatures were to exist at the two surfaces of contact of the conductors.

9.21 Helmholtz on the Contradiction between the Law of Electrical Force and the Law of the Conservation of Force

In his memoir, "Ueber die Bewegungsgleichungen der Elektricität für ruhende leitende Körper," in the *Journal für die reine und angewandte Mathematik* (vol. lxxii, pp. 7 and 8),[166] Helmholtz deduces from the law of electrical force the equation of motion of two electrical particles for motions in the direction of the distance r of the two particles, namely

$$\frac{1}{c^2} \cdot \frac{dr^2}{dt^2} = \frac{C - \frac{ee'}{r}}{\frac{1}{2}mc^2 - \frac{ee'}{r}} \ ,$$

or, putting $C = ee'/r_0$ and $2ee'/mc^2 = \rho$, the equation

[164] [Note by AKTA:] Thomas Johann Seebeck (1770-1831). See [See25] and [See26] with partial Portuguese translation in [FS16].

[165] [Note by AKTA:] Jean-Baptiste Joseph Fourier (1768-1830), [Fou22] with English translation in [Fou52].

[166] [Note by AKTA:] [Hel70].

$$\frac{1}{c^2}\frac{dr^2}{dt^2} = \frac{r - r_0}{r - \rho} \cdot \frac{\rho}{r_0} \ ,$$

that is to say, the same equation as was arrived at in Section 9.9.

If

$$\frac{ee'}{r} > \frac{1}{2}mc^2 > C \ ,$$

that is, if

$$\frac{\rho}{r} > 1 > \frac{\rho}{r_0} \ ,$$

we have dr^2/dt^2 positive and greater than c^2, and dr/dt is therefore real. If the latter is also positive, r will increase until $ee'/r = mc^2/2$, that is till $r = \rho$, and then dr/dt becomes *infinitely great*.

The same will happen if, to begin with,

$$C > \frac{1}{2}mc^2 > \frac{ee'}{r} \ ,$$

that is, if

$$\frac{\rho}{r_0} > 1 > \frac{\rho}{r} \ ,$$

and dr/dt is negative.

These consequences are, according to Helmholtz, in contradiction with the law of the conservation of force.[167]

Now it may be remarked hereupon, in the first place, that two electrical particles are here assumed which begin to move with a *finite* velocity certainly, but one which is greater than the velocity c — greater, that is, than $439450 \cdot 10^6$ millimetre/second. The case of two bodies moving relatively to each other with such a velocity is nowhere recognizable in nature. In all practical cases we are accustomed rather to treat $(1/c^2)(dr^2/dt^2)$ as a very small fraction; and this deserves notice.

For, according to Helmholtz (*loc. cit.* p. 7), a law is in contradiction with the law of the *conservation of force* if two particles, moving in accordance with it and beginning with a *finite* velocity, attain, within a finite distance of each other, *infinite* vis viva, and so are able to do an infinitely great amount of work.

The principle seems to be here announced that, according to the law of the conservation of force, two particles cannot, under any circumstances, possess infinite *vis viva*.

For the above assertion may evidently be inverted, and we may say a law is in contradiction with the law of the conservation of force, if two particles, moving in accordance with it and beginning with *infinite* velocity, attain, at a finite distance from each other, finite *vis viva*, and thus suffer an infinitely great diminution of the work which they are able to perform.

The two particles must therefore always retain an infinite velocity; for if they have not lost it in any finite distance, however great, they would, in accordance with the nature of

[167][Note by AKTA:] This expression "conservation of force" appears in the title of Helmholtz paper of 1847 in which he discussed what is nowadays called the conservation of energy, [Hel47] with English translation in [Hel66].

potential, never lose it even at greater distances. But bodies which always move relatively to each other with an infinite velocity are excluded from the region of our inquiries.

But if two particles never possess more than finite *vis viva*, there must be a finite limiting value of *vis viva* which they never exceed. It is consequently possible that this limiting value for two electrical particles e and e' may be $= ee'/r$; that is, that the square of the velocity, with which the two particles move relatively to each other may not exceed c^2.

The contradiction urged by Helmholtz would, according to this, lie not in the law, but in his assumption, according to which the two particles began to move with a velocity the square of which, namely dr^2/dt^2 was $> c^2$.

If such a determination of the limiting value of *vis viva* is assumed in connexion with the *law of the conservation of force* according to Helmholtz, it may equally well be assumed in connexion with the *fundamental law of electrical action* (see Section 4); that is, the *work* denoted there by U, as well as the *vis viva* denoted by x (in the law $U + x = ee'/\rho$), may both be regarded as being *by their nature positive quantities*.

In the second place, it may be remarked that, though the two electrical particles do attain infinite *vis viva* at a finite distance from each other, this finite distance is

$$\rho = \frac{2ee'}{c^2}\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) ,$$

which, according to our measures, is an *undefinable small distance*, for the same reasons that the electrical masses ε and ε' are themselves undefinable according to our measures. This distance was consequently denominated in Section 9.9 a molecular distance.

The *theory of molecular motions* requires in any case a special development, which as yet is wanting throughout. But as long as such a theory remains excluded from mechanical investigations, any doubts as to *physical admissibility* in relation to *molecular motions* are without foundation.

It may be remarked, in the third place, that the same objection, namely that two particles, which begin with finite velocity, attain infinite *vis viva* at a finite distance from each other, applies also to the law of gravitation, if it is assumed that the masses of ponderable particles are *concentrated in points*. But if this objection is got rid of, in the case of the law of gravitation, by assuming that the masses even of the smallest particles *occupy space*, we must make the same assumption in relation to electrical particles, in which case it results that only a vanishingly small part of such a particle arrives at a given instant at the distance ρ; another vanishingly small part, which arrived at the distance ρ at the previous instant, will have exchanged its infinitely great velocity of approach for an infinitely great velocity of separation. But if these vanishing parts of the smallest particles are solidly connected together, there cannot be any question of such infinite velocities at all.

Even cosmical masses may begin their movements under physically admissible conditions, and, by continuing to move according to the law of gravitation, may come into physically inadmissible conditions, which can be avoided only through the cooperation of *molecular forces confined to molecular distances*. The disregard of this cooperation is, strictly speaking, only temporarily allowable, namely so long as the conditions are such that its influence is either nothing or may be regarded as vanishingly small. But just as little as an objection to the law of gravitation is derived from this fact, ought any objection to the fundamental law of electrical action to be derived from the physically inadmissible conditions to which, according to Helmholtz, this law leads, when it is considered that these inadmissible conditions are connected only with certain molecular distances.

Chapter 10

Editor's Introduction to Tisserand's 1872 Paper

A. K. T. Assis[168]

Here I present the English translation of Tisserand's 1872 paper: "Sur le mouvement des planètes autour du Soleil, d'après la loi électrodynamique de Weber".[169] It has been translated by D. H. Delphenich.[170] François Félix Tisserand (1845-1896) was a French astronomer.

Wilhelm Weber studied the two-body problem with his force law of 1846.[171] Weber mentioned Tisserand's paper which is being presented here in his work written in the 1880's and published posthumously in 1894.[172]

Many scientists considered the two-body problem utilizing Weber's law applied to electrodynamics and gravitation.[173]

[168] Homepage: www.ifi.unicamp.br/~assis

[169] [Tis72] with English translation in [Tis17a].

[170] feedback@neo-classical-physics.info and http://www.neo-classical-physics.info/index.html.

[171] [Web46] with a partial French translation in [Web87] and a complete English translation in [Web07]; and especially in [Web71] with English translation in [Web72], see Chapter 9.

[172] [Web94b, Section 3] with English translation in [Web08, Section 3]. See Section 15.3 of Chapter 15.

[173] [See64] with German translation in [See24], see also [Nor65, p. 46]; [Hol70] with English translation in [Hol17]; [Tis72] with English translation in [Tis17a], [Tis90] with English translation in [Tis17b], see also [Tis96, Volume 4, Chapter 28 (Vitesse de propagation de l'attraction), pp. 499-503] and [Poi53, pp. 201-203]; [Zöl72, p. 334], [Rie74]; [Zöl76b, pp. xi-xii], [Zöl76a, p. 216] and [Zöl83, pp. 126-128]; [Lol83]; [Ser85]; [Rit92]; [Sch97]; [Ger98] with English translation in [Ger], and [Ger17]; [Zen21, pp. 46-47]; [See17a] and [See17b]; [Sch25] with Portuguese translation in [XA94] and English translation in [Sch95]; [Bus26]; [Wie60], [Whi73a, pp. 207-208], [Eby77], [SS87], [Ass89], [CA91], [Ass92a], [AC92], [Ass94], [Ass99a], [AW03], [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18], [Ass13], [Ass14], [Ass15a] and [FW19].

Chapter 11

[Tisserand, 1872] On the Motion of Planets Around the Sun According to Weber's Electrodynamic Law

François Félix Tisserand[174,175,176]

Under that law, the force that produces the motion of the planet around the Sun is:

$$F = \frac{fm\mu}{r^2}\left(1 - \frac{1}{h^2}\frac{dr^2}{dt^2} + \frac{2}{h^2}r\frac{d^2r}{dt^2}\right) \ ,$$

in which f is the constant of universal attraction, m is the mass of the planet, μ the sum of that mass and that of the Sun, r is the distance from the planet to the Sun, and h is the velocity by which the attraction propagates in space.

The integration of the equations of motion is accomplished rigorously with the aid of elliptic function. Upon starting with that solution, one can obtain some approximate formulas that will be convenient for the sake of obtaining numerical values. Nonetheless, one will arrive at the goal more rapidly by setting:

$$F = \frac{fm\mu}{r^2} + F_1$$

and regarding F_1 as a perturbing force. Moreover, it will suffice to vary the constants of the elliptical motion.

Here are the equations of the perturbed motion:

$$\left.\begin{array}{l} \frac{d^2x}{dt^2} + \frac{f\mu x}{r^2} + X = 0 \ , \\ \frac{d^2y}{dt^2} + \frac{f\mu y}{r^2} + Y = 0 \ , \\ \frac{d^2z}{dt^2} + \frac{f\mu z}{r^3} + Z = 0 \ , \end{array}\right\} \qquad (1)$$

in which

174[Tis72] with English translation in [Tis17a].

175Translated by D. H. Delphenich, feedback@neo-classical-physics.info and http://www.neo-classical-physics.info/index.html. Edited by A. K. T. Assis.

176The Notes by A. K. T. Assis are represented by [Note by AKTA:].

$$\begin{aligned}
X &= \frac{f\mu}{h^2}\frac{x}{r^2}\Omega \ ; \\
Y &= \frac{f\mu}{h^2}\frac{y}{r^2}\Omega \ ; \\
Z &= \frac{f\mu}{h^2}\frac{z}{r^2}\Omega \ ; \\
\Omega &= -\frac{dr^2}{dt^2} + 2r\frac{d^2r}{dt^2} \ .
\end{aligned}$$

The equations of elliptic motion are obtained by setting $X = Y = Z = 0$ in equations (1). Suppose that these equations have been integrated and let the elliptic elements be taken to be: a, the semi-major axis, e, the eccentricity, φ, the inclination, θ is the longitude of the node, ϖ is that of the perihelion, and ε is that of the epoch. For the present case, one will have formulas that determine the variation of the constants by taking the well-known formulas and replacing the derivative dR/dp of the perturbing function with respect to an arbitrary element p with $X\frac{dx}{dp} + Y\frac{dy}{dp} + Z\frac{dz}{dp} = R_p$ in them.

Now, one has:

$$R_p = \frac{f\mu}{h^2}\frac{\Omega}{r^3}\left(x\frac{dx}{dp} + y\frac{dy}{dp} + z\frac{dz}{dp}\right) = \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{dp} \ .$$

Since the expression for the radius vector depends upon only a, e, $\varepsilon - \varpi$, one will have:

$$\frac{dr}{d\varphi} = 0 \ , \qquad \frac{dr}{d\theta} = 0 \ , \qquad \frac{dr}{d\varepsilon} = -\frac{dr}{d\varpi} \ ,$$

and as a result:

$$R_\varphi = 0 \ , \qquad R_\theta = 0 \ , \qquad R_\varepsilon = -R_\varpi \ .$$

One will easily find the following formulas:

$$\left.\begin{array}{c}
\frac{da}{dt} = -\frac{2}{na}R_\varepsilon \ , \qquad \frac{d\theta}{dt} = 0 \ , \\
\frac{de}{dt} = -\frac{1-e^2}{na^2e}R_\varepsilon \ , \qquad \frac{d\varpi}{dt} = -\frac{\sqrt{1-e^2}}{na^2e}R_e \ , \\
\frac{d\varphi}{dt} = 0 \ , \qquad \frac{d\varepsilon}{dt} = \frac{2}{na}R_a - \frac{\sqrt{1-e^2}}{na^2e}\left(1 - \sqrt{1-e^2}\right)R_e \ .
\end{array}\right\} \qquad (2)$$

One will remark that these formulas (2) that φ and θ are not altered by the perturbing force, which is obvious *a priori*; however, what is less obvious is that the parameter does not change either. Indeed, one will have:

$$\frac{d[a(1-e^2)]}{dt} = -\frac{2}{na}(1-e^2)R_\varepsilon + 2ae\frac{1-e^2}{na^2e}R_\varepsilon = 0 \ .$$

In order for us to get some idea of the value of the perturbation, we shall develop those perturbations into series that proceed in sines and cosines of multiples of the mean anomaly ζ and neglect the powers of e that are greater than one.

We first address Ω, which contains the term $r\frac{d^2r}{dt^2}$; now, we have:[177]

[177][Note by AKTA:] Due to a misprint, this equation appeared in the original as:

$$r\frac{d^2r}{dt^2} = -\left(\frac{dr}{dt}\right)^2 + x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \frac{dx^2+dy^2+dz^2}{dt^2} \ ,$$

or even, with an approximation that is entirely satisfactory:

$$r\frac{d^2r}{dt^2} = -\frac{dr^2}{dt^2} - \frac{f\mu}{r} + f\mu\left(\frac{2}{r} - \frac{1}{a}\right) \ ;$$

it will then result that:

$$\frac{\Omega}{r^2} = 2f\mu\left(\frac{1}{r^3} - \frac{1}{ar^2}\right) - \frac{3}{r^2}\frac{dr^2}{dt^2} \ ,$$

which is an expression that is developed as follows:

$$\frac{\Omega}{r^2} = n^2 e\left[2\cos\zeta + \frac{e}{2}(1 + 11\cos 2\zeta)\right] + \ldots \ ,$$

One will then have:

$$\begin{aligned}
R_a &= \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{da} = \frac{2f\mu}{h^2}n^2 e\cos\zeta + \ldots \ , \\
R_e &= \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{de} = -\frac{f\mu}{h^2}n^2 ae\left[1 + \cos 2\zeta + \frac{3e}{4}(3\cos\zeta + 5\cos 3\zeta)\right] + \ldots \ , \\
R_\varepsilon &= \frac{f\mu}{h^2}\frac{\Omega}{r^2}\frac{dr}{d\varepsilon} = \frac{f\mu}{h^2}n^2 ae^2\sin 2\zeta + \ldots \ ,
\end{aligned}$$

and upon neglecting e^2, as always, one will deduce that:

$$\begin{aligned}
\frac{da}{dt} &= 0 \ , & \frac{d\theta}{dt} &= 0 \ , \\
\frac{de}{dt} &= -\frac{f\mu}{h^2}\frac{ne}{a}\sin 2\zeta \ , & \frac{d\varpi}{dt} &= \frac{f\mu}{h^2}\frac{n}{a}\left[1 + \cos 2\zeta + \frac{3e}{4}(3\cos\zeta + 5\cos 3\zeta)\right] \ , \\
\frac{d\varphi}{dt} &= 0 \ , & \frac{d\varepsilon}{dt} &= \frac{4f\mu}{h^2}\frac{ne}{a}\cos\zeta \ ,
\end{aligned}$$

so upon integrating those equations:

$$r\frac{d^2r}{dt^2} = -\left(\frac{dr}{dt^2}\right)^2 + x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \frac{dx^2+dy^2+dz^2}{dt^2} \ .$$

The expression $(dx^2 + dy^2 + dz^2)/dt^2$ should be understood as:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \ .$$

$$
\begin{aligned}
\delta\alpha &= 0\ , & \delta\theta &= 0\ , \\
\delta e &= \frac{f\mu}{h^2}\frac{e}{2a}\cos 2\zeta\ , & \delta\varpi &= \frac{f\mu}{h^2}\frac{n}{a}t + \frac{f\mu}{h^2}\frac{1}{a}\left[\frac{1}{2}\sin 2\zeta + \frac{9e}{4}\sin\zeta + \frac{5}{4}e\sin 3\zeta\right]\ , \\
\delta\varphi &= 0\ , & \delta\varepsilon &= \frac{4f\mu}{h^2}\frac{e}{a}\sin\zeta\ .
\end{aligned}
$$

We will then see that the perturbations of the elements are zero or periodic, with the exception of those of $\delta\varpi$, which contain a secular part. Later, we shall confirm that the periodic parts are entirely negligible under the various hypotheses that one can make on the value of h, in such a way that we will arrive at the following conclusion:

Under Weber's law, the elements will remain the same as under Newton's law. Only the longitude of perihelion will be found to have increased by $\frac{f\mu}{h^2}\frac{n}{a}t$, which is a quantity that will get larger as the planet gets closer to the Sun.

Consider the case of Mercury. Upon taking the mean solar day to be the unit of time, and the semi-major axis of the orbit of the Earth to be the unit of distance, one will find that:

$$\delta\varpi = \frac{(1.05160)}{h^2}t\ .$$

If we assume that h has the same value as in Weber's experiments on electricity, namely, $h = 439450 \times 10^6$, with seconds and millimeters for units, then we will first have:

$$\log h = 2.40805 \qquad \text{and} \qquad \delta\varpi = (\bar{4}.23550)t$$

per century, with our units, and then find that:

$$\delta\varpi = +6.28''\ ;$$

for Venus, one will have only:

$$\delta\varpi = +1.32''\ .$$

If one supposes that h is equal to the speed of propagation of light, then one will have:

$$\log h = 2.23948$$

and then

For Mercury and one century: $\delta\varpi = +13.65''$,

For Venus and one century: $\delta\varpi = +2.86''$.

In order to show that these periodic terms are negligible, it will suffice to take the biggest of them, which is $\delta\varpi$, namely, $\frac{f\mu}{2ah^2}\sin 2\zeta$; one will find that its coefficient does not reach $0.003''$.

Chapter 12

[Weber, 1876] Remarks on Edlund's Reply to Two Objections Against the Unitary Theory of Electricity

Wilhelm Weber[178,179,180]

(Contribution made by letter.)

To begin with, I must allow myself the observation that the first objection raised [by Edlund] against Neumann completely misses the mark.[181] In the "Postscript" to his essay in volume 155 of the *Annalen* (page 228) Neumann has *firstly* laid out the facts of the so-called *unipolar induction*, and *secondly* proved thereby, that (if it be at all true to ascribe the action of the electric current to any matter whatsoever, which flows through the conductor with a certain velocity) then at least *two* such types of matter must be supposed.[182]

Now Edlund has made no objection against the latter proof by Neumann. But, he also objected to Neumann's alleged (but in no way established or authenticated) fact, that a current ring of constant strength induces no electromotive force in an *unclosed* linear wire if both are fixed, but that an electromotive force of a certain value is induced if the ring be rotated around its geometric axis with constant velocity — and Edlund, in his second reply, has likewise stated explicitly that this is in correspondence with the general representation of the physical laws of *unipolar induction*. His doubt raised against the *correctness of the fact*, however, strikes Neumann, who has not established it, not at all.

Edlund sets out the facts of the unipolar induction (p. 592) in the following words:

> "Experience teaches that when a *closed* stationary conductor, b, is placed in the neighborhood of a magnet rotating about its own axis, no current is induced in the closed conductor. — The reason is, according to the usual model, that the rotating magnet actually induces a current in one part, b_1, of the closed conductor, but that

178[Web76] with English translation in [Web19b]. This work is related to Edlund's 1875 paper, [Edl75].
179Translated by Laurence Hecht, larryhecht33@gmail.com. Edited by A. K. T. Assis.
180The Notes by Laurence Hecht are represented by [Note by LH:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].
181[Note by AKTA:] Edlund was criticizing C. Neummann's 1875 paper on unipolar induction, [Neu75].
182[Note by LH and AKTA:] That is, two electric fluids, positive and negative.

> an equally large current is induced in the other part, b_2, of the conductor; the two induced currents however flow in opposite directions and cancel one another. But if a portion of the path, b_1, be fixed to the magnet and the magnet set in rotation around its own axis, then the magnet does not act on this portion of the path. Now if the experiment be arranged such that b_1, notwithstanding the rotation, remains in continuous electrical contact with the other portion, b_2, then an induced current is produced in the conductor."

However, Edlund now denies the *correctness* of these facts, which he himself has set forth, namely that the rotation *itself* (be it of a current ring, or a magnet) produces a certain inducing action on the *stationary part* of a closed conductor located in the vicinity. Rather, he claims that the opposite is confirmed by experiment, and cites as proof of this assertion an experiment of Plücker (Vol. 87, p. 352 of this *Annalen*)[183] which he himself has repeated for this purpose.

According to Edlund's description of this experiment, a current is observed in a conductor which remains closed while a copper cylinder containing a part of the current path, b_1, is rotated around the axis of a magnet located in the cylinder. *This current remains unvaried in direction and strength, whether the magnet remains stationary or rotates together with the cylinder.*

Edlund now thinks that this result must have been left out of consideration in the usual formulation of the physical laws of unipolar induction. — Namely, if the cylinder b_1 alone rotates and the magnet is at rest, one can consider the magnet as bound to the galvanometer wire b_2, which is also at rest, so that, according to that mode of representation, induction could take place only in b_1; however, when the magnet and the cylinder b_1 are rotated with equal velocity in the same direction, one can consider the magnet as bound to b_1, and then induction can take place only in b_2.

The induced current, Edlund continues, must therefore, *according to that representation*, alter its direction from one experiment to the other, and, as that does not occur, Edlund concludes that the hitherto accepted model of unipolar induction, must be incorrect *because it runs counter to experiment*.

The following remark will suffice to show Edlund's error in this deduction.

In the *first experiment*, the portion of the conductor, b_1, in which the current is induced rotates (forward), and the magnet stands still; in the *second* experiment, the magnet rotates (also forward) and the portion of the conductor, b_2, in which the current is induced stands still.

A direct comparison of the two experiments is not possible, but an indirect one can easily be made, if one observes that it is all the same whether the *wire rotates forward and the magnet stands still*, or *the wire stands still and the magnet rotates backward*.

In order to make the comparison of the *two experiments* possible, one must look at it as follows: the *backward rotating* magnet (in the first experiment) induces a current of *equal direction and strength* in the stationary portion of the conductor b_1, to that which the *forward rotating* magnet (in the second experiment) induces in the stationary portion of the conductor b_2, which stands in complete correspondence with the general representation of the physical laws of unipolar induction, as Edlund himself set it forth. — What has been overlooked by Edlund is that *oppositely directed currents* are only induced in the two portions, b_1 and b_2, of a permanently closed conductor, if the magnet *rotates in the same direction* relative to the

[183] [Note by AKTA:] [Plü52].

wire, whether it induces it in b_1 or in b_2; on the contrary, *equally directed currents* will be induced in b_1 and in b_2 if, as in the above experiments, the magnet *rotates backwards* relative to the conductor b_1, while it rotates forwards relative to b_2.

Leipzig, 23. December 1875.

Chapter 13

[Weber, 1878a, EM7] Electrodynamic Measurements, Seventh Memoir, relating specially to the Energy of Interaction

Wilhelm Weber[184,185,186]

[184][Web78a] with English translation in [Web21e].

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[186]The Notes by Wilhelm Weber are represented by [Note by WW:]; the Notes by H. Weber, the Editor of Volume 4 of Weber's *Werke*, are represented by [Note by HW:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

Introduction

Helmholtz claimed,[187] with the agreement of William Thomson, Tait and others,[188] that the general fundamental law of electric action set out in 1846 in the "*Electrodynamic Measurements*",[189,190,191] and the subsequently derived *potential* of the electric force (Poggendorff's *Annalen*, 1848, Vol. 73, p. 299),[192,193] would contradict the principle of conservation of energy. But C. Neumann and Maxwell disagreed,[194] pointing out an error by Helmholtz when he asserted that the principle of conservation of energy is only valid for forces that depend *solely* on the distance.[195,196]

Helmholtz then established a completely *new* principle of energy,[197] which differed from the *ordinary* principle of energy as specified by Neumann in the following words:[198]

> "While the *ordinary* principle of energy requires from any material system the existence of an *energy function*, i.e. the existence of a function depending on the momentary state of the system, a function which has the property to increase in any time interval by exactly the quantity of work fed into it during this interval, — the *new* principle established by Helmholtz requires not only the *existence* of such a function, but simultaneously a certain special *behavior* of that function, by claiming that the '*kinetic* part of this function (the part that depends on velocity) must be always positive'."

With respect to this Neumann also comments:

> "There is no doubt that it lies in the nature of the principles of physics that these are extensible and flexible. The *principle of vis viva*[199] has slowly extended to the *principle of energy* and is possibly even further expandable."

In fact, it lies wholly in the essence and progress of *experimental research*, to already utilize such a principle as a *guideline* even when the *ultimate* formulation is still missing

[187] [Note by AKTA:] [Hel47] with English translation in [Hel66].
[188] [Note by AKTA:] [TT67].
[189] [Note by WW:] See *Abhandlungen bei der Begründung der Königl. Sächs. Gesellschaft der Wissenschaften*. Leipzig 1846.
[190] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 25.
[191] [Note by AKTA:] [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].
[192] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 245.
[193] [Note by AKTA:] [Web48] with English translation in [Web52c], [Web66] and [Web19a].
[194] [Note by AKTA:] [Neu68a] with English translation in [Neu20a], see Chapter 5; [Max73a, Chapter XXIII, Articles 852-853, pp. 429-430] and [Max54b, Chapter XXIII, Articles 852-853, pp. 483-484].
I discussed Maxwell's points of view in relation to Weber's electrodynamics in [Ass94, Section 3.6, pp. 73-77].
[195] [Note by WW:] See also Ad. Mayer: "Ueber den allgemeinsten Ausdruck der inneren Potentialkräfte eines Systems bewegter materieller Punkte, welches sich aus dem Princip der Gleichheit von Wirkung und Gegenwirkung ergiebt". [Translation: "On the most general expression of the internal potential forces of a system of moving material points, which results from the principle of the equality of action and reaction".] *Mathematische Annalen*, Vol. 13, p. 20.
[196] [Note by AKTA:] [May78].
[197] [Note by AKTA:] [Hel73].
[198] [Note by AKTA:] [Neu75, pp. 216-217] and [Neu77, p. 322].
[199] [Note by AKTA:] See footnotes 26 and 140 on pages 17 and 74, respectively.

and can only later be extracted from the results of research; but since it is clear that the principle, in order to serve as a guideline for research, must nevertheless be *formulated*, hence meaning that it could only be tentative, it follows therefore that *during* this research, such principle is really extensible and flexible.

If by this, Helmholtz was entitled to formulate the principle of energy tentatively in such a way that my fundamental law, condemned by him, is in contradiction with it, then obviously the opposite is equally legitimate, namely, to formulate the same principle *tentatively* in such a way that it is not only *in agreement* with that fundamental law, but that the latter even results as a necessary consequence of the principle by proving that all *electrodynamic* laws, to which that fundamental law belongs, can be derived via the *tentative* principle from the *electrostatic* laws. To try it this way around is the intention of the present treatise whereby, instead of beginning with the general fundamental law of *electrical interaction* established in the first treatise[200] which simultaneously embraced electrostatics and electrodynamics, the principle of the conservation of energy will be the starting point, from which then, in combination with the fundamental law of *static* interaction between two particles, we shall deduce, *firstly*, that fundamental law of electrical interaction and, *secondly*, the existence of an *energy function* for every pair of particles from which follows the validity of the *ordinary* energy principle, as it was spelled out by Neumann.

13.1 Guidelines for Experimental Research in Electrodynamics

After having obtained the *general laws of motion of bodies* as a base in physics, it essentially only remained to explore the *laws of the interaction of bodies*; because without interaction all bodies would *remain forever* in their original state of rest or motion. All changes in these states and all resulting phenomena are therefore *consequences* of their interactions.

Such interactions take place when the bodies *touch* each other, and also when they are *at a distance* from one another, and it was obvious that one had to start at investigating the latter, in order to extract a *guideline* for the former, which becomes especially necessary whenever the spacial situation of the bodies is not directly observable, as happens in the case of interactions of bodies touching one another. Indeed, it did actually happen this way, as one began with the investigation of the interactions of the *celestial bodies*, that is with the *gravitational interactions*.

After this first area of successful investigation of the *interaction of bodies*, namely *gravitational interactions*, came the investigation of *electric and magnetic interactions*, because apart from the gravitational interactions these were the only ones which were performed by one body *at measurable distance* from another one and which themselves could be determined by measurement.

For a long time almost all theoretical investigations into *electricity* and *magnetism*, in particular those of Coulomb and Poisson,[201] used Newton's theory of gravitation as a guide-

[200][Note by AKTA:] See footnote 191.

[201][Note by AKTA:] Charles Augustin de Coulomb (1736-1806) and Siméon Denis Poisson (1781-1840). See [Cou88a] with partial English translation in [Cou35a], complete English translation in [Cou12] and German translation in [Cou90b]; [Cou88c] with German translation in [Cou90d] and partial English translation in [Cou35b]; [Cou88d] with German translation in [Cou90a]; [Cou88b] with German translation in [Cou90c]; [Cou89]; [Cou91]; [Cou93]; [Pot84]; [Gil71b] and [Gil71a]. See also [Poi12a], [Poi12b], [Poi13], [Poi25a],

line,[202] until finally, as a consequence of Oersted's[203] and Ampère's discoveries of the *equivalence of closed currents and magnets*,[204] a totally new guideline arose. This new guideline firstly involved the *reduction of all magnetic interactions to electric interactions* and, secondly, led to the establishment of a *fundamental law of interaction between any two current elements*.

The general idea of deriving the mutual interaction of all bodies from the interaction of any two served as a *third guideline*, and accordingly the interactions between *current elements* should thus also be reducible to the interactions between *any two electric particles*. From experience, this idea would seem to be confirmed and justified in many circles even apart from the fact that infinite complications would arise if the opposite were the case (namely, interactions of *three* or more bodies which would not be reducible to interactions of *any two*).

The material particles to be taken into account with respect to the interaction between two current elements were now essentially one positive and one negative electric particle in every current element, resulting in four independent pairwise interactions.[205] For the determination of these four interactions, Coulomb-Poisson's fundamental law (modeled on the law of gravitation) offered itself, as it had already been confirmed in electrostatics; but nevertheless the four individual interactions do not result in a net interaction since they cancel each other completely,[206] and therefore Ampère's fundamental law of action at a distance between current elements was not reducible to Coulomb-Poisson's fundamental law of interaction between pairs of electric particles.

Coulomb-Poisson's fundamental law of interaction between pairs of electric particles had been established only for pairs of particles at *rest relative* to each other or, at least, could only be justified as being in agreement with experience for such mutually stationary particles. In contrast, the four electric particles in two current elements form four pairs of particles which are not at relative rest, but in *relative motion*. Therefore this pointed to the conjecture that Coulomb-Poisson's fundamental law of interaction between any two electric particles, *if these particles are in relative motion*, still requires a *correction* which shall be denoted by x. Denoting the corrections of the above four interactions in the order x_1, x_2, x_3, x_4, then the sum of them should be non-zero and equal to the force determined by Ampère's law.

In this way it has now been found that — denoting any two electric particles in absolute measure by e and e',[207] and their relative distance, velocity, and acceleration by r, dr/dt, and d^2r/dt^2, and distinguishing these four values for the four pairs considered in two current elements with the subscripts 1, 2, 3, 4 — the repulsive force of two current elements determined by Ampère's law, namely

[Poi25b], [Poi22a] and [Poi22b].

[202][Note by AKTA:] See footnote 32 on page 19.

[203][Note by AKTA:] Hans Christian Ørsted (1777-1851). See [Oer20b], [Oer20a], [Oer20c], [Oer65], [Ørs86] and [Ørs98]. See also [Fra81] and [Rei13].

[204][Note by AKTA:] See footnote 8 on page 10.

[205][Note by AKTA:] Let us assume that current element 1 is composed of two equal and opposite charges, e_+ and $e_- = -e_+$. Likewise current element 2 is composed of two equal and opposite charges, e'_+ and $e'_- = -e'_+$. The interaction between current elements 1 and 2 is then composed of four independent pairwise interactions, namely, e_+ interacting with e'_+, e_+ interacting with e'_-, e_- interacting with e'_+, and e_- interacting with e'_-.

[206][Note by AKTA:] Due to the charge neutrality of both current elements, the sum of these four Coulomb interactions add up to zero.

[207][Note by AKTA:] That is, e and e' are the values of the electric charges of the two particles expressed in the absolute system of units introduced by C. F. Gauss (1777-1855) and Wilhelm Weber, [ARW04].

$$\frac{\alpha\alpha' ii'}{r^2}(3\cos\vartheta\cos\vartheta' - 2\cos\varepsilon) \ ,$$

(where α and α' are the lengths, i and i' the current intensities of the two current elements, r their distance, ϑ and ϑ' the angles which α and α' form with r, and ε the angle formed by α and α'), is really represented through the sum

$$x_1 + x_2 + x_3 + x_4 \ ,$$

if one sets[208]

$$x = \frac{1}{c^2} \cdot \frac{ee'}{r^2}\left(2r\frac{d^2r}{dt^2} - \frac{dr^2}{dt^2}\right) \ .$$

Here c denotes a *constant*, namely that relative velocity of two electric particles, at which no interaction takes place so long as it remains unchanged.[209]

In order to prove this, it is only necessary to express the quantities α, α', i, i' and the angles ϑ, ϑ' and ε related to the current elements as functions of the quantities e, e', r, dr/dt, d^2r/dt^2 with respect to the four pairs of particles.

This *correction* must hence be added to the repulsive force determined by the fundamental law of Coulomb-Poisson, whenever it should be valid not only for pairs of particles at relative rest, but also for such motions which occur in current elements for which Ampère's law is valid.

But it is clear that those four electric particles can also be set in *diverse other relative motions*, other than those which take place within two current elements for which Ampère's law is valid. Indeed one can easily arrange an apparatus in which two particles of positive and negative electricity are inside a current element and moving, instead of with *equal and constant velocity in opposite directions* (as Ampère assumed), either with equal but *variable* velocity in opposite directions, or with *unequal* velocities in directions which form an *arbitrary angle* with each other. All these different cases can easily be arranged, partly by allowing the existing current in a conductor to *disappear*, then to *arise* again, through opening or closing the circuit, partly by giving, to the oppositely moving electricities flowing inside the conductor, a *joint motion with their conductor*.

If now the *corrected law* of Coulomb-Poisson is really valid in general for two electric particles, not just at relative rest or belonging to constant currents in resting conductors, but also for all their other motions, then from it can be predicted and predetermined the action of current elements just as in the case of individual particles — also in the just listed as well as in all other cases in which Ampère's law does not apply (which for a long time were left unnoticed and unobserved) — which serves as test and confirmation of the general validity of that law. In fact, in that way *all the laws of Voltaic induction* have been found to be,[210] in complete conformity with the phenomena observed by Faraday, and have been universally confirmed through manifold observation and measurements.

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[208][Note by AKTA:] The expression dr^2/dt^2 should be understood as $(dr/dt)^2$.

[209][Note by AKTA:] That is, if dr/dt remains constant and equals to c, then there will be no net force between the two electric particles interacting according to Weber's fundamental law.

[210][Note by AKTA:] The expression utilized by Weber, *Volta-Induktion*, had been first suggested by Faraday himself in paragraph 26 of his first paper on electromagnetic induction of 1831, see [Far32a, § 26] and [Far52, § 26, p. 267 of the *Great Books of the Western World*]. Portuguese translation in [Far11, p. 159]:

> For the purpose of avoiding periphrasis, I propose to call this action of the current from the voltaic battery, *volta-electric induction*.

To this general fundamental law of interaction between two electric particles one can link further considerations about the *essence of the interaction*.

During all changes in the celestial bodies the *masses* of the bodies always remain unchanged, and also the *vis viva* of the bodies would, if there was no *interaction*, remain unchanged by the law of inertia. *Interactions* are therefore the reason for all changes of the vis viva, hence the question very obviously arises, that if not vice versa the reason for all changes of interactions should be searched for in the *vis viva*, so that amplification of the interaction can only be gained, if vis viva is lost, and that inversely vis viva is gained, only if interaction suffers a decrease. *Interaction* of bodies would then be the *equivalent* of the lost vis viva, and *vis viva* the *equivalent* for lost interaction, whereby the *values* of interactions and vis viva would become *dependent* on one another.

The general fundamental law of electrical interaction mentioned above corresponds to this idea, in that it establishes the dependence of the force resulting from the mutual interaction upon the vis viva of the bodies, in contrast to the Coulomb-Poisson law according to which there is no such dependence.

If one now calls the magnitude of the interaction of two particles their *energy of interaction*, and the magnitude of the relative vis viva of two particles their *energy of motion*,[211] there arises obviously the conjecture that when one energy increases in conjunction with the simultaneous decrease of the other one, the gain in one energy being compensated also *quantitatively* by the loss in the other, which presupposes the *homogeneity* of both energy quantities and means that *their sum is constant*. Denoting then by Q the relative vis viva of two particles and by P the energy of their interaction, one accordingly has to set

$$P + Q = a \ ,$$

where a is a constant for each *pair of particles*, just as mass is a constant for each *individual particle*.

It would thus be determined, *how much the interaction of two particles gets changed by their mutual motion*, a foundation for the derivation of the *dynamic* law from the *static* one.

The total constant energy a would at the same time be the limit which could not be exceeded by the energy P, because indeed the energy Q (i.e. the vis viva of the particle) cannot have a value smaller than zero.

The conjecture put forth here has been subject to *several modifications*, and has found various statements, according to the different tentatively formulated expressions of the *principle of the conservation of energy* which have served as *guideline* for the many recent researches,

This phenomenon of Volta-induction is nowadays called Faraday's law of induction.

[211] [Note by AKTA:] In German: *Bewegungsenergie*. This expression can be translated as "energy of motion" or "kinetic energy", see [Web71, p. 258 of Weber's *Werke*] with English translation in [Web72, p. 10]. See also footnote 141 on page 76 of Chapter 9.

particularly in the theory of heat and electricity.[212,213] Given the importance and significance of this newly achieved *guideline*, some differences in the points of view and meanings deserve special attention.

The previous tentative formulation for the *principle of the conservation of energy* is fundamentally different, and it could easily appear to be in contradiction (which on closer inspection is not the case) with the formulation of the *"ordinary principle of energy"* of which C. Neumann says in Vol. XI, p. 320, of the *Mathematische Annalen*:[214]

> "This principle requires that for every material system there exists an *energy function*, i.e. a function depending on the momentary state of the system, which has the property that it increases during any given time interval by precisely the amount of work that is added to the system from the outside. At the same time we notice that this *energy function* (which one simply calls the *energy* of the system),

[212][Note by WW:] It goes back to Thomas Young and W. Thomson to denote the sum of the *vis viva* and *heat* of a system of bodies together with the work determined by its *potential*, with the name of its *mechanical energy*, or shortly its *energy*, and this was then recognized and accepted by Clausius as being very practical.

Thomas Young, *Lectures on Natural Philosophy*, London 1807, Lecture VIII, says on page 78:

> "The term energy may be applied, with great propriety, to the product of the mass or weight of a body, into the square of the number expressing its velocity."

So Young denotes only the vis viva of a body (actually twice its value) with the name energy, but without explicitly adding that the body has only this one but no other energy. Rather this seems to suggest, since on the following page he uses for the vis viva of a body the more complete terminology 'energy of its motion', that a body may have, aside from its motion, another energy.

W. Thomson in *Phil. Magazine and Journal of Science*, IV. Series, [Vol.] 9, London 1855, p. 523, says:

> "A body which is either emitting heat, or altering its dimensions against resisting forces, is doing work upon matter external to it. The mechanical effect of this work in one case is the excitation of thermal motions, and in the other the overcoming of resistances. The body must itself be altering in its circumstances, so as to contain a less store of work within it by an amount precisely equal to the aggregate value of the mechanical effects produced; and conversely, the aggregate value of the mechanical effects produced must depend solely on the initial and final states of the body, and is therefore the same whatever be the intermediate states through which the body passes, provided the *initial* and *final* states be the same. — The total mechanical energy of a body might be defined as the mechanical value of all the effect it would produce in heat emitted and in resistances overcome, if it were cooled to the utmost, and allowed to contract indefinitely or to expand indefinitely according as the forces between its particles are attractive or repulsive, when the thermal motions within it are all stopped."

Herein W. Thomson has simultaneously enunciated with the *name energy*, the principle of the *conservation of energy*; because what a system of bodies looses from its stock of energy is gained by another system of bodies, from this obviously follows the *conservation of energy* in all systems of bodies taken together. — The same principle was essentially formulated earlier, only utilizing another expression, specifically by Helmholtz under the name of the *principle of conservation of force*.

[213][Note by AKTA:] [You07, Lecture 8, p. 78], [Tho53a, p. 475 of the *Transactions of the Royal Society of Edinburgh*, p. 523 of the *Philosophical Magazine* and pp. 222-223 of the *Mathematical and Physical Papers*].

What Helmholtz called *principle of the conservation of force* [*Princip der Erhaltung der Kraft*], [Hel47] with English translation in [Hel66], is nowadays called *principle of the conservation of energy*.

[214][Note by AKTA:] [Neu75, pp. 214-217] and [Neu77, pp. 320-322]. When Weber mentioned from where the quotation of Helmholtz came from, he cited [Hel72a], with English translation in [Hel72b]. However, the correct reference is: [Hel73, p. 36 of the *Journal für die reine und angewandte Mathematik* and p. 649 of the *Wissenschaftliche Abhandlungen*].

based on Weber's law [this is Neumann's terminology for the above described corrected Coulomb-Poisson law], is represented by the sum of vis viva and potential ... Helmholtz meanwhile takes a somewhat different approach to this question ... as stated in the article (*Monatsber. d. Berl. Akad.*, 18. April 1872) in the words:

> "The investigations into the validity of the law of conservation of energy for certain processes of nature usually requires examination into whether or not an infinitely repeated cyclical process creates or destroys work, if I may express the analytical result in practical terms. — Now in this sense Weber's hypothesis does not violate the law of the conservation of energy; but it does so in another sense — — —"

Neumann continues:

> "The following objection no longer concerns the ordinary energy principle, but a completely *new* one, a principle formulated here for the first time. Namely, while the ordinary energy principle requires for each material system the existence of an energy function, i.e. the existence of a function which has the property to increase in any time interval by precisely the amount which equals the work added to the system in the interval, — this new principle not only requires the *existence* of such a function, but also the condition that the *kinetic part* of this function (the part which depends on velocity) must always be *positive*."

Neumann adds in a note the following remark, already cited above:[215]

> "There is no doubt that physical principles are incapable of a fixed formulation, since by their nature these are extensible and flexible. The *principle of vis viva* has slowly been extended to the *principle of energy* and is possibly even further expandable. — Accordingly it is a priori not impossible that this *energy principle* gradually extends to that *new principle* of Helmholtz. It only appears useful to me, at least temporarily, to denote both principles with different names." —

That last remark applies not only to Helmholtz's principle, but also to the one established above which also deviates from the *ordinary* one, and so, to better distinguish it, the name principle of the *conservation* of energy was used, because according to it the *whole energy*, namely the sum of the motion and the interaction, is *conserved*, while according to the ordinary energy principle there exists only an *energy function*, whose magnitude is not at all conserved, but has the property that it increases in any time interval by precisely the amount of work added to the system from the outside. Only in two special cases can the ordinary principle be considered also as a principle of the *conservation* of energy, namely in the case where the system under consideration contains *all* bodies in the world, and also in the case where the system under consideration is to be viewed as *completely isolated*, the reason being that in these two cases there are no *external* influences.

