The Problem of Surface Charges and Fields in Coaxial Cables and its Importance for Relativistic Physics

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We calculate the surface charges, potentials and fields in a long cylindrical coaxial cable with inner and outer conductors of finite conductivities and finite areas. It is shown that there is an electric field outside the return conductor.
Key Words: Surface charges, coaxial cable, classical electrodynamics.
PACS: 41. 20. –q, 41. 20. Gz

1. Coaxial Cable

The possible existence of a second order motional electric field arising from steady conduction currents and its implications to the theory of relativity has been discussed recently by a number of authors: [1-7]. This field is of the second order in v_d/c, v_d being the drifting velocity of the conduction electrons, and is supposed to exist outside the wire. However, most of these authors do not consider the first order coulombian electric field (proportional to the current or to the drifting velocity) which should arise outside resistive wires carrying a steady current. As this first order electric field is relevant to the interpretation of some experiments, we decided to consider it here in a particular geometry. We discussed the second order electric field in [8], [Section 6. 6] [9] and [Section 5. 4] [10].

Before discussing the first order electric field we want to call attention to Ivezic's work: [11], [12], [13] and [14]. Although discussing the second order electric field, he was aware of the coulombian electric field. According to him the second order field might be due to a relativistic contraction of the average

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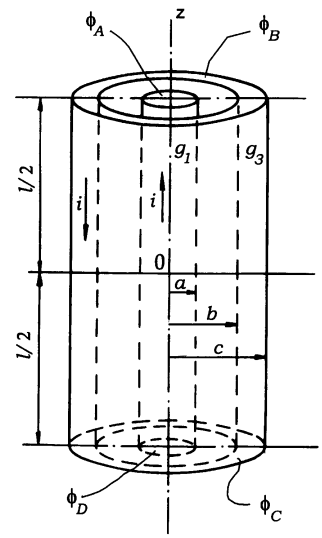

Figure 1. Geometry of the problem.

distance between moving electrons as compared with their average distance when there is no current. He could then explain several experiments based on this approach. In this connection it would be relevant to analyse Selleri's recent proposal of a new set of spacetime transformations between inertial systems in order to see if this new approach might give theoretical support to Ivezic's ideas when there is curvature in the wires (and then centripetal acceleration of the electrons) and explain the same set of experiments, [15] and [16]. Selleri's transformations don't present the discontinuity between accelerated and inertial reference frames which exists in Einstein's theory of relativity.

In the study of dc and low frequency ac circuits, the following subjects are seldom analysed in electromagnetic textbooks: electric fields outside the conductors, surface charges on the wires and energy flow from the sources to the conductors where energy is dissipated. There are two main reasons for this: (I) The scalar electric potential is the solution of Laplace's equation with frequently complicated boundary conditions; and (II) the solution of elementary circuits, based on Ohm's law, is obtained by the application of Kirchhoff's rules. As these rules utilize only the values of current and potential inside the conductors, the discussion of the subjects listed above is unnecessary. However, some authors have treated these topics in the past few years: [17,18] and references therein. The case of a long coaxial cable has been treated by Sherwood, [18], Marcus, [19], Sommerfeld, [20, pp. 125–130] (German original from 1948), Griffiths, [21, pp. 336–337] and a few others. All of these works considered a grounded return conductor either with an infinite area or with an infinite conductivity. Our main contribution in this work is to generalize these assumptions considering a return conductor with finite area, finite conductivity and with a variable electric potential along its length. We calculate at all points in space the scalar and vector potentials, the electric and magnetic fields and analyse the energy flow by means of Poynting vector. We also calculate the surface electric charges.

The geometry of the problem is that of Fig. 1. A constant current I flows uniformly in the z direction along the inner conductor (radius a and conductivity g_1), returning uniformly along the outer conductor (internal and external radii b and c, respectively, and conductivity g_3). The conductors have uniform circular cross sections and a length $l >> c > b > a$ centered on $z = 0$.

The medium outside the conductors is considered to be air or vacuum with $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ C^2 N^{-1}m^{-2}. The potentials at the right extremities (z = l/2) of the inner and outer conductors are maintained by a battery at the constant values ϕ_A and ϕ_B, respectively. The potentials at the left extremities (z = –l/2) of the outer and inner conductors are maintained by another battery at the constant values ϕ_C and ϕ_d, respectively. Instead of two batteries, the solution presented here can also be applied to the situation of one battery at one extremity and a resistor at the other extremity.

In the previous works quoted above the authors considered only a particular case: a grounded outer conductor ($\phi_C = \phi_d = 0$) with an infinite area (Sommerfeld, $c \to \infty$) or with an infinite conductivity (Griffiths, $g_3 \to \infty$).