But given those differences, it must be proved that there is no *contradiction* between the principle of the *conservation* of energy and the *ordinary* energy principle, as defined by Neumann, for which, as is easily seen, it is only necessary to show that the energy of interaction P increases in any time interval by precisely the difference, in that time interval,

[215][Note by AKTA:] [Neu75, pp. 216-217] and [Neu77, p. 322].

of the increase of the potential V and the work S added to the pair of particles from the outside, i.e. that $dP = dV - dS$, which by taking into account the equation given by the principle of the conservation of energy, namely $P + Q = a$, where a denotes a constant, leads to the *ordinary* energy principle, namely

$$d(Q + V) = dS \ ,$$

where $(Q + V)$ denotes Neumann's *energy function.* — One will attempt to provide this proof in Section 13.4 below.

The objective which shall be reached with this new principle, different from both the *ordinary* and the energy principle formulated by Helmholtz, consists essentially in

> *obtaining a principle which determines what gets changed in the interaction of bodies as a result of their motion.*

Interaction only takes place between two bodies and is subject to change *only* through the relative vis viva of their motion. Presupposing this, and in addition that this *interaction* of two bodies or material particles is a quantity *homogeneous* with its relative vis viva, which forms with the magnitude of this vis viva the constant *energy sum* a, then a obviously means the magnitude of the interaction of the both particles *at rest*, i.e. their *static* interaction, and the principle of the conservation of energy is then the law through which it is determined that this *static* interaction decreases by Q, as a consequence of the relative *motion* imparted by any vis viva of magnitude Q.

The *general fundamental law of electrical interaction*, *as such*, would immediately be completely replaced by the principle of the conservation of energy and transformed into a theorem, which would be derived and proved from the fundamental law of *electrostatics* by means of the *principle of the conservation of energy.*

13.2 Interaction Energy Reduced to Absolute Measure

It is clear that from the equation established in the previous Section, in which the principle of the conservation of energy was tentatively formulated, namely

$$P + Q = a \ ,$$

the *energy of motion* Q can be determined, if the *energy of interaction* P is given, and vice versa; at the same time it is clear that the meaning of the equation, as the formulation of a principle, rests on the *physical significance*, which is bound up with the notion of every *individual* energy, from which the possibility of the determination of the magnitude of any individual energy independent of the others must be evident. For the *energy of motion* such a determination has long since been given; it is required therefore only a similar determination for the *interaction energy.*

The interaction of two particles during a change in distance consists in *work* done. Without change of distance, interaction takes place, but not *work*; yet the pair of particles always possesses an *ability to do work*,[216] i.e. the property of being able to perform work through

[216][Note by AKTA:] In German: *Arbeitsvermögen*. This expression can also be translated as "working capacity", "work capacity", "work capability", "energy capability" or "energy capacity".

changes of distance. From this *ability to do work* one recognizes the interaction and its magnitude gives the scale of the *energy* of the interaction.

Quantitative determination of the ability to do work must be built on *work measurement*. But now *work* consists either in *cancellation* of opposing work, or in *creation* (or destruction) of *vis viva*. Works that cancel one another elude direct measurement; in contrast increase or decrease of vis viva is under suitable conditions the subject of direct observation and measurement, to which ultimately all *work measurement* is reduced.

If accordingly *work* is determinable from the measurable vis viva generated by it, when it is not cancelled by any opposing work, then to determine the *ability to do work* of a pair of particles it suffices to determine the magnitude of work which would be achieved by the particle *interaction* during a certain, still to be determined, change of distance. If whether this magnitude of work happens to be positive or negative is not taken into consideration, then the *absolute value*[217] of this magnitude of work serves as a *measure* of the ability to do work.

In contrast, according to the established principle of the conservation of energy, to determine the ability to do work one has to take into consideration the velocity dr/dt with which the change of distance takes place. The reason for that is that the energy of motion Q varies with this velocity, and consequently according to the mentioned principle also the interaction energy P. From this it is clear that the energy P, i.e the ability to do work of a pair of particles, is only exactly measurable for a *given value of the velocity* dr/dt, and that this value has to be assumed *constant* during the corresponding change of distance.

But because at a constant value of dr/dt vis viva neither increases nor decreases, which could be used for *direct* measurement of work, one has to look for an *indirect* method to determine the ability to do work. If during a change of distance one wishes not to cause a change of relative velocity through interaction, then the work performed during the change of distance via *interaction* must be counteracted by that performed through *external* influence, and the latter can be used — if it is of known origin and hence exactly determined, e.g. if it arises from known weights which act on the particles during the change of distance — to *indirectly* measure the work performed by the interaction.

The work performed via interaction during the change of distance dr of two particles e and e' is given in absolute value by $\pm[\partial V/\partial r]dr$,[218] where V denotes the *potential* of the pair of particles and the upper or lower sign is valid depending on whether the product ee' is positive or negative. Similarly, the work performed during a greater change of distance from ρ' to ρ'' is given by

$$\pm \int_{\rho'}^{\rho''} \frac{\partial V}{\partial r} \, dr \ .$$

The task of reducing to absolute measure the energy of interaction of two particles e and e', i.e. the determination of their ability to do work by absolute measure, is thereby reduced to finding the value of the integral $\pm \int_{\rho'}^{\rho''} \frac{\partial V}{\partial r} \, dr$ in which only the integration limits ρ' and ρ'' need to be determined.

[217][Note by WW:] From this it follows that the validity of the principle $P+Q=a$ is limited to the cases in which Q does not exceed the value of a. But the vis viva, Q, of all the bodies known to us is however such a small fraction of a, that most likely the case $Q > a$ does not appear. According to our present knowledge two bodies with a relative velocity $> 439\,450$ kilometer/[second] would be necessary for this to take place.

[218][Note by WW:] The symbol of *partial derivation* has been chosen to indicate that in this differentiation dr/dt should be considered constant.

As by *ability to do work* one understands the value of the work performed via interaction during an *exactly to be determined change of distance*, it is then clear that the limits of distance ρ' and ρ'' in the expression $\pm\int_{\rho'}^{\rho''} \frac{\partial V}{\partial r}\, dr$ must get *exactly determined and be constant values*, from which it follows that these distance limits cannot be the same as those of the potential $\int_{\infty}^{r} \frac{dV}{dr}\, dr$ for which one, namely r, is a *variable*.

Since further this is about determining the *whole ability to do work*, associated with the pair of particles via the interaction of its particles, it is clear that these distance limits should be placed as far apart as possible, without contradicting the principle of the conservation of energy, according to which the *energy sum* of the pair of particles should be a constant a, and this constant should also be the limit which the *energy of interaction* must not exceed and should only be reached when the *energy of motion* is zero.

From this results first the determination of one distance limit $\rho' = \infty$; concerning the other limit ρ'', its value must not be smaller than the one for which $\pm\int_{\infty}^{\rho''} \frac{\partial V}{\partial r}\, dr = a$ would be valid. Let now ρ denote the resulting value of ρ''.

In order to determine the *energy P of interaction* of two particles e and e' whose masses are denoted by ε and ε' and whose energy of motion is

$$Q = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \ ,$$

one then gets the equation

$$P = \pm\int_{\infty}^{\rho} \frac{\partial V}{\partial r}\, dr \ .$$

Here it is to be observed that, firstly, during the change of distance there should take place, apart from the interaction, an *external* influence on the pair of particles which keeps the given value of $\partial r/\partial t$ *constant* in V; secondly, that the upper or lower sign is valid depending on whether the product ee' is positive or negative; thirdly, that $P = a$ when $Q = 0$ which serves to determine ρ.

The formula for P can yet be transformed as follows. One can decompose

$$P = \pm\int_{\infty}^{\rho} \frac{\partial V}{\partial r}\, dr$$

into two parts, namely:

$$P = \pm\int_{\infty}^{r} \frac{dV}{dr}\, dr \pm \int_{r}^{\rho} \frac{\partial V}{\partial r}\, dr \ ,$$

where the *first* part is the absolute value of the *potential* V, and where here we used the ordinary symbols of differentiation since it does not matter if dr/dt is treated as variable, or not. Concerning the *second* part one has to note that the *given* value of the velocity dr/dt must be presupposed constant during the change of distance from r to ρ.

Denoting by s the work performed by *external* influence during the change of distance from r to ρ, then in order to keep dr/dt constant, the following relationship must necessarily be valid:

$$\pm\int_{r}^{\rho} \frac{\partial V}{\partial r}\, dr + s = 0 \ .$$

From this one gets the following formula to determine the energy P, namely

$$P = \pm V - s \ ,$$

where V denotes the *potential* of the particles of charges e and e', and s the work which must be performed during the change of distance from r to ρ by *external* influence, so that the *given* value of the relative velocity dr/dt remains unchanged.

13.3 Derivation of the Electrodynamic Potential Law from the Electrostatic One by Means of the Principle of Energy

According to the definition of both energies in the case of a pair of electric particles e and e' whose masses were denoted by ε and ε' and their distance by r, namely[219,220,221]

$$\text{the } \textit{energy of motion } Q = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\cdot\frac{dr^2}{dt^2}\ ,$$

and

$$\text{the } \textit{energy of interaction } P = \pm\int_\infty^\rho \frac{\partial V}{\partial r}\,dr\ ,$$

— where the upper or lower sign is valid depending on the product ee' being positive or negative, and where with the given value of the magnitude Q the relative velocity dr/dt must be assumed *constant* during the change of distance, and the same should be valid for Q, — where there results from the principle of energy presented in Section 13.1, according to which $P+Q=a$ forms a *constant* sum, the following equation between the two constants a and ρ and the two variables Q and V, namely

[219][Note by WW:] If α and β denote the velocities of the mass ε in the direction r and the one orthogonal to it, while α' and β' denote the same velocities for ε', so that $\alpha-\alpha'=dr/dt$ is the relative velocity of the two particles, then

$$\frac{1}{2}\varepsilon\left(\alpha^2+\beta^2\right)+\frac{1}{2}\varepsilon'\left({\alpha'}^2+{\beta'}^2\right)$$

is the total vis viva belonging to the two particles. Set now for α,

$$\frac{\varepsilon\alpha+\varepsilon'\alpha'}{\varepsilon+\varepsilon'}+\frac{\varepsilon'(\alpha-\alpha')}{\varepsilon+\varepsilon'}\ ,$$

and for α',

$$\frac{\varepsilon\alpha+\varepsilon'\alpha'}{\varepsilon+\varepsilon'}-\frac{\varepsilon'(\alpha-\alpha')}{\varepsilon+\varepsilon'}\ ,$$

then one gets the total vis viva of the two particles as the sum of *two parts*,

$$=\frac{1}{2}\cdot\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\cdot\frac{dr^2}{dt^2}+\frac{1}{2}\left[\frac{(\varepsilon\alpha+\varepsilon'\alpha')^2}{\varepsilon+\varepsilon'}+\varepsilon\beta^2+\varepsilon'{\beta'}^2\right]\ ,$$

from which the *first*, namely $\frac{1}{2}[\varepsilon\varepsilon'/(\varepsilon+\varepsilon')]\cdot[dr^2/dt^2]$ is the *relative vis viva* of the two particles, which was denoted above by Q. — See *Abhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften*, Vol. X, p. 12.

[220][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 257.

[221][Note by AKTA:] [Web71, pp. 256-257 of Weber's *Werke*] with English translation in [Web72, p. 9]. See also footnote 140 on page 74 of Chapter 9.

$$\pm\int_{\infty}^{\rho}\frac{\partial V}{\partial r}\,dr = a - Q\ . \qquad (1)$$

Now according to the fundamental law of *electrostatics* the potential for $Q = 0$ is given by $V = ee'/r$. Inserting these values for the variables Q and V into Equation (1), one obtains the following equation between the two constants a and ρ, reducing one to the other, namely

$$\pm\int_{\infty}^{\rho}\frac{d\frac{ee'}{r}}{dr}\,dr = a\ ,$$

from which one finds the value of the constant ρ, namely

$$\rho = \pm\frac{ee'}{a}\ . \qquad (2)$$

Inserting now this value of ρ into Equation (1), there results in the following equation between only *one* constant, namely a, the given value of the variable Q, and the value of the variable V which we are looking for, namely

$$\pm\int_{\infty}^{\pm\frac{ee'}{a}}\frac{\partial V}{\partial r}\,dr = a - Q\ , \qquad (3)$$

from which V is to be determined.

One can easily observe that this Equation (3) is satisfied by V defined as

$$V = \frac{ee'}{r}\left(1 - \frac{Q}{a}\right)\ ;$$

indeed, substituting this value in the first term of Equation (3) and taking into consideration that, according to the definition given in Section 13.2, in the formula $P = \pm\int_{\infty}^{\rho}\frac{\partial V}{\partial r}\,dr$, the given value of the relative velocity dr/dt, hence also $Q = \frac{1}{2}[\varepsilon\varepsilon'/(\varepsilon+\varepsilon')]\cdot[dr^2/dt^2]$, must be assumed *constant* during the change of distance, then one finds for one limit, $r = \pm ee'/a$, where the value $V = \pm a(1 - Q/a)$, and for the other limit $r = \infty$ where the value $V = 0$, and consequently the difference of these values

$$\int_{\infty}^{\pm\frac{ee'}{a}}\frac{\partial V}{\partial r}\,dr = \pm a\left(1 - \frac{Q}{a}\right)\ ,$$

therefore,

$$\pm\int_{\infty}^{\pm\frac{ee'}{a}}\frac{\partial V}{\partial r}\,dr = a\left(1 - \frac{Q}{a}\right) = a - Q\ ,$$

completely in agreement with Equation (3).

This formula of the *law of the electrodynamic potential* derived from the fundamental law of *electrostatics* with the help of the *principle of energy*, namely

$$V = \frac{ee'}{r}\left(1 - \frac{Q}{a}\right)\ , \qquad (4)$$

can yet be rewritten in the following way.

The *constant sum of energy* a is according to the principle of energy the limit of the kinetic energy $Q = \frac{1}{2}[\varepsilon\varepsilon'/(\varepsilon+\varepsilon')]\cdot[dr^2/dt^2]$ for decreasing values of the interaction energy

P, i.e. one has $Q = a$ when $P = 0$. Denoting then by c the relative velocity dr/dt of the two particles for this *limit* of the kinetic energy a, there results

$$a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot c^2 \ .$$

Substituting now these values of $Q = \frac{1}{2}[\varepsilon\varepsilon'/(\varepsilon+\varepsilon')]\cdot[dr^2/dt^2]$ and $a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'}\cdot c^2$ in Equation (4), one obtains for the law of the electrodynamic potential the following expression:

$$V = \frac{ee'}{r}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2}\right) \ .$$

Between the three *constants* a, ρ, c appearing in this derivation of the law of the electrodynamic potential of a pair of electric particles e and e', with masses ε and ε', the following relations finally take place, namely

$$a = \pm\frac{ee'}{\rho} = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon+\varepsilon'} \cdot c^2 \ .$$

For the *electrodynamic potential* V one gets by interchanging these constants the following formula:

$$V = \frac{ee'}{r}\left(1 - \frac{Q}{a}\right) = \frac{ee'}{r}\left(1 - \frac{1}{c^2}\cdot\frac{dr^2}{dt^2}\right) = \frac{ee'}{r} \mp \frac{\rho}{r}Q \ ,$$

where the upper or lower sign is valid depending on whether the product ee' is positive or negative.

13.4 Derivation of the Ordinary Principle of Energy from the Principle of the Conservation of Energy

The *ordinary* principle of energy, as it was formulated by Neumann, requires that for every material system there exists an *energy function*, i.e. a function depending only on the momentary state of the system, which has the property to increase in each time interval by precisely the amount of work added to the system from the outside during that interval. This *energy function* one has often simply been called the *energy*.

In case of a system of two particles at a distance r from each other, on which, during the change of distance dr, is exerted by mutual interaction the *internal* work Rdr, and by *external* influence the *external* work dS, the increase of the vis viva Q is according to a well known *general theorem of mechanics* precisely equal to the sum of all *internal* and *external* works exerted on the system, namely

$$dQ = Rdr + dS \ .$$

Hence if there is a function depending on the present state of the pair of particles and which has the property to increase during the change of distance dr by the amount $dQ - Rdr = dS$, then for such a pair of particles the *ordinary* principle of energy is valid.

Since now for a pair of electric particles e and e' it was proved using the *potential law* developed in the previous Section, under the assumption of the principle of the *conservation of energy*, that the *internal* work Rdr is the total differential of the function $-(ee'/r)(1 -$

dr^2/c^2dt^2), which depends just like Q, only on the present state of the pair of particles, it is clear that the difference of the two quantities which also depends only on the present state of the pair of particles, namely

$$Q + \frac{ee'}{r}\left(1 - \frac{dr^2}{c^2dt^2}\right) ,$$

has the property to increase during the change of distance dr by $dQ - Rdr = dS$, whereby consequently for such a pair of particles, not only is the principle of the *conservation of energy* valid, but also the *ordinary* principle of energy, and

$$Q + \frac{ee'}{r}\left(1 - \frac{dr^2}{c^2dt^2}\right) ,$$

is its *energy function*.

The simultaneous validity of both principles, namely the principle of the *conservation* of energy, stating that $P+Q = a$, and the *ordinary* principle of energy, stating that $d(Q+V) = dS$, where S denotes the work performed by *external* influence, presupposes, as already mentioned at the end of Section 13.1, firstly that

$$dP = dV - dS ,$$

or, since $P = \pm V - s$ according to Section 13.2, then consequently one has

$$dP = \pm dV - ds ,$$

and also $\pm dV - ds = dV - dS$, which can easily be proved with the help of the equations obtained in Sections 13.2 and 13.3:

$$\pm \int_r^\rho \frac{\partial V}{\partial r}\,dr + s = 0 , \qquad (1)$$

$$V = \pm\frac{\rho}{r}\,(a - Q) , \qquad (2)$$

and with the help of the probative formulation of the *ordinary* principle of energy, already mentioned in Section 13.1, equation

$$dS = d(Q+V) \qquad (3)$$

can be easily proved as follows. From (1) and (2) results the equations

$$s = (a - Q)\left(\frac{\rho}{r} - 1\right) ,$$

$$-ds = \rho(a - Q)\frac{dr}{r^2} + \left(\frac{\rho}{r} - 1\right)dQ ,$$

$$\pm dV = -\rho(a - Q)\frac{dr}{r^2} - \frac{\rho}{r}\,dQ ,$$

from which follows

$$\pm dV - ds = -dQ .$$

But now $dV - dS = -dQ$ according to (3), consequently $\pm dV - ds = dV - dS$, which was to be proved.

13.5 The General Law of Electric Force

The *potential* of two electric particles e and e' at a distance r,

$$V = \frac{ee'}{r}\left(1 - \frac{Q}{a}\right) ,$$

which was found in Section 13.3, is interpreted as the *work* performed during the interaction between the two particles of charges e and e' possessing relative vis viva Q, whenever they are displaced from infinity to distance r. The differential quotient dV/dr then denotes the *force* exerted by the two particles on each other at a separation distance r, which will be attractive or repulsive depending on whether this expression is positive or negative.

The relative vis viva Q of the two particles with masses ε and ε' is represented by

$$Q = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} ,$$

which indicates that Q is a function of time t (except when dr/dt is explicitly supposed to be constant), as well as r, and that consequently any of these two variables r and Q can also be considered as a function of the other.

From this results the *repulsive force*

$$-\frac{dV}{dr} = \frac{ee'}{r^2}\left(1 - \frac{Q}{a}\right) + \frac{ee'}{ar} \cdot \frac{\frac{dQ}{dt}}{\frac{dr}{dt}} ,$$

or, if one substitutes herein the values

$$Q = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr^2}{dt^2} \quad \text{and} \quad a = \frac{1}{2}\frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot c^2 ,$$

from which follows

$$\frac{dQ}{dt} = \frac{\varepsilon\varepsilon'}{\varepsilon + \varepsilon'} \cdot \frac{dr}{dt} \cdot \frac{d^2r}{dt^2} = \frac{2a}{c^2} \cdot \frac{dr}{dt} \cdot \frac{d^2r}{dt^2} ,$$

this then results in the *repulsive force*

$$-\frac{dV}{dr} = \frac{ee'}{r^2}\left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2} + \frac{2r}{c^2} \cdot \frac{d^2r}{dt^2}\right) .$$

But now the relative acceleration d^2r/dt^2 is composed of *two parts*, namely the part *depending* on the interaction between the two particles, and another part which is *independent* of it. If the *latter* is denoted by f, then the *former* multiplied by $\varepsilon\varepsilon'/(\varepsilon+\varepsilon')$ gives the repulsive force $-dV/dr$, and hence it can be represented by the quotient $-[(\varepsilon+\varepsilon')/\varepsilon\varepsilon'] \cdot [dV/dr]$. Therefore one has

$$\frac{d^2r}{dt^2} = f - \frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{dV}{dr} .$$

Substituting this value for d^2r/dt^2 into the above equation, and setting $\rho = \pm 2[(\varepsilon+\varepsilon')/\varepsilon\varepsilon'] \cdot [ee'/c^2]$ according to Section 13.3, where the upper or lower sign depends on whether the product ee' is positive or negative, then one gets

$$-\frac{dV}{dr} = \frac{ee'}{r^2}\left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2} + \frac{2r}{c^2} f\right) \mp \frac{\rho}{r} \cdot \frac{dV}{dr} ,$$

and finally from here, we obtain the following expression for the *repulsive force*:

$$-\frac{dV}{dr} = \frac{ee'}{r(r \mp \rho)} \cdot \left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2} + \frac{2r}{c^2} f\right) ,$$

where the upper or lower sign depends on whether the product ee' is positive or negative. One can also write this expression in the form

$$-\frac{dV}{dr} = \frac{ee'}{r(r - \frac{ee'}{a})} \cdot \left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2} + \frac{2r}{c^2} f\right) .$$

The expression for the electric force in this form can now serve as a better overview of the forces that each of the two particles exert upon one another, but one has to bear in mind that in this form the forces cannot be composed according to the parallelogram law.[222,223] One can see that for positive values of the product ee' this force would be infinitely large, not only for $r = 0$, but also for $r = \rho$; but one can also see that in reality the case $r = \rho$ will never happen, because, no matter how large the relative velocity might be at any distance that differs from ρ by a finite value, then r will never be equal to ρ.

Since ee' is positive, the force for $r > \rho$ is *repulsive*, and it increases to *infinity* while r decreases towards ρ, from which it is clear that the approximate velocity, which although very large is not yet infinite, must be cancelled by this *repulsive force* increasing to infinity, before r reaches ρ, and that then r immediately starts increasing again. It follows from this that r never equals ρ, but that the two particles necessarily have to stay apart at a distance larger than ρ.

For $r < \rho$ the force is *attractive* and it increases with r to a value which approximates the limit ρ, *to infinity*, from which it is clear that the distant velocity, which although is very large is not yet infinite, must be neutralized by the *attracting force* increasing to infinity, before r reaches ρ, and that then r immediately starts decreasing again. Hence in this case the two particles always will stay apart by a distance smaller than ρ.

If a similar restriction of motion, like the one for two particles whereby they had to stay at a distance less that ρ once they were that close, should be encountered for a larger number of particles in a small region, such that all these particles were confined within that region, then such particles would together form a *molecule*, just as in the case of two, and in the same way and under appropriate circumstances, the particles outside this molecule could also be joined together into *molecules*. It is clear that all these molecules, must be separated from one another by gaps of at least the size ρ, and that they would repel one another. But further investigations would be needed to decide if, and under which circumstances, a *system of such molecules* could rest in a stable equilibrium, and, if such would be the case, according to which laws small perturbations of the equilibrium would propagate, in order to decide the question of whether the *light ether* and the *light waves* in space could not be based on and

[222][Note by WW:] Since the components of the acceleration, by forces of the kind whose potential depends on the velocities of the moving points, are given by expressions which contain the accelerations themselves in such a form that the values of the latter can only be obtained by resolving the equations, then according to Carl Neumann, one must observe that while *before* the resolution one may compose the expressions of the accelerations in the presence of simultaneous action of several forces according to the common rules, that the latter property gets lost after reformation of the expressions due to resolving the equations. Here accelerations getting infinite is characterized by the vanishing of the determinant formed from the coefficients of the accelerations in the individual equations. Cf. *Mathematische Annalen*, Vol. 11, p. 323, Note.

[223][Note by AKTA:] [Neu77, Note on p. 323].

explained by a stable aggregate state of such *molecules* distributed in the celestial space and composed of electric particles.

It is common to refer to that force which is exerted by two electric particles e and e' at distance r from one another, when they are at *relative rest*, as the *electrostatic force* and determine it according to the *electrostatic law*, namely $= ee'/r^2$. But two particles are at *relative rest*, only if their relative velocity $dr/dt = 0$ vanishes. But it now follows from the above obtained *general law* for the repulsive force acting between two electric particles, that the magnitude of the force is given not by $= ee'/r^2$, but by

$$= \frac{ee'}{r(r \mp \rho)} \cdot \left(1 + \frac{2r}{c^2} f\right) ,$$

where f is that part of its relative acceleration which is *independent* of the *interaction* between the two particles, i.e. the sum of that acceleration $= \alpha^2/r$ which arises, firstly, from the relative velocity α between the particles in some direction orthogonal to r and, secondly, from that acceleration $= [(\varepsilon + \varepsilon')/\varepsilon\varepsilon'] \, \Delta$ which arises from the difference Δ of the *external* forces decomposed according to r and exerted on the two particles e and e', where ε and ε' are the masses of the two particles.

But even in the case where both particles are at relative rest, and also the part f of their acceleration that is *independent* of their interaction is equal to zero, their repulsive force still turns out different from the value ee'/r^2 determined by the *electrostatic law*; according to the above *general law* the repulsive force is $= [ee'/r(r \mp \rho)]$.

In order to get the value ee'/r^2 determined by the *static law*, in accordance with this *general law*, the part of the acceleration denoted by f must not be $= 0$, but must be opposite to the other part that depends on the interaction of the two particles, namely equal to $[(\varepsilon + \varepsilon')/\varepsilon\varepsilon'] \cdot [ee'/r^2]$, i.e.

$$f = -\frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{r^2} = \mp \frac{\rho c^2}{2r^2} .$$

With this value of f one finds according to the *general law*, in the case $dr/dt = 0$, the magnitude of the repulsive force:

$$-\frac{dV}{dr} = \frac{ee'}{r(r \mp \rho)} \cdot \left(1 + \frac{2r}{c^2} f\right) = \frac{ee'}{r(r \mp \rho)} \cdot \left(1 \mp \frac{\rho}{r}\right) = \frac{ee'}{r^2} ,$$

that is, equal to the value determined by the *electrostatic law*. Therefore a real static equilibrium between two particles at relative rest only happens, if the acceleration resulting from their interaction gets neutralized by the acceleration that is independent of their interaction.

13.6 Laws of Motion for Two Electric Particles Impelled Only by Their Action on Each Other

The laws of motion for two electric particles impelled only by their action on each other, which were already developed in the *Electrodynamic Measurements*, Vol. X of these *Abhandlungen*,[224,225] shall here only be considered in detail and represented graphically for the case

[224] [Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 268.

[225] [Note by AKTA:] [Web71, Section 8, p. 268 of Weber's *Werke*] with English translation in [Web72, Section 8, p. 119]. See Section 9.8 on page 85 of Chapter 9.

in which these particles have no relative motion orthogonal to their connecting segment, since this serves to refute erroneous conclusions drawn from the fundamental law.

According to Section 13.3 the *general potential* of two electric particles e and e' at a distance r from one another was

$$V = \frac{ee'}{r}\left(1 - \frac{1}{c^2} \cdot \frac{dr^2}{dt^2}\right) ,$$

where the *repulsive force* between the two particles, as mentioned in the previous Section, was represented by $-dV/dr$; but if this repulsive force is to be represented by $+dV/dr$, then the potential

$$V = \frac{ee'}{r}\left(\frac{dr^2}{c^2\,dt^2} - 1\right)$$

has to be set.

According to the latter, the acceleration of the particle e in the direction r is given by $= [1/\varepsilon] \cdot [dV/dr]$, and the acceleration of the particle e' in the opposite direction by $= [1/\varepsilon'] \cdot [dV/dr]$, where ε and ε' are the masses of the particles e and e', from which results the *relative acceleration* of the two particles,

$$\frac{d^2r}{dt^2} = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \cdot \frac{dV}{dr} .$$

Multiplying this by $2dr$, we obtain the differential equation

$$2\frac{dr}{dt} \cdot \frac{d^2r}{dt^2} = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \cdot \frac{dV}{dr}\, dr ,$$

and by integration from $r = r_0$ to $r = r$, where r_0 denotes the value of r for which the relative velocity $dr/dt = 0$, we get

$$\frac{d^2r}{dt^2} = 2\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon'}\right) \cdot \left[\frac{ee'}{r}\left(\frac{dr^2}{c^2\,dt^2} - 1\right) + \frac{ee'}{r_0}\right] ,$$

or, if we set $dr/c\,dt = u$ and $2([1/\varepsilon] + [1/\varepsilon']) = \pm\rho c^2/ee'$,

$$u^2 = \pm\rho\left(\frac{1}{r}\left(u^2 - 1\right) + \frac{1}{r_0}\right) ,$$

where the upper or lower sign depends on whether the product ee' is positive or negative.

Considering now the case where ee' is positive, and expressing the distances r and r_0 in terms of the constant ρ associated with the pair of particles above, one gets

$$u^2 = \frac{1}{r}(u^2 - 1) + \frac{1}{r_0} .$$

If for such a pair of particles the distance r and the velocity u are determined at a particular time, then from the above equation we obtain,

$$r_0 = \frac{r}{1 - (1 - r)u^2} .$$

When the particles in the pair are only moving as a result of their mutual interaction, then, at a distance r_0 where $u = 0$, from the equation

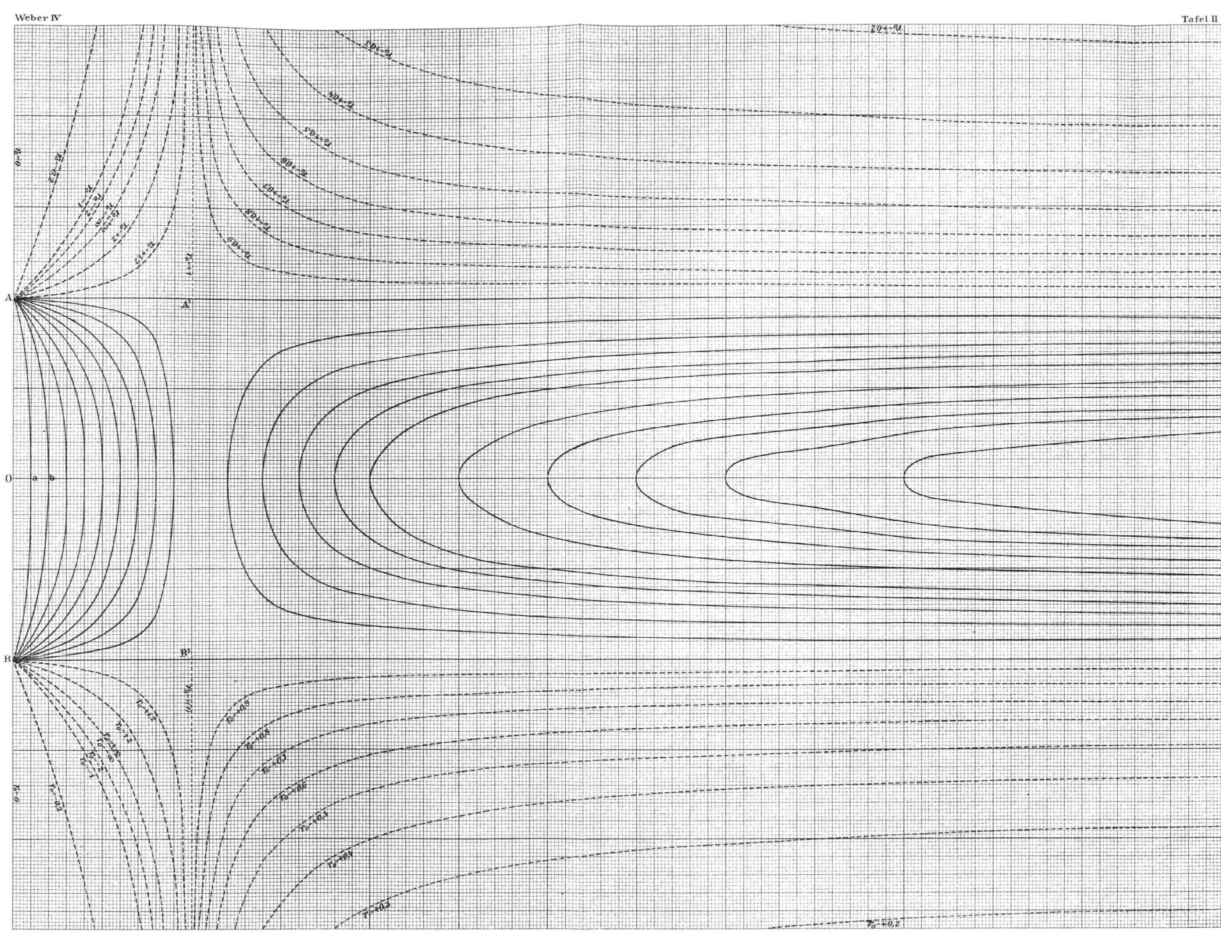
$$u^2 = \left(1 - \frac{r}{r_0}\right) \frac{1}{1 - r} \ ,$$

one can find all values of r and u for the given value of r_0 if one inserts for r any arbitrary series of increasing values from $r = 0$ to $r = \infty$.

Such a series of related values of r and u is graphically represented by a curve, whose abscissa and ordinate represent the related values of r and u.

But the value of r_0 for the same pair of particles can be very different at different times, if in the intermediate period, external influences occurred on top of their mutual interaction. For every other value of r_0, after eliminating any external influences, another series of relationships between r and u exists, which is represented graphically by another curve.

Hence we obtain a Table of related values of r and u for different values of r_0, and a corresponding system of curves, shown in the Figure.[226]



Here it is to be remarked that in this graphic representation, the values of r in the following Table are shown as abscissas associated to ordinates $\pm u$, more precisely those of $+u$ as *positive* and those of $-u$ as *negative*, in order to distinguish the *distance* of the particles from their *approach*. The system of curves corresponding to the *first* section of the Table

[226][Note by AKTA:] Ordinate $\pm u$ as function of abscissa r. A larger image appears on page 164.

fills the space $AA'B'B$, the one corresponding to the *second* section fills the space $A'A_0B_0B'$, which has to be extended to infinity on the side of A_0B_0.

Values of $u = \sqrt{\frac{1}{1-r}\left(1 - \frac{r}{r_0}\right)}$ for values of r and r_0 between 0 and 1:

$r_0 =$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$r = 0.0$	0.00	±1.00	±1.00	±1.00	±1.00	±1.00	±1.00	±1.00	±1.00	±1.00	±1.00
$r = 0.1$		0.00	±0.75	±0.86	±0.91	±0.95	±0.97	±0.98	±0.99	±0.99	±1.00
$r = 0.2$			0.00	±0.65	±0.79	±0.86	±0.91	±0.94	±0.97	±0.98	±1.00
$r = 0.3$				0.00	±0.60	±0.75	±0.84	±0.90	±0.94	±0.97	±1.00
$r = 0.4$					0.00	±0.57	±0.75	±0.84	±0.91	±0.96	±1.00
$r = 0.5$						0.00	±0.57	±0.75	±0.86	±0.94	±1.00
$r = 0.6$							0.00	±0.60	±0.79	±0.91	±1.00
$r = 0.7$								0.00	±0.65	±0.86	±1.00
$r = 0.8$									0.00	±0.75	±1.00
$r = 0.9$										0.00	±1.00
$r = 1.0$											±0/0

Values of $u = \sqrt{\frac{1}{1-r}\left(1 - \frac{r}{r_0}\right)}$ for values of r and r_0 between 1 and ∞:

$r_0 =$	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0	5.0	6.0	∞
$r = 1.0$	±0/0												
$r = 1.2$	±1.00	0.00											
$r = 1.4$	±1.00	±0.65	0.00										
$r = 1.6$	±1.00	±0.75	±0.49	0.00									
$r = 1.8$	±1.00	±0.79	±0.60	±0.40	0.00								
$r = 2.0$	±1.00	±0.82	±0.65	±0.50	±0.33	0.00							
$r = 2.5$	±1.00	±0.85	±0.72	±0.61	±0.51	±0.41	0.00						
$r = 3.0$	±1.00	±0.86	±0.75	±0.66	±0.57	±0.50	±0.32	0.00					
$r = 3.5$	±1.00	±0.88	±0.77	±0.68	±0.62	±0.55	±0.40	±0.26	0.00				
$r = 4.0$	±1.00	±0.88	±0.79	±0.71	±0.64	±0.57	±0.45	±0.33	±0.22	0.00			
$r = 5.0$	±1.00	±0.89	±0.80	±0.73	±0.66	±0.61	±0.50	±0.41	±0.33	±0.24	0.00		
$r = 6.0$	±1.00	±0.89	±0.81	±0.74	±0.68	±0.63	±0.53	±0.45	±0.37	±0.31	±0.20	0.00	
∞	±1.00	±0.91	±0.84	±0.79	±0.74	±0.71	±0.63	±0.57	±0.54	±0.50	±0.45	±0.41	0.00

The graphic representation of these numerical values in the Figure now gives a clear insight into the meaning of the result that arises from the formula, namely that the mutual acceleration of two particles *is infinite at the so-called critical distance* ρ. This raised diverse concerns against the general law of electric force that underlies the formula.

One can indeed see from the graphic representation, that at distance ρ simultaneously with the infinite acceleration, there occurs a jump of the relative velocity of both particles from $-c$ to $+c$, or the other way around, which occurs so abruptly that the distance ρ does not change at all.

By becoming *infinite*, the acceleration changes sign, and as a consequence the velocity changes, at the same moment and without any loss of time, from $-c$ to $+c$, or vice versa. Before the distance ρ can experience even the smallest finite change, the transition of the velocity c to the opposite has already happened.

The formula above tells us that in principle, there is an abrupt reflection of the particles from one another at the moment when they get to the distance ρ, just like in the case of the formula in mechanics for two colliding *elastic* balls, which also reflect from one another, with the reflection being the more abrupt, the smaller the balls and the larger their elasticity coefficient. *Instantaneous* reflection is the *limiting case*, which in reality never happens, but which to date is considered to be neither odd nor absurd according to the principles of mechanics. Finally, special attention should be drawn to the fact, that according to the formula, for the case of $r = \rho$, the acceleration would be infinite, but that the case $r = \rho$ never *really happens* just as an elastic body with infinite elasticity coefficient never really exists.

Further, one can see that the totality of all curves representable according to the mentioned formula, forms two groups completely separated from one another, namely, (1), a group in which all mutual distances of the particles are smaller than ρ and, (2), a group in which they are larger than ρ. The two groups differ from one another in that, as shown above, the case in which the distance would be $= \rho$ neither really ever occurs, nor can it occur.

Both groups together cover the whole range over all abscissa values from $r = 0$ to $r = \infty$ and for all ordinate values from $u = -c$ to $u = +c$, and none of these curves allows for an extension beyond the limits of this space. From this it follows that two electric particles, which are impelled only by mutual interaction and whose relative velocity is never larger than $+c$ and never smaller than $-c$, stay inside the mentioned limits.[227]

13.7 Electric Rays, Especially Reflection and Scattering of Rays

The motion of two electric particles impelled only by mutual interaction, moving relative to one another both along their connecting line and orthogonal to it, were considered in the *Electrodynamic Measurements*, Vol. X of these *Abhandlungen*, and for their determination the following equations have been found:[228,229]

$$\frac{u^2}{c^2} = \frac{r - r_0}{r - \rho}\left(\frac{\rho}{r_0} + \frac{r + r_0}{r} \cdot \frac{{\alpha_0}^2}{c^2}\right) , \qquad (1)$$

$$r\alpha = r_0\alpha_0 , \qquad (2)$$

where r is the separation distance between the two particles, and u and α are their relative velocities in the direction of r and orthogonal to it; furthermore r_0 denotes the value of r for

[227][Note by WW:] It would be different for the case that was excluded by the definition of ability to do work, namely that two electric particles would possess already initially a velocity $> +c$ or $< -c$, or if the two particles would move not only by mutual interaction, but in addition impelled by an *external* influence, and thereby would have acquired such a velocity, either $> +c$ or $< -c$. Suppose such a case should really happen, then the motions of two such particles, if they would be impelled only by mutual interaction from this very moment on, would be represented by completely different curves which would be excluded from the region of the system of curves considered above. All these other curves would form a closed system which would fill up the whole of space. All curves of this second kind are represented in the Figure by dotted lines. Also these types of motion, or their representative curves, decompose into two groups separated from each other at the spot determined by the critical distance ρ. Namely, one group in which all distances of the particles are always smaller than ρ, and another group in which they are larger than ρ. Moreover, again there is a *complete symmetry* between the curve arms with ordinates $> +c$ and $< -c$. Furthermore, at $r = \rho$ both curve arms are connected with one another as a consequence of the abrupt change of the velocity from $\pm\infty$ to $\mp\infty$.

[228][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 273.

[229][Note by AKTA:] [Web71, p. 273 of Weber's *Werke*] with English translation in [Web72, p. 124]. See Section 9.11 of Chapter 9.

which $u = 0$, α_0 the value of α, for which $r = r_0$, and finally ρ is a constant that depends on the nature and the masses ε and ε' of the two particles e and e', namely

$$\rho = 2\frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{c^2} \ ,$$

where ρ is positive or negative as like in the case of the product ee'. — If ρ shall refer to the distance between two particles, and which can *only be positive*, like r and r_0, then one has to set

$$\rho = \pm 2\frac{\varepsilon + \varepsilon'}{\varepsilon\varepsilon'} \cdot \frac{ee'}{c^2} \ ,$$

where the upper or lower signs depend on whether the product ee' is positive or negative. Then for Equation (1) one has to set

$$\frac{u^2}{c^2} = \frac{r - r_0}{r \mp \rho}\left(\frac{r + r_0}{r} \cdot \frac{{\alpha_0}^2}{c^2} \pm \frac{\rho}{r_0}\right) \ ,$$

with the same determination for the signs. — Since hereinafter only electric particles of the *same type* will be considered,[230] it will always be the upper signs that will be valid. — The equation $r\alpha = r_0\alpha_0$ tells us that $\alpha = 0$ for $r = \infty$ whenever r_0 and α_0 are given and finite, and this is related to the existence of a straight asymptote with which the trajectory coincides at infinity.

Now we shall consider the case of two *equal* electric particles e and e' approaching each other with a large velocity u which is decreasing as a consequence of mutual repulsion, and where the largest value of u, namely for $r = \infty$, shall be denoted by u_0. For simplicity e shall be considered as stationary during this relative motion. Suppose that along the trajectory of e' and with the same velocity relative to e, there is a whole sequence of *equal* particles e'', e''', ... that follow e', and moreover suppose they follow in such intervals that the mutual perturbations can be neglected.

From the law represented by the above Equation (1) there results the value of u for $r = \infty$, namely

$$u_0 = c\sqrt{\frac{{\alpha_0}^2}{c^2} + \frac{\rho}{r_0}} \ . \qquad (3)$$

Since ρ is now equal for equal particles, and also u_0 has been assumed to be equal, a difference can only occur with respect to the value of r_0 and the resulting value of α_0 according to Equation (3).

The system of all these particles is called an *electric ray*,[231] and the asymptote, in which the particles are located when they are very far away from e, serves to determine the *direction* of the ray.

If for all particles one had $r_0 = \rho[c^2/{u_0}^2]$, from which it would follow $\alpha_0 = 0$, then they all would move along the same line up to a distance r_0, from which they would all return again along the same line. But if α_0 is non-zero and simultaneously with r_0, different for all particles e', e'', ..., for which u_0 is non-zero, but very small for all of them, then each particle will deviate from each asymptote as r approaches r_0. The angle which is then formed by

[230] [Note by AKTA:] That is, both positive or both negative.

[231] [Note by AKTA:] In German: *Elektrischer Strahl*. This expression can also be translated as "electric beam" or a "beam of electrified particles".

the line, e.g. ee', as a consequence of the deviation from the direction of the ray, shall be denoted by φ. Set $\varphi = \varphi_0$ if the decreasing distance between $e'e$ becomes equal to r_0, where in fact the velocity in the direction e' to e is equal to zero and in the orthogonal direction it is equal to α_0.

From that moment on, when the distance becomes $r = r_0$, the two particles e and e' start moving away from each other and their connecting line approaches another line which, with the direction that ee' had when it became equal to r_0, forms an angle $= \varphi_0$. And with the direction of the original ray, it forms the angle equals $2\varphi_0$, which shall be called the *angle of reflection*. But this angle of reflection is quite different for the different pairs of particles ee', ee'', ..., which belong to the same ray, according to the difference of the values of α_0 or r_0/ρ, which tells us that such a reflected ray also gets *scattered* simultaneously. This *scattering* of electric rays shall now be determined in more detail according to the previous laws.

To begin with, from the above law (2), this results in the growth of the angle φ, namely

$$d\varphi = \frac{\alpha\, dt}{r} = \frac{\alpha_0 r_0}{r^2}\, dt \,. \qquad (4)$$

Furthermore, if we substitute into Equation (1) the value of ${\alpha_0}^2 = {u_0}^2 - [\rho/r_0]\, c^2$ resulting from Equation (3), then one gets

$$\frac{u^2}{c^2} = \frac{dr^2}{c^2\, dt^2} = \frac{r - r_0}{r - \rho}\left(\frac{r + r_0}{r} \cdot \frac{{u_0}^2}{c^2} - \frac{\rho}{r}\right) \,, \qquad (5)$$

consequently, if r decreases with increasing t,

$$dt = -dr\sqrt{\frac{r(r-\rho)}{(r - r_0)({u_0}^2(r + r_0) - \rho c^2)}} \,. \qquad (6)$$

From this it follows that

$$d\varphi = \frac{\alpha_0 r_0}{r^2}\, dt = -\frac{\alpha_0 r_0}{u_0} \cdot \frac{dr\sqrt{r^2 - r\rho}}{r^2\sqrt{r^2 - \frac{c^2}{{u_0}^2}\,\rho r - \left({r_0}^2 - \frac{c^2}{{u_0}^2}\,\rho r_0\right)}} \,,$$

or, if one sets $1/r = s$,

$$d\varphi = +\frac{\alpha_0 r_0}{u_0} \cdot ds\sqrt{\frac{1 - \rho s}{1 - \frac{c^2}{{u_0}^2}\,\rho s - \left({r_0}^2 - \frac{c^2}{{u_0}^2}\,\rho r_0\right) s^2}} \,, \qquad (7)$$

from which one sees that φ can be represented by *elliptic functions*.

If one now restricts attention to those cases where the value of ${\alpha_0}^2$ equals ${u_0}^2 - [\rho/r_0]\, c^2$, where the value of $({r_0}^2 - [c^2/{u_0}^2]\,\rho r_0)s^2$, *either* vanishes completely or is yet very small, then the above equation reduces in the *former* case to

$$d\varphi = +\frac{\alpha_0 r_0}{u_0} \cdot ds\sqrt{\frac{1 - \rho s}{1 - r_0 s}} \,, \qquad (8)$$

and in the *latter* case, where α_0 is supposed to be very small, although not vanishing completely, we set $r_0\,[{\alpha_0}^2/{u_0}^2] = r_0 - [c^2/{u_0}^2]\,\rho = \beta$. And if β is so small that in Equation (7), which via introduction of β turns into

$$d\varphi = \frac{\alpha_0 r_0}{u_0} \cdot ds \sqrt{\frac{1-\rho s}{(1-r_0 s)(1+\beta s)}} \ ,$$

one can write $(1-\frac{1}{2}\beta s)$ in place of the factor $\sqrt{1/(1+\beta s)}$, then Equation (7) turns into

$$d\varphi = \frac{\alpha_0 r_0}{u_0} \cdot \left(1 - \frac{1}{2}\beta s\right) ds \sqrt{\frac{1-\rho s}{1-r_0 s}} \ , \tag{9}$$

from where, if one sets $S = 1-(\rho+r_0)s + \rho r_0 s^2$, it follows that

$$\int d\varphi = \frac{\alpha_0 r_0}{u_0}\left[\int \frac{ds}{\sqrt{S}} - \left(\frac{1}{2}\beta + \rho\right)\int \frac{s\,ds}{\sqrt{S}} + \frac{1}{2}\beta\rho \int \frac{s^2\,ds}{\sqrt{S}}\right] \ .$$

If one sets $b = -(\rho + r_0)$ and $c = \rho r_0$, then carrying out the integration one gets:[232]

$$\frac{u_0}{\alpha_0 r_0}\int d\varphi =$$

$$\left[1 + \frac{b}{4c}(\beta + 2\rho) + \frac{\beta\rho}{4c}\left(\frac{3b^2}{4c} - 1\right)\right]\frac{1}{\sqrt{c}} \cdot \log\left(\sqrt{S} + s\sqrt{c} + \frac{b}{2\sqrt{c}}\right)$$

$$-\frac{1}{c}\left(\rho + \frac{\beta}{2}\left(1 + \frac{3b}{4c}\rho\right) - \frac{\beta\rho}{4}s\right)\sqrt{S} \ ,$$

from which, setting $m = \rho/r_0$, $n = c/u_0$ and so $\alpha_0/u_0 = \sqrt{1-mn^2}$ according to Equation (3), one gets:

$$\varphi_0 = \int_{s=0}^{s=\frac{1}{r_0}} d\varphi =$$

$$\sqrt{1-mn^2} \cdot \left[\frac{1-m}{2}\left(1 - \frac{1+3m}{8m}(1-mn^2)\right)\sqrt{\frac{1}{m}} \cdot \log\frac{1+\sqrt{m}}{1-\sqrt{m}}\right.$$

$$\left. + 1 + \frac{1-3m}{8m}(1-mn^2)\right] \ . \tag{10}$$

Based on this the following Table of values of φ_0, for different values of m and n, has been calculated:[233]

[232][Note by WW:] Namely

$$\begin{aligned}
\int \frac{ds}{\sqrt{S}} &= \frac{1}{\sqrt{c}} \cdot \log\left(\sqrt{S} + s\sqrt{c} + \frac{b}{2\sqrt{c}}\right) \ , \\
\int \frac{s\,ds}{\sqrt{S}} &= -\frac{b}{2c\sqrt{c}} \cdot \log\left(\sqrt{S} + s\sqrt{c} + \frac{b}{2\sqrt{c}}\right) + \frac{\sqrt{S}}{c} \ , \\
\int \frac{s^2\,ds}{\sqrt{S}} &= \frac{3b^2-4c}{8c^2\sqrt{c}} \cdot \log\left(\sqrt{S} + s\sqrt{c} + \frac{b}{2\sqrt{c}}\right) + \frac{1}{2c}\left(s - \frac{3b}{2c}\right)\sqrt{s} \ .
\end{aligned}$$

[233][Note by HW:] The itemized values 1.2500 in the original Memoir for $m = 0$, $n = 1$ and $n = 2$ have later been changed by W. Weber to the ones shown in the Table.

	$n = 1$	$n = 2$
$m = 1$	0	
$m = 1/2$	0.9658	
$m = 1/3$	1.1269	
$m = 1/4$	1.1479	0
$m = 1/5$	1.2272	0.7776
$m = 1/6$	1.2486	0.9688
$m = 1/7$	1.2629	1.0690
$m = 1/8$	1.2732	1.1302
$m = 0$	1.3750 (see footnote 233)	1.3750 (see footnote 233)

From here it follows that for all particles of an electric ray e', e'', ... which *approach* the particle e from a large distance, with velocity u_0, that once they have reached the distance r_0 they turn around and move away again from e with a velocity which again increases back up to u_0 again. However, the two directions, in which the two particles first approached each other with velocity u_0 and then turned backward, form an angle $2\varphi_0$ which for the different pairs is very different according to the difference of the value of r_0.

The diversity of the angle $2\varphi_0$, which is called the *angle of reflection*, for the different pairs of particles according to the different values of r_0, forms the phenomenon which is given the name *scattering* of electric rays by reflection.[234] Moreover, the law of dependence of the angle of reflection $2\varphi_0$ on m and n, which was discovered earlier, gives a precise determination of this scattering, if one takes into account that n has the same value for all particles of the same ray, which depends on u_0 according to the equation $n = c/u_0$. Furthermore, that for each pair of particles, m can be determined for any distance r — using the three equations $m = \rho/r_0$, ${\alpha_0}^2 = {u_0}^2 - [\rho/r_0]\, c^2$, and $\alpha_0 r_0 = \alpha r$, after the elimination of r_0 and α_0 — from the relative velocity α of the two particles in the direction orthogonal to their connecting line, namely via the equation:

$$m^2 + \frac{\rho^2 c^2}{r^2 \alpha^2} m = \frac{\rho^2 {u_0}^2}{r^2 \alpha^2} \ .$$

13.8 Application of the Theory of Reflection and Scattering of Electric Rays to the Light Ether and Gases According to the Theory of Molecular Collisions Due to Krönig and Clausius

The reflection and scattering of electric rays consisting of pairs of like electric particles[235] which approach and move away from each other in empty space with equal velocity, leads to a similar aggregate state of the whole system of particles as that ascribed to gases in the theory of Krönig and Clausius,[236] with the simple difference that the particles of the gases in ballistic motion[237] are ponderable particles, while the electric particles are usually called

[234][Note by AKTA:] In German: *Zerstreuung elektrischer Strahlen durch Reflexion.*

[235][Note by AKTA:] In German: *Elektrisch gleichartigen Theilchenpaaren.* Each pair of particles is composed of charges of the same sign.

[236][Note by AKTA:] [Krö56], [Cla57b] with English translation in [Cla57a], and [Cla79].

[237][Note by AKTA:] In German: *Wurfbewegung.* This expression can also be translated as "throwing motion" or "in a state of motion arising from collision".

imponderable, because the applicability of the law of gravitation to them has, until now at least, not been proven. Only according to Mossotti's theory of gravitation (see Zöllner, *Wissenschaftliche Abhandlungen*, Vol. 1, No. 2, Leipzig 1878),[238] wherein all gravitational forces are a result of electrical repulsions and attractions, would all interactions, for both ponderable masses as well as electric particles, fall under a common determination, in which each ponderable particle hereafter would be an *electric double particle* (like a double star), namely a positive and negative electric particle which would orbit each other.

It is a natural result of Mossotti's representation, that when these ponderable particles are found in empty space in ballistic motion, as is assumed to be the case for gases in the theory of Krönig and Clausius, the laws of electrical interaction would result in similar laws for the reflection and scattering of these ponderable particles moving in empty space as was found in the previous Section for moving electric particles of like charge. This is easily recognized, when one observes that those laws apply particularly for pairs of particles of like charge which approach each other, as a result of their relative motion, up to a distance r_0, greater than ρ. After all, two ponderable molecules contain two pairs of electric particles of the same sign,[239] and for each of these pairs there is a distance ρ which the particles of the pair cannot reach, because their repulsive force would become infinite, which is only prevented by the fact that the ever-increasing repulsive force will bring both particles to a standstill (well before they reach the distance ρ), whereby, due to this persisting repulsive force that results from their interaction, they again begin to distance themselves from one another, as they earlier had come together.

This permits us to carry over the laws of reflection and scattering for the rays of like electric particles, found in the previous Section, to the rays of ponderable particles brought together according to Mossotti's representation. And if these ponderable molecules are now the molecules of a gas, an aggregate state of the gas will form which wholly corresponds to the aggregate state ascribed to the gas in the theory of Krönig and Clausius — except that there is no need to ascribe a special form and elasticity to these ponderable gas molecules, as did Krönig, nor must one ascribe special repulsive forces inversely proportional to a higher power of distance, as did Clausius and Maxwell.[240]

However, if there exists a space, e.g. outer space,[241] wherein no ponderable molecules are found, then the possibility arises that this space might contain one of the two constituent parts of these ponderable molecules, that is either the positive or negative electric particle, which would likewise form in their motions of collision a body of a specific aggregate state. However, because it only consists of like electric particles, it should not be called a ponderable body, but rather an imponderable *ether*, to which the laws of motion for *dynamic media* developed by Maxwell (Philos. Transact. 1867),[242] namely the laws for a *wave propagation*

[238][Note by AKTA:] Ottaviano-Fabrizio Mossotti (1791-1863). See [Mos36] with English translation in [Mos66]; [Zöl78] and [Zöl82]. Weber wrote Mossotti's name as Mosotti. I corrected this misprint.

[239][Note by AKTA:] Each ponderable molecule would be composed of two particles orbiting around one another. These particles would have opposite electric charges of the same magnitude. See [Web94b, Section 1] with English translation in [Web08, Section 1], see also Section 15.1 of Chapter 15.

Ponderable molecule 1 has two particles with charges $q_1 > 0$ and $-q_1 < 0$. Ponderable molecule 2 has two particles with charges $q_2 > 0$ and $-q_2 < 0$. These two ponderable molecules contain two pair of particles with charges of the same sign, namely, $(q_1,\ q_2)$ and $(-q_1,\ -q_2)$.

[240][Note by AKTA:] See footnote 236 on page 148. See also [Max67] and [Max65].

[241][Note by AKTA:] In German: *den Weltraum*. This expression can also be translated as the universe, deep space or the cosmos.

[242][Note by AKTA:] See footnote 240.

in accordance with the laws for the propagation of light waves, would likewise apply. The idea of a space-filling medium, composed of mutually repelling particles, would only seem to be possible in the absence of fixed spatial borders, under a hypothesis whereby the medium would extend all the way to infinity. It would seem however that restricting such a medium to a finite space without fixed borders is indeed possible according to Mossotti, because this medium surrounds a ponderable body of Mossotti's type, which would exert a force of attraction on the medium thereby holding it together.

13.9 Laws of Motion for Two Electric Particles Impelled by Mutual Interaction and *External* Influence

Only the simple case will be considered where the *external* influence on particle e consists of a constant force in the direction of the prolonged line $e'e$, which divided by the sum $\varepsilon + m$ of the own mass of the particle e and the ponderable mass tightly connected to it, provides the quotient g.[243] — Let the *external* influence on the other particle e' consist of a force which is *equal and opposite* to the force that acts on e' as a result of the *mutual interaction* of e and e'.

According to Section 13.5 the *potential* V of the two particles e and e' at distance r — where V is that function whose differential quotient dV/dr represents the *repulsive force* — is given by

$$V = \frac{ee'}{r}\left(\frac{dr^2}{c^2 \cdot dt^2} - 1\right) .$$

From this now follows the *acceleration due to the interaction* of the particle e with respect to r, namely $= [1/(\varepsilon + m)] \cdot [dV/dr]$, and that of the particle e' in the opposite direction, namely $= [1/\varepsilon'] \cdot [dV/dr]$, from where results the relative acceleration of the two particles:

$$= \left(\frac{1}{\varepsilon + m} + \frac{1}{\varepsilon'}\right) \cdot \frac{dV}{dr} ,$$

due to their mutual interaction. In addition to this, the acceleration *due to external influence* is to be taken into account. For e, this is $= g$ in the direction r, and for e' it is $= [1/\varepsilon'] \cdot [dV/dr]$ in the same direction, hence it follows that the *relative* acceleration of the two particles *due to external influence*:

$$= g - \frac{1}{\varepsilon'} \cdot \frac{dV}{dr} .$$

And hence the total relative acceleration is:

$$\frac{d^2r}{dt^2} = \frac{1}{\varepsilon + m} \cdot \frac{dV}{dr} + g .$$

Multiplying this equation by $2dr$ one obtains

$$2\frac{dr}{dt} \cdot \frac{d^2r}{dt^2} = \frac{2}{\varepsilon + m} \cdot \frac{dV}{dr} + 2g\, dr ,$$

[243][Note by AKTA:] Consider a particle with charge e and mass ε connected to a ponderable mass m. If there is a constant force F acting on this system, then according to Newton's second law of motion this system will move with acceleration g relative to an inertial frame of reference given by $g = F/(\varepsilon + m)$.

and from here by integration from $r = r_0$ to $r = r$, where r_0 denotes the value of r at the time where $dr/dt = 0$, one gets:

$$\frac{dr^2}{dt^2} = \frac{2}{\varepsilon + m}\left[\frac{ee'}{r}\left(\frac{dr^2}{c^2 \cdot dt^2} - 1\right) + \frac{ee'}{r_0}\right] + 2g(r - r_0) \ .$$

Denoting $dr/c\,dt$ by u and taking into account that $\pm[ee'/\rho] = a = \frac{1}{2}[\varepsilon\varepsilon'/(\varepsilon + \varepsilon')]c^2$, then one gets

$$u^2 = \pm\frac{\varepsilon\varepsilon'\rho}{(\varepsilon + m)(\varepsilon + \varepsilon')}\left(\frac{1}{r_0} - \frac{1}{r} + \frac{1}{r}\,u^2\right) + \frac{2g}{c^2}(r - r_0) \ ,$$

and from here

$$u^2 = \frac{\pm\frac{\varepsilon\varepsilon'\rho}{(\varepsilon+m)(\varepsilon+\varepsilon')}\left(\frac{1}{r_0} - \frac{1}{r}\right) + \frac{2g}{c^2}(r - r_0)}{1 \mp \frac{\varepsilon\varepsilon'\rho}{(\varepsilon+m)(\varepsilon+\varepsilon')} \cdot \frac{1}{r}} \ .$$

Setting now $\varepsilon\varepsilon' \cdot [\rho/(\varepsilon + m)(\varepsilon + \varepsilon')] = \rho'$, then it follows that:

$$\text{for positive values of } ee', \qquad u^2 = \frac{\rho'}{\rho' - r}\left(1 - \frac{r}{r_0}\right)\left(1 + \frac{2g}{c^2} \cdot \frac{rr_0}{\rho'}\right) \ , \tag{1}$$

$$\text{for negative values of } ee', \qquad u^2 = \frac{\rho'}{\rho' + r}\left(1 - \frac{r}{r_0}\right)\left(1 - \frac{2g}{c^2} \cdot \frac{rr_0}{\rho'}\right) \ , \tag{2}$$

or, if one expresses r and r_0 as parts of ρ':

$$\text{for positive values of } ee', \qquad u^2 = \frac{1}{1 - r}\left(1 - \frac{r}{r_0}\right)\left(1 + \frac{2g\rho'}{c^2} \cdot rr_0\right) \ , \tag{3}$$

$$\text{for negative values of } ee', \qquad u^2 = \frac{1}{1 + r}\left(1 - \frac{r}{r_0}\right)\left(1 - \frac{2g\rho'}{c^2} \cdot rr_0\right) \ . \tag{4}$$

When one sets $g = 0$, we obtain the equations found in Section 13.6 for two particles impelled only *through mutual interaction*. If, in contrast, one sets $\varepsilon = 0$ or $\varepsilon' = 0$, in which case $\rho' = 0$, then one obtains from (1) and (2) the relationship

$$u^2c^2 = \frac{dr^2}{dt^2} = 2g(r - r_0) \ ,$$

i.e. the *law of the free fall*, where $(r - r_0)$ denotes the *fall space*.

From the positive values of ee' in the case of pairs of electric particles impelled to motion *only by mutual interaction* (Figure on page 142), we can obtain the *graphic* representation for the motion of a pair of particles that is impelled by *both mutual interaction and external influence*. One only needs to enlarge by the ratio $1 \colon \sqrt{1 + [2g\rho'/c^2]\,r_0 r}$ in the Figure, all ordinates $\pm u$ of any of the curves determined by a certain value of the constant, with unchanged abscissas r, in order to get the particular curve which represents the motion of the pair of particles under the given *external* influences, where the only thing to be noted, is that r_0 and r are represented as parts of ρ', instead of ρ, and that $\rho' \colon \rho = [\varepsilon\varepsilon'/(\varepsilon+\varepsilon')] \colon \varepsilon + m$, i.e. it behaves nearly equal to $\varepsilon \colon m$ for small values of ε.

One can see from this that, even with *external influence* as mentioned above, for a *certain distance* ρ' between similar particles, their relative acceleration due to mutual interaction is infinite according to the formula, but that the distance ρ' could never occur for the reason mentioned in the former Section, which is also valid here. Indeed, should the distance ρ' ever actually occur, then the particles would have to be *approaching* each other or *moving apart*. If there is a *repulsion* during approach and an *attraction* during departure, and if this repulsion and this attraction grow according to the mentioned law in such a way that it would become infinite for the distance ρ', then in none of the two cases will the particles reach the distance ρ', but are forced to stop and to return before they get there. This is the situation according to our law and the reason why the case of infinite acceleration can never really happen according to this law should be clear.

Hence nothing discussed in this Section regarding the combination of *mutual interaction* and *external* influence is in contradiction with the established law. Therefore it is not necessary, in order to defend this law in view of the case of infinite acceleration in the molecular distances ρ or ρ', to seek refuge in the hypothesis that, as ρ and ρ' are molecular distances, *special molecular forces* could yet come into consideration.

It is important to note that the distances ρ and ρ' will always remain molecular distances, because, although ρ can be increased by increasing the mutually interacting electric masses, at least one of the two mutually interacting electric masses will be bound to a *ponderable* mass, which must be moved on with it, as in the case just considered, whereby a *reduction* of ρ takes place, in the ratio of the total mass $\varepsilon + m$ to the electric [mass] ε, where m denotes a ponderable mass in relation to which ε vanishes.

The case considered in the present Section of two electric particles impelled by both mutual interaction and *external* influence is the one, to which the objection raised by Helmholtz refers to. This objection was subjected to a closer inspection by Neumann in his Memoir, page 91 ff. of this Volume.[244]

On page 92, in relation to what Helmholtz considered to be the "absurd result of infinite acceleration" at the so-called critical distance,[245] Neumann adopted the alternative argument that my electric fundamental law would merely require (similar to what happens with Newton's law) a certain modification for extraordinary small distances. Meanwhile he raised the argument that this case of infinite acceleration could be arranged such that only *large* distances come into consideration which would exclude a modification of the result for molecular forces.

Neumann remarks that for these cases, in which only large values of critical distances come into consideration, neither *reality* nor *feasibility* have been proved, without which these cases can not be used as a test of a physical law, and that Helmholtz' objection would not *be taken seriously* until that proof had been provided.

In contrast, in the present and in the previous Section, proof has been provided, that the *possibility of the case* in which according to Helmholtz the "absurd result of infinite acceleration" would occur, will be excluded *by means of the fact* that the two particles, before they can get to the critical distance, must have approached each other beforehand, either from *smaller* or from *larger* distance. But because of the *backwards* acceleration increasing to infinity upon the particles getting closer, i.e. deceleration, which happens when the particles

[244][Note by AKTA:] [Neu74, p. 91 and the following pages].

[245][Note by WW:] An *infinite acceleration* occurs frequently when considering colliding bodies and is not considered absurd in mechanics, but as a *limiting case* of growing elasticity. If this *limiting case* never happens, then the same is valid in our present case, as will now be shown.

approach both from *smaller* and from *larger* distance, they *can never reach* the critical distance. From this it follows, that the "absurd result of infinite accelerations" criticized by Helmholtz *does not exist at all*, and that only an error committed by Helmholtz and not yet contradicted has led to this. — The magnitude of the critical distance is completely irrelevant here. —

However, in the next two Sections, some cases will be discussed in which it was believed that a significant enlargement of the so-called critical distance could be obtained, and these cases have attracted special interest through the related conclusions.

13.10 Laws of Motion for an Electric Particle Inside an *Electrified Spherical Shell* that is Impelled by Both Mutual Electrical Interaction and *External* Influence

On pages 103–106 of this Volume of these *Abhandlungen*,[246] C. Neumann has drawn attention to the following case:

> "Consider a fixed spherical shell (of radius α) uniformly covered with electricity. In the interior of this shell let there be a cylinder covered with electricity (of radius $= a$ and moment of inertia $= \mathfrak{M}$), rotatable about its firmly situated horizontal axis. Let a thread be wound around this cylinder at whose free end is fixed a weight Mg. — The goal is to examine more closely the motion achieved by the cylinder under the influence of the electrified spherical shell on the one hand, and under the influence of the weight Mg on the other hand.
>
> Hereby let us suppose that the cylinder is connected *rigidly* and *indissolubly* with its existing electric matter, and that the same is also the case for the spherical shell."[247]

[246][Note by AKTA:] [Neu74, pp. 103-106].

[247][Note by AKTA:] Carl Neumann presented the Figure shown in this footnote in order to describe this configuration, [Neu74, pp. 103-106]:

For this case, on page 106,[248] assuming the electric charges of the cylinder and the spherical shell to be *constant*, Neumann arrived at the following equation:

$$L\vartheta'^2 = Mga\vartheta + \text{constant} \ ,$$

or, differentiated with respect to t,

$$2L\vartheta'' = Mga \ ,$$

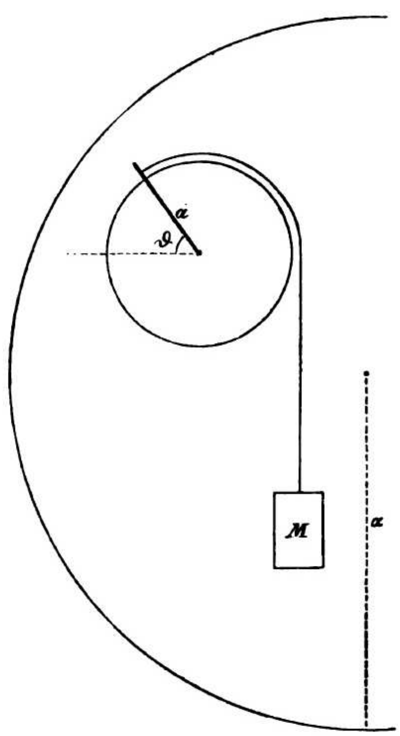
where ϑ denotes the rotation angle, ϑ' the rotational velocity, and ϑ'' the rotational acceleration of the cylinder, and, when H denotes the [surface] density of the electricity on the spherical shell and Σe the charge of the cylinder surface, one sets

$$L = \frac{Ma^2 + \mathfrak{M}}{2} - \frac{4\pi\alpha H \cdot a^2 \Sigma e}{3c^2}$$

and supposes that it is *constant*.

Neumann linked to this the following statement:

> "If the *constant* $L =$ positive, then the attached weight Mg will *sink* with accelerated velocity. If $L = 0$ there arises an infinitely large acceleration. If L is negative, any weight will be lifted up with accelerated velocity. In this last case the weight could, if one supposes an infinitely long thread, be lifted infinitely high, hence an infinitely large work will be done.
>
> However, if one investigates whether or not the cases $L = 0$ and L being negative can really occur, then one encounters the same difficulties as earlier." — —



[248][Note by AKTA:] [Neu74, p. 106].

Of these difficulties, indicated here by Neumann, the following two shall be especially emphasized, *firstly* those that arise from the limits posed on the electric separating forces[249] through the *nature of bodies*, *secondly* the difficulties linked with the assumption of a *constant value of* L, which consist in the fact that this hypothesis is linked to the assumption of *certain unchangeable charges on the spherical shell and on the cylinder.*

1. Difficulties Arising from the Limits Imposed on the Electric Separating Forces through the Nature of the Bodies

Denoting the electricity of the cylinder and the spherical shell by e and e' for short (instead of Σe and $4\pi^2\alpha H$), and setting the moment of inertia of the cylinder as $\mathfrak{M} = ma^2$, then

$$L = \left(\frac{M+m}{2} - \frac{ee'}{3\alpha c^2}\right) a^2 \ ;$$

consequently for $L = 0$:

$$ee' = \frac{3}{2}\alpha c^2 (M+m) \ .$$

Now $2e'/\alpha$ is the required *separating force* [acting] on the charge e' of a spherical shell;[250] however the magnitude of this *separating force* is limited and depends on the *separating means* present in nature, because although the variety of these means is huge there is yet no means for [generating] *infinitely large separating forces.*

If now $L = 0$, then according to the equation above it follows that

$$\frac{2e'}{\alpha} = 3c^2 \cdot \frac{M+m}{e} \ ,$$

[249][Note by AKTA:] In German: *Elektrischen Scheidungskräften*. See also footnote 156 on page 100.

[250][Note by WW:] The separating force exerted by a spherical shell of radius a covered uniformly with electricity e' acting on an *external* linear conductor of unlimited length ℓ, which lies in the extension of a radius, is the *difference* of the *repulsive force* $= e'\int_0^\ell \frac{dx}{(\alpha+x)^2}$, exerted on the unit *positive* electricity contained in each unit length interval of the conductor, and the *attractive force* $= -e'\int_0^\ell \frac{dx}{(\alpha+x)^2}$, exerted on the unit *negative* electricity contained in each unit length interval of the conductor; consequently

$$= 2e'\int_0^\ell \frac{dx}{(\alpha+x)^2} = 2e'\left(\frac{1}{\alpha} - \frac{1}{\alpha+\ell}\right) \ ,$$

from where for unlimited value of ℓ it follows that the *separating force* $= 2e'/\alpha$, as stated above.

If in this conductor a pillar is interposed, through which the charge on the sphere remains stationary, then this proves that the separating forces exerted by the charge on the sphere and by the pillar on the conductor are equal and opposite to each other, whereby also the separating force of the pillar is determined, namely $-2e'/\alpha$.

But it is also clear that, when the spherical shell was not yet charged, it would get charged from the pillar, and that this charge would grow, until it got to e', assuming a sphere of radius $= \alpha$, i.e. until the *separating force of the charge on the sphere* would have become $= 2e'/\alpha$ and *cancelled the separating force of the pillar.*

Further from here it follows that two spherical shells with charges e' and ne', whose radii are α and $n\alpha$, and whose *potentials* for all points in the interior, namely e'/α and $ne'/n\alpha$, are consequently equal, could be connected by a conductor, without any part of the charge going from one shell to the other, in conformity with the theorem, that in the case of the equality of the *potentials* in the interior of two conductors, there occurs no transfer of electricity. —

It is yet to be remarked, that the above *separating forces* are expressed in *mechanical measure* and are to be multiplied by $155\,370 \cdot 10^6 = [c/2\sqrt{2}]$, in order to express them in *magnetic measure.*

or, since $c = 439\,450 \cdot 10^6$, it must be true that

$$\frac{2e'}{\alpha} \cdot \frac{e}{2(M+m)} = 289\,670 \cdot 10^{18} \ .$$

— It will be difficult to present a cylinder with fixed axis of rotation, charged, where its charge e is in *absolute measure* larger than the ponderable mass $2m$ expressed *in milligram*; but if we add to $2m$ twice the mass of the weight, namely $2M$, one can surely assume that $e/2(M+m)$ will be a *proper* fraction. From which it follows that to charge the spherical shell, when L equals zero, it would be necessary a separating force which *in mechanical measure* should be $= 2e'/\alpha > 289\,670 \cdot 10^{18}$, i.e. a separating force which supersedes at least 261 *trillion* times the largest of the measured ones in the *Electrodynamic Measurements*, Vol. V of these *Abhandlungen*, pages 243-250,[251,252] namely $= 2\frac{6410.5}{11.567} = 1108$. It is doubtful that there exist yet unknown bodies in nature which permit the possibility for such large separating forces. Enlargement of the two coefficients $e/(M+m)$ and e'/α by a factor 10 or 100 would not help at all; if the *nature of bodies* does not allow for the enlargement of the product of these two coefficients by many *trillion* times, then the production for the case $L = 0$ always remains *impossible*.

But also, if the nature of bodies is such that the possibility of such large separating forces could exist, then even with such separating forces, the required charges could not be produced, because there exists no *insulator* stable enough to resist the expansive forces of such charges, which would explode more powerfully than gunpowder charges and destroy everything.

But even if such stable and perfect insulators would exist, which could resist the tremendous expansive forces of such charges, and supposing one could bring the charge of the sphere to the required magnitude, then one would have achieved $L = 0$, but even then the acceleration ϑ'' would *not become infinite*, at least not according to the law underlying the above calculation, as shall be proved in the following Section.

13.11 Continuation

2. Difficulties Linked to the Assumption of a Constant Value of L

Aside from the doubt explained in the previous Section, if, under the barriers imposed on the *electric separating forces* by the *nature of bodies*, [the case] $L = 0$ could actually occur, it would then remain open to discuss the question, if the precisely determined product of the two charges e and e' could be *kept constant* for $L = 0$, which must be assumed if L shall be *constant*, in particular $= 0$. Furthermore, there is the question of what effect this would have, if one of the two charges was *variable*.

The value of L depends on the charges e and e' of the cylinder and the spherical shell, and more precisely, if the value of $L = 0$ is *constant*, it is not only the *magnitude* of the charges e and e' that is relevant, but also the *method of production* of a precisely defined value.

If in addition, one of the two charges, namely the cylinder charge e, should remain *constant*, the charge on the spherical shell e' should remain *variable*, in order to get through

[251] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, pp 631-638.

[252] [Note by AKTA:] [KW57, Sections 9-11, pp. 631-638 of Weber's *Werke*] with English translation in [KW21, Sections 9-11, pp. 28-36].

its steady growth to the point where $L = 0$. But even then the charge e' would not abruptly stop changing and remain *completely constant.* Without doubt it would proceed to oscillate within certain limits, because the production of the required charge *with absolute precision* is not possible at all, but only within certain wider or narrower limits. Consequently the charge e' would always have to be considered as a function of time, which for any short time interval can be represented by $e' = p + qt$.

Now *in this case, in which* e' *is variable,* the equation established by Neumann on page 105 of this Volume is valid,[253,254] namely

$$T = P - U + Mga\vartheta + \text{constant} ,$$

where $T = [(M + m)/2] \cdot a^2\vartheta'^2$, $P = [ee'/3\alpha c^2] \cdot a^2\vartheta'^2 + \text{constant}$, and $U = ee'/\alpha$, which therefore changes [its magnitude] simultaneously with e'.

Setting now $e' = p + qt$ and $L = ([(M + m)/2] - [ee'/3\alpha c^2])\, a^2$, then one gets

$$L\vartheta'^2 = Mga\vartheta - \frac{e}{\alpha}(p + qt) + \text{constant} ,$$

or, differentiated with respect to t,

$$2L\vartheta'' = Mga - \frac{eq}{\alpha\vartheta'} ,$$

from where for $L = 0$ it follows that *either* $\vartheta'' = \infty$, *or* (if ϑ'' is not infinite), $\vartheta' = eq/Mga\alpha$.

This alternative will be chosen, if one takes into account, that L varies with time t. One then calculates the time t from that instant, where according to the equation $e' = p+qt$ one would have $L = 0$, from where it follows that

$$\frac{M + m}{2} = \frac{ep}{3\alpha c^2} .$$

It follows that after the time element δ,

$$L = -\frac{eq\delta}{3\alpha c^2}\, a^2 ,$$

consequently, if one puts this value for L into the above equation,

$$2L\vartheta'' = -\frac{2eq\delta}{3\alpha c^2}\, a^2 \cdot \vartheta'' = Mga - \frac{eq}{\alpha\vartheta'} .$$

Since for the case of $L = 0$ and $t = 0$ at a finite value of ϑ'' it has now been found that

$$\vartheta' = \frac{eq}{Mga\alpha} ,$$

and the value of ϑ' for $t = \delta$, if δ is *vanishingly small*, is not noticeably different from the value of ϑ'' for $t = 0$, thus it yields:

$$2L\vartheta'' = -\frac{2eq\delta}{3\alpha c^2}\, a^2 \cdot \vartheta'' = Mga - \frac{eq}{\alpha} \cdot \frac{Mga\alpha}{eq} = 0 ,$$

[253] [Note by HW:] *Abhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften*, mathematisch-physische Klasse, Vol. 11.

[254] [Note by AKTA:] [Neu74, p. 105].

according to which $\vartheta'' = 0$.

Now since this value $\vartheta'' = 0$ is valid, no matter how small q may be, it shall consequently also apply, when $q = 0$.

One sees from this, that whenever the case $L = 0$ occurs, the only transitions are from smaller values to larger ones or vice versa, so that the acceleration ϑ'' for $L = 0$ is not at all infinite, but $= 0$, which eliminates all objections that are based on the claimed *infinite acceleration*.

13.12 Conclusion

Even before Neumann noticed and investigated the case considered in the previous Section, Helmholtz had already drawn attention to a similar case, namely where an electric mass point ε is located in the *interior* of an electrified spherical shell, and he found the surprisingly simple result that the *components of the force exerted on ε by the electrified spherical shell are equal to the acceleration x'', y'', z'' multiplied by a constant factor*.

Furthermore, Helmholtz (Borchardt's *Journal*, Vol. 75)[255] developed the equation of the vis viva from the fundamental law of electrical interaction, which results for the case of just *one* mass point μ with the electric quantum ε that moves in some space, that is limited by a spherical shell of radius R uniformly covered with electricity, namely the equation:[256,257,258]

$$\frac{1}{2}\left(\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'\right) q^2 - V + C = 0 \ ,$$

where ε' denotes the quantum of electricity per unit area on the spherical shell, q the velocity of the mass point μ on its trajectory s, hence $q = ds/dt$, and V the potential of the *non-electric* forces. It follows from this equation via differentiation with respect to s, that:

$$\mu q \frac{dq}{ds} - \left(\frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon' \cdot q\frac{dq}{ds} + \frac{dV}{ds}\right) = 0 \ ,$$

where $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' \cdot q[dq/ds]$ is the *electric* force exerted on μ along the direction of the trajectory s, and dV/ds the *non-electric* force exerted on μ in the same direction.

Since $q = ds/dt$ denotes the velocity of the point μ on its trajectory s and $q[dq/ds] = dq/dt = d^2s/dt^2$ the acceleration of μ on its trajectory, it follows, that the *electric* force $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' \cdot q[dq/ds]$ exerted on μ found from the equation above, is the product of this acceleration $q[dq/ds]$ multiplied by the constant factor $[8\pi/3c^2] \cdot R\varepsilon\varepsilon'$, completely in agreement with the result stated above.

If now the *electric* force exerted on μ is proportional to the acceleration $q[dq/ds]$ of the point μ, on which *two* forces are exerted, namely the stated *electric* force $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' \cdot q[dq/ds]$, and the *non-electric* force dV/ds, then it is clear that $q[dq/ds]$ is obtained by dividing the sum of these *two* forces by μ, namely:

$$q\frac{dq}{ds} = \frac{\frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon' \cdot q\frac{dq}{ds} + \frac{dV}{ds}}{\mu} \ ,$$

[255][Note by AKTA:] [Hel73].

[256][Note by WW:] The factor $\frac{1}{2}q^2$ in the above equation is not $(\mu - [4\pi/3c^2]R\varepsilon e)$, as Helmholtz stated, but $(\mu - [8\pi/3c^2]R\varepsilon\varepsilon')$. Cf. Neumann, §§ 3 and 7 of his Memoir in this Volume.

[257][Note by HW:] *Abhandlungen bei der Begründung der Königl. Sächs. Gesellschaft der Wissenschaften*, mathematisch-physische Klasse, Vol. 11.

[258][Note by AKTA:] [Neu74, §§ 3 and 7].

from where it is found

$$q\frac{dq}{ds} = \frac{\frac{1}{\mu} \cdot \frac{dV}{ds}}{1 - \frac{1}{\mu} \cdot \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'} \ .$$

Substituting this value for $q[dq/ds]$ into the expression for the *electric* force exerted on μ, then one obtains an expression for this force independent of the acceleration, namely

$$\frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon' \cdot q\frac{dq}{ds} = \frac{\frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'}{\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'} \cdot \frac{dV}{ds} \ ,$$

from where one easily recognizes, that the *electric* force in the trajectory s is directed towards the same side, as the *non-electric* force dV/ds, as long as $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' < \mu$, but that it has the opposite direction, as soon as $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' > \mu$. But while $[8\pi/3c^2] \cdot R\varepsilon\varepsilon'$ becomes $= \mu$, *as it grows*, the *electric* force grows simultaneously up to $+\infty$, then jumps abruptly over from $+\infty$ to $-\infty$, and grows with $[8\pi/3c^2] \cdot R\varepsilon\varepsilon'$, which becomes $> \mu$, again continuously from $-\infty$ to 0. If on the other hand $[8\pi/3c^2] \cdot R\varepsilon\varepsilon'$ becomes $= \mu$, *as it decreases*, then simultaneously the *electric* force decays continuously down to $-\infty$, then jumps abruptly over from $-\infty$ to $+\infty$, and decreases, after $[8\pi/3c^2] \cdot R\varepsilon\varepsilon'$ becomes $< \mu$, again continuously from $+\infty$ to 0.

Both the growth of such a force to infinity, and the change of its direction, at the moment when it becomes infinite, could appear as a violation of the continuity found in nature and might be considered as a basis of objection to the general validity of that law, from which such violations of continuity are derived. However, it can be easily proved that these conclusions can not be justifiably drawn from that law, because these conclusions are linked to completely unrealizable conditions, as was already remarked in Poggendorff's *Annalen*, Vol. 156, p. 29,[259,260,261] which requires a proof, which should finally be given here.

The acceleration of μ by the previously mentioned *electric* and *non-electric* forces obtained by Helmholtz from the fundamental law of electric action, resulted in the following equation:

$$q\frac{dq}{ds} = \frac{dV/ds}{\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'} \ ,$$

in which the acceleration from μ, instead of $q[dq/ds]$, can also be represented by dq/dt or d^2s/dt^2.

The acceleration is infinitely large, when the value of ε' is $= 3c^2\mu/8\pi R\varepsilon$. The determination of this value of ε', which shall be denoted by η, presupposes that the value of ε has already been determined before. It might appear that conversely η could also have been previously determined, by making the determination of ε depend on the knowledge of η. However it is clear that, after the spherical shell has been charged and η determined, no

[259][Note by WW:] It is stated at the mentioned location:

> Such a jump of the electric force in *magnitude* and *direction*, namely from $+\infty$ to $-\infty$, never really occurs within the law, since the mass μ with its charge e cannot, in consequence of the always growing acceleration, remain long enough in the interior of the spherical shell, before $[8\pi/3c^2] \cdot R\varepsilon\varepsilon' = \mu$ occurs, but would have already been impelled earlier towards the spherical shell formed by the *rigid* insulator, through whose resistance rest would have been restored, and so the relations presupposed in the calculation would no longer apply.

[260][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 333.

[261][Note by AKTA:] [Web75, pp. 333-334 of Weber's *Werke*].

charge could be carried to the interior of the spherical shell, hence neither could the charge ε of the particle μ.

Hence if the charge ε of the particle μ in the interior of the spherical shell is given, then the surface charge of the spherical shell, for which the force becomes infinite, can be calculated in advance, namely for any unit surface, as was already stated above:

$$\eta = \frac{3c^2}{8\pi} \cdot \frac{\mu}{R\varepsilon} \ ;$$

however the actual *production* of this charge would necessarily be connected with a *gradual* growth of the charge from $\varepsilon' = 0$ to $\varepsilon' = \eta$.

Assuming this, we will denote the time at which $\varepsilon' = \eta$ occurs by $t = 0$, and the time at which $\varepsilon' = 0$ had occurred by $t = -\vartheta$. If one now further sets the growth of the charge ε' proportional to time, namely

$$\varepsilon' = \eta \left(1 + \frac{t}{\vartheta}\right) \ ,$$

and if one assumes, in order to simplify the analysis, the center of the spherical shell as the initial point of the trajectory s, where the particle μ at time $t = -\vartheta$ (i.e. at the time where $\varepsilon' = 0$) is at rest, then with $\varepsilon' = 0$ one has simultaneously $s = 0$ and $q = 0$, and if one finally assumes the *non-electric* force $dV/ds = a$ exerted on μ to be *constant*, then from the stated equation, namely from

$$\frac{dq}{dt} = \frac{\frac{dV}{ds}}{\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'} \ ,$$

it follows that after the substitutions $\varepsilon' = \eta(1 + [t/\vartheta])$, $\mu = [8\pi/3c^2] \cdot Re\eta$, and $dV/ds = a$, the equation

$$dq = -\frac{a\vartheta}{\mu} \cdot \frac{dt}{t} \ .$$

The integral of this equation can be written as:

$$q = -\frac{a\vartheta}{2\mu} \cdot \log c^2 t^2 \ .$$

From here it follows, since $q = 0$ for $t = -\vartheta$, that $c^2 = 1/\vartheta^2$.

If one substitutes this value for c^2 into the previous equation and puts ds/dt for q, then one obtains

$$ds = -\frac{a\vartheta}{2\mu} \cdot \log \frac{t^2}{\vartheta^2} \cdot dt \ .$$

From here it follows by integration that

$$s = \frac{a\vartheta t}{\mu} \left(1 - \tfrac{1}{2} \log \frac{t^2}{\vartheta^2}\right) + C \ .$$

Since s now equals zero for $t = -\vartheta$, it then results that $C = a\vartheta^2/\mu$; consequently

$$s = \frac{a\vartheta^2}{\mu} \left[1 + \frac{t}{\vartheta} \left(1 - \tfrac{1}{2} \log \frac{t^2}{\vartheta^2}\right)\right] \ .$$

Both obtained formulas, which, when one denotes the *non-electric* force acting on μ by $a = g\mu$, can be written as:

$$
\begin{aligned}
q &= -\frac{g\vartheta}{2} \cdot \log \frac{t^2}{\vartheta^2} \;, \\
s &= g\vartheta^2 \left[1 + \frac{t}{\vartheta} \left(1 - \tfrac{1}{2} \log \frac{t^2}{\vartheta^2} \right) \right] \;,
\end{aligned}
$$

they can now be represented easily in a tabular overview as follows, where e is the base of the natural logarithm:[262]

$\frac{t}{\vartheta}$	$\frac{s}{g\vartheta^2}$	$\frac{q}{g\vartheta}$	$\frac{\varepsilon'}{\eta} = \left(1 + \frac{t}{\vartheta}\right)$
-1	0	0	0
$-e^{-1}$	$1 - 2e^{-1}$	1	$1 - e^{-1}$
$-e^{-2}$	$1 - 3e^{-2}$	2	$1 - e^{-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	1	∞	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$+e^{-2}$	$1 + 3e^{-2}$	2	$1 + e^{-2}$
$+e^{-1}$	$1 + 2e^{-1}$	1	$1 + e^{-1}$
$+1$	2	0	2
$+e$	1	-1	$1 + e$
$+e^2$	$1 - e^2$	-2	$1 + e^2$

One can see from this overview, that the particle μ, which would have traveled the distance $\frac{1}{2}g\vartheta^2$ in time ϑ under the influence of the acceleration g originating from the *non-electric* force, doubles its distance in the presence of the electric force, and, while without the electric force it would have reached the velocity $g\vartheta$, with the electric force it reaches infinite velocity.

However, having obtained infinitely large velocity, it does not cover the smallest finite distance element, as a consequence of the fact that the then infinite *positive* acceleration abruptly turns into infinite *negative* acceleration, and that as a consequence of this, the velocities are equal to one another for the same time period *before* and *after* this instant, according to which the velocity q at time $t = +\vartheta$ (i.e. after it has passed the time period 2ϑ counted from the start of the motion) is equal to the velocity at the starting time $t = -\vartheta$, namely $q = 0$, where the distance s, when the spherical shell is sufficiently large so that s fits in, would have increased again by $g\vartheta^2$, hence reaching $s = 2g\vartheta^2$. The charge ε' would in the process have reached 2η. But from now on, with continued growth of time and charge, the distance s of the particle μ from the center of the spherical shell would quickly decrease again until $s = 0$, and then become negative until $s = -R$, when the particle μ would hit the spherical shell, at time t, which can be determined from the equation $-R = g\vartheta^2[1 + [t/\vartheta](1 - \frac{1}{2}\log[t^2/\vartheta^2])]$, and with velocity q which, after having determined t, is found to be $= [g\vartheta/2]\log[t^2/\vartheta^2]$.

So far, as already remarked, it was assumed that the radius R of the spherical shell is larger than the maximal value $2g\vartheta^2$, which s reaches at time $t = +\vartheta$. If R were smaller, then it is clear, that the particle μ would hit the spherical shell sooner, namely at the moment

[262] [Note by AKTA:] Due to a misprint, the number in the first column of the second line appeared as 1 instead of the correct value -1 presented here.

when s had become $= R$, at which time t which could be determined from the equation $R = g\vartheta^2[1 + [t/\vartheta](1 - \frac{1}{2}\log[t^2/\vartheta^2])]$.

The case will now be considered in which the electric charge does not actually grow, but instead remains *constant* after superseding the value η. For instance, suppose the constant charge is given by $\varepsilon' = \eta(1 + [1/e^2])$, where μ has the velocity $q = 2g\vartheta$ and is located at distance $s = (1 + [3/e^2])g\vartheta^2$ from the center of the spherical shell.

If one inserts into Helmholtz's equation

$$\frac{dq}{dt} = \frac{\frac{dV}{ds}}{\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\varepsilon'}$$

the value $\eta(1 + [1/e^2])$ for ε', then

$$\frac{dq}{dt} = \frac{\frac{dV}{ds}}{\mu - \frac{8\pi}{3c^2} \cdot R\varepsilon\eta\left(1 + \frac{1}{e^2}\right)} .$$

Now set herein as earlier $dV/ds = a$ and $[8\pi/3c^2] \cdot R\varepsilon\eta = \mu$, to obtain the differential equation

$$dq = -\frac{ae^2}{\mu} \cdot dt ,$$

and through its integration:

$$q = -\frac{ae^2}{\mu}t + C .$$

If we start counting the time from the instant at which $\varepsilon' = \eta(1 + [1/e^2])$, then for $t = 0$ the value of $q = 2g\vartheta$ has already been found above, and consequently, $a = g\mu$, and

$$C = 2g\vartheta ,$$

hence

$$q = 2g\vartheta - e^2 gt ,$$

or

$$ds = \left(2g\vartheta - e^2 gt\right) dt ,$$

from where by integration

$$s = 2g\vartheta t - \frac{e^2}{2}gt^2 + C' .$$

Now it was found, if one counts the time from the instant at which $\varepsilon' = \eta(1 + [1/e^2])$, that for $t = 0$ the corresponding value for $s = (1 + [3/e^2])g\vartheta^2$, consequently

$$C' = \left(1 + \frac{3}{e^2}\right) g\vartheta^2 ,$$

hence

$$s = \left(1 + \frac{3}{e^2}\right) g\vartheta^2 + 2g\vartheta t - \frac{e^2}{2}g \cdot t^2 .$$

This formula together with the preceding one

$$q = 2g\vartheta - e^2 gt ,$$

can now be easily and clearly arranged, just like the previous formulas for s and q, in the form of the following Table:

$\frac{t}{\vartheta}$	$\frac{s}{g\vartheta^2}$	$\frac{q}{g\vartheta}$	$\frac{\varepsilon'}{\eta}$
0	$1+\frac{3}{e^2}$	2	$1+\frac{1}{e^2}$
1	$3+\frac{3}{e^2}-\frac{e^2}{2}$	$2-e^2$	$1+\frac{1}{e^2}$
2	$5+\frac{3}{e^2}-2e^2$	$2-2e^2$	$1+\frac{1}{e^2}$

This Table can be easily extended; but one can already see from here, that from $t = 2\vartheta/e^2$ onwards, after the charge becomes constant, the distance s of the particle μ from the center of the spherical shell decreases and very soon becomes negative, until eventually the particle μ, when $s = -R$ has been reached, hits the spherical shell at time t, and with the velocity q, both of which can be determined from the two equations

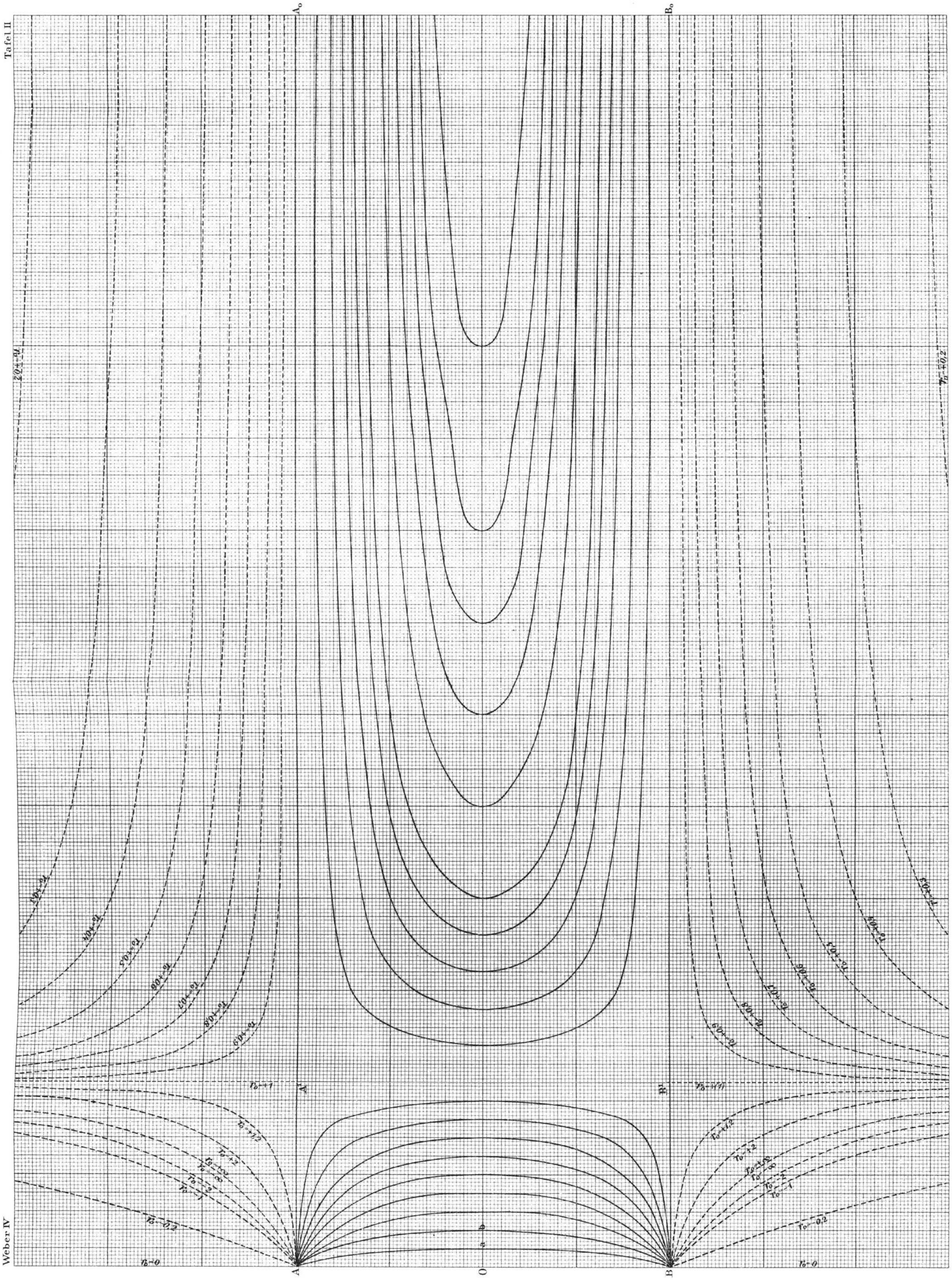
$$\begin{aligned}
-R &= \left(1+\frac{3}{e^2}\right)g\vartheta^2 + 2g\vartheta t - \frac{e^2}{2}gt^2 \ , \\
q &= 2g\vartheta - e^2 gt \ .
\end{aligned}$$

One can see from this presentation of the whole process in its *context*, that none of the "inconsistent and absurd" consequences, with which Helmholtz wanted to disprove the established fundamental law, ever actually occur.

Helmholtz's objection (A) in Borchardt's *Journal*, Vol. 72, p. 61 and Vol. 75, p. 38 has not yet been discussed.[263] It consists in the claim, that the established fundamental law of electrical interaction, or rather the differential equations of Kirchhoff originating from this law,[264] would lead to an unstable equilibrium of the electric matter, or rather to a motion of this matter, whose velocity would grow in time to infinity. But Neumann has already proved in the *Berichte der Königl. Sächs. Gesellschaft*, October 1871, p. 477,[265] that the differential equations of Kirchhoff rest, apart from that fundamental law, on yet various other peripheral assumptions, and that consequently this law cannot be doubted based on general concerns presented against these differential equations.

After this clarification by Neumann, which is discussed in more detail in his Memoir, pages 128-149 of the present Volume,[266] a further discussion of this objection is no longer required. Such discussion, which would mainly only concern those peripheral assumptions, lies completely outside the scope of the present Memoir.

[263][Note by AKTA:] [Hel70, p. 61] and [Hel73, p. 38].
[264][Note by AKTA:] [Kir57c] with English translation in [GA94].
[265][Note by AKTA:] [Neu71, pp. 477-478].
[266][Note by AKTA:] [Neu74, pp. 128-149].


Tafel II
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Chapter 14

[Weber, 1878b] On the Energy of Interaction

Wilhelm Weber[267,268,269]

(Excerpt by the author from the Treatise on *Elektrodynamische Maassbestimmungen* in Volume XVIII of the *Königl. Sächs. Gesellschaft der Wissenschaften.*)[270,271,272]

[267][Web78b] with English translation in [Web21f].

[268]Translated and edited by A. K. T. Assis, www.ifi.unicamp.br/~assis. I thank Frederick David Tombe for relevant suggestions.

[269]The Notes by H. Weber, the Editor of Volume 4 of Weber's *Werke*, are represented by [Note by HW:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[270][Note by HW:] *Annalen der Physik und Chemie*, edited by G. Wiedemann, Vol. 4, Leipzig, 1878, pp. 343-373.

[271][Note by HW:] As § 1-5 of the excerpt coincides in content and wording with § 1-5 of the previous treatise, in fact up to page 382 line 10 from above, only the last Section of the excerpt, § 6, has been printed here.

[272][Note by AKTA:] This work is an excerpt from Weber's Seventh major Memoir on *Electrodynamic Measurements*, "Elektrodynamische Maassbestimmungen", [Web78a] with English translation in [Web21e], see Chapter 13. Pages 343-365 of [Web78b] coincide with pages 645-664 line 12 from above of the *Abhandlungen der mathematisch-physischen Classe der Königlich Sächsischen Gesellschaft der Wissenschaften (Leipzig)*, [Web78a], and with pages 364-382 line 10 from above of Volume 4 of Weber's *Werke*, [Web94c].

6. A Particle Driven by both an Electric and a Non-Electric Force while Enclosed in an Electrified Spherical Shell

Regarding the applications of the fundamental electric law, in order to show that none of the "inconsistent and absurd" consequences occur, through which Helmholtz wished to refute this fundamental law, we will only consider here the application to the motion of a mass point μ (with an electric quantum ε) enclosed in an *electric spherical shell*, when acted on by both an *electric* force and a *non-electric* constant force a.[273]

From this fundamental law, Helmholtz deduced in Borchardt's *Journal*, [Volume] LXXV,[274] the equation of the *vis viva*[275] for this mass point μ with electric quantum ε, [inside] a spherical shell of radius R uniformly covered with electricity, which appears as follows:[276]

$$\frac{1}{2}\left(\mu - \frac{8\pi}{3c^2}\cdot R\varepsilon\varepsilon'\right)q^2 - V + C = 0 \ ,$$

where ε' denotes the quantum of electricity per unit area on the surface of the spherical shell, q the velocity of the mass point μ and V the potential of the *non-electric* force.[277]

From this equation it has been concluded that when, with an existing *difference between the potential V of the non-electric force and the constant C*, ε' would have increased from 0 to $[8\pi/3c^2]R\varepsilon\cdot\varepsilon' = \mu$, then the *vis viva* of the point mass μ would have increased from $\frac{1}{2}\mu q^2 = V - \ C$ up to $\frac{1}{2}\mu q^2 = \infty$, which would be an *infinitely large work output*.[278] The removal of this objection can now be obtained from the complete presentation of the whole process of motion in its context, as indicated earlier in these *Annalen*, [Volume] XLVI, p. 29.[279,280]

Let us denote by η that charge ε' on the unit area of the spherical shell for which the velocity q of the mass μ would be infinite, then set $\eta = [3c^2\mu/8\pi R\varepsilon]$, and assume that ε has a certain constant value, while ε' grows uniformly from 0 at time $t = -\vartheta$ up to η at time $t = 0$, the latter value being gradually attained. Furthermore, to simplify the analysis, take the center of the sphere as the starting point of the path s[281] where the particle μ at time $t = -\vartheta$ (where $\varepsilon' = 0$) is at rest, that is, with $\varepsilon' = 0$[282] we have $s = 0$ and $q = 0$. Then with

[273][Note by AKTA:] Weber is referring here to his electrodynamic force law which he presented in 1846, [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

Weber studies in this paper of 1878 the motion of a particle with mass μ and electric charge ε moving inside a uniformly electrified spherical shell. He considers two forces acting on this particle, namely, the electric force exerted by the shell and a non-electric constant force a. He considers this constant force a to be the weight of the particle near the surface of the Earth, namely, $a = \mu g$. He is replying to Helmholtz's criticisms presented in 1873, [Hel73], see also [Hel72a] with English translation in [Hel72b].

[274][Note by AKTA:] [Hel73]; see also [Hel72a] with English translation in [Hel72b].

[275][Note by AKTA:] See footnote 26 on page 17.

[276][Note by AKTA:] [Hel73, Section 12, pp. 48-54], see also [Neu74, §§ 3 and 7].

[277][Note by AKTA:] ε' is the surface charge density. The total charge spread over the whole surface of the spherical shell of radius R is then given by $4\pi R^2\varepsilon'$.

[278][Note by AKTA:] In German: *Arbeitsleistung*. This expression can also be translated as "work performed".

[279][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 333.

[280][Note by AKTA:] [Web75, p. 29 of the *Annalen der Physik und Chemie* and p. 333 of Weber's *Werke*].

[281][Note by AKTA:] Weber will consider the motion of the particle along a straight line beginning at the center of the shell. We can represent this motion as taking place along the x axis, with $x = 0$ at the center of the shell, so that the path or trajectory $s = x$ might have positive or negative values. When $s = \pm R$, the particle would reach the spherical shell of radius R.

[282][Note by AKTA:] Due to a misprint, this expression appeared in the original as $\varepsilon = 0$.

the help of the values

$$\varepsilon' = \eta\left(1 + \frac{t}{\vartheta}\right) \ , \qquad \mu = \frac{8\pi}{3c^2}\cdot R\varepsilon\eta \qquad \text{and} \qquad \frac{dV}{ds} = a \ ,$$

(see Article 12 of the *Abhandlung*)[283,284] the following equation is obtained:

$$dq = -\frac{a\vartheta}{\mu}\cdot\frac{dt}{t} \ .$$

The integral of this equation can be written as:[285]

$$q = -\frac{a\vartheta}{2\mu}\cdot\log C^2t^2 \ ,$$

in which $C^2 = 1/\vartheta^2$, because $q = 0$ should take place for $t = -\vartheta$. Therefore, as $q = ds/dt$:

$$ds = -\frac{a\vartheta}{2\mu}\cdot\log\frac{t^2}{\vartheta^2}\cdot dt \ .$$

From this it follows through integration:

$$s = \frac{a\vartheta}{\mu}\left(1 - \frac{1}{2}\log\frac{t^2}{\vartheta^2}\right)\cdot t + C' \ .$$

Since now $s = 0$ for $t = -\vartheta$, it results $C' = a\vartheta^2/\mu$, therefore:

$$s = \frac{a\vartheta^2}{\mu}\left(1 + \frac{t}{\vartheta}\left(1 - \frac{1}{2}\log\frac{t^2}{\vartheta^2}\right)\right) \ .$$

When we set the *non-electric* force acting on μ as $a = g\mu$, with q' being the ratio of the velocity q to $g\vartheta$, and with s' being the ratio of s [the path] to $g\vartheta^2$, then these formulas can be written as:[286]

$$\frac{dq'}{dt} = -\frac{1}{t} \ ,$$

$$q' = -\frac{1}{2}\log\frac{t^2}{\vartheta^2} \ ,$$

283[Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 333.

284[Note by AKTA:] [Web75, p. 29 of the *Annalen der Physik und Chemie* and p. 333 of Weber's *Werke*].

285[Note by AKTA:] What Weber writes here as "log" of a magnitude θ, $\log\theta$, should be understood as the natural logarithm of θ to the base of Euler's constant $e = 2.718...$, namely, $\log\theta = \log_e\theta = \ln\theta$. His integration can be expressed as follows:

$$\int_{q=0}^{q} dq = -\frac{a\vartheta}{\mu}\int_{t=-\vartheta}^{t}\frac{dt}{t} = -\frac{a\vartheta}{\mu}\left[\ln|t|\right]_{t=-\vartheta}^{t} = -\frac{a\vartheta}{\mu}\ln\sqrt{\frac{t^2}{\vartheta^2}} \ ,$$

such that

$$q = -\frac{a\vartheta}{2\mu}\ln\frac{t^2}{\vartheta^2} \ .$$

286[Note by AKTA:] Weber is assuming here that the constant force a is the weight of the particle of mass μ near the surface of the Earth, namely, $a = \mu g$. Moreover, he is defining the dimensionless displacement $s' = s/(g\vartheta^2)$ and the dimensionless velocity $q' = q/(g\vartheta) = (ds/dt)/(g\vartheta)$.

$$s' = 1 + \frac{t}{\vartheta}\left(1 - \frac{1}{2}\log\frac{t^2}{\vartheta^2}\right) \ .$$

Now they can be used for the construction of all motions of the particle μ with an uniformly growing charge ε' and can be represented in a tabular overview, where e is the base of the natural logarithm:[287]

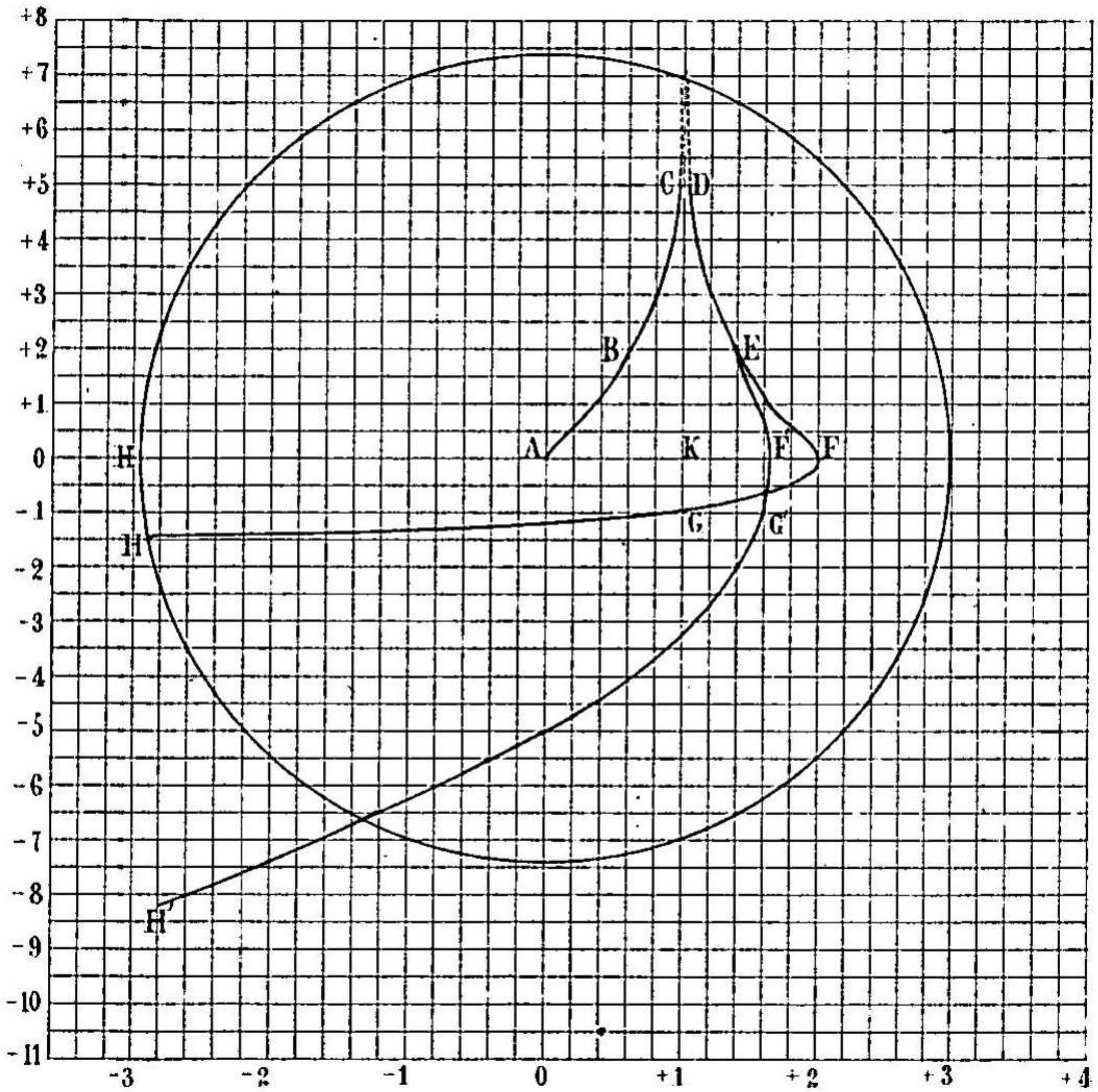
$\frac{t}{\vartheta}$	s'	q'	$\frac{dq'}{dt}$	$\frac{\varepsilon'}{\eta}$
-1	0	0	$+1$	0
$-e^{-1}$	$1-2e^{-1}$	1	$+e$	$1-e^{-1}$
$-e^{-2}$	$1-3e^{-2}$	2	$+e^2$	$1-e^{-2}$
$-e^{-3}$	$1-4e^{-3}$	3	$+e^3$	$1-e^{-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	1	∞	$\pm\infty$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$+e^{-3}$	$1+4e^{-3}$	3	$-e^3$	$1+e^{-3}$
$+e^{-2}$	$1+3e^{-2}$	2	$-e^2$	$1+e^{-2}$
$+e^{-1}$	$1+2e^{-1}$	1	$-e$	$1+e^{-1}$
$+1$	2	0	-1	2
$+e$	1	-1	$-e^{-1}$	$1+e$
$+e^2$	$1-e^2$	-2	$-e^{-2}$	$1+e^2$

The curve $ABCDEFGH$ in the next Figure represents, according to this information, the dependence of the velocity q' as a function of the path length s', namely, s' as abscissa and q' as ordinate. This curve goes from the center A of the sphere as the starting point of the coordinates out to B, C and approaches asymptotically the *ordinate* for $s' = 1$, then returning from there to D, E, F, where it intersects the axis of abscissas at the point $s' = 2$, and then goes on to G and H, where s becomes $= -R$ and μ hits the spherical shell.[288]

[287][Note by AKTA:] Instead of dq'/dt, the expression in the fourth column of the first line in the next Table should be the dimensionless acceleration given by

$$\frac{1}{g}\frac{d^2s}{dt^2} = \frac{1}{g}\frac{dq}{dt} = \vartheta\frac{dq'}{dt} = -\frac{\vartheta}{t} \ .$$

[288] [Note by AKTA:] Ordinate q' as function of abscissa s'. When the ordinate $q' = 0$, the letters from left to right along the abscissa s' should read as follows: H°, A, K, F' and F. Due to a misprint, the first point H° was printed as H. When the ordinate q' is equal to -1, the letters along the abscissa from left to right are G and G'. Close to $q' = -1.5$ and $s' = -3$ we have letter H, while close to $q' = -8.5$ and $s' = -2.8$ we have letter H'.



One can see from this overview that the particle μ, which would have covered the distance $\frac{1}{2}g\vartheta^2$ in the time ϑ due to the acceleration g coming from the *non-electric* force, covers twice this path under the joint action of the *electric* force; moreover, while it had reached the velocity $g\vartheta$ without the electric force, it now reaches an *infinitely large velocity* with [the joint action of] the electric force.

However, with this attained *infinitely large velocity*, the particle μ does not cover the *smallest finite path element*, due to the fact that at the same moment the *acceleration* dq/dt, which became equally infinitely large, suddenly jumps from $+\infty$ to $-\infty$, that is, changes to an infinitely large *deceleration*, causing the *velocities* to become equal long *before* and *after* this moment. For instance, the velocity q at time $t = +\vartheta$ (that is, after the time interval 2ϑ calculated from the beginning of the motion) is equal to the velocity in the beginning, at time $t = -\vartheta$, namely $q = 0$, where the path s, when the spherical shell is large enough for s to still have room inside it, would have grown again by $g\vartheta^2$, so that s would become $= 2g\vartheta^2$. The charge ε' would thereby have grown up to 2η. From now on, however, with time and charge [of the spherical shell] continuing to increase, the displacement of the particle μ

from the center of the shell would decrease quickly up to $s = 0$, and then become negative up to $s = -R$, where the particle μ would hit the spherical shell at time t, which can be determined through the equation

$$-R = g\vartheta^2 \left[1 + \frac{t}{\vartheta}\left(1 - \frac{1}{2}\log\frac{t^2}{\vartheta^2}\right)\right] \ ,$$

and with the velocity q which, after t has been determined, is found from the equation $q = [g\vartheta/2]\log[t^2/\vartheta^2]$.

It has been assumed up to now, that the radius R of the sphere is larger than the largest value which s has reached at time $t = +\vartheta$, namely, $2g\vartheta^2$. If R were smaller, then it is evident that the particle μ would have collided earlier against the spherical shell, namely, at the moment in which s would become $= R$, which can be determined from the equation

$$R = g\vartheta^2 \left[1 + \frac{t}{\vartheta}\left(1 + \frac{1}{2}\log\frac{t^2}{\vartheta^2}\right)\right] \ .$$

Now, finally, when there is no continuous increase in the electric charge ε', as previously assumed, but instead of this the charge ε' remains *constant* after it reaches the value η and surpasses it by any assumed arbitrarily small value, then let us designate this constant charge as $\eta(1 + e^{-n})$, and consequently the time at which this occurred as $t = +e^{-n}\vartheta$, the velocity of the particle μ at this moment as $q = ng\vartheta$, and the distance of the particle from the center of the sphere as $s = (1 + (1 + n)e^{-n})g\vartheta^2$. This results in the differential equation:

$$dq = -\frac{ae^n}{\mu} \cdot dt \ ,$$

and from it through integration:

$$q = -\frac{ae^n}{\mu}t + C \ .$$

Now if the time is calculated from the moment in which the charge [on the spherical shell] has become constant, where the velocity $q = ng\vartheta$, thus yielding $C = ng\vartheta$, therefore, as [the constant force] a has been set $= g\mu$, [we obtain]:

$$q = \frac{ds}{dt} = -ge^n \cdot t + ng\vartheta \ .$$

From this one obtains through a second integration:

$$s = ng\vartheta t - \frac{1}{2}ge^n \cdot t^2 + C' \ ,$$

and, as has already been mentioned, for $t = 0$ we have the value from $s = (1+(1+n)e^{-n})g\vartheta^2$, yielding consequently:

$$C' = \left(1 + (1 + n)e^{-n}\right) g\vartheta^2 \ ,$$

therefore:

$$s = ng\vartheta \cdot t - \frac{1}{2}ge^n \cdot t^2 + \left(1 + (1 + n)e^{-n}\right) g\vartheta^2 \ .$$

This formula for the displacement s and the obtained formula for the velocity, namely:

$$q = -ge^n \cdot t + ng\vartheta$$

are now used, *for a constant remaining charge* ε', to determine all motions of the particle μ. They can be represented in a tabular overview, for instance in the following Table for the case in which $n = 2$, when $s/(g\vartheta^2) = s'$ and $q/(g\vartheta) = q'$ are set as above:

$\frac{t}{\vartheta}$	s'	q'	$\frac{\varepsilon'}{\eta}$
0	$1 + \frac{6}{2e^2}$	2	$1 + \frac{1}{e^2}$
$\frac{1}{e^2}$	$1 + \frac{9}{2e^2}$	1	—
$\frac{2}{e^2}$	$1 + \frac{10}{2e^2}$	0	—
$\frac{3}{e^2}$	$1 + \frac{9}{2e^2}$	-1	—
$\frac{4}{e^2}$	$1 + \frac{6}{2e^2}$	-2	—
$\frac{5}{e^2}$	$1 + \frac{1}{2e^2}$	-3	—
$\frac{6}{e^2}$	$1 - \frac{6}{2e^2}$	-4	—

This Table can easily be continued; but one can see already from it that, after the charge [on the spherical shell] has become constant, from the time $t = 2\vartheta/e^2$ onwards, the displacement of the particle μ from the center of the shell decreases and very soon becomes negative, until finally the particle μ, when s becomes $= -R$, collides against the spherical shell, at time t and with the velocity q, which can be determined from the two equations:

$$-R = \left(1 + \frac{3}{e^2}\right) g\vartheta^2 + 2g\vartheta \cdot t - \frac{e^2}{2} g \cdot t^2 \ ,$$

$$q = 2g\vartheta - e^2 g \cdot t \ .$$

One can see from this presentation of the whole process in its *context*, that none of the "inconsistent or absurd" consequences, by which Helmholtz wanted to refute the established fundamental law, actually occur.

The curve $ABCDE$ on page 169 represents the dependence of the velocity q as a function of the displacement s of the particle μ from the center of the sphere, *with a uniformly increasing charge* ε', up to the moment when this charge becomes greater than η, namely, $= \eta(1 + [1/e^2])$. This curve can now be continued in two ways, *either* for a *charge* [on the spherical shell] *continuing to grow* uniformly as before, which is represented by the curve $EFGH$ and which has already been considered, or for a charge $\varepsilon' = \eta(1 + [1/e^2])$ which remains *constant* from now on, which is related to the determinations in the Table mentioned above, after which the curve $EF'G'H'$ forms the continuation of curve $ABCDE$.

In both cases the particle μ moves in a continuous path, namely, in the first case along a straight line from A up to F and from there back to A and further to H°,[289] where the particle hits the spherical shell; in the second case along a straight line from A up to F' and from there back to A and H°.

Also the velocity of the particle along its path changes always continuously, except at *one* point K, in the middle of the path AF, where the velocity of the particle becomes infinitely large, and at the same time with it the work performed from the beginning of the motion onwards. But if we represent this performed work as *positive*, this is immediately followed by a *negative* case which is also infinitely large.

Each of these two performed works can be divided into two parts, namely, the *first* or *positive* case of the work performed along the path from A to a point at a distance $= [(n+1)/e^n] \cdot g\vartheta^2$ before K, and in the work performed along this last distance *before* $K = [(n+1)/e^n]g\vartheta^2$; the *latter* or *negative* case of the work performed on the way through the distance *after* $K = [(n+1)/e^n]g\vartheta^2$, and on the *rest of the way up to* F or F'.

Of these four performed works, the two on the path $= [(n+1)/e^n]g\vartheta^2$ *before* and *after* K are *infinitely large*, but *oppositely equal*, while the other two are also oppositely equal, but have *finite* values. Since n can now be considered so large, that the time [interval] of the first two, infinitely large performed works, namely, $2\vartheta/e^n$, can be regarded as negligible, one has two infinitely large, but oppositely equal performed works taking place in an infinitely small period of time, which, as is self-evident, have no physical effect or meaning at all.

Instead of the example above, where n was $= 2$, one can choose another example, where n is much larger, so that the difference of the charge ε', which became constant, from η becomes vanishingly small; no substantial change is brought about by this and one can see from the presentation of the whole process in context, that none of the "inconsistent and absurd" consequences, by which Helmholtz wanted to refute the established fundamental law, ever really take place.

[289][Note by AKTA:] See Footnote 288 on page 168.

Chapter 15

[Weber, 1894a, EM8] Electrodynamic Measurements, Eighth Memoir, relating specially to the Connection of the Fundamental Law of Electricity with the Law of Gravitation

Wilhelm Weber[290,291,292,293]

[290] [Web94b]

[291] The English version presented in this book is based on the translation by George Gregory (1998), [Web08]. It was edited by Laurence Hecht and A. K. T. Assis.

[292] Wilhelm Weber's Notes are represented by [Note by WW:]; the Notes by H. Weber, the editor of the fourth volume of Weber's *Werke*, are represented by [Note by HW:]; the Notes by L. Hecht are represented by [Note by LH:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[293] [Note by LH and AKTA:] This is the last of Weber's eight major Memoirs with the general title of *Electrodynamic Measurements*. This eighth Memoir, thought to be written in the 1880s, was published posthumously in 1894.

15.1 On Deriving Qualitative Differences of Bodies from Quantitative Ones, on the Hypothesis that All Ponderable Molecules are Connections of Positive and Negative Electric Molecules

A distinction has been made between the *ponderable bodies* in space in solid, liquid, and gaseous aggregate states, and *imponderable bodies*, in particular *light*, *heat material*, *two electric* and *two magnetic* substances[294] and, accordingly, all of physics has been categorized into the Mechanics of ponderable *solid*, *liquid*, and *gaseous* bodies, and the Theories of *Light*, of *Heat*, of *Electricity*, and of *Magnetism*.

In the course of the development of these theories, however, the theory of *magnetism* along with its extension into the theory of diamagnetism, has been entirely absorbed by the *theory of electricity*, so that the *assumption of two special magnetic materials* has been discarded. Likewise, the differentiation of a *light ether* for light radiation from a *heat ether* for heat radiation, has also been dropped; *an ether*, however, for light and heat radiation remains, and a *heat material* as a conductor for heat in the interior of ponderable bodies, in particular metallic heat conductors.

It is interesting to pursue this present development of physical research further and indeed, one observes:

Firstly, that the *theory of magnetism* can only be absorbed by the theory of electricity under the *assumption of movable parts in the interior* of all magnetic and magnetizable bodies, i.e., *positive electric molecules*, which form *molecular currents* around the *negatively* electrically charged ponderable molecules, in the interior of all magnetizable bodies.[295]

Secondly, by the further consideration, that the *theory of galvanism and of heat*, in order likewise to be absorbed by the theory of electricity, must also presuppose movable parts in the interior of all galvanic conductors and heat conductors; that, however, these need by no means be *different parts* which produce *in the interior of ponderable bodies*, *magnetism*, and *others* which produce the motion of *galvanism*, and still others, whose movements produce *heat*; rather that *the same parts*, according to the difference of their motions, can produce

[294][Note by AKTA:] In German: *Lichtstoff*, *Wärmestoff*, *zwei elektrische* und *zwei magnetische* Stoffe.

The German word *Stoff* can be translated as *substance*, *material* or *matter*. Therefore, *Lichtstoff* can be translated as *light*, *light substance* or *light material*. *Wärmestoff* can then be translated as *heat material*, *heat substance* or *caloric*.

The two electric substances would be positive and negative electricities, positive and negative electric fluids, positive and negative electrified particles, or positive and negative electric charges.

The two magnetic substances would be North and South magnetic fluids, or austral and boreal magnetic fluids.

According to Möller, [Möl20, Table 1.2], A.-L. de Lavoisier (1743-1794) presented a French nomenclature of simple substances or such as have not hitherto been decompounded including *Lumière* and *Calorique*. These two expressions were translated into German as *Lichtstoff* and *Wärmestoff*. In English they were translated as *light* and *caloric*. At the time of Lavoisier, light (Lichtstoff, lumière) and caloric (Wärmestoff, calorique) were still assumed to be matter by many scientists.

[295][Note by LH and AKTA:] The signs of the charge are reversed in Weber's notation compared to modern usage. Thus the particle of positive charge, orbiting about a negatively charged central body, corresponds to the negatively charged electron of modern parlance orbiting about a positively charge nucleus. It should be noted that Weber is 30 to 40 years ahead of his time in proposing an electric atom. See also the book *Weber's Planetary Model of the Atom*. In English, [AWW11], in Portuguese, [AWW14], and in German, [AWW18].

magnetism, *galvanism*, and *heat*, sometimes together, sometimes separately, and that these moving parts in the interior of ponderable bodies are *molecules of one electricity*, which ought to be called *positive electricity*.

Thirdly, it is to be considered, that the movements of these positive electric molecules around the negatively electrically charged *ponderable molecules* of bodies, *either* form closed orbits, or spiral orbits differing only slightly from circular orbits with *periodically* increasing and attenuating diameter, *or* spiral-shaped orbits with continuously increasing diameter, whereby they ultimately pass over into a ballistic trajectory, thus effecting the transfer of this electric molecule from *one ponderable molecule* to *another neighboring ponderable molecule*, whereupon in part *heat conduction*, and in part *galvanic currents* in metallic conductors, are based.

Fourthly, and finally, it is furthermore to be considered, that by means of *magnetic* or *electrodynamic induction from the outside*, circular currents around the *ponderable molecules* of a body may be excited, or circular currents already in existence may be *enhanced*, *weakened*, or *changed in direction*.

It is self evident, that *all metallic heat- and electricity- conductors* belong to the class of ponderable bodies, around whose molecules *positive electric molecules* move in circular orbits, albeit with increasing diameters, which make a transition into a ballistic trajectory and thus a transition from those *ponderable* molecules about which they revolve, to neighboring *ponderable molecules*; that, on the other hand, all transparent bodies, such as glass and crystals, belong to the class of *ponderable bodies* about whose molecules positive electric molecules indeed move, but only in *tighter circles without transition into a ballistic trajectory* (which, therefore, are neither conductors of heat nor electricity), while the greater remaining part of the body consists of the space (like the cosmic space)[296] between the *ponderable molecules*, with positive electric molecules forming the light ether, and characterized by ballistic or wave motion.

As concerns the molecular currents formed by *electric molecules* around *ponderable molecules*, it is evident, that an attractive force issuing from the ponderable molecules is required for the persistence of such circular currents, and the question is merely, where this attractive force comes from? Is a contrary electric charge of the ponderable molecules necessary for it, or can each ponderable molecule found in each center exert this attractive force for itself alone? It turns out, that this force of attraction can be exerted by the *ponderable* molecules for themselves alone, without an additional electric charge, and indeed upon a *positive* electric molecule circling about it, as well as on a *negative* electric molecule, presupposing that the following two assumptions, first clearly and definitively expressed by Zöllner,[297] are met:

1. *That all ponderable molecules are mere connections of equal quantities of positive and negative electricity,* and that

2. *The force of attraction of equal quantities of different kinds of electricity is greater than the repulsive force of the same quantities of similarly charged*

[296][Note by AKTA:] In German: *Weltenraume*. This word can also be translated as outer space, deep space or cosmos, see footnote 241 on page 149.

[297][Note by LH and AKTA:] Johann Karl Friedrich Zöllner (1834-1882), [Zöl78]. See also [Zöl82] and [Mos36] with English translation in [Mos66].

electricity.[298]

These two assumptions form the foundation for that *theory of ponderable bodies*, according to which the *law of gravitation* which is valid for all of these bodies, is yielded as a necessary consequence of the fundamental law of electricity.

It is easy to appreciate the great importance, which the confirmation of the above assumptions would have for all of physics, if one considers the extraordinary multiplicity of *qualitative differences* of ponderable bodies, all of which, accordingly, would have to be reducible to mere *quantitative* differences, which differences would have to be derivable from the fundamental law of electricity.

15.2 The Derivation of the Law of Gravitation from the Fundamental Law of Electricity According to Zöllner

The derivation of the law of gravitation from the fundamental law of electric action according to Zöllner, requires closer examination in order to be able to build further upon it.

According to Zöllner, it is assumed, that every *ponderable molecule* consists of one or more molecules of positive electricity and one or more molecules of negative electricity, where the first is denoted with $+e$ or $+ne$, the latter with $-e$ or $-ne$. The numerical value of e (aside from the sign) serves to determine the *quantity of electricity* of a molecule, independent of the *kind of electricity*, which may be positive or negative, for e is made dependent merely upon the choice of the *unit of length* and of the *unit of force*, whereas ee denotes the *repulsive force* of a positive or negative electric molecule $\pm e$ *on a molecule equal to it in the unit of distance.*

It is furthermore assumed, that the magnitude e, which is called *quantity of electricity*, and is distinguished from the *mass* ε of the molecule, is *equal for all electric molecules*, and that, consequently, the ponderable molecule composed of $+e$ and $-e$ is always *neutral*, i.e., that it behaves the same with respect to a $+e$ as to a $-e$. The same holds as well for ponderable molecules composed of $+2e$ and $-2e$, or $+3e$ and $-3e$, etc.

From this equality of quantities of electricity, which, according to Zöllner, holds for *all simple electric molecules*, whereupon the *neutrality* of *ponderable molecules* composed of an equal number of positive and negative electric molecules is based, it does indeed follow, that there is an *equality of mass* of all *positive electric* molecules among themselves, as well as of all *negative electric* molecules among themselves, but it by no means follows, that *the masses of positive and negative molecules are the same*, rather the decision about the equality or inequality of their masses must remain for experiment to determine, be it by direct measurements of mass, or by an indirect route by investigating their connection with other measurable phenomena. It then results from the *fundamental law of electric action*,[299]

[298] [Note by LH and AKTA:] That is, the attractive force between the charges $+e$ and $-e$ is greater than the repulsive force between $+e$ and $+e$, and also greater than the repulsive force between $-e$ and $-e$.

[299] [Note by LH and AKTA:] What Weber calls the *fundamental law of electric action* is the expression he introduced in 1846 for the force between the electric charges e and e' separated by a distance r, i.e.:

$$\frac{ee'}{r^2}\left[1-\frac{a^2}{16}\left(\frac{dr}{dt}\right)^2+\frac{a^2}{8}r\frac{d^2r}{dt^2}\right] ,$$

that the force which *two ponderable molecules* (where each is composed of $+e$ and $-e$) exert upon each other, are the sum of four forces, which the two constituent parts $+e$ and $-e$ of *one ponderable molecule* exert, from an arbitrary distance r, in relative rest or motion, upon the two constituent parts $+e$ and $-e$ of the *other ponderable molecule*, namely, *firstly* the two *repulsive forces* of the *molecules of similar kind*[300] contained in the ponderable molecules:

the *repulsive force* of $+e$ and $+e$

$$= \frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) \ ,$$

the *repulsive force* of $-e$ and $-e$

$$= \frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) \ ;$$

secondly, the *attractive forces* of the *dissimilar electric molecules* contained in the same ponderable molecules, which, according to Zöllner's assumption, ought to be larger in a relationship of $1 : 1 + \alpha$, namely,

the *attractive force* of $+e$ and $-e$

$$= -(1+\alpha)\frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) \ ,$$

the *attractive force* of $-e$ and $+e$

$$= -(1+\alpha)\frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) \ .$$

This yields *a negative value of the repulsive force* (i.e., an attractive force) of *two ponderable molecules*, each of which consists of one $+e$ and one $-e$, i.e., the value

$$-2\alpha\frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) \ ,$$

see [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

In 1852, he replaced the constant $a^2/16$ by $1/c^2$ obtaining:

$$\frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2}{c^2}r\frac{d^2r}{dt^2}\right] \ ,$$

see [Web52b, p. 366 of Weber's *Werke*] with English translation in [Web21a].

In Weber's formulation, the force between particles was thus dependent upon their *relative velocity* and *relative acceleration*. This constant c would represent the uniform relative velocity at which the force between particles would fall to zero. Weber's c (known throughout the 19th century as the *Weber constant*) is not the same as the modern $c = 2.998 \times 10^8\ m/s$, but $\sqrt{2}$ times this last value (or, $c = 4.24 \times 10^8\ m/s$). The *Weber constant*, c, was first measured by Weber and Kohlrausch in 1854-1856. They obtained $c = 4.39 \times 10^8\ m/s$. See [Web55] with English translations at [Web21g]; [WK56] with English translation in [WK03] and Portuguese translation in [WK08]; and [KW57] with English translation in [KW21].

[300][Note by AKTA:] In German: *gleichartig elektrischen Moleküle*. That is, particles with charges of the same sign.

where the unknown value of α can be determined by the consideration, that the above force, of which α is a factor, is set equal to the known *gravitational force* of the two ponderable molecules upon each other.

In this it has been assumed, that the two *electric molecules* which belong to a *ponderable molecule* always remain at a negligibly small distance from each other.

If V is the *potential* of the two ponderable molecules, and consequently their repulsive force is

$$\frac{dV}{dr} = -2\alpha\frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) ,$$

then that *potential* is

$$V = -2\alpha\frac{ee}{r^2}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) .$$

If now the mass of $+e$ is represented by ε, the mass of $-e$ is represented by $a\varepsilon$, then the acceleration of *one* ponderable molecule in the direction r, turns out to be $= 1/\varepsilon \cdot [dV/dr]$, that of the *other* in the opposite direction is $= 1/a\varepsilon \cdot [dV/dr]$; consequently, the relative acceleration of the first molecule with respect to the other is

$$\frac{d^2r}{dt^2} = \frac{1+a}{a\varepsilon} \cdot \frac{dV}{dr} .$$

Multiplying this equation by $2dr$, one obtains the following differential equation:

$$2\frac{dr}{dt} \cdot \frac{d^2r}{dt} = 2\frac{1+a}{a\varepsilon}dV ,$$

and by integration from $r = r_0$ to $r = r$, if r_0 represents that value of r, for which $[dr/dt] = 0$,

$$\frac{dr^2}{dt^2} = \left(1 + \frac{1}{a}\right) \cdot \frac{4\alpha ee}{\varepsilon}\left[\frac{1}{r}\left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2}\right) - \frac{1}{r_0}\right] ,$$

or, when $dr^2/dt^2 = c^2u^2$ and the constant

$$\frac{4(1+a)}{a\varepsilon} \cdot \alpha ee = c^2\rho ,$$

is established, then

$$u^2 = \rho\left[\frac{1}{r}\left(1 - u^2\right) - \frac{1}{r_0}\right] ,$$

the same equation which was found in the 7th Memoir on "Electrodynamic Measurements", p. 668,[301,302] for two dissimilar electric molecules e and e', only that here, where the issue is

[301][Note by HW:] Wilhelm Weber's Werke, Vol. IV, p. 385.

[302][Note by LH and AKTA:] [Web78a, p. 385 of Weber's *Werke*] with English translation in [Web21e], see Chapter 13.

the interaction of *ponderable* molecules, the factor 2α is added to the value of ρ and $-e$ and $a\varepsilon$ are applied for e' and ε'.[303,304]

This thus yields, for the interaction of two molecules at relative rest, consisting of *equal amounts* of positive and negative electricity, the same law as for two molecules subjected to the law of gravitation for all distances in comparison to which ρ is negligible; but for *molecular distances*, for which this is not the case, there are divergences from the Newtonian Law,[305] which, were they corroborated, would serve as the best proof, that the ponderable molecules *really* do consist of *equal amounts* of positive and negative electricity.

Such *molecular distances* come into special consideration in the case of *ponderable gas molecules* according to the *dynamic gas theory*. Maxwell ("On the Dynamical Theory of Gases," Philos. Transact., Vol. 157, Part I, page 49 ff.)[306] has already found, that the law of reflection and dispersion for the ballistic motion of the gas molecules in their encounter, necessarily assumed to explain the behavior of gases according to this theory (but which can not be founded upon the Newtonian law of gravitation), could be based upon an assumption conceived particularly for the purposes of this explanation, of a repulsive force proportional to the 5th power of the distance of the molecules, an assumption, however, which would otherwise be in no way justifiable. — Any such arbitrary assumption may now be entirely avoided, if all ponderable molecules, consequently also all gas molecules, are connections of *equal amounts of positive and negative electricity*, since the law of gravitation holds for such molecules only at greater distances, but, for molecular distances, the *law of reflection and dispersion* is similarly yielded, as for two *similar electric* molecules in ballistic motion, which encounter one another according to the theory developed in the 7th Memoir

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[303][Note by WW:] From the above equation, where the velocity u is expressed in parts of the velocity c known from the electric fundamental law, i.e., from

$$u^2 = \rho\left(\left[\frac{1}{r}\right](1-u^2) - \frac{1}{r_0}\right) ,$$

one obtains

$$u = \pm\sqrt{\left[\frac{\rho}{r_0}\right]\cdot\left[\frac{r_0-r}{r+\rho}\right]} ,$$

i.e., for each distance r, two opposite equal values of u, one *positive* for the case of mutual separation of the molecules, and one *negative* for the case of mutual approach. If one considers here $r_0 - r = s$ as the *space of fall* downwards from the point of rest, in that one conceives the one ponderable molecule as like the mass of the Earth concentrated in one point, the other molecule conceived as represented by the falling stone, and u as the velocity of fall v of the stone expressed in parts of c, i.e., $cu = v$, this yields

$$\frac{v^2}{s} = \frac{\rho c^2}{[r_0(r_0+\rho)]} ,$$

i.e., the *Galilean law of falling bodies*, where the *constant*

$$\frac{\rho c^2}{[r_0(r_0+\rho)]}$$

has the significance of the constant usually denoted by $2g$ in the Galilean law of falling bodies.

[304][Note by AKTA:] See [Gal54] and [Gal85].

[305][Note by AKTA:] See footnote 32 on page 19.

[306][Note by LH and AKTA:] See [Max67]. For a discussion of all quotations of Maxwell made by Weber, see [AW03].

on "Electrodynamic Measurements."[307,308]

An issue of special consideration in this interaction of *two ponderable molecules* consisting of equal amounts of positive and negative electricity, is that *equal forces* are exerted by the one molecule upon *both constituent parts of the other*, both upon the positive as well as the negative, and these are *forces of attraction*, the sum of which yields the *gravitational force* exerted by the one molecule upon the other.

By virtue of the here postulated force of attraction exerted by every *ponderable* molecule, not merely upon another *equal* molecule, but upon each of its two *constituent parts*, all of those ponderable molecules which had *first* met up with *positive electric* molecules, would have bound them as *positive electric satellites*, and, on the other hand, other *entirely identical* ponderable molecules, which had first met up with *negative electric molecules*, would have bound them as *negative electric satellites*; and, therefore, *all ponderable molecules* would fall into three classes, which can be distinguished as *positive ponderable*, *negative ponderable*, and *neutral*, of which the latter would be such ponderable molecules, which had not yet drawn satellites to themselves.

If all of these satellites remained bound with the ponderable molecules in the same way, they would have to be considered as belonging to them, and therefore *their mass would be the mass of the ponderable molecules* to which they belonged, and the mutual *gravitational force of the molecular pair* exerted by the satellites of *two* ponderable molecules, as well as that exerted by each of the two satellites upon that one of the two ponderable molecules to which it itself does not belong, would be either added or subtracted, depending upon the difference of the sign.

15.3 On the Inadequacy of Direct Attempts to Decide the Question, Whether in the Case of Equal Amounts of Electricity, the Attractive Force of Two Dissimilar Electric Molecules would Really Be Larger Than the Repulsive Force of Two Similar Electric Molecules

Metal conductors may be taken, for example two hollow spheres of copper, which are charged with equal amounts of positive or negative electricity, and the repulsive or attractive force exerted by two of these, respectively, at equal distances, can be measured with great accuracy. Were the precision of this measurement in no way limited, it is apparent, that it would have to be determinable thereby, whether, given equally strong charges, the *attractive force of dissimilarly charged conductors* were larger than the *repulsive force of similarly charged conductors*, or not.

The most precise instruments and experiments, which have been carried out to the purpose of similar measures, have been described in the 4th Memoir on "Electrodynamic

[307][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV. p. 389.

[308][Note by AKTA:] [Web78a, Section 7, p. 389 of Weber's *Werke*] with English translation in [Web21e, Section 7]. See, in particular, Section 13.7 of Chapter 13.

Measurements",[309,310] and the question is thus posed, whether, with the same instruments, measurements may be carried out to decide the above question.

With such a torsion balance as described in that place, the *repulsive forces* of *two similarly electrically* charged spheres, as well as the *attractive force* of *two dissimilarly charged* electric spheres, could indeed be measured; it is, however, easily seen, that in the latter case, when the one sphere is positively charged, the other negatively, the *equality of the strength* of the two charges, could only be measured by completely discharging the two spheres in a condition of contact with one another. In order, therefore, to be certain of the *equal strength of the charge* immediately prior to the discharge, which is the essential point here, one can only bring two *entirely identical torsion-balances* into action simultaneously, by charging the *revolvable sphere* of the one torsion-balance *positively*, and the other *negatively*. But for each revolvable sphere of the torsion balance, there is a completely identical *fixed sphere*, which is in contact with the revolvable sphere when it is charged, which is the guarantee for the equality of the charge of both spheres of each pair. Now, the strength of the charge of the *positively* charged pair may be different from that of the *negatively* charged; this difference is precisely determinable, however, by measurement of their *force of repulsion*. If these repulsive forces of the positively charged [pair] are found to be the same as those of the negatively charged, it follows that *the absolute values of that positive and this negative charge are equal*.

These two *fixed spheres* are now, however, to be firmly bound to each other by a *well-insulating rod of shellac*, and let this rod of shellac be equipped with a pivot at its mid-point, about which it may be so rotated, that after a half rotation both fixed spheres have exchanged their positions, so that the distances of the two fixed spheres from the revolvable spheres of the two torsion-balances have remained unchanged.

If one denotes the equal charges of the revolvable sphere and the fixed sphere of the *first torsion-balance* with $+e$, and their distance from each other with r, and the same magnitudes for the *second torsion-balance* with $-e'$ and r', then *prior to the exchange of the fixed spheres* one obtains the two *repulsive forces* f and f', measured with the two revolvable scales, equal to ee/r^2 and $e'e'/r'^2$; *after the exchange of the fixed spheres*, however, the measured *attractive forces* g and g' equal to $-(1+\alpha)\cdot ee'/r^2$ and $-(1+\alpha)e'e/r'^2$, from which the ratio of the product of the two *measured forces of repulsion* to the two *measured forces of attraction*, yields $ff' : gg' = 1 : (1+\alpha)^2$, where α can be determined by the measured magnitudes $ff'gg'$, i.e., $\alpha = \sqrt{gg'/ff'} - 1$.

But even with the greatest perfection of the *torsion-balances* produced for these measurements and with highest precision in the conduct of all measurements, it will not be possible to demonstrate with certainty, for equal charges, a difference in the magnitude of the *force*

[309] [Note by HW:] Wilhelm Weber's *Werke*, Vol. III, p. 670.
[310] [Note by LH and AKTA:] [KW57] with English translation in [KW21], see also [WK68].

of repulsion of similar electric charges, because this difference is much too small.[311,312]

If, however, in the future, it turns out, even with the measurements conducted with the most perfect instruments, that the value of α is much too small to allow of a secure determination, there would remain still one other factor contained in the fundamental law of electric action, which could be used for the purpose of testing and confirming the Zöllner derivation of the law of gravitation from the electric law, namely, the factor

$$\left(1 - \left[\frac{1}{c^2}\right]\left[\frac{dr^2}{dt^2}\right] + \left[\frac{2r}{c^2}\right]\left[\frac{d^2r}{dt^2}\right]\right) \ ,$$

for which the proof would be of great interest, that its influence upon the movement of some heavenly bodies, although very small, were yet measurable.

C. Seegers[313] first dealt with this in the essay *De motu perturbationibusque planetarum secundum legem electrodynamicam Weberianam solem ambientium. Scripsit C. Seegers, Gott. 1864*, following which Prof. Scheibner in Leipzig found eight years later,[314] that, while maintaining the numerical value of c for the Weber constant, at best the difference that could be observed in the movement of Mercury was a secular variation of the perihelion of 6.73 arc-seconds. — Finally, Tisserand, on 30 September 1872, provided an essay to the French Academy: "Sur le mouvement des planetes autour du Soleil d'après la loi electrodynamique de Weber." *Compt. rend.* 1872, Sept. 30,[315] where he finds the value 6.28 for the secular variation of the perihelion for the case of Mercury, and the value of 1.32 seconds for Venus. As small as these corrections might be, it is evident that it is possible to confirm or refute them with more precise observations.

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[311] [Note by WW:] Should it turn out, that out of all of the measurements conducted with the most perfect *torsion-balances*, the value of α is far too small to allow a secure determination from such observations, the question would still remain, whether it might be possible, on the basis of *other phenomena and observations*, to determine the magnitude of α, for example on the basis of the so manifold, most interesting phenomena and observations provided by Crookes' *light mills*.

One does not yet know exactly the difference which would be necessary between the front and back sides of the vanes of these light mills, so that the mills might be set into rotational motion by means of beams of light. Without a difference, however slight, between the two sides of the vanes, be it in their constitution or form (convex or concave), no rotation occurs.

It were, for example, possible, that, in the Crookes' light mills, the *necessary difference* between the front and back sides of each vane, is based on an *electric difference*, for example, that the front side were more *positively electric*, the back side *negatively electric*, and that the beam of light would produce the rotation, by acting like a positively charged conductor, which *repulsed* the *positive* electric *front side* of one vane turned toward it, and would *attract* the *negative* electric *back side* of the other vane turned toward it.

Were this the case, it is evident, that one would only need to combine a second pair with the first pair of vanes, which would rotate with the latter about the same vertical axis. If, now, the second pair stood at right angles above the first, but turned toward the opposing side of the beam of light, then, given perfect symmetry, no rotation would occur, if the *force of attraction of dissimilar* electric charges, given *equal strengths of charge*, were equal to the repulsive force of similar electric charges; but there would be a rotation, if the force of attraction of *dissimilar electric charges*, given equal strengths of charge, were larger than the force of repulsion of *similar electric charges*.

With the extraordinary precision and sensitivity, of which the light mills are capable, one may hope to be able to observe the *rotational action* of this, however slight, preponderance of that force of attraction, and from that be able to determine the magnitude of α.

[312] [Note by AKTA:] See [Cro74] and [Woo66].

[313] [Note by AKTA:] [See64] with German translation in [See24].

[314] [Note by AKTA:] W. Scheibner's result was presented by Zöllner, [Zöl72, p. 334], [Zöl76a, p. 216] and [Zen21, p. 46]. Scheibner's work was published only in 1897, [Sch97].

[315] [Note by LH and AKTA:] [Tis72] with English translation in [Tis17a]; [Tis90] with English translation in [Tis17b]; and [Tis96, Chapter XXVIII, Sections 225 and 226, pp. 499-503].

For comparison with the Newtonian law: the *law of gravitation derived by Zöllner from the fundamental law of electric action*, two identical *ponderable molecules*, each of which consists of a $+e$ and a $-e$ (where each would exert the unit of force upon the molecule identical to it, at relative rest, at a given unit of distance), would exert a *force of attraction* upon each other at any arbitrary relative velocity and acceleration of

$$= 2\alpha \frac{ee}{r^2}\left(1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) .$$

While according to the *Newtonian law*, *two equal ponderable molecules of mass* m, expressed in parts of that mass which exerts the unit of accelerating force at the given unit of distance, would exert a *force of attraction*

$$= \frac{mm}{r^2} ,$$

to which one can add, that if n denotes the *number* of ponderable molecules constituted each of one $+e$ and one $-e$, contained in the *same unit of mass* according to which m is expressed, $2\alpha nnee/r^2$, thus

$$n = \frac{m}{e}\sqrt{\frac{1}{2\alpha}} .$$

15.4 Electric Filling of Space, Particularly Concerning the Existence of an Ether Consisting of Similar Electric Molecules — Light Ether — in All Spaces Not Occupied by Ponderable Bodies. — Manifold of Ponderable Bodies

The assumption of the existence of a dispersed *imponderable ether* consisting of similar electric molecules, in all spaces not occupied by *ponderable bodies*, depends, as already remarked in the foregoing Section, essentially upon the assumption, that all *ponderable molecules* are constituted of positive and negative electric molecules, and indeed in *equal amounts.* From that, it would follow that, if negative electric molecules existed at all in the world, but not in amounts equal to the positive [molecules], *the surplus of positive electric molecules* (apart from the possibly existent *satellite* ponderable molecules, mentioned at the conclusion of the first Section, which, however, may be formed by *both kinds* of electric molecules, without there being any reason for a greater number of one kind) would necessarily be dispersed in all spaces or intervals not occupied by ponderable bodies, and indeed, in consequence of the mutual repulsions, reflections and dispersions in all of their encounters, in *nearly uniform proliferation.*

But if now each *ponderable* molecule contains *equal amounts of positive and negative* electricity, which can be denoted with $+e$ and $-e$, the *equality of the amounts* by no means entails the *equality of the masses*, which must be distinguished from them, and thus the mass of the *amounts* of $+e$ may be denoted as ε, the mass of the *amounts* of $-e$ with $a\varepsilon$.

But even if the two *equal amounts* of two dissimilar electric molecules $+e$ and $-e$ can combine into a ponderable molecule, there will be no combination *into a point*, rather, as close to one another as the two molecules may come, they will still always remain separated from one another, in that they rotate around one another; the two, which together have the mass of $(1 + a)\varepsilon$, will, however, always remain in a very small space, which does not change under conditions of unchanged angular velocity, so that a certain *density* $d = [(1 + a)/v] \cdot \varepsilon$ can be ascribed to such a *ponderable molecule*.

Were now such a combination *of all negative electric* molecules with *positive electric* molecules to have occurred, then all of the thus generated *ponderable* molecules, taken together, would occupy a certain *space*, and the entire remaining space would be empty, were not one of the two electricities, which can be taken as *positive*, existent in considerable surplus. In the case of such a *surplus*, however, all of the remaining space would be filled by an extant electricity of mutually *repelling* molecules, everywhere present in uniform distribution.

Let all of these electric molecules uniformly filling the empty space be denoted as *imponderable ether*, while all of the *pair-wise combined* molecules, reduced into a closed space, form the world of *ponderable* bodies, accordingly, therefore, since *mass* is an attribute of all molecules, *ponderable masses* and *imponderable masses* must be distinguished from one another.

The *law of gravitation* has been called the *law of indifferent mass attraction*, which, according to the forgoing, would not be correct. For if positive electric molecules also exist, which are not combined with negative electric molecules into *ponderable* molecules, and thus do not belong to the world of *ponderable bodies*, which do, however, possess *mass*, then the Newtonian *law of gravitation* does not hold *for the masses of these latter molecules*, but an entirely different law, i.e., the *fundamental law of electric interaction*, according to which the action is not always that of attraction, but just as often that of repulsion, and the validity of the law of gravitation must be limited to the masses of those *pair-wise combined* positive and negative electric molecules, called *ponderable molecules*. Those *positive electric molecules*, which exist distinct from negative electric molecules, and which fill, in uniform distribution, the empty space of the ponderable bodies, form the so-called *ether* — light ether.

If the *law of gravitation* holds neither for positive electric molecules as such, nor for the negative as such, but for all *ponderable molecules* formed by combination of these two kinds of molecules, it is evident, since all characteristics of *combined* molecules must be based upon the characteristics of *non-combined* molecules, that the law of gravitation for all ponderable molecules must be based, in general, upon the law of electric reciprocal action, as already demonstrated in Section 15.2.

But were all ponderable bodies really only combinations of positive and negative electric molecules, the issue would be, given the essentially identical constitution of all ponderable bodies, how to explain the *infinite multiplicity and difference* of these ponderable bodies. The *reason for all of these differentiations* could only be found in different *numbers*, *spatial arrangements* and *kinetic energy* of the electric molecules of both type combined in smaller groups, which need not be subjected to changes by external influences. The influence of the *number* and *arrangement*, as long as these were to remain unchanged, would be easier to observe and determine than the influence of different *kinetic energies* of the molecules acting upon one another in a group, whose laws have been impossible to completely develop out of the known basic laws, even under the limitation of only three molecules.

15.5 Classification of Material-Molecules According to Their Composition and Differentiation

If there are two kinds of *simple* material particles, viz., simple *positive electric* and simple *negative electric*, then it is possible that, by combination of a plurality of electric particles[316] of either type with one another or of particles of one type with those of the other type, many kinds of different *composite* molecules can be formed, and indeed, first of all *indivisible molecules*.[317]

Let the initial assumption be, that all positive and negative electric particles possess the same *amounts* e and the same *mass* ε, so that the force is ee with which two similar particles at rest at a given unit of distance, repel each other, and ee/ε is the velocity which this force would impart to each of the two particles in a given unit of time.

Such a composite *indivisible molecule* is formed of *two simple positive electric* or *negative electric* particles, which are at a smaller distance from one another than their critical distance ρ; for these attract one another with a force which would become *infinitely* large, given a distance growing up to ρ, from which it is evident, that no finite external force would be capable of pulling them apart to a distance of ρ, and thus also not beyond ρ. Both particles must therefore always remain within distances smaller than ρ.

Many more forms of such composite *indivisible* molecules can exist, because, if a simple, for example positive electric particle came together with more than one other such particle in a space so close, that the distances of these from one another were all smaller than ρ, so that *all of these particles together would form just one such indivisible molecule*, as is the case for two. If of that plurality, for example of *three* particles, a, b, c, two of them, a and b, were to approach the distance ρ, where their mutual *force of attraction* would become infinitely large, it would only be possible for the third particle, c, to annul this force of attraction's becoming infinitely great, if this third particle were upon the opposite side at the same distance, thus at a distance $= 2\rho$ from a, which would be contrary to the assumption.

Under the same assumption for *negative electric* particles as that just made for *positive electric* particles, just as many cases of indivisible molecules could exist, composed of simple *negative* electric particles, as cases of indivisible molecules composed of simple *positive electric* molecules.

Additionally, not only two or three, but a far larger number of *similar electric particles* could be together in such a small space, without the distance of any particle from another being greater than or equal to ρ, so that all of these particles together, also form an *indivisible molecule which remains together for ever.* And finally, it should be noted that these particles enclosed in a small space of a molecule, have as little need to be at rest as the particles originally dispersed in larger spaces, but they can have the most manifold movements, partly *together*, in close connection with one another in space, partly *against* one another within the small space in which they are, without thereby ceasing to form an *indivisible group* or a single *composite molecule.* Each such composite molecule forms an enclosed world for itself, and according to the difference of the number of simple electric particles which it contains, and their mutual movements, such a composite molecule can exert quite diverse actions upon all other molecules lying outside of it, according to which *very diverse characteristics* may obtain for that molecule. If one further considers, that the *number* of simple electric

[316][Note by AKTA:] In German: *elektrischen Theilchen.*

[317][Note by AKTA:] In German: *unscheidbarer Moleküle.* This expression was translated as *indissoluble molecules* in [Web08]. Here we replaced everywhere this expression by *indivisible molecules.*

particles which can be combined in this way, although not unlimited, can yet be very large, it is conceivable, that such eternally unchangeable, partly positive, partly negative electric particles or molecules can recombine themselves to quite *different ponderable bodies*, for example of very different density or hardness, etc., for each *group* consisting of a larger number of *similar electric particles*, *partly positive*, *partly negative*, of which each occupies only a spherical space of diameter ρ, must obviously attract each other and combine with a *force much larger* than a simple positive electric molecule with a simple negative electric molecule.

For all such *indivisible molecules* composed of more than two similar electric particles, three cases can be distinguished, *firstly* the case where all of these molecules lie so close to one another, that, by actions at a distance, they can be considered as united at one point; *secondly*, the case where two molecules rotate around one another; and *thirdly*, the case where a larger number of molecules move in different orbits about each other in the space they occupy. Different characteristics of the molecules may be based upon these differences.

This yields the following classification of material molecules, first of the *indivisible electric* molecules, then of the *ponderable* molecules. The *positive or negative particles* contained in a molecule, are denoted by their number and + or − value sign, and that these together form an *indivisible molecule* is indicated by enclosure in brackets.

1. Simple electric molecules:

positive electric molecule $(+1)$,

negative electric molecule (-1).

2. Composite indivisible electric molecules

of *positive electric*:

of two $(+2)$,

of three $(+3)$,

of four $(+4)$,

of five $(+5)$,

of six $(+6)$, etc.

of *negative electric*:

of two (-2),

of three (-3),

of four (-4),

of five (-5),

of six (-6), etc.[318]

3. *Ponderable* molecules, composed of the same number of *positive and negative electric* molecules, which may be arranged graphically as follows.

[318][Note by WW:] Molecules $(+n)$ and $(-n)$, where n were a larger number, as a consequence of their larger reciprocal force of attraction, will seldom appear singly, but mostly only in composites $\begin{bmatrix} +n \\ -n \end{bmatrix}$, i.e., as *ponderable molecules*.

$$\begin{bmatrix}+1\\-1\end{bmatrix}$$

$$\begin{bmatrix}-2\\+1\\+1\end{bmatrix}\begin{bmatrix}+2\\-2\end{bmatrix}\begin{bmatrix}+2\\-1\\-1\end{bmatrix}\begin{bmatrix}+1\\+1\\-1\\-1\end{bmatrix}$$

$$\begin{bmatrix}-2\\-1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-3\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-3\\+2\\+1\end{bmatrix}\begin{bmatrix}+3\\-3\end{bmatrix}\begin{bmatrix}+3\\-2\\-1\end{bmatrix}\begin{bmatrix}+3\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+2\\+1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+1\\+1\\+1\\-1\\-1\\-1\end{bmatrix}$$

$$\begin{bmatrix}-2\\-1\\-1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-2\\-2\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-3\\-1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-4\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-4\\+2\\+1\\+1\end{bmatrix}\begin{bmatrix}-4\\+2\\+2\end{bmatrix}\begin{bmatrix}-4\\+3\\+1\end{bmatrix}\begin{bmatrix}+4\\-4\end{bmatrix}\begin{bmatrix}+4\\-3\\-1\end{bmatrix}\begin{bmatrix}+4\\-2\\-2\end{bmatrix}\begin{bmatrix}+4\\-2\\-1\\-1\end{bmatrix}\begin{bmatrix}+4\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+3\\+1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+2\\+2\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+2\\+1\\+1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+1\\+1\\+1\\+1\\-1\\-1\\-1\\-1\end{bmatrix}$$

$$\begin{bmatrix}-2\\-1\\-1\\-1\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-2\\-2\\-1\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-3\\-1\\-1\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-3\\-2\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-4\\-1\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-5\\+1\\+1\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-5\\+2\\+1\\+1\\+1\end{bmatrix}\begin{bmatrix}-5\\+2\\+2\\+1\end{bmatrix}\begin{bmatrix}-5\\+3\\+1\\+1\end{bmatrix}\begin{bmatrix}-5\\+3\\+2\end{bmatrix}\begin{bmatrix}-5\\+4\\+1\end{bmatrix}\begin{bmatrix}+5\\-5\end{bmatrix}\begin{bmatrix}+5\\-4\\-1\end{bmatrix}\begin{bmatrix}+5\\-3\\-2\end{bmatrix}\begin{bmatrix}+5\\-3\\-1\\-1\end{bmatrix}\begin{bmatrix}+5\\-2\\-2\\-1\end{bmatrix}\begin{bmatrix}+5\\-2\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+5\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+4\\+1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+3\\+2\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+3\\+1\\+1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+2\\+2\\+1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+2\\+1\\+1\\+1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}+1\\+1\\+1\\+1\\+1\\-1\\-1\\-1\\-1\\-1\end{bmatrix}$$

etc.

Here, each of the numbers enclosed in the same brackets refers to a number of like electric particles moving about each other, whose distance from one another remains smaller than ρ. These indivisible particles move together in *one* orbit, and a particular orbit corresponds to each number. The orbits of dissimilar electric particles are held together by mutual attraction. The molecules comprised in each number are, accordingly, indivisible and likewise also all of the molecules of the *second* of the *three classes enumerated above*, also those, which are composites of many simple molecules, because they are similar and their distances from one another are $< \rho$.

In all molecules of the *third class*, on the other hand, those listed as *positive electric* under $+$ are *possibly* always divisible from the *negative electric* listed under $-$, even if no force sufficient to cause their dissolution[319] exists. In reality, no such dissolution, whereby a ponderable body were broken up into its imponderable constituent parts, has been observed. But since the dissolution of ponderable bodies into ponderable constituent parts is often observed, by continued dissolution, however, one finally arrives at ponderable bodies which have not been further divisible, one has indeed called these latter ponderable bodies *elemental*

[319][Note by AKTA:] In German: *Scheidung*. It can be translated as "dissolution" or "separation".

bodies, whereby however the *possibility* of their dissolution into positive and negative electric molecules is not excluded.

Those ponderable molecules will be the most difficult to break down, where many *similar electric* particles are at distances from one another smaller than ρ, thus all molecules denoted as $\begin{bmatrix} +n \\ -n \end{bmatrix}$, where n is a larger number. Ponderable molecules $\begin{bmatrix} +n \\ -n \end{bmatrix}$ with large number values n will accordingly behave as *ponderable elementary bodies*, while on the other hand ponderable molecules where n is a small number, for example 1, will be most easily divisible into *electric elements*.

If one now takes the weight of the ponderable molecule $\begin{bmatrix} +1 \\ -1 \end{bmatrix}$ as the unity of atomic weight, n would be the atomic weight of the molecule $\begin{bmatrix} +n \\ -n \end{bmatrix}$. The smallest ponderable molecule known to us is that of hydrogen, and is usually given the value = 1. Accordingly, the atomic weights of the other previously not divisible ponderable bodies and the composition out of positive and negative electric elementary particles, would be as follows:

	Atomic weight	Electric composition
Hydrogen	1	$\begin{pmatrix} +1 \\ -1 \end{pmatrix}$
Carbon	12	$\begin{pmatrix} +12 \\ -12 \end{pmatrix}$
Lithium	13	$\begin{pmatrix} +13 \\ -13 \end{pmatrix}$
Beryllium	14	$\begin{pmatrix} +14 \\ -14 \end{pmatrix}$
Nitrogen	14	$\begin{pmatrix} +14 \\ -14 \end{pmatrix}$
Oxygen	16	$\begin{pmatrix} +16 \\ -16 \end{pmatrix}$
Fluorine	19	$\begin{pmatrix} +19 \\ -19 \end{pmatrix}$
Bromine	20	$\begin{pmatrix} +20 \\ -20 \end{pmatrix}$
Boron	22	$\begin{pmatrix} +22 \\ -22 \end{pmatrix}$
Magnesium	25	$\begin{pmatrix} +25 \\ -25 \end{pmatrix}$
Aluminum	27	$\begin{pmatrix} +27 \\ -27 \end{pmatrix}$
etc.	etc.	etc.

The case where two entirely different ponderable molecules have the same atomic weight, occurs five times, and there is even one case where three such bodies have the same atomic weight, i.e.,

1. Beryllium and Nitrogen $\begin{pmatrix} +14 \\ -14 \end{pmatrix}$

2. Cobalt and Nickel $\begin{pmatrix} +59 \\ -59 \end{pmatrix}$

3. Rhodium and Ruthenium $\begin{pmatrix} +104 \\ -104 \end{pmatrix}$

4. Thorium and Uranium $\begin{pmatrix} +119 \\ -119 \end{pmatrix}$

5. Barium and Vanadium $\begin{pmatrix} +137 \\ -137 \end{pmatrix}$

and finally Gold, Platinum, and Iridium, all $\begin{pmatrix} +197 \\ -197 \end{pmatrix}$.

In what does the difference of such ponderable elementary bodies of the same atomic weight consist? This difference could, according to this hypothesis, only consist in the difference of the orbits and velocities, in and with which the united *positive electric* particles of a ponderable molecule, whose distances from one another are smaller than ρ, move, and of the orbits and the velocities in and with which the united *negative electric* particles of the same ponderable molecules, whose distances from one another are smaller than ρ, move. The more rapidly these orbits are traversed, the greater the molecule's resistance will be to the penetration of other particles, and thus the greater the hardness it will possess.

As far as the divergence of some atomic weights from the multiple of the atomic weight of hydrogen is concerned, they may, at least in part, be due to *satellites* of some ponderable molecules, whose existence seems to be bound to certain relationships, which shall be more closely examined.

A *ponderable* composite molecule of $+e$ and $-e$ (where for e also a plurality of similar electric molecules can be posited) exerts two forces upon a *positive electric* molecule $+e'$, i.e., one *repulsive force* from $+e$ upon $+e'$

$$= +\left(\frac{ee'}{r^2}\right) \cdot \left(1 - \left[\frac{1}{c^2}\right]\left[\frac{dr^2}{dt^2}\right] + \left[\frac{2r}{c^2}\right]\left[\frac{d^2r}{dt^2}\right]\right) ,$$

and one *attractive force* from $-e$ upon $+e'$

$$= -(1+\alpha)\left(\frac{ee'}{r^2}\right) \cdot \left(1 - \left[\frac{1}{c^2}\right]\left[\frac{dr^2}{dt^2}\right] + \left[\frac{2r}{c^2}\right]\left[\frac{d^2r}{dt^2}\right]\right) ,$$

thus, in sum, one *attractive force*

$$= -\alpha\frac{ee'}{r^2} \cdot \left(1 - \frac{1}{c^2}\frac{dr^2}{dt^2} + \frac{2r}{c^2}\frac{d^2r}{dt^2}\right) .$$

By means of this *force of attraction* exerted by a *ponderable molecule* on the *positive electric molecule*, the latter can be continuously maintained in a *rotational motion* around the *ponderable molecule.*

The same *ponderable molecule* consisting of $+e$ and $-e$ also exerts two forces upon a *negative electric* molecule $-e'$, i.e., a *force of attraction*

$$= -(1+\alpha)\left(\frac{ee'}{r^2}\right) \cdot \left(1 - \left[\frac{1}{c^2}\right]\left[\frac{dr^2}{dt^2}\right] + \left[\frac{2r}{c^2}\right]\left[\frac{d^2r}{dt^2}\right]\right) ,$$

and a *force of repulsion*

$$= + \left(\frac{ee'}{r^2}\right) \cdot \left(1 - \left[\frac{1}{c^2}\right]\left[\frac{dr^2}{dt^2}\right] + \left[\frac{2r}{c^2}\right]\left[\frac{d^2r}{dt^2}\right]\right) ,$$

thus, in sum, also a *force of attraction*, and indeed of the *same magnitude* as the *force of attraction* exerted upon $+e'$, where this *negative electric* molecule can also be maintained in a *rotational movement around the ponderable molecule*.

In this way, most *ponderable molecules* will have obtained, over the course of time, either a *positive electric* or a *negative electric* molecule as a satellite, and accordingly the ponderable molecules would fall in *three classes*, i.e., into the class accompanied by *positive satellites*, into the class of those accompanied by *negative electric satellites*, and into the class of those remaining *without satellites*.

If now *composites of ponderable molecules out of electric particles* could occur up to the number of five positive particles with five negative particles, this would accordingly yield, according to the above scheme, 53 ponderable basic materials,[320] from which the possibility of $53 \cdot 54/2 = 1431$ *binary composite ponderable bodies* would result.

If one further considers the extraordinary multiplicity which can occur in each of these *ponderable* molecules in relationship to the orbits and the *vires vivae*[321] of particular electric particles, out of which they are composed, there is the possibility of *infinitely many different kinds of such molecules*.

15.6 Electricity in Metallic Conductors

In the discussion "On Galvanometry" in Vol. 10 of the *Abhandlungen der Königl. Gesellschaft der Wissenschaften zu Göttingen* (1862),[322,323] Section 33 deals with "*the transformation of work of the electric current into heat*." It is there said: The work of the electric current is related to the movement of electric fluids, and according to the mechanical theory of heat, *heat* is also connected to the movement of a body, which, however, one usually distinguishes from the electric fluids, and calls *heat material*,[324] without however, determining this difference more closely. A closer examination of the way in which *work of the electric current* is transformed into *heat*, would accordingly seem to require, that either the identity of heat-material with the electric fluid is demonstrated, or, if this is not the case, that the movements of the electric fluid would have to be pursued up to the point, where the transition of the movements from *electric fluid* to *heat-material* occurs. In the latter case, however, the coexistence of many substances in the smallest parts of the space of the conductor would have to be assumed, i.e., the *ponderable conductor substance* along with *both electric fluids*, and also the so-called *heat-material*. To avoid such an accumulation of material in the same space, one therefore initially attempted to eliminate the *ponderable conductor substance* to the extent possible [by assuming] that, for example, the copper, instead of being *uniformly* distributed *throughout the entire space*, is in *particular spatially separate* points, i.e, it is assumed that the so-called *ponderable molecules* are concentrated, and by further assuming the surface

[320][Note by AKTA:] In German: *Grundstoffe*. This word can also be translated as raw material, basic substance or chemical element.

[321][Note by AKTA:] See footnote 26 on page 17.

[322][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 17.

[323][Note by LH and AKTA:] [Web62].

[324][Note by AKTA:] In German: *Wärmestoff*. See footnote 294 on page 174 for a discussion of this expression.

of each such molecule to be firmly connected to a layer of negative electric molecules on its surface, and also that *positive electric molecules* flow around it, which, in the case of a galvanic current, move successively from *one ponderable molecule to another*.

Whatever the forces upon which the *work* depends, generated by the exit of one such *positive electric* molecule out of the sphere of attraction of a *negatively charged* ponderable molecule, an *opposite and equal work* will always be performed by the same electric molecule *upon entrance* into the sphere of attraction of the next, also *negatively charged* ponderable molecule, so that these *two magnitudes of work* always compensate each other. Once, however, the electric molecule is separated from *one* ponderable molecule, it will pass through the spatial interval α until the *next* ponderable molecule, driven by the electromotive force f, and thus perform the work $f\alpha$. The sum of all of these work-magnitudes $\sum f\alpha$ forms the electric work in the conductor. Every electric molecule, therefore, enters, upon transition *from one ponderable molecule to another*, with a *vis viva increased by* $f\alpha$ into the area of the latter, in comparison to the *vis viva* with which it exited from the area of the previous [molecule], whereby therefore the value of the *vis viva* in the total closed circuit must be increased by an equivalent amount with the total *electric work*. An *increase of the vis viva equivalent to this electric work* in all parts of the closed conductor taken together, however, is now, according to the *mechanical theory of heat*, also the *heat* produced by the current, and the question is only, whether it is itself identical to it, or whether that *vis viva* belonging to the electric fluids must first be transferred to another medium (to the so-called *heat-material*), in order to appear as *heat*.

It has been demonstrated above, that there is no reason to assume such a transfer, that, however, with the omission of this transposition, every reason for assuming a particular *heat-material* or a *heat medium* also falls away, because it would be represented by *electricity*.

But now, if this representation of a so-called heat-material by electricity is to be complete, the laws of *heat conduction*, *of heat radiation*, and of *heat absorption*, as well as the dependent law of *temperature equalization* in ponderable bodies, must be derivable from the laws of movement of *electricity* in ponderable bodies and in empty space.

In the essay[325,326] "On the Movement of Electricity in Bodies of Molecular Constitution" in Poggendorff's *Annalen*, 1875, Vol. 156, the attempt was actually made to trace all phenomena of *heat*, as well as of *magnetism* and *galvanism* to movements of *electricity* in these bodies.

Statics and dynamics of ponderable bodies are distinguished, depending upon whether one considers them in a state of rest or in movement; but in speaking of the state of rest in these bodies in statics, one has by no means characterized a state of rest of *all* parts encompassed within the bounds of these bodies, but only of those *ponderable* parts encompassed within these bounds. Without this restriction, it would never be possible to speak of the state of rest of a ponderable body, because other parts are contained in each such body in addition to its ponderable parts, which never come to rest.

Firstly, as we have seen, precise research into the observed *electric* phenomena in *ponderable* bodies, has led to the result, that there are movable parts, i.e. *electricity particles*, in all of these bodies, whose dislocation and movement on the surface and in the interior of these bodies, is the reason for all phenomena of *electric charge* and *galvanic currents*, as well as all *electrodynamic effects* as a whole.

Likewise, precise research of the *magnetic* phenomena in ponderable bodies, has led to

[325] [Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 312.

[326] [Note by LH and AKTA:] [Web75].

the result, that in the interior of all of these bodies movable parts exist, which one had attempted to distinguish for a long time under the name of *magnetic fluids* from the first, i.e., from the *electric*. Of this *magnetic fluid* it was claimed, that it could be differently distributed in the interior of the body, depending upon different conditions, but that under persistent conditions it would come to rest and equilibrium. The reason for magnetic phenomena was sought in the *distribution* of these magnetic fluids, without requiring their continuous movement. But further research showed, that magnetic fluids at rest, however they were distributed, could not be the explanation for *all magnetic* (paramagnetic and diamagnetic) phenomena; that, however, *all* of these phenomena could be explained on the basis of the existence of *continuously moved* parts in the interior of ponderable bodies, and indeed, movement of the same parts, whose movements are the reason for all galvanic phenomena, viz., *electric particles*.

Thirdly, there is the further point, that research into the *temperature* ascribable to each ponderable body, showed, that, in the interior of all of these bodies, movable parts exist, and that the reason for the observed temperature-phenomena, i.e., of *heat*, was to be sought in the movement of these particles. The suspected *identity* of these particles with the *electric* [particles] has also been confirmed by facts, particularly by the *actual equality of vis viva produced by electromotive forces in the electrical and heat conductors, with that of the heat produced by current*.

In particular, in the investigations of heat production by the galvanic current in a conductor, it has been shown, that the *mechanical equivalent of the produced heat* in time-elements dt is equal to the product of dt into the current intensity i and into the electromotive force e, where for e the product of i may also be posited into the conductor-resistance w, thus $eidt = wi^2dt =$ the *mechanical equivalent of the heat produced*.[327]

But now, $eidt$ is the product of the force acting upon the electricity flowing in the unit-length of the conductor into the distance traveled by this force in the time dt in the direction of this force, i.e., the work performed *by the moved electricity* contained in the unit length of the conductor in the time dt, which is the same as the *heat* produced in the unit length of the conductor in the time dt. Consequently, this *heat* is the same as the *work performed by the moved electricity*, and the *heat-material itself is identical to the moved electricity* in the conductor.

We limit ourselves here to considerations of the behavior of *electricity*, *galvanism*, *and heat* in *metallic conductors*, and leave to one side, whether their behavior in *moist conductors*, for example in diluted acids, would be the same or different; to distinguish the two cases, it may be noted, that the current in the first case, i.e., in *metallic conductors*, is formed merely by the electric fluids, without any participation of ponderable molecules, while in the latter (i.e., in *moist conductors*), *ponderable materials*, such as hydrogen and oxygen, take part in the movement.

A clear insight into this relationship of the electric fluids to the ponderable molecules in *metallic conductors during galvanic currents* requires, however, previous better knowledge of the behavior of the electric fluids in *metallic conductors without galvanic current*. If there is no galvanic current in a metallic conductor, the electricity within it is by no means in a state of rest, rather it is in movement, and, accordingly, there exists a *vis viva* in metallic conductors, which is denoted with the name *heat*. Clear insight into the behavior of electric fluids in metallic conductors, however, requires, therefore, a separation and precise

[327][Note by AKTA:] This result is due to James Prescott Joule (1818-1889), [Jou41]. A detailed analysis of Joule's paper can be found in [MS20].

distinction of *those movements* of the electricity present in metallic conductors, which are merely the reason for *heat phenomena*, from those, which form the *galvanic current* in metallic conductors.

As far, *firstly*, as the movement of the extant electricity in metallic conductors is concerned, which contain the reason for *heat phenomena*, we distinguish two parts in the space in this conductor, namely, that occupied by *ponderable molecules*, and that not occupied by ponderable molecules, so-called *empty spaces in-between*. In the latter there are *positive electric molecules* which move, while all *negative electric molecules* exist partly as persistent constituent parts of *ponderable molecules*, and partly as *charges* assumed to be temporarily bound to them.

The movements of the *positive electric molecules* in the empty space surrounding a *ponderable metal molecule*, are not, however, limited to this space, rather such a molecule can pass from the environment of each *ponderable* molecule into the environment of a neighboring *ponderable* molecule; but this transition must take place from all ponderable molecules (if there is no galvanic current) indifferently in all directions, albeit not simultaneously, but successively. In bodies where that were not the case, there would be, as is easily seen, no *reciprocal heat radiation*, upon which, as is known, the law of *heat conduction* rests, i.e., the law of the *transfer* of the *vis viva* of heat from one ponderable molecule to the surrounding [molecules], which is the characteristic quality of *metallic conductors*. By *metallic conductors*, such bodies are understood, around whose *ponderable negative electrically charged molecules*, positive electric molecules rotate, and are thrown out in all directions without differentiation.

All of these movements of electric molecules in empty space between the ponderable *metallic molecules* follow laws, which are derivable from the *fundamental law of electric action*. This derivation has already been provided in the cited essay in Poggendorff's *Annalen*, 1875, Vol. 156, Art. VI,[328,329] "On the movement of electricity in conductors," pp. 39 ff.[330]

From the equation cited at the bottom *of the footnote*[331] for a *dissimilar electric molecular pair*, which is derived from the *fundamental law of electric action*, it follows, as cited there, that for $u = 0$, either $r = r_0$ or $r = [n/(1-n)]\,r_0$.

[328] [Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, page 339.

[329] [Note by LH and AKTA:] [Web75, p. 339 and following of Weber's *Werke*].

[330] [Note by WW:] This derivation was based upon consideration of the movement of two *dissimilar electric* molecules, of which the *one* (negative electric) is bound to a *ponderable* molecule, while *the other* (positive electric) can move around it freely, and upon the resulting differences in the *molecular material constitution*.

Let us limit ourselves here to such systems which consist of pairs of molecules, of which the one $(-e)$ is *negative electric* and bound to a ponderable molecule, the other $(+e)$ is *positive electric* and moves around the first; in the 6th Memoir on "Electrodynamic Measurements" (*Abh. d. Kgl. Sächs. Ges. d. Wiss*, Leipzig 1871, Art. 11, p. 32) [Note by HW: Wilhelm Weber's *Werke*, Vol. IV, p. 273] [Note by LH and AKTA: [Web71, p. 273 of Weber's *Werke*] with English translation in [Web72, p. 125], see Section 9.11 of Chapter 9], the following equation was found for this:

$$\frac{u^2}{c^2} = \frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0} + \frac{r+r_0}{r}\cdot\frac{\alpha_0^2}{c^2}\right) ,$$

where ρ, r_0, and α_0 are given constants, r is the distance of the two molecules from each other, and u is their relative velocity. If one substitutes $\alpha_0^2/c^2 = -n\rho/r_0$, one obtains

$$\frac{\rho-r}{\rho}\cdot\frac{u^2}{c^2} = \left(\frac{r}{r_0}-1\right)\left[n\left(\frac{r_0}{r}+1\right)-1\right] ,$$

from which it follows, that for $u = 0$, either $r = r_0$ or $r = [n/(1-n)]\,r_0$.

[331] [Note by AKTA:] See footnote 330.

From that the further result is, that such molecular pairs fall into two classes, i.e., into molecular pairs whose rotation around one another is *persistent*, and into such, whose rotation around each other *is not persistent*. Whether a molecular pair belongs to the one or other of these classes, depends upon its value for $n = -r_0\alpha_0^2/\rho c^2$, where r_0 denotes the smallest value of r, for which the relative velocity of the two molecules $u = 0$, and α_0 is the angular velocity of the electric molecule around the ponderable [molecule] at distance r_0.

Persistence of the rotation of the electric particle around the ponderable molecule occurs, when the value denoted by n lies between 1/2 and 1; *it does not occur*, on the other hand, if the value denoted by n lies between 0 and 1/2.

Since we consider here only pairs of dissimilar electric molecules which can rotate around each other, where ρ always has a *negative* value, it follows, that n must always have a *positive* value. If this value is larger than 1, then $r_0 = -n\rho \cdot [c^2/\alpha_0^2]$ is the only value for r for which $u = 0$. If this value, on the other hand, is smaller than 1, but larger than 1/2, then, in addition to the value $r_0 = -n\rho \cdot [c^2/\alpha_0^2]$, there is another value for r, for which $u = 0$, i.e., that value for which $n\,([r_0/r] + 1) = 1$, where the value $[n/(1+n)] \cdot r_0 = r^0$, from which it follows, that, if n is smaller than 1, but larger than 1/2, the distance of the two molecules from one another will alternately grow and recede, from r_0 to r^0 and then from r^0 to r_0, etc., whereby a *continuous rotation* of the two molecules around each other is given.

This *persistent rotation* of the two molecules around each other, if n is smaller than 1, but larger than 1/2, is now contrasted to the rotation of the molecules around each other, when n is smaller than 1/2, but greater than 0, where there is only a single value of $r = r_0$, for which $u = 0$, of which r would continuously grow *into infinity*, were it not prevented from doing so by external influences.

This growth of r (or the reduction of the curvature of the molecular orbit) is accelerated greatly upon approach, in a ballistic trajectory transition, of the *positive electric molecule* toward a neighboring *ponderable* molecule, where the molecular orbit initially makes a transition into a straight-line ballistic trajectory, and then, with continued approach to the next *ponderable* molecule, finally enters again into a circular orbit around the latter molecule.

As a consequence of the different directions of the ballistic trajectories, through which the transition of this *positive electric* molecule from one *ponderable* molecule to the *ponderable* neighbor molecules is mediated, a distribution of the same occurs to all *ponderable* neighbor molecules, just as, in reverse, the *positive electric* molecules thrown out by all ponderable neighbor molecules reach the first *ponderable* molecule.

This ballistic trajectory of positive electric molecules from each ponderable [molecule] to all ponderable neighbor molecules and the reverse, from all of these latter to the first, is denoted as *reciprocal radiation*. Fourier has shown,[332] that the laws of *heat conduction* result from such *reciprocal radiation* between all *ponderable molecules* of a heat conductor, whereby the phenomena of distribution and movement of electricity are closely connected to the phenomena of distribution and movement of heat.

This is the basis, *firstly*, for the assumption, that *metals* are bodies, whose *negatively* charged *ponderable molecules* are surrounded by a flow of *positive electric molecules*, which, however, do not find themselves in a *persistent rotation* around these, but rather in a *rotation, which transposes into a ballistic trajectory*, whereby these *positive electric molecules* are dispersed in all directions. For these *positive electric molecules*, that is to say, only the above cited value of n in metallic conductors smaller than 1/2 and larger than 0 is to be assumed.

[332][Note by AKTA:] See footnote 165 on page 109.

This is, *secondly*, the basis for the assumption, that *solid ponderable bodies*, which distinguish themselves from metals in that they are *not conductors of electricity and heat*, for example, glass or crystals, are bodies, whose *ponderable molecules* indeed also have a *negative electric* charge and are surrounded by a flow of *positive electric particles*, but which find themselves in a *persistent* rotation around those *ponderable molecules*, [and] thus do not transpose into ballistic trajectories, since, that is, for them the above cited value n is larger than 1/2 but smaller than 1.

In place of the propagation of electricity and heat in *metallic conductors* through ballistic trajectories, in *glass-like and crystalline* bodies a propagation occurs through a *wave movement* of the *ether or light-medium* present in them, which is formed by the positive electric molecules existing between the ponderable molecules.

Now the *laws of galvanic currents in metallic conductors* must be derived from the general *fundamental law of electric action* according to the above determinations. Such a derivation has already been provided in the cited essay, "On the movement of electricity in bodies of molecular constitution" (Poggendorff's *Annalen*, 1875, Vol. 156, Art. VI),[333,334] which shall be further elaborated here.

To this purpose, a *molecular constitution of metallic conductors* is assumed, i.e., a system of *ponderable* and *negative electrically* charged molecules, separated from each other by intervening spaces and in a stable equilibrium. This *stable equilibrium* of the *ponderable molecular system* forming the *metallic conductor*, is said to result in the way specified by Mossotti,[335]

1. from the mutual *repulsion* of these ponderable molecules as a consequence of their similar, i.e., *negative electric charges*, while their mutual attraction, through gravitation, disappears;

2. from the mutual *repulsion* of all *positive electric molecules* rotating around the ponderable molecules;

3. from the mutual *attraction of those ponderable molecules* with their negative electric charges and *these positive electric molecules* which fill the spaces in between. Mossotti attempted to prove the *possibility* of such a ponderable molecular system in stable equilibrium in his essay: Sur les forces qui regissent la constitution intérieuse des corps, aperçu pour servir à la détermination de la cause et des lois d l'action moléculaire. Turin 1836.

To this purpose, Mossotti assumes, that there are *ponderable molecules* at certain distances from one another in the space of a ponderable body, which reciprocally repulse each other, — just as the above considered *negatively charged ponderable molecules of a metallic conductor* — and that their intervening spaces are filled by an *elastic fluid*, whose atoms also mutually repulse each other, but are attracted by the *ponderable* molecules, all of which also holds for the above *positive electric* [molecules], which fill the intervening spaces of a metallic conductor, transposed into the ballistic trajectory, insofar as these also repulse each other, but are attracted by the *negatively* electrically charged *ponderable* molecules.

Mossotti then proves, that, given a certain relationship of the repulsive and attractive forces, those *ponderable molecules*, at *greater distances from one another*, behave just as if

[333][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 339.
[334][Note by LH and AKTA:] [Web75].
[335][Note by LH and AKTA:] See footnote 238 on page 149.

they alone existed, and attracted each other according to the law of gravitation; on the other hand, at *smaller* so-called *molecular distances*, they behave as if they were alone in space, and maintained each other in a *stable equilibrium* by the combined effect of the forces of attraction and repulsion.

The analogy of our case with that considered by Mossotti seems to lead to the same conclusion, i.e., to the possibility of the occurrence of a stable equilibrium also of our *ponderable molecular system*, which forms *metallic conductors*.

It would accordingly also be true of *metallic conductors*, that two of their *ponderable molecules* at greater distances from one another, would behave as if they alone existed in space, and would attract each other by the law of gravitation, but at smaller so-called *molecular distances*, they would behave as if they alone formed a molecular system in *stable equilibrium*.

15.7 Theory of the Galvanic Resistance of Metallic Conductors

(See Poggendorff's *Annalen*, Vol. 156, pp. 49-55.)[336,337]

Instead of the assumption of Mossotti on the interior constitution of solid bodies, mentioned in the previous Section, for the purpose of simplification of the consideration of *galvanic resistance of metallic conductors*, the assumption is made here of a *solid bonding* of the *negative electric molecules* of the metallic conductor, separated from each other by intervening spaces, and represented by *fixed lines*, which do not hinder the movements of *positive electric molecules* around all particular ponderable molecules and their transition into ballistic trajectories.

The really very *solid cohesion* of the ponderable molecules of *metallic conductors* is probably due to the reason, that each *positive electric* molecule in its circular orbit, encompasses not only the one *negative electric* molecule of the *one* ponderable neighboring molecule, but also the *other negative electric* molecule of the *other* ponderable neighboring molecule. The same holds for the circular orbit of every *negative electric* molecule and two positive electric neighboring molecules.

Let us assume, that one such negative electrically charged *ponderable molecule* of a metallic conductor is at point A, Figure 1, around which *imponderable positive electric molecules* move, as described above.

[336][Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 348-353.

[337][Note by LH and AKTA:] [Web75, pp. 49-55 of the *Annalen der Physik und Chemie* and pp. 348-353 of Weber's *Werke*].

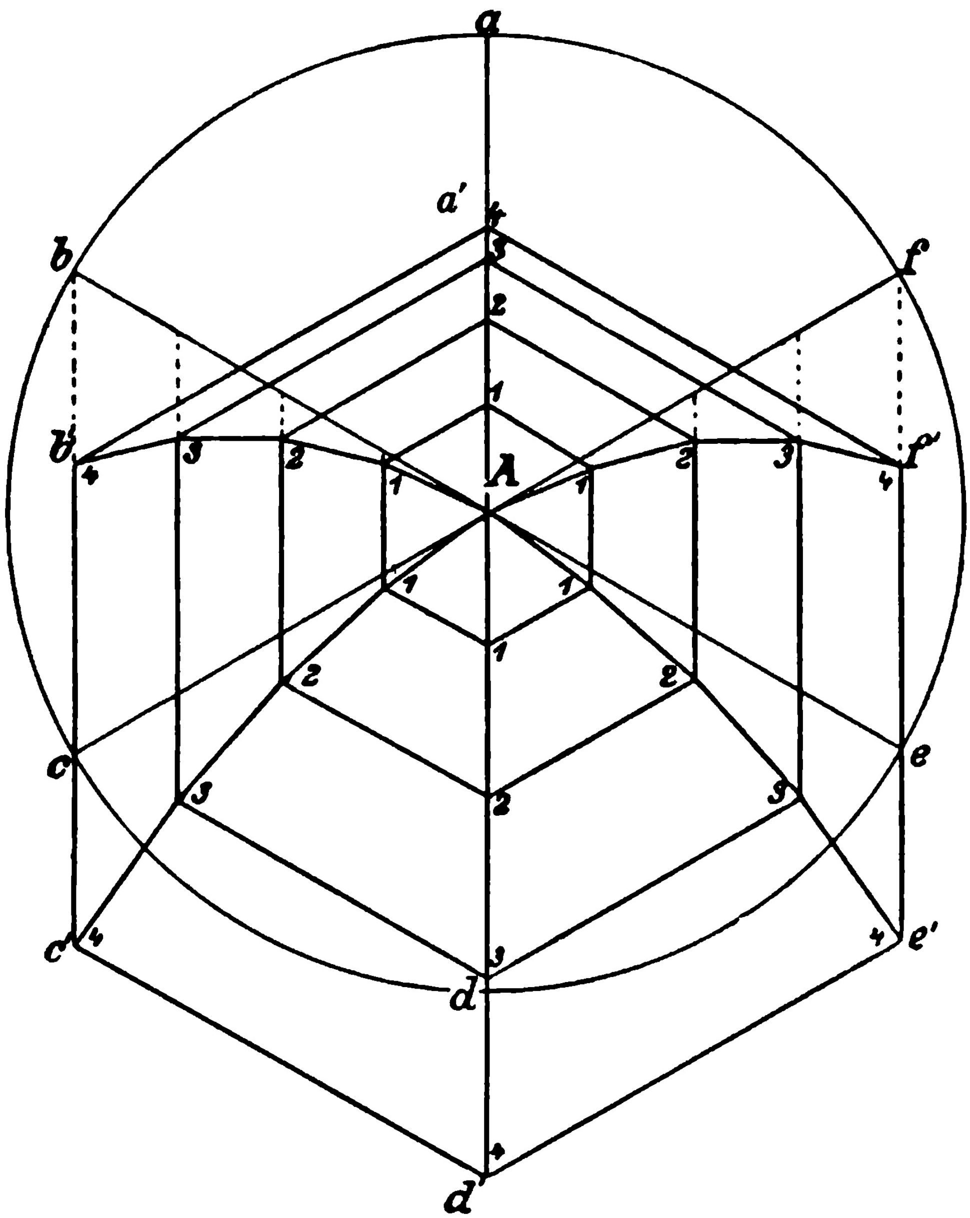

Fig. 1.

Were there *no electromotive force*, then, according to the hypothesis, the *positive electric* molecules, having made the transition to ballistic trajectory, would move, seen from the position A, in all directions indifferently, indicated in the Figure by six cardinal directions, i.e., by the radii Aa, Ab, Ac, Ad, to which are added in thought the two radii Ae and Af perpendicular to the plane of the Figure upwards and downwards.[338] The velocity of these ballistic motions would be assumed to be identical in all of these directions, and *each such orbit* would extend into the area of the next ponderable molecule, whose mean length would be $= r'$ and taken to be along the *length of the conductor element*.

But if now an *electromotive force* acts upon all of these *positive electric* molecules moving in different directions in ballistic motion, for example, Figure 1 in the vertical direction from top to bottom, then all of these molecules will be deflected from their straight-line ballistic

[338][Note by HW:] In the above Figure, drawn by Weber himself, the directions Ae and Af are drawn laterally.

orbit, in a way similar to [the trajectory] of a thrown stone by the force of gravitation, and must [then] describe curved orbits, which are represented in the Figure beneath the straight-line ballistic orbits, i.e., Ab' beneath Ab, Ad' beneath Ad, where Ae' and Af' are to be conceived as beneath Af.

As far as the resulting movement of current is concerned, it is evident, that the initial *ballistic motions* contribute nothing to that *movement of current* without the additional movement of the *electromotive force*, because they occur symmetrically in all directions, so that one may, therefore, completely abstract from those ballistic motions in determining the latter [current movement]. For the current movements of the molecules thrown out, only the orbits $aa' = bb' = cc' = dd' = ee' = ff' = r'$ remain.

One can also conceive for the purpose of determining the current, all of the *positive electric molecules* ejected from the other *ponderable molecules* contained in the same *conductor element* $= r'$ as united in A with that [*positive electric molecule*] ejected from A, and likewise all [ejected positive electric molecules] of the *ponderable molecules* contained in the *following conductor element* $= r'$ [as united] in A', and so forth, where $AA' = A'A''... = r'$ denoted as the *mean distance of two ponderable* molecules, between which *immediately reciprocal radiation* occurs. r' is accordingly the mean path distance, which the positive electric molecules ejected by the ponderable molecules must travel, until they reach into the area of the next ponderable molecule, around which they once again rotate.

If one denotes the amount of positive electricity with E, which is ejected or radiated by a ponderable molecule in a unit of time, and n is the number of such molecules contained in the *unit of length* of a *closed* conductor, then r' is the mean path length which each particle of the amount of electricity E travels from the site of the *radiating* molecule to the site of the *absorbing* molecule in the conductor l, consequently, $nr'E$ is the *limit-value of the current intensity* for growing *electromotive force*, expressed according to *mechanical measure*, and the *strength of a galvanic current*, which is generated in one such closed conductor by a *weaker electromotive force*, will be equal to only a *fraction of* $nr'E$.

In order now to determine this fraction, let the *ballistic velocity* of the positive electric molecules which issue from a *ponderable* molecule in A, Figure 2, be denoted as α, AB would be the direction in which such a molecule moves, and ξ is the path traveled with this velocity in time t, $\xi = \alpha t$.

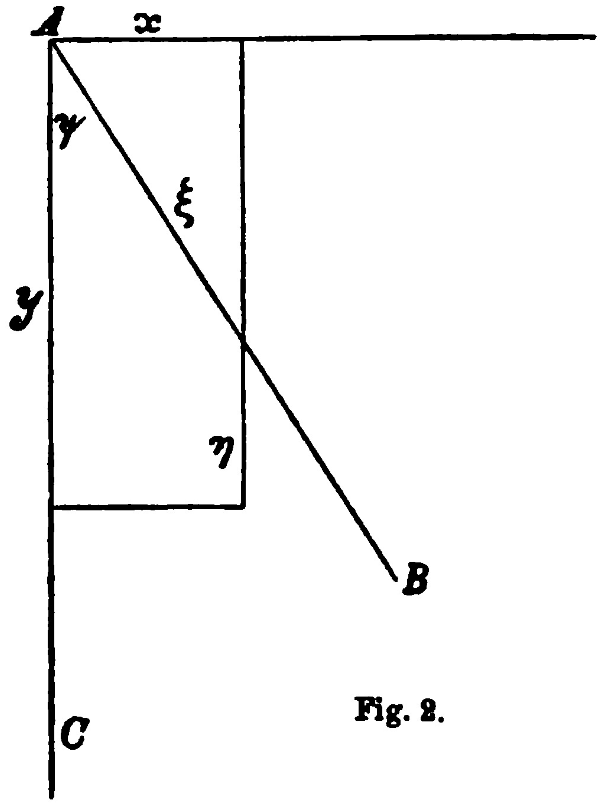

Fig. 2.

If, however, a constant electromotive force $= e^0$ acts upon this molecule, according to mechanical measure in the direction parallel to AC, which describes the angle ψ with AB, then this molecule would, through this force alone, travel in time t a growing path η proportional with t^2 or ξ^2. Accordingly,

$$\eta = a\xi^2 \ ,$$

$$x = \xi \sin\psi \ ,$$

$$y = \xi\cos\psi + \eta = x\cot\psi + \frac{a}{\sin\psi^2}x^2 \ ,$$ [339]

$$r^2 = x^2 + y^2 \ ,$$

which yields

$$y = \cot\psi \cdot \sqrt{r^2 - y^2} + \frac{a}{\sin\psi^2} \cdot \left(r^2 - y\right) \ .$$ [340]

[339] [Note by AKTA:] This equation would be written nowadays as

$$y = \xi\cos\psi + \eta = x\cot\psi + \frac{a}{\sin^2\psi}x^2 \ .$$

The same replacement of $\sin\psi^2$ by $\sin^2\psi$, and of $\cos\psi^2$ by $\cos^2\psi$, should be considered in the next equations.

[340] [Note by AKTA:] This equation should be written as

$$y = \cot\psi \cdot \sqrt{r^2 - y^2} + \frac{a}{\sin^2\psi} \cdot \left(r^2 - y^2\right) \ .$$

On this hypothesis, this ballistic motion reaches its end, when the molecule ejected from A travels the path r' and thus arrives in the area of the *next ponderable molecule of the metallic conductor*.

If one denotes with y' the value for $r = r'$, one obtains the equation:

$$y' = \cot\psi \cdot \sqrt{r'^2 - y'^2} + \frac{a}{\sin\psi^2} \cdot \left(r'^2 - y'^2\right) \ ,$$

from which the result, that, for growing values of the *electromotive force* a, y' approaches a *limiting value*, i.e., the value r'.

Let E denote the amount of positive electricity radiated by the *ponderable molecule* in A in the unit of time, and n the number of *ponderable molecules* contained in the unit length of the conductor, where $nr'E$ is the amount of positive electricity, which would pass through, in the given *limit case*, the cross-section of the conductor in the unit-time, and would be the maximum of the current strength for electromotive force growing into infinity, according to *mechanical measure*. The *current intensity* would, accordingly, not always grow proportionally with the *electromotive* force, but it would approach the limiting value $nr'E$ as the electromotive force grew into infinity.

If, however, the *electromotive force* or the magnitude a proportional to it, is very small, in which case $r'\cos\psi$ is an approximate value of y', which can be applied in the *last member multiplied with* a of the above cited equation, i.e.,

$$y' = \cot\psi\sqrt{r'^2 - y'^2} + \frac{a}{\sin\psi^2}\left(r'^2 - y'^2\right) \ ,$$

can be substituted for y', then one obtains the equation

$$y' = \cot\psi\sqrt{r'^2 - y'^2} + ar'^2 \ ,$$

or

$$\left(y' - ar'^2\right)^2 \sin\psi^2 = \left(r'^2 - y'^2\right)\cos\psi^2 \ .$$

If $y'^2\cos\psi^2 - a^2r'^4\sin\psi^2 = y'^2\cos\psi^2 - a^2r'^4\sin\psi^2$ is added, then one obtains

$$y'^2 - 2y'ar'^2\sin\psi^2 = r'^2\cos\psi^2 - a^2r'^4\sin\psi^2 \ ,$$

or

$$y' - ar'^2\sin\psi^2 = \pm r'\cos\psi \ .$$

Accordingly, one obtains *in the mean* for each two molecules, which are ejected from A in the directions determined by the two angles ψ and $\pi - \psi$,

$$y' = ar'^2\sin\psi^2 \ .$$

The mean value of the paths of all molecules ejected from A, in the direction of the electromotive force acting upon it, is accordingly

$$\frac{1}{2\pi}\int_0^{\pi/2} 2\pi y'\sin\psi d\psi = ar'^2\int_0^{\pi/2}\sin\psi^3 d\psi = \frac{2}{3}ar'^2 \ .$$

Were this value $= r'$, then the *current intensity* would be equal to the previously considered *limit value*, i.e., $= nr'E$, according to mechanical measure; but the *real current intensity*, at small values of a, as assumed here, is only a small fraction, i.e., $\frac{2}{3}ar'$, according to which, therefore, the *real current intensity according to mechanical measure* is obtained, i.e.,

$$i^0 = \frac{2}{3}ar' \cdot nr'E \cdot \left[\sqrt{\frac{MR^3}{T^4}}\right] ,$$

if the posited *measure of the mass*, *length and time* are denoted with M, R, and T. In this equation, only a requires to be more precisely determined, and this is achieved in the following way. The *electromotive force, according to mechanical measure*, acting upon each *electrostatic unit* of the electricity issuing from A, is denoted as e^0. If the mass of the electrostatic unit is posited $= [1/\sigma]M$, then the accelerating force $= \sigma e^0$ and the path traveled in time t as a consequence of this acceleration from A

$$\eta = \frac{1}{2}\sigma e^0 t^2 = a\xi^2 = a \cdot \alpha^2 t^2 ;$$

consequently

$$a = \frac{1}{2}\frac{\sigma e^0}{\alpha^2} .$$

According to mechanical measure, the *current intensity* is accordingly[341]

$$i^0 = \frac{1}{3}\frac{\sigma e^0 r'}{\alpha^2} \cdot nr'E \left[\sqrt{\frac{MR^3}{T^4}}\right] .$$

Following this determination of the *current intensity* i^0 *according to mechanical measure*, the *electromotive force according to mechanical measure* still remains to be determined, which acts upon the entire closed circuit, whose length is denoted with l, and every unit length of which contains n ponderable molecules, of which each is identical to the molecule A, which ejects E positive electric units each second, and e^0, as indicated, denotes the *electromotive force according to mechanical measure* acting upon each *electrostatic* unit (whose mass $=$ $[1/\sigma]M$).

nlE is, accordingly, the number of *electrostatic units* existing in the entire circuit, which are in ballistic motion. The *electromotive force* e^0 acts upon each unit of electricity, but not one second long, in which this particle, on account of its already extant ballistic-velocity, would travel the path α, but only during the fraction r'/α of a second, i.e., during the time in which the same would travel the path r' with the velocity α.

This yields the electromotive force for the entire circuit according to mechanical measure $= nlEe^0$, but which does not act *continuously* upon the totality of ejected particles of all nl ponderable molecules of the closed circuit simultaneously at any given moment, but only for r'/α seconds, which however, repeats itself at each following ejection, i.e., E-times each second, which is equivalent in its effect to the *electromotive force for the entire circuit according to mechanical mass* being

[341][Note by WW:] This value of i^0 is the same given in Poggendorff's *Annalen*, Vol. 156, p. 53 [Note by HW: Wilhelm Weber's *Werke*, Vol. IV, p. 352] [Note by LH and AKTA: [Web75]], but where the limit-value cited here as $nr'E$ is denoted as $n\varepsilon\sigma$. The then following determination of the electromotive force $e^0 = \gamma/\sigma \left[\sqrt{MR^{-1}T^{-2}}\right]$, on the other hand, requires the here following correction, i.e., $e^0 = 2\alpha^2 a/\sigma \left[\sqrt{MR^{-1}T^{-2}}\right]$.

$$E^0 = nlE \cdot \frac{r'}{\alpha} e^0 \ .$$

The quotient of this *electromotive force*, divided by the *current intensity according to mechanical measure* $i^0 = \frac{1}{3}[\sigma e^0 r'/\alpha^2] \cdot nr'E$, then yields the *resistance of the circuit according to mechanical measure*:

$$w^0 = nlE \cdot \frac{r'}{\alpha} \cdot \frac{e^0}{\frac{1}{3}\frac{\sigma e^0 r'}{\alpha^2} \cdot nr'E} \ ,$$

$$w^0 = \frac{3\alpha l}{\sigma r'} \ ,$$

where r'/α denotes the time tT in the unit-measure of time T, which each ejected particle needs in order to travel its orbit r'. Moreover, l is the expression for the length, according to the length-measure L, of the entire closed circuit $= lL$. Finally, $3/\sigma$ is a pure number, i.e., σ the pure number ratio of the mass of the electrostatic unit to the mass of one milligram. According to established measures, one obtains therefore the *resistance of the circuit according to mechanical measure*

$$w^0 = \frac{3}{\sigma} \cdot \frac{\alpha l}{r'} \left[\frac{R}{T}\right] \ ,$$

i.e., the resistance of a circuit according to mechanical measure is *directly proportional length l of the circuit and inversely proportional* to the time r'/α in which an electric molecule ejected by a ponderable molecule with a ballistic-velocity of α travels the mean length r' of the orbit until the next ponderable molecule, where the citation in Poggendorff's *Annalen*, Vol. 156, p. 54, is to be accordingly corrected.[342,343]

15.8 Some Problems Still to be Solved According to the Fundamental Law of Electric Action in Connection with the Hypothesis of the Composition of Ponderable Molecules Out of Positive and Negative Molecules

All *persistent aggregate states* of electric and ponderable molecules must be derivable from the *fundamental law of electric action*, on the hypothesis that ponderable molecules are connections of positive and negative electric molecules, which then must yield the *mechanics of all bodies in such an aggregate state.*

Accordingly,

1. the *mechanics of expandable fluids* (gases),
2. the *mechanics of non-expandable fluids*, and
3. the *mechanics of solid elastic bodies*

[342] [Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 352.
[343] [Note by LH and AKTA:] [Web75].

are to be explained.

In the explanation of the mechanics of bodies in the *first of the three cited aggregate states*, i.e., of the *gases*, the hypothesis would be that these consist of *ponderable nuclei*, which exist in larger spaces from one another, each accompanied by a positive electric *satellite*. In *each gaseous form, all of the nuclei* would be hypothesized to be *identical*, but different from those of every other gaseous form; the *satellites*, on the other hand, would be the same for *all gaseous forms*, i.e., all of them identical positive electric molecules. Two molecules of any gas with its satellites would, therefore, attract each other on account of the *force of gravitation of their nuclei*, but on account of the *electric repulsive force of their satellites*, they would repulse each other, and indeed, at *equal distances of the gas molecules*, there would be equal *repulsive forces of the satellites for all forms of gas, but different attractive forces of the nuclei for different forms of gas.* Furthermore, for each two gas molecules, there would be additionally a mutual force of attraction of the core of the one molecule and of the satellite of the other, which is the same for all molecule-pairs, but in relationship to the previously mentioned forces of repulsion, these are very small.

The thus occurring equality of the forces of repulsion of the gas molecules at equal distances for all gases, is of great importance for the mechanics of expandable fluids (gases) and deserves closer examination in the future.

For the explanation of the *mechanics of bodies in the second of the three cited aggregate states*, i.e., that of the *non-expandable fluids*, the hypothesis would be, that these consist of *ponderable molecules without satellites* which, on account of the reciprocal *force of gravitation* exerted upon each other, would rotate around each other.

Finally, for the explanation of the *mechanics* of bodies in the third of the three cited aggregates states, i.e., of solid elastic bodies, the difference between *electrical* conductors and *non-conductors* ought to be the first point to consider, a difference which must fundamentally be based on the interior constitution of these bodies, as is evident from the Sections 15.6 and 15.7, dealing with *metallic conductors*.

A well-founded explanation, of the connection of the fundamental electric law with the law of gravitation of ponderable bodies requires, first of all, an explanation based on the fundamental electric law of the *three aggregate states*, i.e., of the *solid*, *liquid* and *gaseous* states and their dependence upon heat, since the entire world of ponderable bodies is to be resolvable according to this connection into positive and negative molecules, according to which, therefore, not only the *law of gravitation* of ponderable bodies, but also all of their *aggregate states* must be derivable, where the chief issue would be the *explanation of heat* and its influence upon the *aggregate state*.

Differentiations are made between the *solid*, *liquid* and *gaseous* aggregate states of ponderable bodies, and among the *solid bodies*, between *metals and crystals*,[344] where the *first* are characterized by conduction of galvanic currents, and the *latter* by propagation of light.

As for the bodies of the first, i.e., solid aggregate state, the consideration of the associated metals and their *electric conductivities closely connected to their thermal conductivities*,[345] has led to the hypothesis of *molecular* constitution, according to which positive electric molecules rotate in these bodies around the individual ponderable molecules with continuously changing radii, each for so long, until it is transposed into a ballistic motion, and is

[344][Note by AKTA:] In German: *Krystalle*. In the context of this paper, this word can also be translated in general as "transparent solids".

[345][Note by AKTA:] In German: *ihres mit Wärmeleitungsvermögen eng verbundenen elektrischen Leitungsvermögens*.

thereby led out of the sphere of action of *one ponderable molecule into that of another*. The *heat conduction* of metals is based on the reciprocal *radiation* of ponderable molecules and the *galvanic conductivity* of metals. Since, however, the positive electric molecules rotating around the ponderable molecules are not in a stable equilibrium, no wave-movement can occur in them, and therefore no propagation of light.

Moreover, as for the *solid aggregate states*, the consideration of *crystals* with the *propagation of light* through them and as *electric insulators*, leads to the *solid aggregate states* as explained by Mossotti, according to which molecules at certain distances from each other are in stable equilibrium, which equilibrium comes into being through the *repulsive forces* of these molecules themselves, also through *the repulsive forces* of molecules of a (positive electric) fluid contained in the intervening spaces, and finally through the *forces of attraction* between the ponderable and the (positive electric) molecules.

The normal equilibrium state of the electric fluid filling the intervening spaces is likewise the light-ether, through which light is propagated, for which, on account of its molecular constitution, three axes of elasticity obtain. A further consequence of the normal stable equilibrium state is [an explanation of] why no conduction of heat and electricity occurs in crystals by means of mutual radiation of the molecules, but as a consequence of the disruption of the equilibrium, there are wave movements, i.e., light propagation in the imponderable medium filling the intervening spaces of the ponderable molecules, viz., the *light-ether*.

15.9 Continuation

In addition to the *mechanics of bodies according to the difference of their aggregate states*, on the same hypothesis as in the foregoing Section, the dependency of *all chemical properties of bodies* upon their molecular constitution must be derivable from the *fundamental law of electric action*, for example, *all chemical properties of hydrogen, oxygen* and of *water*.

It would be assumed, for example, that each molecule of *oxygen* is composed of 160 positive electric [simple molecules], and just as many negative electric *simple* molecules (whose distances from one another, the totality of positive as well as the totality of negative, are smaller than the critical distance ρ), where, in the gaseous aggregate state, one positive electric molecule also exists as a *satellite*, that furthermore each molecule of *hydrogen* is composed of 10 simple positive electric [molecules], and just as many simple negative molecules, where likewise, in the gaseous aggregate state, there is one additional positive electric molecule as a *satellite*; and, finally, likewise, that a molecule of *nitrogen* consists of 140 positive electric and just as many negative electric molecules, where in the gaseous state there is one additional positive electric molecule as a *satellite*.

At the same pressure (with which these differently composed molecules occupy the same space, so that the densities would behave as the numbers of the simple electric molecule-pairs which they contain, apart from their *satellites*, i.e., 160:10:140 = 16:1:14), whereby the numbers of molecules of these gases is the same in the same volume, the equal pressure would result merely from the interaction of the *satellites*, whose distances from one another would be the same under the same pressure. To be precise, this pressure resulting from the interaction of the satellites, would have to be added to a correction resulting from the mutual gravitation of the ponderable gas molecules and from the interaction of each gas molecule with the satellite of the neighboring gas molecule, but this can be considered to be vanishingly small.

Upon combination of hydrogen and oxygen to form water, the positive electric *satellites* of the ponderable oxygen and hydrogen molecules would escape; but these ponderable molecules would themselves be set into rotation around each other, whereby water would be formed; if, on the other hand, no escape of the positive electric satellites occurred, the same combination would form *steam*.

The *vis viva*, which each molecule of ponderable oxygen and hydrogen possesses by virtue of their rotation around each other, is the *latent heat of the water*; if this is withdrawn from the water, these ponderable molecules no longer form *water*, but *ice*. The molecules which rotated around each other in water, arrange themselves in ice in a sequence as a consequence of the firm connection of the *positive electric* molecule of the one ponderable molecule with the *negative electric* molecule of a ponderable neighboring molecule, etc.

All ponderable bodies, through which light- and heat radiation go, consist of isolated *ponderable molecules*, whose intervening spaces are filled by an imponderable *light- or heat-ether*. There is no reason not to assume, that this *light-* or *heat-ether* is formed by positive electric molecules, which also fill empty space of the universe,[346] even if the enclosed ether in those intervening spaces of ponderable molecules, undergoes a modification of its aggregate state as a result of the later.

All *vires vivae* are products of masses into the squares of their velocities, and resolve into those of which we can observe the masses as well as their velocities, from the observation of which we obtain immediately neither knowledge of the masses nor of the velocities whose products they are. The *vires vivae* of the latter kind are called *light* and *heat*, because all perceptions of light and heat are the effects of *vires vivae*, from which we obtain immediately neither knowledge of the masses nor of the velocities, whose products they are.

At distances from each other smaller than ρ, identical electric molecules can have diverse movements, without surpassing the distance ρ, and these movements can also exert a multiplicity of actions on the outside, so that these identical electrically composed molecules acquire heat, which is sometimes transferred to them from the outside, sometimes transferred from them to the outside.

15.10 On Diverse Movements in the Ponderable Material-Molecules Formed of Positive and Negative Electric Molecules, and on the Heat Characteristics Dependent upon Them

If there really exists in the world only electric molecules, which, by their connections, form all ponderable molecules, and, unconnected, form the imponderable media — which is usually denoted by the name of electric charges, or as light- and heat-ether — it is evident, that all laws of equilibrium and of movement, as well as all phenomena of light and heat, of those ponderable bodies as well also of these imponderable media, must be derivable from the fundamental law of electric action, if the position and the motion of all electric molecules, from which those ponderable bodies and these imponderable media are formed, were given at any time.

[346][Note by AKTA:] In German: *von welchen auch der leere Weltenraum erfüllt wird*. This expression can also be translated as "which also fill the empty cosmic space".

Even if no general solution to this problem can be hoped for or expected, the possibility would yet exist, given the infinite multiplicity of ponderable bodies, to guess the composition of *one or some* of these bodies, as well as their mutual position and motion at a certain time, and from the then resulting development to compare their laws of equilibrium and motion, as well as their light and heat phenomena, with observed phenomena of these bodies and then to test the laws. By means of particular fortunate experiments, a breakthrough would be made to be able to decisively test and firmly establish, or to refute, the hypothesis made here.

It would represent progress in that direction, for example, if it were possible to demonstrate the existence of a positive electric imponderable medium in space from the indeed extant surplus of positive electricity in the world as a whole, and to derive the laws of the wave movement of such a medium from the fundamental law of electric action, and to demonstrate the agreement of that law with the laws of propagation of light and the radiation of heat in space.

It would also represent progress in this direction, if it were possible to consider bodies in the *gaseous* aggregate state as consisting of ponderable molecules bound with positive electric molecules as satellites, so that the ponderable molecules for diverse gases would be different, but their electric satellites would be the same for all forms of gas, and if it were possible to derive all laws of equilibrium and of movement, including the propagation of light and heat in all gases, from the fundamental law of electric action. Without some such decisive results, there is no reason to expect a firm foundation for the theory of the electric composition of ponderable molecules. In particular, there seems to be no sufficient reason for the great differences of chemical relationships between different kinds of ponderable molecules.

The chief issue would be the difference of the ponderable molecules, on which the difference of the specific weight of gases under equal pressures, would depend. A differentiation would be made in each gas molecule, apart from the satellites which are the same for all, between a positive electric and a negative electric molecule, in which the former can be either simple or itself composed of many, and the latter *either divisibly or indivisibly* constituted,[347] which is probably the reason for a multiplicity of differences among gases.

The additional consideration would be, that the movements of electric molecules around each other, which form the *ponderable nuclei* of gas molecules, would be dependent upon external influences according to the laws of induction, and thus variable, but after removal of these influences, would be re-constitutible, so that they, apart from such transient changes, form the persistent differences of the gases.

15.11 Ice, Water, Steam

In ice, water, and steam, on the hypothesis, that ponderable molecules are connections of positive and negative electric molecules, the interesting case occurs, where merely through heat, i.e., merely the differences of the movement of the molecules, such fundamental differences as the aggregate states of ice, water, and steam are produced.

The first point to be considered, is that a force of mutual repulsion must be attributed to the ponderable molecules of steam, which does not obtain for the ponderable molecules of ice and water. This mutual force of repulsion, however, can hold for these ponderable

[347][Note by AKTA:] In German: *scheibar oder unscheidbar zusammengesetzt*. This expression can also be translated as "divisibly or indivisibly composed" or "separably or inseparably constituted".

molecules only as a consequence of similar (positive) electric satellites accompanying them, which the ponderable molecules of water must have obtained upon being transformed into steam. It would also result from this, that steam would have to behave toward the water out of which it emerged, *electrically positive*, which seems actually to be confirmed by the electric effects of the represented electric machines [*Elektrisirmaschinen*] with steam boilers. It becomes therefore unnecessary for the explanation of the efficacy of these steam-electric machines [*Dampf-Elektrisirmaschinen*] to have recourse to a *friction of the steam* on the walls of the exhaust pipes.

The[348] ponderable molecules [ponderable nuclei] of gases and steam exert no repulsive force upon each other, from which it follows, that the cause of the expansive force of gases and steam cannot lie in their ponderable molecules. Every ponderable molecule, thus also every ponderable gas or steam molecule, attracts, however, a *positive electric* molecule, which can thus only remain in the sphere of action of a ponderable molecule if it rotates around it with a certain velocity, and thus forms a *satellite* of the ponderable molecule. The ponderable molecules of gases and steam, when they are accompanied by one such formed *satellite* of such molecules formed from imponderable positively electrically charged molecules, exert by means of their satellite, *repulsive forces* upon each other, which are far greater than the *attractive forces* exerted by the ponderable molecules upon each other. The expansive forces of all gases and steam come from these repulsive forces of such satellites. And, at equal pressure and temperature, the specific weights of the gas and steam are nearly proportional to the weights of their ponderable molecules, because the weight of the satellites in comparison to the weight of the ponderable molecules, is small.

When *water* is transformed *into ice* by withdrawal of heat, i.e., by slowing down the rotation of the positive and negative electric molecules around one another, the reason for the *formation of a thread* must result, where, when the rotation is *slowed*, the actions of the positive electric and negative electric poles which every ponderable molecule possesses, are greater than with *rapid rotation*. As a consequence of this stronger action of the electric poles under conditions of slowed rotation, two neighboring molecules will close upon one another in rows with their dissimilar poles, where they are connected like parts of a thread.

But where do such satellites in water come from, if the water evaporates upon being subjected to heat?

15.11.1 The Melting Point of Ice and the Boiling Point of Water

Even if every ponderable molecule is composed of one positive and one negative electric molecule, there may occur great differences both of the positive as well as the negative electric molecules in different ponderable molecules, depending upon whether the electric molecules are simple or a multiplicity, but if the latter, then, according to Section 15.5, they must be *indivisible*, i.e., they may consist of an arbitrary number of similar electric molecules, of which, however, none of them are at a distance from one another $> \rho$. Accordingly, a ponderable molecule can be formed from an n-fold, but indivisible positive electric molecule and an n-fold also indivisible negative molecule, which are held together by their reciprocal force of attraction by rotating around one another. Such molecules are called *ponderable elemental bodies*, or *chemical atoms*, whose *weight* is proportional to the number n.

[348] [Note by HW:] The following paragraph, concerning the "expansive forces of gases and steam," is at the end of the original essay, but inserted here because of its connection to the discussion.

These must include all previously discovered chemical atoms. If, for example, *hydrogen* were a body consisting of *ponderable elementary particles*, and indeed that one for which $n = 1$, it would easily be shown, that the *ponderable elemental body* for which $n = 12$, would have to be *carbon*, that for which $n = 14$, *nitrogen*, that for which $n = 16$, *oxygen*, etc., up to gold, for which $n = 197$, and *silver*, for which n would have to be 216. There would thus result a large number of ponderable elementary materials, which can not yet be dissolved into other ponderable elemental materials, but possibly into positive electric and negative electric molecules, and indeed in n simple *similar* and indivisible composite electric molecules, where n denotes a whole number, which would indeed have no *weight*, but a *mass*, which might not always be considered negligibly small.

The question now posed, however, is, in what does the change actually consist, which occurs with *ice at the melting point*, and also in what does the change consist, which occurs with *water at the boiling point*.

Crucially important here is, that heat flows into the body without changing its temperature. The heat flowing in, increases the *vis viva* in the body; the ponderable particles, however, do not participate in this increase of the *vis viva*; that increase of the *vis viva* must, therefore, occur in the similar electric particles existing between the ponderable particles and independent of them.

In *metallic conductors*, it is assumed, that the positive electric particles are in a circular movement around the ponderable molecules, by which they are attracted, and that this circular motion would be accelerated by the influx of heat, and propagated ballistically from the environment of the one ponderable molecule to the other.

In *moist conductors*, especially in water, the same assumption is made, with the difference, that the *accelerated* circular motion of positive electric particles around each ponderable molecule, under the influx of heat, does not transpose into a ballistic motion, and thus *in no way* propagates from the environment of one ponderable molecule to that of another, but rather persists with the first molecule, but by means of increased centrifugal forces, the firm bond of this and the neighboring *ice-molecule*, according to Mossotti, would loosen, whereby the transformation of ice into water is effected.

The ponderable water molecules, with their *satellites* formed from positive electric molecules, repel each other like molecules of air, and as a consequence would spread out in a wider space, were not a certain *external pressure* exerted upon them. But if this *external pressure* remain constant, while the centrifugal forces of the satellites continuously grow due to the continued influx of heat, the *external pressure* is overcome, and the water, transformed into steam, expands itself like air.

15.11.2 Crystallization of Solid Bodies

All ponderable molecules with their satellites exert collisional-, directional- or rotational-forces upon each other, which are of particular importance for the crystallization.[349]

The great multiplicity and differences of these crystals are probably due chiefly, however, to the differences of those ponderable molecules themselves, i.e., in the differences of the *number* of positive and negative electric molecules from which they are formed. Ponderable molecules formed of one positive and negative electric molecule, are quite different from those formed of 10 or from 100 positive and negative electric molecules.

[349][Note by AKTA:] In German: *Krystallbildung*. This word can also be translated as "crystal formation" or "formation of crystals".

Let it be assumed that all the positive as well as all the negative electric molecules which form a ponderable molecule, are contained in the space of a sphere of a diameter $< \rho$, so that the distance between any two is always smaller than ρ. The number of similar electric molecules which can be contained in one such space, depends, accordingly, obviously on the relationship of their diameters to ρ. If this relationship is a very small fraction, the number of identical electric molecules in one such space can be very large, from which the possibility is evident, that this number may be greater than 10 or 100.

In addition to this difference in the *number*, both of positive as well as negative electric molecules, from which the ponderable molecules are formed, there is also the multiplicity of different orbits and velocities of all of these molecules in their orbits, which they may travel in the spherical space constrained to the diameter of ρ, which of course must have a great influence upon the interactions of the ponderable molecules to which they belong, and upon the crystallization which depends upon them. For an electric molecule rotating in a circle around a ponderable molecule, represents, in its action upon similar molecules, a magnet, and this action, even if it disappears at measurable distances, can be very large at molecular distances, quite in agreement with chemical forces, for which it can be substituted here, since constraint within molecular distances is characteristic for all chemical forces.

15.12 The Light-Ether is a Static Medium Formed from Positive Electric Molecules

Positive electric molecules may be at rest, when they are enclosed in a space of *fixed boundary* and are so distributed, that each molecule, thus surrounded by other molecules, lies at the center of many molecule-pairs, so that, therefore, both molecules of each pair exist symmetrically at equal distances at opposite sides.

Were such a molecule slightly shifted in any direction, for example, from North to South, and thereby [moved] closer to the molecules on the southside, and more distant from the molecules on the northside, it would be driven back from South to North, from which it is evident, that such a shift would be in *stable equilibrium*.

That same which holds for molecules in a space of *fixed boundary*, also holds for countless molecules filling an unbounded space in the same way, and each disruption of the equilibrium position of these molecules would, as one can easily see, be propagated by wave motion.

In space, however, only light and heat waves are propagated, if, therefore, only positive electric molecules can be distributed in this manner in space, then it seems that these positive electric molecules would have to form the *light-ether* in space.

The velocity with which light waves are propagated in such an electric medium, depends, at any given molecular mass, upon the magnitude of the force which acts in a given displacement upon the displaced molecule. This force is greater, the greater the number of molecules in the unit of volume.

Each such wave, even if it issued from one single point, spreads out to a surface, and the direction of oscillation of the individual molecules in this surface can be either perpendicular to the surface (longitudinal waves) or coincide with the surface (transversal waves). Since the length of all light waves is very short, but must still extend over a large number of molecular layers, the result is that the number of molecules must be very large even in small volumes, from which a very small space-content of molecules must be concluded, if, that is, only *action at a distance* is supposed to occur between molecules of light-ether, among which the dimensions of the bodies acting upon one another are negligible with respect to their distances.

The known great velocity of the propagation of light waves proves furthermore, that the mean distance of molecules in the light-ether can only be slightly larger than ρ, where, that is, the smallest change of this distance is connected with a very large change of the forces of repulsion, because this repulsive force becomes infinitely large for the distance ρ.

With the resulting very large number of molecules even in a very small segment of space, the further result, out of the totality of the very small mass of all of these molecules, is that the mass of *each particular* positive electric molecule *is very small to a much higher degree.*

15.13 NOTE (by Heinrich Weber, the Editor of Vol. 4 of Wilhelm Weber's *Werke*)

In addition to the Sections presented here, the original manuscript of Wilhelm Weber also included four other Sections, with the following titles:

1. The theory of the reflection and scattering of electric rays[350] is not applicable for the foundation of a theory of dynamic media.
2. The theory of the light-ether in space as a static medium.
3. On the wave-theory of so-called dynamic media.
4. Laws of repulsion and scattering of gas molecules in ballistic trajectories according to the dynamic theory of gases upon their collision, on the hypothesis, that gas molecules are compounds of positive and negative electric molecules.

which were located between Sections 15.4 and 15.5. Wilhelm Weber, however, later edited these Sections out of the discussion, for which reason they are not published here. It may be noted, however, that the entire estate [posthumous works] has been given to the Königlichen Bibliotek zu Göttingen (Royal Library of Göttingen), so that access to these Sections is still possible.

[350][Note by AKTA:] In German: *Die Theorie der Zurückwerfung and Zerstreuung elektrischer Strahlen.* These words can also be translated as "The theory of reflection and scattering of electric beams", see footnote 231 on page 145. This subject was discussed in 1878 by Weber, [Web78a, Sections 7 and 8, pp. 389-395], with English translation in [Web21e, Sections 7 and 8]. See, in particular, Sections 13.7 and 13.8 of Chapter 13. Each electric ray would be a system of electrified particles (with equal masses and equal charges) following one another along the same orbit. Weber studied the reflection and scattering of two electric rays.

Chapter 16

[Weber, 1894b] Aphorisms

Wilhelm Weber[351,352,353]

Among the categories Number, Space, and Time, *Number* alone belongs to pure logic or pure science, while Space already contains something hypothetical or derived from visual imagination (for example, the Euclidean hypothesis of the theory of parallels,[354] and in addition, the concepts of left and right, which cannot be defined by logic). The case of *Time* is surely similar to that of *Space*. According to the conception of Time, which we have framed for the *physical world*, the relationship between past, present, and future assuredly also contains a hypothetical element, which has no absolute validity for the *mental world* (for the world of thoughts, and emphatically for thinking itself).

When we conceptualize the world in the framework of Number, Space, Time, Motion, etc., the assumption of continuity and simplicity of motion (that no object can simultaneously carry out two different motions) is essential to the relationships so framed.

This framework is no longer adequate, when we extend and expand our thinking to the world of mental processes and to Divinity.

The motion of thought cannot be subjected to the same limitations of continuity and simplicity, as we do for a physical motion (which would mean that no thinking being could have two thoughts at the same time). The possibility of reaching a conclusion, requires having three propositions present in the mind at once.

According to the first framework, the one conceived for the physical world, the present is really nothing at all, namely a mere boundary between past and future, without any content of its own.

In the mental world, the present contains consciousness, which has a significant content (including all memory). In the mental domain, therefore, the present is something real, and is not merely the boundary between past and future; it has a real content.

Without consciousness as the content of the present, there could be no mental life, and for the Divinity, the content of the present, existing in consciousness, must in fact be infinitely extended.

Such a content, however, requires time; in mental life, the present, being filled with

351[Web94a] with English translation in [Web97].

352Translated by J. Tennenbaum. See also [Ten97].

353The Notes by A. K. T. Assis are represented by [Note by AKTA:].

354[Note by AKTA:] See [Euc56] with Portuguese translation in [Euc09].

consciousness, is no mere boundary between past and future, but is a *boundary layer* between past and future; it is a real element of time.

In the Divinity, the consciousness-filled present must be infinitely extended.

A Divinity, which were assumed to exist only in the present, insofar as the present is understood in the manner assumed for the physical world (where the present is merely a boundary between past and future)—that is, a present which were understood to have existed in the past and which still has to achieve existence in the future—were only an empty notion or a mere illusion.

Our power of thought, our power of sensation and our power of memory, are like a sum loaned to us, which we must produce with. We create a world of cognition, which stands in a wondrous relation to our sensations. Thereby we learn to value the "loan" extended to us, to honor and trust the one who gave that loan. This conceptual world, formed in connection with our sensations, embraces also a conception of ourselves, which, however, is confined, through our notion of our birth and our death, to the interval of time between those events. Although this world of cognition contains many notions—including notions of the time before our birth and after our death—of these notions none is of ourselves. As long as we live, we work with the loan granted to us, and continually strive to perfect and to complete our world of cognition; but, what we have already achieved, suffices, that we be filled with the loftiest appreciation of the loan, and the highest trust in the one who gave it; and especially the confidence, that He, who granted that loan, will continue to care for us. Upon this trust is based our conviction, that the true ordering of the world, exceeds by far the ordering of our cognitive world.

When material entities, which are separated from each other spatially and temporally, interact with one another, then the reason for this interaction lies in the nature of both together *as a single whole*. The mutually dependent parts of this *whole* exist in different points of space and time. If there exist *material* entities, which as *wholes* cannot be confined to a single point in space and time, then this holds all the more for spiritual entities.

Granted to us are our powers of sensation, thought, and memory. We thereby gain not only a world of thoughts, but also a hypothesized world, where the material and the mental stand in causal connection. This causal connection leads to a final cause—God. The possibility of a deeper insight into the How and Why seems not to be given through the faculties of sensation and thought. A hypothesized world in causal ordering, which were based, not on a final cause, but only on many final causes—the properties of all things in space—would be confined to the hypothesized world of physical bodies; all that which *thinks*, is excluded from that domain, and thereby we would exclude also any *explanation* of the hypothesized world of physical bodies *(Körperwelt)*. For, a world of physical bodies might exist, without it being thought.

In descriptive natural science (including chemistry) a relationship, given through a law, is accepted as an *explanation*, without having located the *reason* for the law in the nature of the hypothesized entity, not to speak of the reason for all sensation and thought.

Chapter 17

Overview and Future Developments of Weber's Law Applied to Electromagnetism and Gravitation

A. K. T. Assis[355]

Weber's law was the leading electrodynamics during the second half of the XIXth century.

In the last few years there has been a renewed interest in Weber's law applied to electromagnetism and gravitation due to novel experiments and important theoretical developments.[356]

In this final Chapter of the book with Weber's main works on electrodynamics translated into English I wish to give an overview of what has been accomplished so far. I will also present my personal view on some important aspects and new developments of Weber's law applied to electromagnetism and gravitation.[357]

17.1 Ampère's Unification of Magnetism, Electrodynamics and Electromagnetism

In 1687 Isaac Newton (1642-1727) proposed in the *Principia* his law of universal gravitation.[358] According to Newton, the force of attraction between two particles is proportional to the product of their masses m and m', varies inversely as the inverse square of their distance

[355] Homepage: www.ifi.unicamp.br/~assis

[356] [PK74], [Wes87], [SS87], [Gra90d], [Gra90a], [Gra90c], [Gra90e], [Gra90b], [Wes90b], [Wes90c], [Wes90d], [Wes90a], [Phi90a], [Phi90b], [Cor90], [Wes91], [She91], [Ass92a], [Rag92], [Phi92], [GG93, Chapter 3: The Riddle of Inertia], [Gal93], [Ass94], [Zyl94], [Bue94], [Ass95a], [GM95], [KF96], [Phi96], [GV97], [BC97], [Dru97], [FK97], [GV98], [Ass98], [BA98], [Mik99], [GV99a], [GV99b], [Cos99], [CL99], [Phi99], [Ass99a], [Ass99b], [LL00], [GV01], [GVM01], [Bun01], [BA01], [Mik01], [Wes02], [Fuk03], [Cos03], [Mik03], [GV04], [JP04], [GVAB05], [AGV07], [AH07], [AH09], [AC11], [AWW11], [Här12a] with Portuguese translation in [Här12b], [War13, Chapter 5: The Logic of Relational Physics], [Taj13], [Phi13], [AH13], [Ass13], [AWW14], [Ass14], [Ass15a], [Taj15], [Pry15], [STM15], [SJM15], [BA15], [AC15], [TSLBLVRR15], [SJY$^+$16], [PPL16], [Mon17], [LT17], [SM17], [Ano18], [BT18], [Pry18], [Här18], [Tra18], [CL18], [AWW18], [WT19], [Ano19], [FW19], [Lim20], [Här20], [BSM20] etc.

[357] [Ass94, Section 8.4: Weber's law and plasma physics, quantum mechanics, nuclear physics, etc.].

[358] See footnote 32 on page 19.

r, acts along the straight line connecting the particles and follows the principle of action and reaction:

$$\frac{mm'}{r^2} \ . \tag{17.1}$$

In 1785 Charles Augustin de Coulomb (1736-1806) obtained an analogous expression describing the interaction between two electrified particles at rest relative to one another.[359] In this case we assume the existence of two kinds of particles, namely, those electrified positively and negatively. Particles electrified with charges of the same sign repel one another, while particles electrified with charges of opposite sign attract one another. The force between two particles is proportional to the product of their electric charges e and e', varies inversely as the inverse square of their distance r, acts along the straight line connecting the particles and follows the principle of action and reaction:

$$\frac{ee'}{r^2} \ . \tag{17.2}$$

By working with long and thin artificial magnets with well-defined poles, Coulomb also obtained a similar expression describing the force between magnetic poles. In this case we assume the existence of two magnetic fluids, namely, austral and boreal. They are also called North magnetic fluid and South magnetic fluid. The poles (or centers of action) of his uniformly magnetized bars were concentrated very close to their extremities. Poles of the same type repel one another, while poles of opposite type attract one another. Coulomb's force between two magnetic poles is proportional to the product of the strengths p and p' of the poles, varies as the inverse square of their distance r, acts along the straight line connecting them and follows the principle of action and reaction:

$$\frac{pp'}{r^2} \ . \tag{17.3}$$

An isolated magnetic pole has never been found in nature. The simplest magnetic entity we can work with is a magnetic dipole, that is, two equal and opposite magnetic poles separated by a small distance. In any event, the forces and torques between two dipoles may be obtained utilizing Coulomb's law between magnetic poles. Likewise, the forces and torques between two magnets may be obtained with an appropriate distribution of dipoles inside each magnet. The orientation of compass and dip needles by the Earth can also be obtained with Coulomb's law between magnetic poles, coupled with appropriate distributions of magnetic dipoles on the compass and on the Earth.

I now discuss the work of André-Marie Ampère (1775-1836) which, together with the works of Newton and Coulomb, was so influential to Wilhelm Weber. His main book on electrodynamics was published in 1826, being fully available in French, Portuguese and English.[360]

Ampère's unification of the physics of his time has been discussed in detail in Section 22.2 of our book "Ampère's Electrodynamics — Analysis of the Meaning and Evolution of Ampère's Force between Current Elements, together with a Complete Translation of His Masterpiece: Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience".[361] In that book we utilized the following nomenclature:

[359]See footnote 201 on page 125.

[360]See footnote 8 on page 10.

[361][AC15, Section 22.2, Ampère's Unification].

- *Electrostatic phenomena:* Forces and torques between electrified bodies which are at rest relative to one another.

- *Magnetic phenomena:* Forces and torques between magnets, together with the torques exerted by the Earth on magnets (orientation of compass and dip needles).

- *Electromagnetic phenomena:* Forces and torques between a current-carrying conductor and a magnet, together with the forces and torques exerted by the Earth on current-carrying conductors.

- *Electrodynamic phenomena:* Forces and torques between conductors carrying steady currents.

Between 1820 and 1827 Ampère was able to unify three of these four branches of physics, namely, magnetostatics, electrodynamics and electromagnetism.

Magnetic phenomena were known many centuries before Ampère. These phenomena include the orientation of a compass needle by terrestrial magnetism; the orientation of dip needles by the Earth; the attractions and repulsions between magnets depending on their distances and relative orientations; the torques between magnets depending on their distances and orientations; etc. These magnetostatic phenomena may be described theoretically utilizing Coulomb's law for magnetism. That is, utilizing his force between magnetic poles, together with appropriate distributions of magnetic dipoles inside magnets and inside the Earth.

The discovery of electrodynamic phenomena taking place only between current-carrying wires, without the influence of any magnet, is due totally to Ampère in the period 1820-1827. He was the first person to observe the attractions and repulsions between flat spirals carrying steady currents. He also observed the attractions and repulsions between current carrying parallel conductors. This led to his famous current balance. He also discovered the torque between current-carrying conductors; the continuous rotation of a current-carrying conductor due to its interaction with another current-carrying conductor; the so-called Ampère's bridge experiment (also called Ampère's floating wire experiment or Ampère's hairpin experiment) etc. He discovered many equilibrium experiments in which a mobile circuit was kept at rest due to equal and opposite forces and torques exerted by two other current-carrying conductors. Some examples are the cases of equilibrium of the sinuous wire, of anti-parallel currents, of the nonexistence of continuous rotation, of the nonexistence of tangential force, of the law of similarity, etc.

Ampère was able to explain all of these electrodynamic phenomena utilizing his force between current elements. His force is proportional to the product of the current intensities i and i' of the two elements. It is also proportional to the product of the infinitesimal lengths ds and ds' of the elements. Moreover, it varies inversely as the square of their distance r. His force always complies with Newton's action and reaction law and points along the straight line connecting the elements, no matter the directions of the two interacting current elements. In these aspects his force is also similar to Newton's law of gravitation, just like Coulomb's laws for electrostatics and magnetostatics. However, Ampère's force depends also on the angle ε between the directions of the two current elements and on the angles θ and θ' between each current element and the straight line connecting their centers:

$$\frac{ii'dsds'}{r^2}\left(\cos\varepsilon-\frac{3}{2}\cos\theta\cos\theta'\right)=\frac{ii'dsds'}{r^2}\left(\sin\theta\sin\theta'\cos\omega-\frac{1}{2}\cos\theta\cos\theta'\right) . \qquad (17.4)$$

Here ω is the angle between the planes drawn through each current element and the straight line joining them. We discussed the meaning of these angles and presented several figures representing them in Section 2.8 (The Angles Appearing in Ampère's Force) of our book on Ampère's Electrodynamics.[362]

By integrating this expression he showed that a closed circuit of arbitrary form exerts a force on a current element of another circuit which is always orthogonal to this element and to a certain straight line passing through the midpoint of this element. This straight line was called directrix or normal to the directing plane. Ampère and his former student F. Savary (1797-1841) introduced the concept of the electrodynamic solenoid. They obtained the force and torque between a current element and an electrodynamic solenoid, between a closed circuit of arbitrary form and an electrodynamic solenoid, and also between two electrodynamic solenoids.

Many scientists contributed to the discovery of electromagnetic effects. In 1820 H. C. Ørsted (1777-1851) discovered the first electromagnetic phenomenon, namely, the deflection of a compass needle by a nearby current-carrying wire, removing the needle from its natural orientation along the magnetic meridian.[363] Soon after that he showed the inverse phenomenon, namely, the orientation of a current-carrying loop by a nearby magnet. In this way he discovered the torque and counter-torque acting between a magnet and a current-carrying wire. Between 1820 and 1822 Ampère discovered many new electromagnetic phenomena: forces of attraction and repulsion between a magnet and a current-carrying wire; the orientation of current-carrying loops due to the influence of the Earth (experiment analogous to the orientation of a compass needle by terrestrial magnetism); etc. J.-B. Biot (1774-1862) and F. Savart (1791-1841) obtained the torque exerted by a straight wire acting on a small magnet as a function of their distance. They also obtained the torque of a bent wire acting on a small magnet as a function of the opening angle of the bent wire. In 1821 M. Faraday (1791-1867) discovered a new electromagnetic phenomenon, namely, the rotation of the extremity of a magnet around a fixed current-carrying wire, together with the rotation of one extremity of a current-carrying wire around a fixed magnet. Between 1821 and 1822 Ampère discovered many new phenomena related to this topic like the rotation of a current-carrying wire due to the influence of the Earth; the rotation of a magnet around its axis etc. He also obtained some equilibrium experiments involving magnets and current-carrying conductors like the case of equilibrium of orthogonal currents.

Ampère also obtained one of the first unifications in the history of science. He succeeded in combining magnetic, electrodynamic and electromagnetic phenomena into a single theoretical framework. When he first heard of Ørsted's experiment, he had an original and extremely fruitful insight, namely, he supposed the existence of electric currents flowing inside magnets and also inside the Earth. Moreover, he assumed that all magnetic and electromagnetic interactions were due essentially to electrodynamic forces. That is, he interpreted the electromagnetic experiments of Ørsted, those due to himself and also those of Faraday as being due to interactions between the electric current flowing in the wire and the supposed microscopic currents flowing around the particles of the magnets. He interpreted the terrestrial orientation of a compass needle and of a dip needle as being due to torques exerted by the supposed microscopic electric currents flowing around the particles of the Earth acting on the microscopic electric currents flowing around the particles of the compass or dip needle. Likewise, he interpreted the forces and torques acting between two magnets

[362]In English: [AC15]. In Portuguese: [AC11].

[363]See footnote 203 on page 126.

as being due to electrodynamic forces acting between the supposed microscopic currents of both magnets.

Utilizing his force between current elements, Ampère unified theoretically these three branches of science. He and Savary modeled a magnetic pole as the extremity of a simply indefinite electrodynamic solenoid. They obtained the force between a current element and this solenoid. This expression is analogous to Biot and Savart's formula for the interaction between a current element and a supposed magnetic pole. Ampère also obtained the force exerted by a closed circuit of arbitrary shape carrying a steady current acting on a simply indefinite electrodynamic solenoid. He could then unify electromagnetism with electrodynamics through the mathematical identification of the extremity of a simply indefinite electrodynamic solenoid with a magnetic pole placed at this extremity. With his formulas it was possible to explain quantitatively the electromagnetic experiments of Ørsted, Biot, Savart, Faraday and Ampère.

Ampère also obtained the analytical formula expressing the electrodynamic interaction between two simply indefinite electrodynamic solenoids. It represents a force pointing along the straight line connecting these two extremities and varying as the inverse square of their distance. This force is mathematically analogous to the action between two magnetic poles given by Coulomb. Ampère could mathematically unify magnetism with electrodynamics. That is, a pair of interacting magnetic poles was identified with a pair of indefinite electrodynamic solenoids interacting with one another. Ampère and Savary obtained also the force and torque between two definite electrodynamic solenoids. This interaction might be reduced to four forces, each one pointing along the straight line connecting one extremity of a solenoid to one extremity of the other solenoid, varying as the inverse square of their distance. This force was proportional to the product of the current intensities of both solenoids. They were then able to identify a magnet as a definite electrodynamic solenoid.

Ampère was also able to show that any given closed circuit of arbitrary form carrying a steady current was equivalent to a set of two surfaces very close to one another, terminated by this circuit, and over which were spread the two magnet fluids of opposite type and the same intensity, the so-called magnetic shell or magnetic dipole layer. The equivalence here refers to the fact that any one of these systems (the closed circuit or the magnetic shell) exerts the same force and torque on another closed circuit or on another magnetic shell.

He also obtained the forces and torques exerted between two small planar closed loops of areas λ and λ' of arbitrary shapes, carrying constant currents of intensities i and i', respectively, supposing their typical dimensions being much smaller than the distance between their centers. Following Poisson, he also calculated the forces and torques between two small magnetic dipoles of lengths $\delta\rho$ and $\delta\rho'$, supposing their lengths are much smaller than the distance between the centers of these dipoles. Let μ and $-\mu$ be the intensities of the magnetic poles of one dipole, while μ' and $-\mu'$ are the corresponding intensities of the other dipole. Ampère showed that the forces and torques between the two current-carrying loops are equivalent to the forces and torques between the two dipoles when the loops are replaced by the dipoles, provided the axis connecting the North and South pole of one dipole was normal to the area of one loop, while the axis connecting the North and South pole of the other dipole was normal to the area of the other loop. Therefore, the electrodynamic equivalent of a small magnetic dipole is a small plane loop of arbitrary shape, carrying a constant current, with the plane of the loop being orthogonal to the dipole axis. In this way Ampère obtained a complete mathematical equivalence of the magnetic phenomena with the electrodynamic phenomena.

Until Ampère's time, scientists explained magnetic phenomena supposing the existence of austral and boreal fluids, that is, supposing the existence of North and South poles inside magnets and also inside the Earth. One of the basic concepts was that of a magnetic dipole, that is, two opposite poles of the same intensity separated by a small distance. With Ampère's unification, these concepts of magnetic poles and magnetic dipoles became superfluous and unnecessary, as he could explain all magnetic phenomena while dealing only with the interaction of electric currents. This explanation was not only qualitative and conceptual, but also quantitative, through his expression for the force between two current elements, together with the assumption of electric currents flowing around the particles of magnetized bodies and also around the particles of the Earth.

Ampère's force between current elements has been completely forgotten during the whole of the XXth century. The force between current elements appearing in the textbooks is due to Hermann Grassmann (1809-1877) in 1845.[364] Ampère's force always complies with Newton's action and reaction law and points along the straight line connecting the elements, no matter their orientation in space. Grassmann's force, on the other hand, in general does not comply with Newton's action and reaction law. According to Grassmann's law, there are situations in which the force exerted by current element 1 acting on current element 2 is different in magnitude and direction from the force exerted by current element 2 acting on current element 1.

Maxwell knew not only Ampère's force, but also Grassmann's one. In his *Treatise on Electricity and Magnetism* he made an analysis of four formulas expressing the forces between two current elements, namely, those of Ampère, Grassmann and two others which were created by Maxwell himself. Maxwell's final judgement:[365]

> Of these four different assumptions that of Ampère is undoubtedly the best, since it is the only one which makes the forces on the two elements not only equal and opposite but in the straight line which joins them.

Despite Maxwell's clear defense of action and reaction along the straight line connecting the interacting bodies, Ampère's force disappeared from the textbooks. The main reason is that modern theories of physics are based on Einstein's theories of relativity. And Einstein's theories are based not only on Maxwell's equations, but also on Lorentz' force. If we begin with Lorentz' force, then we deduce only Grassmann's force, but not Ampère's force between current elements. On the other hand, if we begin with Weber's force, then we deduce only Ampère's force, but not Grassmann's force between current elements. I presented the proofs of these facts in Section 4.2 (Derivation of Ampère's Force from Weber's Force) and 4.4 (Derivation of Grassmann's Force from Lorentz's Force) of the book *Weber's Electrodynamics*.[366] Ampère's force between current elements is not compatible with Einstein's theory of relativity. Due to this fact, textbook authors deleted Ampère's force from their manuals. Consequently Ampère's force has not been taught at high-school and Universities for more than a century. The first complete translation of his masterpiece of 1826 to any language happened only in 2009 when it was translated into Portuguese.[367] Partial English translations of his works were only published in 1965 and 1969, while complete English translations

[364] [Gra45] with English translation in [Gra65], and [Gra77] with English translation in [Gra21].
[365] [Max54a, vol. 2, articles 526 and 527, p. 174].
[366] In English: [Ass94]. In Portuguese: [Ass92a], [Ass95a] and [Ass15a].
[367] [Amp26] and [Amp23]. Complete Portuguese translation in [Cha09] and [AC11].

of his masterpiece appeared only in 2012 and 2015.[368]

Textbooks attribute to Ampère the so-called "Ampère's circuital law", that is, the line integral of a magnetic field around a closed loop is proportional to the electric current passing through the loop. However, Ampère never derived such a law, he did not write it down in any format. As a matter of fact, he never worked with the magnetic field concept. *Moreover, he did fight explicitly against anything circulating around a current carrying wire*![369] The first to write the circuital law, even without the displacement current, was Maxwell in his first paper dealing with electromagnetism of 1855, twenty years after Ampère's death.[370] The so-called "Ampère's circuital law" is a misnomer and should not be attributed to Ampère.

Maxwell was really impressed with the power of Ampère's force between current elements. His admiration for Ampère's work and for his force given by Equation (17.4) has been expressed in the following words:[371]

> The experimental investigation by which Ampère established the law of the mechanical action between electric currents is one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped, full grown and full armed, from the brain of the 'Newton of Electricity'. It is perfect in form, and unassailable in accuracy, and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electro-dynamics.

I hope that in the XXIst century Ampère's force between current elements will become once again the most important formula of electrodynamics, as advocated by Maxwell.

Detailed descriptions of these phenomena, experiments, theoretical concepts, conflicts of paradigms, together with many quotations of original sources and a large bibliography, can be found in our book "Ampère's Electrodynamics — Analysis of the Meaning and Evolution of Ampère's Force between Current Elements, together with a Complete Translation of His Masterpiece: Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience".[372]

17.2 Weber's Unification of the Laws of Coulomb, Ampère and Faraday

Weber obtained the unification of all branches of electrodynamics known during the XIXth century. Beginning with Ampère's force between current elements and supposing each current element as being composed of positive and negative electrified particles moving in opposite directions relative to the body of the conductor, he proposed in 1846 a *fundamental law of electric action*.[373] It was a force combining Coulomb's law for the interaction between

[368] Partial English translations: [Amp65], [Amp69b], [Amp69a] and [Amp69c]. Complete and commented English translations of Ampère's masterpiece: [Amp12] and [AC15].

[369] A detailed discussion of this topic can be found in Chapter 16 (Ampère Against His Main Opponents) of [AC11] and [AC15].

[370] [Max58, p. 66 of Maxwell's original paper and p. 206 of Niven's book], [Whi73a, pp. 242-245], [Ass94, Section 2.5: Maxwell's Equations] and [Erl99].

[371] [Max54a, vol. 2, article 528, p. 175].

[372] In English: [AC15]. In Portuguese: [AC11].

[373] [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

electrified particles at rest relative to one another, together with a component depending on the relative velocity between the particles and another component depending on their relative acceleration.

In 1852 he presented his law utilizing a constant c representing the uniform relative velocity at which the force between particles would fall to zero. Weber's c (known throughout the 19th century as the *Weber constant*) was first measured by Weber and Kohlrausch in 1854-1856. They obtained $c = 4.39 \times 10^8 \ m/s$.[374]

In 1857 Weber and Kirchhoff were the first to derive theoretically the complete telegraph equation working independently from one another and arriving simultaneously at the same result. Utilizing the modern concepts and usual terminology of circuit theory, we can say that they were the first to take into account not only the capacitance and resistance of the wire, but also its self-inductance. For a circuit of negligible resistance, they concluded that the velocity of propagation of an electric wave along the wire would be given by $c/\sqrt{2} = 3 \times 10^8 \ m/s$. This value coincided with the known light velocity in vacuum, v_L, as deduced from astronomical observations and from terrestrial optical experiments. That is, $c/\sqrt{2} = v_L$ or $c = \sqrt{2} \cdot v_L$. This result was independent of the cross section and conductivity of the wire, and also independent of its surface density of electricity. Kohlrausch, who was collaborating with Weber on some experiments related with the propagation of electromagnetic waves, died in 1858. Weber's work has been delayed in publication and appeared only in 1864.[375]

Weber's 1846 fundamental force can then be expressed in terms of his 1852 constant c and of light velocity v_L as:[376]

$$\frac{ee'}{r^2}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 + \frac{2}{c^2} r \frac{d^2r}{dt^2}\right] = \frac{ee'}{r^2}\left[1 - \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{v_L^2} r \frac{d^2r}{dt^2}\right] . \tag{17.5}$$

The force between two particles was thus dependent upon their *relative velocity*, dr/dt, and *relative acceleration*, d^2r/dt^2. Velocities of the particles relative to the observer or frame of relevance are not relevant here. Weber's force points along the straight line connecting the interacting particles and follows the principle of action and reaction.

Weber had also introduced in 1848 a potential energy from which he could deduce his force law.[377] Weber's potential energy can be written in terms of Weber's constant c and light velocity v_L as:

$$\frac{ee'}{r}\left[1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] = \frac{ee'}{r}\left[1 - \frac{1}{2v_L^2}\left(\frac{dr}{dt}\right)^2\right] . \tag{17.6}$$

In 1868 Carl Neumann (1832-1925) obtained a Lagrangian and Hamiltonian formulation of Weber's law.[378]

When there is no motion between the particles, $dr/dt = 0$ and $d^2r/dt^2 = 0$, Weber's force reduces to Coulomb's law. Therefore, the whole of electrostatics is contained in Weber's electrodynamics, including Gauss' flux law.

[374] [Web55] with English translations at [Web21g]; [WK56] with English translation in [WK03] and Portuguese translation in [WK08]; and [KW57] with English translation in [KW21].

[375] [Kir57b] with English translation in [Kir57a], [Pog57] with English translation in [Pog21], and [Web64] with English translation in [Web21b].

[376] [Web52b, p. 366 of Weber's *Werke*] with English translation in [Web21a].

[377] [Web48] with English translation in [Web52c], [Web66] and [Web19a].

[378] [Neu68a] with English translation in [Neu20a], see Chapter 5.

Consider now two conductors carrying steady currents i and i'. Two current elements of lengths ds and ds' are represented by ids and $i'ds'$. We can consider a neutral current element ids as composed of equal and opposite charges, e_+ and $e_- = -e_+$, moving relative to the matter of this conductor with velocities v_+ and v_-. Likewise, we can consider a neutral current element $i'ds'$ as composed of equal and opposite charges, e'_+ and $e'_- = -e'_+$, moving relative to the body of this conductor with velocities v'_+ and v'_-. The interaction of ids with $i'ds'$ is then composed of four terms, namely, the force between e_+ and e'_+, the force between e_+ and e'_-, the force between e_- and e'_+, and the force between e_- and e'_-. By adding these four expressions, Weber was able to deduce in 1846 Ampère's force between current elements from his fundamental force between electrified particles. Therefore, the whole of Ampère's electrodynamics is contained in Weber's force law. As Ampère had unified magnetostatics, electrodynamics and electromagnetism with his force law, all of these three branches of physics are contained in Weber's electrodynamics (including the so-called magnetic circuital law).

In his deduction of Ampère's force between current elements from a force between electrified particles, Weber had utilized in 1846 Fechner's hypothesis. According to Fechner, in a conductor carrying a steady current the positive and negative particles move relative to the matter of the conductor with equal and opposite velocities. However, it should be emphasized that it is not necessary to impose this condition. As a matter of fact, Ampère's force is deduced from Weber's force even with arbitrary and independent velocities of the positive and negative particles relative to the matter of the conductors. Therefore Ampère's force is deduced from Weber's law even for the case of metallic conductors in which the positive charges are fixed in the lattice and only the conduction electrons are responsible of the currents.[379]

In 1831 Faraday discovered electromagnetic induction.[380] That is, the induction of a current in a secondary circuit by varying the current intensity in a nearby primary circuit. He also showed that it was possible to induce a current in a secondary circuit when there was a relative motion between this secondary circuit and a primary circuit carrying a steady current (or when there was a relative motion between this secondary circuit and a nearby magnet). These phenomena became known as Faraday's law of induction. In 1846 Weber was able to deduce quantitatively Faraday's law of induction from his fundamental force law.[381] James Clerk Maxwell (1831-1879) could only obtain a mathematical formulation of Faraday's law many years after Weber.

Therefore, essentially all electrodynamic phenomena known during the working periods of Maxwell and Weber could be explained quantitatively from Weber's law given by Equations (17.5) and (17.6).

Newton and Coulomb's force laws depend on the distance r between the interacting particles. As it deals with a much broader class of phenomena, Weber's force depends also on the *relative* velocity, dr/dt, and *relative* acceleration, d^2r/dt^2, between the interacting particles. These are *relational magnitudes* which have the same values for all observers and for all frames of reference. These magnitudes have the same value even when comparing an inertial frame of reference with a non inertial frame of reference. They are intrinsic magnitudes related only to the interacting bodies.[382]

[379] See [Ass90], [Wes90b] and Section 4.2 (Derivation of Ampère's Force from Weber's Force) of [Ass94].

[380] See footnote 9 on page 10.

[381] [Web46] with partial French translation in [Web87] and a complete English translation in [Web07].

[382] [Ass13] and [Ass14].

There are many areas in electrodynamics which should be explored utilizing Weber's force.

Weber and many other scientists studied the two-body problem with Weber's force or potential energy.[383] It will be relevant to study in the future this same problem considering now the spin for one of these particles or for both of them.[384] That is, considering a charged particle which can not only translate in space relative to the other particle and relative to the distant bodies in the cosmos, but which can also rotate around its axis. Obviously in the future scientists should also consider the three, four and many body problems with Weber's law.

Consider an electric circuit of self-inductance L and resistance R connected to an applied electromotive force $V(t)$. The equation describing the flow of current $I = dQ/dt$ is a second-order ordinary differential equation with constant coefficients. As mentioned in most textbooks, this equation is mathematically equivalent Newton's second law of motion when a mass m is subject to a damping force $-bv$ proportional to its velocity v, where b is the constant of friction, and to an external applied force $F(t)$. However, these two equations are not only mathematically equivalent. After all, the equation of circuit theory can be derived from Newton's second law of motion together with Weber's force. *As a matter of fact, it is possible to show that the equation of circuit theory is Newton's second law of motion applied to the conduction electrons coupled with Weber's force.*[385] In particular, the self-inductance L is a measure of the effective inertial mass of the conduction electrons. The effective inertial mass of a conduction electron due to its interaction with the positive charges of the metal lattice is much larger than the usual free electron mass. This Weberian interpretation of the self-inductance offers a new insight to the microscopic theory of conduction and should be further extended to other domains.

The Nobel prize winner Hannes Alvén (1908-1995) developed in the 1940's the theory of magnetohydrodynamics, or MHD, in order to deal with a plasma (like a gas composed of positively and negatively electrified particles, or positively charged ions and electrons) as a conducting fluid. It would be important to develop a Weberian plasma physics. This theory might be applied, for instance, for laser plasma physics and inertial confinement fusion.

The physics of particle accelerators might benefit enormously from a new approach based on Weber's law. Weber himself had given some suggestions of a transmutation of one chemical element into another through his electrodynamic force.[386]

The search for controlled thermonuclear fusion began in the 1950's. No real breakthrough has been attained in the last 70 years. By applying Weber's law to this area of research it may be possible to finally reach the desired goal of a controlled thermonuclear fusion reactor.

Space plasma physics should also benefit enormously from this paradigm change. When considering it from the point of view of Weber's law, we may better understand different phenomena like the Earth's ionosphere, Van Allen radiation belts, auroras (polar lights), radio galaxies, quasars etc.

I can illustrate the power of Weber's law when applied to new phenomena not known during his lifetime by mentioning superconductivity, that is, materials exhibiting zero electric resistance. It was discovered in 1911 by H. K. Onnes (1853-1926). The so-called Meissner effect was discovered in 1933 by F. W. Meissner (1882-1974) and R. Ochsenfeld (1901-1993).

[383] See footnote 173 on page 113.

[384] [Ass89].

[385] [Ass97] and [AH06].

[386] Section 1.11 of [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18].

It is normally described as the expulsion of a magnetic field from a superconductor material during its transition to the superconducting state. The so-called London moment is the magnetic moment acquired by a rotating superconductor. When a superconducting material is rotated relative to an inertial frame of reference with an angular velocity $\vec{\Omega}$, a magnetic field $\vec{B} = 2m\vec{\Omega}/e$ is developed throughout its interior, where $m > 0$ is the electron's mass and $e > 0$ is the magnitude of its charge. It was predicted by R. Becker (1887-1955), F. London (1900-1954) and others. By applying Weber's original 1846 force law to these phenomena, it is possible to deduce the London moment, the Meissner effect and the London penetration depth (the characteristic length of the exponential decrease of an external applied magnetic field from the surface inwards of superconductors). These deductions are not only qualitative, but also quantitative without any adjustable or free parameters.[387]

This specific example illustrates the power of Weber's electrodynamics when applied to new phenomena.

17.3 Unification of Optics with Electrodynamics through Weber's Force

As discussed in Section 17.2, Weber introduced in 1852 the constant c in his 1846 force. This constant was first measured by Weber and Kohlrausch in 1854-1856 yielding $c = 4.39 \times 10^8\ m/s$. In 1857 Weber and Kirchhoff were the first scientists to succeed in deducing the complete telegraph equation by taking into account not only the capacitance and resistance of the wire, but also its self-inductance. Both of them worked in the framework of Weber's electrodynamics and utilized Weber's force as the basis of their calculations. For a circuit of negligible resistance, the telegraph equation reduced to the wave equation with the disturbance propagating along the wire with velocity $c/\sqrt{2}$. It is worth while quoting Kirchhoff's words:[388]

> The velocity of propagation of an electric wave is here found to be $= c/\sqrt{2}$, hence it is independent of the cross section, of the conductivity of the wire, also, finally, of the density of the electricity: its value is 41950 German miles in a second, hence very nearly equal to the velocity of light *in vacuo*.

That is, $c/\sqrt{2} = 3 \times 10^8\ m/s$, which has essentially the same value as the known measure of light velocity in vacuum, v_L, as obtained by astronomical observations or terrestrial experiments. Utilizing the International System of Units we can say that in 1846-1852 Weber introduced for the first time in physics the electromagnetic constant $c/\sqrt{2} = 1/\sqrt{\varepsilon_0\mu_0}$, where ε_0 is the so-called vacuum permittivity constant (or permittivity of free space) and μ_0 is the so-called vacuum permeability constant (or permeability of free space). Then in 1854-1856 Weber and Kohlrausch measured this constant in a purely electromagnetic experiment. In particular, they measured the force between two charged spheres and then the torque produced on a nearby magnetic needle when a portion of the charge of one of these spheres was discharged to the ground. From these two force measurements they obtained the value of $1/\sqrt{\varepsilon_0\mu_0}$. They did not measure the velocity of any body, they did not study any property of light in this experiment, nor anything related specifically with optics. In

[387] [AT17] and [Pry18].

[388] [Kir57b, pp. 209-210] and [Kir57a, 406].

any event they obtained $c/\sqrt{2} = 1/\sqrt{\varepsilon_0\mu_0} = 3 \times 10^8\ m/s$, that is, the same value as light velocity in vacuum. In 1857 Weber and Kirchhoff obtained that an electric wave propagates along a telegraphic cable of negligible resistance exactly with this velocity. These results of Weber, Kohlrausch and Kirchhoff showed a direct quantitative connection between optics and Weber's electrodynamics many years before Maxwell.

As a matter of fact, these results inspired many electromagnetic theories of light as those of Maxwell himself in 1861-1873,[389] Georg Friedrich Bernhard Riemann (1826-1866) in 1858 (paper published posthumously in 1867),[390] and Ludvig Valentin Lorenz (1829-1891) in 1867.[391]

Is it possible to deduce a complete electromagnetic theory of light working only in the framework of Weber's electrodynamics? I believe this is possible. Here I present a few hints in this direction.

Weber, Carl Neumann and J. K. Friedrich Zöllner (1834-1882) pointed out different connections between light and Weber's law, as discussed in Section 1.9 (Optical properties of Weber's planetary model of the atom) of our book *Weber's Planetary Model of the Atom*.[392]

Weber always defended a wave theory of light. For instance, in his joint book with his brother, the physiologist Ernst Heinrich Weber (1795-1878), Weber compared the wave theory and Newton's corpuscular theory of light, showing the advantages of a wave propagation through an ether.[393]

He discussed alternating currents in Section 16 of his first major Memoir of 1846. He distinguished the steady galvanic current from the alternating current, which in very short sequential time intervals constantly changes its direction. He then advanced a bold hypothesis that incident light waves might create electric vibrations upon the electric fluids of a material substance! To our knowledge this was the first time he pointed out a possible connection between light and electricity:[394]

> Since the progressive motion of electricity occurs so abundantly in Nature, it is not obvious why, given such great mobility occasional conditions should not also occur, which favor a vibrating movement. If, e.g., light undulations exert an effect on the electrical fluids, and have the power to disturb their equilibrium, it would certainly be expected that these *effects* of light undulations would be structured in time with the same periodicity as the *light undulations themselves*, so that the result would consist of an *electrical vibration*, which, however, we are unable to discover with our instruments.

The electromagnetic influence upon optical phenomena was known since 1845, when Faraday discovered the magnetic rotation of the plane of polarization of light.[395] He observed this rotation for light traversing a piece of heavy glass immersed in a strong magnetic field along the direction of light propagation, or with the glass surrounded by a galvanic current flowing along an helix surrounding the glass, that is, with the current in the helix flowing in planes almost orthogonal to the light beam. Weber was well aware of this discovery and

[389] [Max62], [Max65] and [Max54a].
[390] [Rie67b] with English translation in [Rie67a] and [Rie77a].
[391] [Lor67b] with English translation in [Lor67a].
[392] [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18].
[393] [WW94, Paragraphs 306 to 313 of Weber's *Werke*].
[394] [Web46, p. 124 of Weber's *Werke*] with English translation in [Web07, p. 76].
[395] [Far46, Series XIX, Articles 2146-2242].

mentioned it in his work of 1846, when advancing the suggestion that the ether believed to propagate light vibrations might be a neutral electric medium:[396]

> The *idea of the existence* of such a transmitting medium is already found in the *idea of the all-pervasive neutral electrical fluid*, and even if this *neutral fluid*, apart from conductors, has up to now almost entirely evaded the physicists' observations, nevertheless there is now hope that we can succeed in gaining more direct elucidation of this all-pervasive fluid in several new ways. Perhaps in other bodies, apart from conductors, no currents appear, but only *vibrations*, which can be observed more precisely for the first time with the methods discussed in Section 16. Further, I need only recall Faraday's latest discovery of the influence of *electrical currents on light vibrations*, which make it not improbable, that the all-pervasive neutral electrical medium is itself that all-pervasive ether, which creates and propagates light vibrations, or that at least the two are so intimately interconnected, that observations of light vibrations may be able to explain the behavior of the neutral electrical medium.

Carl Neumann treated mathematically the rotation of the plane of polarization of light by magnetism from the point of view of Weber's electrodynamics in his Dissertation of 1858.[397] Five years later he presented a more detailed account of his theory.[398] He proposed an interaction between ether particles and the molecules of the body depending upon external magnetic forces. But these forces would act only upon mobile ether particles which had been previously excited, but not upon stationary ether particles. He supposed this force to be produced by Ampèrian molecular currents induced in the body by these magnetic forces, in analogy with Weber's explanation for diamagnetism. However, it must be emphasized here that Neumann utilized an *ideal* model of an Ampèrian molecular current, that is, based upon a continuous current flow around the molecule, like the rings of Saturn. These induced molecular currents would act upon the ether particles according to Weber's force law. This interaction would be analogous to the mutual interaction of two electric currents. Neumann's theory was a first tentative to apply Weber's law to optical phenomena.[399]

In 1862 Weber postulated the excitation of heat- or light-waves through molecular currents.[400] Neumann's *ideal* model of an Ampèrian molecular current could not excite these waves in the ether, as just mentioned. In this work of 1862, on the other hand, Weber presented once more a model for *discrete* or *corpuscular* Ampèrian molecular currents similar to the model which he had presented in 1852.[401] This crucial change made it possible the excitation in the ether of light-waves through molecular currents. The only difference of this model as regards Weber's model of 1852, is that now Weber reversed the signs of the mobile and stationary electric charges. In this work of 1862 Weber endowed his planetary model of the atom with optical properties, namely, the possible production of light waves through the ether.

He made the following comments when discussing Neumann's work related to Faraday's rotation:[402]

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[396] [Web46, pp. 213-214] with English translation in [Web07, pp. 141-142].
[397] [Neu58].
[398] [Neu63].
[399] [Wie60, pp. 194-195] and [Wie67].
[400] [Web62, Weber's *Werke*, pp. 94-96].
[401] [Web52b] with English translation in [Web21a]; see also [Rie92, p. 25].
[402] [Web62, Weber's *Werke*, p. 95].

> Neumann found, according to his assumptions, that there could be no action of electric molecular currents upon *stationary ether particles*; however, it should be observed that these assumptions were in agreement with Neumann's goals, which were limited to the action of the molecular currents upon the wave trains propagating in the ether and already existing in the middle of the molecules, indeed to the actions of the molecular currents attaining to very small distances, it afforded the admission of an *ideal* conception of molecular currents, in which these were considered as a *superposition of opposite and equal currents of positive and negative electricity*, but which apparently is not appropriate for the production of new wave trains through the electric molecular currents, which can only happen in the *immediately adjoining layer between the ether* and the molecular currents. For these ether boundaries the considered electric particles moving in opposite directions should no longer be considered as coincident. When we suppose, for instance, the negative fluid as rigidly connected with the molecule, and consider only the positive fluid in molecular current, or vice versa (a conception which recommends itself, as it is consistent with the persistence of the molecular currents without electromotive forces), it is then clear, that the difference in position and behavior of both electric fluids in the domain of the molecule, indeed already by very small distances (as Neumann considered them) does not need any longer to be considered, based upon the admissibility of that *ideal* conception about the molecular currents, that it, however, for the *immediately adjoining ether layer* can have a significance, especially when the electric fluids composing the molecular currents were *not continuous and evenly distributed* around the molecule.

Carl Neumann had an image of the molecular currents as composed of both electric fluids moving in opposite directions in continuous closed orbits around the molecule, in analogy with the rings of Saturn. But this picture was not appropriate for the production of new waves through the ether. As can be seen from this quotation, Weber modified Neumann's conception. The continuous distribution of positive and negative mobile charges moving around the molecule were now considered as concentrated in particles, like the Moon orbiting around the Earth. That is, Weber's transformed Ampère's molecular currents into a planetary system![403]

In 1852 he had already a similar idea, but at that time with the positive charge considered as stationary with the molecule, while the negative charge orbited the positive molecule.[404] In this work of 1862 he reversed the signs of the charges. At that time it was not yet possible to decide which sign of the charge should be connected with Ampère's molecular current.

In the sequence of this work of 1862, Weber even pointed out that the orbital frequency of the charged particles of his planetary model should be identical with the frequency of the excited heat- or light-waves. The relevant quotation runs as follows:[405]

> When a perturbation of the equilibrium in the immediate border of the ether and, consequently, a production of an ether-wave, really takes place, then it is clear that it will repeat itself in each orbit of the electricity around the molecule, in such a way that the period of the wave must be identical with the period of the orbit of the electric particle which is in molecular current.

[403] [Wie67, pp. 157-161].
[404] [Web52b] with English translation in [Web21a].
[405] [Web62, p. 95 of Weber's *Werke*].

Weber did not discuss the consequences of energy conservation in this production of heat- or light-waves by his planetary molecular current.

Weber also believed that it would be possible to utilize the optical properties of his planetary model in order to obtain information about the internal constitution of molecules. The wavelengths of the emitted light, in particular, might yield the key to draw conclusions about electric molecular processes. It might be possible, for instance, to obtain information about these molecular currents from the properties of light emitted by the molecules:[406]

> However, the wavelength of the wave train emitted by *glowing molecules* is well known from optical experiments; therefore, if the supposed relation between electrical molecular currents and the light ether, according to Neumann's ideas, are corroborated, then it would be possible to obtain, from optical experiments, a better information about the behavior of the electricity generating a molecular current.

In 1876 Zöllner advanced the other side of this reasoning, namely, to utilize the internal properties of a planetary model in order to derive the spectral lines of the chemical elements! The relevant quotation runs as follows:[407]

> The laws developed by Weber about the oscillations of an atomic pair will probably lead to an analytical determination of the number and position of the spectral lines of the chemical elements and their connections with the atomic weights.

This is a remarkable passage indicating a possible theoretical explanation of the known spectral lines of the elements. At that moment there was no detailed explanation for these spectral lines. The spectral analysis of the chemical elements had been developed by R. W. Bunsen (1811-1899) and G. R. Kirchhoff (1824-1887) in 1859. The full quantitative understanding of the specific spectral series for each atom was obtained only in the XXth century. In any event, it is amazing how far ahead of their time were Weber and Zöllner with these reasonings.

In this quotation, Zöllner was referring to Weber's work of 1871. Weber had estimated the period of oscillation of two charges of the same sign orbiting around one another separated by distances smaller than the critical distance. That is, separated by a distance r such that $r < \rho$. He found that this period of oscillation was approximately between $2r_0/c$ and $4r_0/c$. He then made the following comment, trying to connect this period of oscillation with that of visible light:[408]

> If we put $c = 439450 \cdot 10^6$ millimetre/second, it follows from this last determination that the value of ρ must lie approximately between 1/4000 and 1/8000 of a millimetre in order that these oscillations may be equal in rapidity to those of light.

As discussed by Hecht, it was with this model that Weber first attempted to find the basis for the production of oscillations at the frequency of light.[409]

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406 [Web62, pp. 95-96 of Weber's *Werke*].

407 [Zöl76b, Vorrede, p. XXI].

408 [Web71, p. 278 of Weber's *Werke*] with English translation in [Web72, p. 129]. See, in particular, page 93 in Section 9.14 of Chapter 9.

409 [Hec96].

The connection between electromagnetic waves and Weber's electrodynamics might also be developed along other directions. For instance, Weber and Kirchhoff calculated in 1857-64 the propagation of electric signals along conducting telegraph cables. They showed that the waves propagate at light velocity when the resistance of the wire was negligible. These calculations based on Weber's force should be extended for electric waves propagating in insulating media composed of positive and negative electrified particles (like air, glass, the medium of outer space etc.). Weber himself presented some ideas along this direction in Section 12 of his posthumous work published in 1894.[410]

Other ideas of how it might be possible to obtain a finite velocity of propagation for electric disturbances beginning with Weber's law were presented in the book *Weber's Electrodynamics*.[411]

17.4 Unification of Nuclear Physics with Electrodynamics through Webe's Force

In 2011 we published the book *Weber's Planetary Model of the Atom*.[412] I present here some of the main properties of Weber's fascinating atomic model.

This book presents the planetary model of the atom developed by Wilhelm Weber in the second half of the XIXth century, before the atomic model of Niels Bohr (1885-1962) which was created between 1911 and 1913. Weber's atomic model is based on his 1846 force law, which depends on the distance between the interacting charges, their relative velocity and their relative acceleration. Weber showed that two interacting charges of the same sign could behave as if they had negative inertial masses when they were accelerated relative to one another, provided they were moving at very close distances smaller than a critical distance r_c. When this condition is fulfilled, these two charges of the same sign will attract one another, instead of repelling each other as usually observed. Weber then predicted that atoms might be composed of negative charges describing elliptical orbits around the positive nucleus, being attracted by the nucleus, while the positive particles composing the nucleus would also attract one another due to their negative inertial masses. Three main remarkable aspects of his model should be emphasized:

- Weber's prediction was made before the discovery of the electron (1897) and also before Rutherford's scattering experiments (1911). His model was also developed before the discovery of Balmer (1885) and Paschen (1908) series describing the spectral line emission of the hydrogen atom. Bohr's atomic model of the atom (1913), on the other hand, was invented in order to be compatible with these experimental findings. While Bohr's model was ad hoc (that is, designed for this purpose), Weber's model was a real prediction.

- Nuclear forces are not necessary in Weber's model in order to stabilize the positive nuclei. After all, the positive particles of the nuclei are held together by purely electrodynamic forces. In modern physics, on the other hand, scientists had to postulate the existence of nuclear forces because they were no longer aware of Weber's electrodynamics. Therefore, after the existence of the positive nuclei was established, they were

[410] [Web94b] with English translation in [Web08], see Section 15.12.
[411] [Ass94, Section 8.2].
[412] [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18].

faced with the problem of explaining the stability of the nuclei against the repulsive Coulombian forces between its positively charged components. To this end they postulated the nuclear forces. With Weber's planetary model of the atom, on the other hand, we have an unification of electromagnetism with nuclear forces. This unification took place before these two branches of physics were separated from one another. Even the stability of the nuclei was predicted and explained by Weber's electrodynamics.

- When Weber developed his model in the 1850s to 1880s, the electron and the positron were not known, as these two particles were only discovered in 1897 and 1932, respectively. Therefore, he could make only qualitative and algebraic calculations relative to his critical distance r_c. But when we utilize the known mass and charge of two positrons, for instance, and calculate Weber's critical distance below which these two particles begin to attract one another, we obtain a number of the order of magnitude of 10^{-15} m, that is, essentially the known size of the nuclei. The similarity between Weber's r_c for these fundamental particles and the size of the nuclei should not be a coincidence.

From my point of view, Weber presented the essence of the correct atomic model. However, Weber's theory was forgotten during the whole of the XXth century. Nuclear physics was developed exactly during this period. Therefore it would be extremely relevant if modern nuclear scientists could analyze their data from the point of view of Weber's electrodynamics. Many new phenomena might even be predicted by exploring Weber's force and potential energy applied for the electrified particles composing the nuclei.

Weber's planetary model of the atom presents an unification of electromagnetism with nuclear physics. Moreover, this unification took place before these two branches of physics were separated from one another in the beginning of the XXth century. I believe that Weber's electrodynamics presents not only the essence of the correct explanation for the stability of the nuclei, but also a justification for their measured size.

17.5 Weber's Law and the Periodic Table of Elements

Weber's mature planetary model of the atom was presented in his posthumous work of 1894.[413] It is thought to be written in the 1880s. It is fragmentary and was not completed in Weber's lifetime. In this work he tried to explain the manifold of ponderable bodies from his fundamental force law of 1846.

He classified the material molecules into three categories. (a) The simple positive particle would have charge $+e$ and mass ε, while the simple negative particle would have charge $-e$ and mass ε' which might be different from ε. (b) The indissoluble positive particle would have m simple positive particles moving very close to one another, below the critical distance r_c, in such a way that they would attract one another. Likewise the indissoluble negative particle would have n simple negative particles moving very close to one another, below their critical distance, in such a way that they would attract one another. (c) The ponderable molecules composed of equal numbers of positive and negative particles.

Beyond these three categories, he envisioned the possibility that an electrified particle might orbit around a ponderable molecule, being dynamically connected to it through his force law. An electrified particle orbiting around a neutral ponderable molecule is essentially

[413][Web94b] with English translation in [Web08], see Chapter 15.

equivalent to the modern ions of physics and chemistry. The ponderable atoms or molecules might then be classified into three groups, namely, neutral molecules, positive ions and negative ions.

By developing his ideas, he considered the periodic table of the chemical elements in Section 15.5 of his work: Classification of material-molecules (*Körpermoleküle*) according to their composition and differentiation.

In 1871 and in his posthumous work he also advanced the great idea that the chemical atomic bonds between atoms might have an electric origin! We discussed some of his ideas in our book on Weber's planetary model of the atom.[414]

He tried to explain the formation of ice, water and steam and presented some qualitative considerations on how these different states of matter might change from one another. He also presented some fundamental ideas trying to explain the distinction between conductors and insulators, together with some of their main properties.

Most of his ideas were only qualitative. In any event, he presented important insights which might be further developed with modern analytical tools.

17.6 Applications of Weber's Law for Gravitation

In the book *Relational Mechanics and Implementation of Mach's Principle with Weber's Gravitational Force* I discussed the works of some scientists who applied Weber's law for gravitation.[415]

Weber's electrodynamics was extremely successful in explaining electrostatics (through Coulomb's force) and electrodynamic phenomena (Ampère's force between current elements, Faraday's law of induction, the telegraph equation describing the propagation of electromagnetic signals with light velocity along conducting wires, etc.). Due to this great success, some scientists tried to apply an analogous expression for gravitation. The pendulum swung back: after the great influence of Newton's gravitational force on Coulomb and Ampère, it was gravitation's turn to be influenced by electromagnetism.

The idea is that the force between two gravitational masses should have the same format as Weber's force between two electrified particles. The first to propose that the gravitational force might also depend on the velocity and acceleration between the interacting masses was Weber himself in his original work of 1846 in which he proposed his force law for electromagnetism:[416]

> Assuming the correctness of the results which we achieved, a case would arise here, in which the force, with which two masses act upon one another, would depend, not simply upon *the magnitude of the masses and their distance* from one another, but also on their *relative velocity and relative acceleration*. [...]

He mentioned this suggestion in several other works.[417]

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[414] Section 1.10.4 (Application to chemical bondings) of [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18].

[415] [Ass14, Section 25.3] and [Ass13, Section 24.3].

[416] [Web46, p. 149 of Weber's *Werke*] and [Web07, p. 92].

[417] [Web55, p. 595 of Weber's *Werke*] with English translation in [Web21g]; [KW57, p. 652 of Weber's *Werke*] with English translation in [KW21]; [Web82]; and [Web94b, pp. 481-488 of Weber's *Werke*] with English translation in [Web08, pp. 4-15]. See also [Woo81] and [Wis81].

Beyond Weber, the first scientists to propose a Weber's law for gravitation seem to have been C. Seegers in 1864 and G. Holzmüller in 1870.[418] Then in 1872 Tisserand studied Weber's force applied to gravitation and its application to the precession of the perihelion of the planets.[419] The two-body problem in Weber's electrodynamics had been solved by Seegers in 1864 in terms of elliptic functions, but Tisserand solved the problem iteratively. Hans Reissner (1874-1967) also worked with a Weberian potential energy for gravitation, although he did not mention Weber's law.[420]

Other people also worked with Weber's law for gravitation applying it to the problem of the precession of the perihelion of the planets: Paul Gerber in 1898 and 1917, Erwin Schrödinger (1887-1961) in 1925, Eby in 1977 and ourselves in 1989.[421] Curiously, none of these authors were aware of Weber's electrodynamics, with the exception of our work. Gerber was dealing with ideas of retarded time and worked in the Lagrangian formulation. Schrödinger was trying to implement Mach's principle with a relational theory. Eby was following the works of Barbour and Bertotti on Mach's principle and also worked with the Lagrangian formulation.

Poincaré discussed Tisserand's work on Weber's law applied to gravitation in 1906-1907.[422] Gerber's works were criticized by Seeliger, who was aware of Weber's electrodynamics.[423] Other scientists applied Weber's law to gravitation in the second half of the XXth century.[424]

Weber's electrodynamics was the leading theory during the second half of the XIXth century. However, Weber's law was essentially forgotten during the whole of the XXth century with the advent of the electromagnetic theory of light due to Maxwell, the experiments on electric waves due to H. R. Hertz (1857-1894) in 1887, the force on a charge in the presence of electric and magnetic fields due to Lorentz in 1895, and the theories of relativity due to A. Einstein (1879-1955) in 1905 and 1916. It is relevant to mention here some aspects of Schrödinger's 1925 paper in order to show how quickly Weber's law disappeared from physics teaching and had already been almost completely forgotten in the beginning of the XXth century.

In his article of 1925 Schrödinger utilized a potential energy for gravitation which is essentially the same potential energy which Weber had introduced in physics in 1848. However, Schrödinger was not aware of Weber's force and potential energy. He mentioned that he had obtained his potential energy "heuristically." The word heuristic refers to experience-based techniques for problem solving, typically by trial and error. He did not quote Weber, Seegers, Holzmüller, Tisserand, Reissner, nor any other author. However, if indeed he did achieve this equation heuristically, he should have arrived at this expression all by himself. Let me quote the relevant passage, my emphasis:[425]

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[418] [See64] with German translation in [See24]; [Hol70] with English translation in [Hol17]; [Nor65, p. 46] and [Jam00, p. 153].

[419] [Tis72] with English translation in [Tis17a], see Chapter 11. See also [Tis90] with English translation in [Tis17b]; [Tis96, Volume 4, Chapter 28 (Vitesse de propagation de l'attraction), pp. 499-503] and [Poi53, pp. 201-203].

[420] [Rei14] with English translation in [Rei95b]; [Rei15] with partial English translation in [Rei95a].

[421] [Ger98] with English translation in [Ger]; [Ger17]; [Sch25] with Portuguese translation in [XA94] and English translation in [Sch95]; [Eby77] and [Ass89]. See also [Meh87, p. 1157], [MR87, pp. 372-373 and 459] and [BP95, p. 51], [AP01] and [Bun01].

[422] [Poi53, pp. 125 and 201-203].

[423] [See17a] and [See17b].

[424] References in [Ass94, Section 7.5], [Ass14, Section 25.3] and [Ass13, Section 24.3].

[425] [Sch25], [XA94] and [Sch95, pp. 148-149].

One must therefore see if it is possible in the case of the kinetic energy, just as hitherto for the potential energy, to assign it, not to mass points individually, but instead also represent it as an energy of interaction of any two mass points and let it depend only on the separation and the rate of change of the separation of the two points. In order to select an expression from the copious possibilities, we use *heuristically* the following analogy requirements:

1. The kinetic energy as an interaction energy shall depend on the masses and the separation of the two points in the same manner as does the Newtonian potential.

2. It shall be proportional to the square of the rate of change of the separation.

For the total interaction energy of two mass points with the masses μ and μ' with separation r we then obtain the expression

$$W = \gamma \frac{\mu\mu'\dot{r}^2}{r} - \frac{\mu\mu'}{r} \ . \qquad (1)$$

The masses are here measured in a unit such that the gravitational constant has the value 1. The constant γ, which for the moment is undetermined, has the dimensions of a reciprocal velocity. Since it should be universal, one will expect that, apart from a numerical factor, this will be the velocity of light, or that γ will be reduced to a numerical factor when the light second is chosen as the unit of time. We shall have cause later to set this numerical factor equal to 3.

As a matter of fact, this is not the whole history of how Schrödinger arrived at this potential energy. The collected works of Schrödinger have been published recently. At the end of the reprint of this article, there is a typewritten note, signed by Schrödinger, where he expressed apologies for Reissner for plagiarizing his ideas, unconsciously.[426] Schrödinger said in this note that he knew Reissner's first paper of 1914, but was not certain as regards the second one of 1915. He considered Reissner's papers very interesting and expected that his own work would also have some interest for presenting a different approach of the subject. Perhaps the fact that he utilized Reissner's ideas without quoting him, and the embarrassment he may have felt when he had to admit this fact to Reissner, influenced him not to deal with this subject any further (other scientists may have perceived the similarities between their works).

In any event, it is a great irony that Weber's force and potential energy for electromagnetism had been published in 1846 and 1848, respectively, some 70 years before Reissner (80 years before Schrödinger). Detailed applications of Weber's law to gravitation dates back at least to the 1860's, some 50 years before Reissner. Weber published in German, like Reissner and Schrödinger. Many of Weber's paper were published in the *Annalen der Physik*, just like some of Reissner's papers and Schrödinger's specific paper of 1925.[427] Weber's work was discussed in the last Chapter of Maxwell's main book published in 1873, *A Treatise on Electricity and Magnetism*, which had been translated into German in 1883.[428] Weber's works were also discussed by many other important scientists. It is amazing that Reissner

[426] [Sch84, p. 192] with Portuguese translation in [XA97].

[427] [Rei16] and [Sch25].

[428] [Max73b] and [Max73a] with German translations in [Max83b] and [Max83a]. A Portuguese translation of the last Chapter of Maxwell's *Treatise* can be found in [Ass92c].

and Schrödinger did not know about Weber's pioneering papers and that even after their publications in 1915 and 1925, no one called their attention to Weber's earlier works.

This simple fact illustrates quite well how science teaching usually presents only the dominating paradigm, relegating other worldviews to total oblivion, as pointed out by Kuhn.[429] The replacement of paradigms sometimes takes place very quickly. When this takes place, then unfortunately the students are deprived of learning extremely important lines of reasoning.

17.7 Unification of Gravitation with Electrodynamics through Weber's Force

Weber's force between two electrified particles is proportional to the product of their charges, depending also on their distance, relative velocity and relative acceleration. In Weber's gravitational force, on the other hand, we simply replace in Weber's original 1846 electrodynamic force the product of the charges with the product of the gravitational masses of the interacting particles. Applications of Weber's force for gravitation were discussed in Section 17.6.

However, is it possible to <u>deduce</u> Newton's law of gravitation from Weber's electrodynamic force? Going a step further, is it possible to deduce Weber's gravitational force (depending also on the relative velocity and relative acceleration) from Weber's electrodynamic force or from a generalization of Weber's electrodynamic law? The answer to these questions seems to be positive. In this Section I will discuss two different approaches which might lead to the unification of gravitation with electrodynamics.

The first approach is due to Weber and was presented in his posthumous work published in 1894.[430]

Mossotti assumed in 1836 that the attractive force between particles electrified with charges e and $-e$ was slightly larger than the repulsive force between e and e, and also slightly larger than the repulsive force between $-e$ and $-e$. Consider now a molecule M_1 composed of n_1 positive and n_1 negative charges, and another molecule M_2 composed of n_2 positive and n_2 negative charges.[431] By considering now the net electric force between molecules M_1 and M_2 with this assumption, then this net force will be attractive, despite the charge neutrality of both molecules. In this way he could derive or deduce an attractive force law analogous to Newton's law of gravitation, beginning only with electrostatic forces.

Weber considered a similar idea, but applied it to his fundamental force of 1846 which depends not only on the distance, but also on the relative velocity and relative acceleration of the interacting electrified particles. He then assumed that all ponderable molecules are connections of equal quantities of positively and negatively electrified particles. Like Mossotti, he assumed that the attractive force of equal quantities of different kinds of electricity is slightly larger than the repulsive force of the same quantities of similarly charged electrified particles. In this way he derived or deduced a force analogous to Newton's law for gravitation, but now including a component depending on the relative velocity between

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[429] [Kuh62] with Portuguese translation in [Kuh82].

[430] [Web94b] with English translation in [Web08], see Chapter 15. See also [Whi73a, pp. 51-52]; [Whi73b, p. 150]; [Woo81]; [Wis81, pp. 282-283]; [Aep79, pp. 119-120 and 223-224]; Section 7.5 (Weber's law applied to gravitation) of [Ass94]; Section 1.10.1 (Deriving a gravitational force law from Webe's electric force law) of [AWW11] with Portuguese translation in [AWW14] and German translation in [AWW18].

[431] [Mos36] with English translation in [Mos66].

the ponderable molecules and another component depending on the relative acceleration between the ponderable molecules.

In essence, Weber obtained the unification of gravitation with electrodynamics utilizing his fundamental force of 1846.

There is second and completely independent approach to deduce a Weberian force for gravitation from electrodynamics.[432] Equations (17.5) and (17.6) suggest that Weber's law may be an expansion up to second order in $1/c$ of a more complex function. A more complete model might include time derivatives of all orders of the distance r between the interacting particles. That is, the force between the particles might include not only r, $dr/dt = \dot{r}$ and $d^2r/dt^2 = \ddot{r}$, but also all other time derivatives like d^3r/dt^3, d^4r/dt^4, etc. Moreover, the general expression of this complete force law might include as well different powers of these derivatives, like $(dr/dt)^m$, $(d^2r/dt^2)^n$, $(d^3r/dt^3)^o$, $(d^4r/dt^4)^p$, ... Here the powers m, n, o, p etc. should be considered as integers still to be determined.

In 1992, I proposed a possible way to deduce Weber's law for gravitation from a generalized Weber's force for electrodynamics including terms of the fourth and higher orders in $\dot{r}/c$. I studied the interaction between two neutral dipoles in which the negative charges oscillate around the positions of equilibrium. It was shown that these extra terms yield an attractive force between the neutral dipoles which can be interpreted as the usual Newtonian gravitational interaction, but now with a component depending on the relative velocity between the interacting dipoles and another component depending on the relative acceleration between the interacting dipoles. The justification for this interpretation is that this net attractive force between the interacting dipoles has the order of magnitude of the Newtonian gravitational force, points along the straight line connecting the dipoles, following Newton's action and reaction law. Moreover, the component which does not depend on the relative velocity and relative acceleration between the dipoles falls as the inverse square of the distance between the dipoles, just like Newton's universal law of gravitation.

17.8 Unification of Inertia with Gravitation and Implementation of Mach's Principle: Deduction of Newton's Second Law of Motion from Weber's Force

Bodies interact with one another through electric, magnetic or gravitational forces. Bodies have also inertial properties (their inertial masses, kinetic energies, linear and angular momenta, inertial forces acting on them, etc.). Ernst Mach (1838-1916) proposed the idea that the inertial properties of a body might be due to its interaction with distant matter in the cosmos.[433]

Mach suggested that the inertia of a body should be connected with distant matter and especially with the fixed stars (in his time the external galaxies were not yet known). He did not discuss or emphasize the proportionality of the inertial mass with the gravitational mass. He did not say that inertia should be due to a *gravitational* interaction with distant masses. He did not propose any specific force law to implement his ideas quantitatively. Newton, for instance, in his famous bucket experiment showed that centrifugal forces are

[432] [Ass92b]; [Ass94, Section 8.5]; [Ass95b] and [BT18].
[433] [Mac60].

produced by the rotation of a body around its axis, as shown by the parabolic shape of the water surface. To Newton this effect proved the existence of absolute space not related with the distant matter in the cosmos. Mach, on the other hand, suggested that this effect was due to the relative rotation between the water and the distant matter. If it were possible, for instance, to keep the bucket and water at rest relative to the ground, while the whole set of distant stars and galaxies were rotated together once a second around the axis of the body, it might be possible to know who was right. If Newton was right, then the surface of the water should remain horizontally flat. On the other hand, if Mach was right, then the water should acquire a parabolic shape. His ideas became known as Mach's principle. However, Mach was not able to show mathematically that the set of stars spinning together around the axis of the bucket should generate centrifugal forces on the water. In any event, his book *The Science of Mechanics* was extremely influential as regards physics. It was published in 1883, and from that time onwards people began trying to implement experimentally or mathematically his intuitive ideas, which were very appealing.

In my book on *Relational Mechanics* I quoted several people who suggested Weber's law for gravitation in order to implement Mach's principle.[434] These scientists include the brothers Gottfried Immanuel Friedlaender (1871-1948) and Benedict Friedlaender (1866-1908), Alois Höfler (1853-1922), Wenzel Hofmann, Reissner, Schrödinger, P. B. Eby etc.[435]

In this book I showed how to implement mathematically Mach's principle utilizing Weber's law for gravitation together with the principle of dynamic equilibrium. According to this principle, the sum of all forces acting on any body is always zero in all frames of reference. I then showed how to deduce many things: (I) an analogous to Newton's second law of motion, (II) the kinetic energy as a gravitational energy arising from the interaction between the test body and the distant bodies in the cosmos, (III) the quantitative explanation of the proportionality between inertial and gravitational masses, (IV) the fact that the best inertial frame known to us is the universal frame of reference defined by the set of distant galaxies, (V) an explanation of Newton's bucket experiment and Foucault's pendulum utilizing Weber's law for gravitation, etc. Details of all these facts can be found in the book *Relational Mechanics and Implementation of Mach's Principle with Weber's Gravitational Force.*[436]

Weber's force and potential energy, Equations (17.5) and (17.6), are relatively simple. It is amazing that from such humble beginnings we can deduce such an amazing amount of facts: (a) The whole of electrostatics (including Coulomb's force and Gauss' flux law), (b) magnetostatics (the interaction between two magnets, or the interaction between a magnet and the Earth), (c) the interaction between current carrying conductors (including Ampère's force between current elements and the so-called magnetic circuital law), (d) electromagnetism (the interaction between a current carrying conductor and a magnet, or the interaction between a current carrying conductor and the Earth), (e) Faraday's law of induction, (f) the quantitative connection between electrodynamics and optics by the introduction and first measurement of the magnitude $c/\sqrt{2} = 1/\sqrt{\varepsilon_0\mu_0} = v_L = 3 \times 10^8\ m/s$, (g) the complete telegraph equation by taking into account the self-inductance of the wire, (h) the wave equation with the electromagnetic signal propagating at light velocity, etc.

[434] [Ass14, Section 25.4] and [Ass13, Section 24.4].

[435] [FF96] with partial English translation in [FF95] and with complete English translation in [FF07]; [Hof00, Note on p. 126], [Nor95, pp. 21 and 41] and [BP95, pp. 21, 24, 34-35, 40-41, 46, 53 and 164]; [Hof95]; [Rei14] with English translation in [Rei95b]; [Rei15] with partial English translation in [Rei95a]; [Sch25] with Portuguese translation in [XA94] and English translation in [Sch95]; [Eby77], [Eby79] and [Eby97].

[436] [Ass13] and [Ass14].

When applied to gravitation Weber's law leads to the precession of the perihelion of the planets. It is possible even to deduce an analogous to the gravitational laws of Newton or Weber either from Mossotti's idea or from a generalization in higher order of $1/c$ of Weber's electrodynamic law. Weber concluded based on his force that electrified particles of the same sign attract one another when they are moving inside a very small critical distance. This led to his prediction of a planetary model of the atom with stable nuclei, that is, to an unification of nuclear physics with electrodynamics. Moreover, Weber's law applied for gravitation leads to a mathematical implementation of Mach's principle showing the unification of inertia with gravitation! We <u>deduce</u> Newton's second law of motion, the proportionality between inertial and gravitational masses, a relational explanation of Newton's bucket and Foucault's pendulum experiments, etc.

In essence, everything seems to be contained in Weber's law. We only need to open our eyes to this paradigm, be inspired by it and let fire our imagination. Weber's force is a scalar formula with ordinary derivatives. It is not only extremely simple, but also philosophically appealing. It contains only relational magnitudes like the distance r between the interacting bodies, their *relative* radial velocity dr/dt and their *relative* radial acceleration d^2r/dt^2. It acts along the straight line connecting the particles and follows Newton's action and reaction law. Moreover, it complies with the conservation of linear momentum, angular momentum and energy.

If you want to make a real difference in the life of your students, I hope that you will begin to teach them Ampère's force between current elements and Weber's electrodynamics, together with their brilliant insights and revolutionary ideas. In this way they will begin to appreciate the history of science, will learn about the conflicts of paradigms, will develop a critical reasoning in physics and will begin to think by themselves. Moreover, by teaching Weber's electrodynamics to your students, they will have the opportunity to participate effectively in the experimental and theoretical developments of this fascinating and remarkable theory.

Newton, Coulomb, Ampère and Weber paved the way to past and present generations. I have no doubt that the future of physics will belong to the students and scientists who will follow their footsteps. By standing on the shoulders of these giants, they will see further and will lead us to a brighter future.

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- Page 7, the 1st line should be replaced by:

The picture on the cover of Volume 4 comes from a 1885 portrait of Weber made by the

- Page 8, the 3rd line from bottom to top should be replaced by:

electrodynamics; and (f) unification of inertia with gravitation. I discuss, in particular, the

- Page 18, the 3rd line of the 3rd paragraph should be replaced by:

latter of which is known for an elementary force of the first type, *but is completely unknown*

- Page 21, the 2nd line below Equation (5.1) should be replaced by:

The *receptive potential* on the other hand, is that which *is received by each of the two points at time t* and which

- Page 60, the 20th line should be replaced by:

surements presented in Volume 2 of these English translations of his main works on elec-

- Page 71, the 3rd line of the 3rd paragraph should be replaced by:

does not depend exclusively upon these particles themselves, on their distance and relative velocity, but also upon the portion of

- Page 77, the 3rd line from bottom to top should be replaced by:

where $u = dr/dt$ is the relative velocity of the two particles, and α the difference of their

- Page 105, the 7th line should be replaced by:

tion 9.16, at the same distance during their rotation around each other), we obtain

- Page 107, the 2nd line of the 4th paragraph should be replaced by:

of the electrical particles in the Ampèrian molecular currents contained in the same body might possibly

- Page 107, the 6th line from bottom to top should be replaced by:

fact consist in *an increase in the strength of the Ampèrian molecular currents formed by the electrical*

- Page 118, the 16th, 17th and 18th lines should be replaced by:

$h = 439450 \times 10^6$, with seconds and millimeters for units, then we will first have with our units:

$$\log h = 2.40805 \qquad \text{and} \qquad \delta\varpi = (\bar{4}.23550)t \ ;$$

after a century, we find that:

- Page 126, the 3rd line should be replaced by:

firstly involved the *reduction of all magnetic interactions to electric current interactions* and, sec-

- Page 142, the last two lines should be replaced by:

$+u$ as *positive* and those of $-u$ as *negative*, in order to distinguish the *moving away* of the particles from their *coming closer*. The system of curves corresponding to the *first* section of the Table

- Page 144, the 11th line of footnote 227 should be replaced by:

particles are always smaller than ρ, and another group in which they are always larger than ρ. Moreover, again

- Page 145, the penultimate line should be replaced by:

particles e', e'', ..., for which u_0 is the same, but very small for all of them, then each particle

- Page 149, the 6th line should be replaced by:

each ponderable particle hereafter would be an *electric double particle* (like a binary star),

- Page 155, the 2nd line of Section 1. should be replaced by:

of Σe and $4\pi\alpha^2 H$), and setting the moment of inertia of the cylinder as $\mathfrak{M} = ma^2$, then

- Page 155, the 6th line of Section 1. should be replaced by:

Now $2e'/\alpha$ is the required *separating force* for the charge e' of a spherical shell;[250]

- Page 155, the 1st line of footnote 250 should be replaced by:

[250][Note by WW:] The separating force exerted by a spherical shell of radius α covered uniformly with

- Page 155, the 8th, 9th and 10th lines of footnote 250 should be replaced by:

If a battery is inserted in this conductor, through which the charge on the sphere remains stationary, then this proves that the separating forces exerted by the charge on the sphere and by the battery on the conductor are equal and opposite to each other, whereby also the separating force of the battery is determined, namely

- Page 155, the 12th up to 14th lines of footnote 250 should be replaced by:

But it is also clear that, when the spherical shell was not yet charged, it would get charged from the battery, and that this charge would grow, until it got to e', assuming a sphere of radius $= \alpha$, that is, until the *separating force of the charge on the sphere* would have become $= 2e'/\alpha$ and *cancelled the separating force of the battery.*

- Page 155, the last line of footnote 250 should be replaced by:

be multiplied by $155\,370 \cdot 10^6 = [c/(2\sqrt{2})]$, in order to express them in *magnetic measure*.

- Page 162, the 16th line should be replaced by:

value of $q = 2g\vartheta$ has already been found above, consequently, as it was set before $a = g\mu$,

- Page 167, the 3rd line should be replaced by:

(see Section 12 of the *Abhandlung*)[283,284] the following equation is obtained:

- Page 167, footnote 283 should be replaced by:

[Note by HW:] Wilhelm Weber's *Werke*, Vol. IV, p. 408.

- Page 167, footnote 284 should be replaced by:
[Note by AKTA:] [Web78a, p. 408 of Weber's *Werke*].

- Page 172, the 1st line of the 2nd paragraph should be replaced by:
The velocity of the particle along its path also changes continuously in both cases, except at *one*

- Page 179, the last line should be replaced by:
motion, which encounter one another according to the theory developed in the 7th Section of the 7th Memoir

- Page 180, the 4th paragraph should be replaced by:
If all of these satellites remained bound with the ponderable molecules in the same way, they would have to be considered as belonging to them, and therefore *their mass would have to be added to the mass of the ponderable molecules* to which they belonged, and the force exerted reciprocally by the satellites of *two* ponderable molecules, as well as the force exerted by each of the two satellites on the one of the two ponderable molecules to which it does not itself belong, would have to be added or subtracted from the *gravitational force of the pair of molecules* according to the difference in sign.

- Page 181, the last line should be replaced by:
to demonstrate with certainty, for equal charges, a difference in the magnitude of the *force of attraction of dissmilar electric charges* and the *force*

- Page 183, the 1st line should be replaced by:
For comparison with the Newtonian law: according to the *law of gravitation derived by Zöllner from*

- Page 183, the 13th line should be replaced by:
$2\alpha nnee/r^2 = mm/r^2$, thus

- Page 186, the 2nd line of the 3rd paragraph should be replaced by:
molecules, then of the *ponderable* molecules. The *positive or negative electric particles* contained in

- Page 195, item 1 should be replaced by:
1. from the mutual *repulsion* of these ponderable molecules as a consequence of their similar, i.e., *negative electric charges*, in contrast to which disappears their mutual attraction due to gravitation;

- Page 200, the 1st line of the 2nd paragraph should be replaced by:
If one denotes with y' the value of y for $r = r'$, one obtains the equation:

- Page 204, the first three lines should be replaced by:
thereby led out of the sphere of action of *one ponderable molecule into that of another*. The *heat conduction* of metals through mutual *radiation* of the ponderable molecules and the *galvanic conductivity* of metals are based on this. Since, however, the positive electric molecules rotating

- Page 222, the 3rd line below Equation (17.5) should be replaced by:
of reference are not relevant here. Weber's force points along the straight line connecting

This is the fourth of 4 volumes of the book “Wilhelm Weber’s Main Works on Electrodynamics Translated into English”.

This fourth Volume begins with the English translation of Gauss' posthumous paper published in 1867. He arrived at a force law depending on the positions and velocities of the interacting electrified particles from which he could deduce not only electrostatics, but also the force between current elements. Then comes Carl Neumann's 1868 paper on the principles of electrodynamics. In this work Neumann introduced the Lagrangian and Hamiltonian formulations of Weber's electrodynamics. Moreover, he also showed that Weber's law was compatible with the principle of the conservation of energy.

Then comes Weber's 1871 Sixth major Memoir on Electrodynamic Measurements. He showed once more in detail that his force law was compatible with the principle of the conservation of energy. Moreover, he studied the two-body problem according to his electrodynamics. He showed that in some conditions two particles electrified with charges of the same sign might attract one another. This Volume also contains the English translation of Tisserand's 1872 paper on the motion of planets around the Sun according to Weber's law. He calculated, for instance, the precession of the perihelion of the planets.

Weber's Seventh major Memoir on Electrodynamic Measurements was published in 1878. It is devoted to the energy of interaction. His Eight major Memoir, thought to be written in the 1880's, was published only posthumously in 1894 in his collected works. It is related to the connection of Weber’s force with the law of gravitation. Moreover, it contains his mature planetary model of the atom in which the nucleus is held together by purely electrodynamic forces according to Weber's law.

This work closes with an overview of Weber's law applied to electromagnetism and gravitation, together with some possible future developments of his theory. The main topics are the unifications in physics: (a) Ampère's unification of magnetism, electrodynamics and electromagnetism; (b) Weber's unification of the laws of Coulomb, Ampère and Faraday; (c) unification of optics with electrodynamics; (d) unification of nuclear physics with electrodynamics; (d) applications of Weber's law for gravitation; (e) unification of gravitation with electrodynamics; and (e) unification of inertia with gravitation, in particular, the implementation of Mach's principle and the deduction of Newton's second law of motion from Weber's force.

About the Editor: Prof. Andre Koch Torres Assis has been working on Weber’s law applied to electromagnetism and gravitation for more than 30 years: https://www.ifi.unicamp.br/~assis

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