We are interested in calculating the potentials and fields in a point $\vec{r} = (\rho, \varphi, z)$ such that $l >> \rho$ and $l >> |z|$, so that we can neglect border effects (ρ, φ and z are the cylindrical coordinates). All solutions presented here were obtained in this approximation. With this approximation and geometry we then have the potential as a linear function of z, [22]. In order to have uniform currents flowing in the z direction along the inner and outer conductors, with a potential satisfying the given values at the extremities, we have:

$$\phi(\rho \leq a, \varphi, z) = \frac{\phi_A - \phi_D}{\ell} z + \frac{\phi_A + \phi_D}{2}, \tag{1}$$

$$\phi(b \leq \rho \leq c, \varphi, z) = \frac{\phi_B - \phi_C}{\ell} z + \frac{\phi_C + \phi_B}{2} \tag{2}$$

where, by Ohm's law (R_1 and R_3 being the resistances of the inner and outer conductors, respectively):

$$\phi_d - \phi_A = R_1 I = \frac{\ell I}{\pi g_1 a^2}, \tag{3}$$

$$\phi_B - \phi_C = R_3 I = \frac{\ell I}{\pi g_3 (c^2 - b^2)}. \tag{4}$$

In the four regions ($\rho < a$, $a < \rho < b$, $b < \rho < c$ and $c < \rho$) the potential ϕ satisfies Laplace's equation $\nabla^2 \phi = 0$. The solutions of this equation for $\rho <$ a and for $c < \rho$ in cylindrical coordinates satisfying the boundary conditions above, Eqs. (1) and (2), and imposing the value $\phi(\rho = l, \varphi, z) = 0$ to complete the boundary conditions, yield:

$$\phi(a \leq \rho \leq b, \varphi, z) = \left[\frac{\phi_A - \phi_D + \phi_C - \phi_B}{\ell} z + \frac{\phi_A + \phi_D - \phi_C - \phi_B}{2}\right] \frac{\ln(b/\rho)}{\ln(b/a)} + \left[\frac{\phi_B - \phi_C}{\ell} z + \frac{\phi_C + \phi_B}{2}\right] \tag{5}$$

$$\phi(\leq \rho,\varphi,z)=\left[\frac{\phi_B-\phi_C}{\ell}z+\frac{\phi_C+\phi_B}{2}\right]\frac{\ln(\ell/\rho)}{\ln(\ell/c)}. \tag{6}$$

The electric field $\vec{E}=-\nabla\phi$ is given by

$$\vec{E}(\rho<a,\varphi,z)=\frac{\phi_D-\phi_A}{\ell}\hat{z}, \tag{7}$$

$$\begin{aligned}\vec{E}(a<\rho<b,\varphi,z)=&\left[\frac{\phi_A-\phi_D+\phi_C-\phi_B}{\ell}z+\frac{\phi_A+\phi_D-\phi_C-\phi_B}{2}\right]\frac{1}{\ln(b/a)}\frac{\hat{\rho}}{\rho}\\&+\left[\frac{\phi_C-\phi_B}{\ell}+\frac{\phi_D-\phi_A+\phi_B-\phi_C}{\ell}\frac{\ln(b/\rho)}{\ln(b/a)}\right]\hat{z}\end{aligned}, \tag{8}$$

$$\vec{E}(b<\rho<c,\varphi,z)=\frac{\phi_C-\phi_B}{\ell}\hat{z}, \tag{9}$$

$$\begin{aligned}\vec{E}(c<\rho,\varphi,z)=&\left[\frac{\phi_B-\phi_C}{\ell}z+\frac{\phi_C+\phi_B}{2}\right]\frac{1}{\ln(\ell/c)}\frac{\hat{\rho}}{\rho}\\&+\frac{\phi_C-\phi_B}{\ell}\frac{\ln(\ell/\rho)}{\ln(\ell/c)}\hat{z}.\end{aligned} \tag{10}$$

Eqs. (5) and (8) had been obtained by Jefimenko, [pages 509–511] [23], who also discussed the flow of energy in this system.

The surface charges densities σ along the inner conductor ($\rho = a$, $\sigma_a(z)$) and along the inner and outer surfaces of the return conductor ($\rho = b$, $\sigma_b(z)$ and $\rho = c$, $\sigma_c(z)$) can be obtained easily utilizing Gauss's law:

$$\oiint_S \vec{E}\cdot d\vec{a}=\frac{Q}{\varepsilon_o}, \tag{11}$$

where $d\vec{a}$ is the surface element pointing normally outwards the closed surface S, Q is the net charge inside S. This yields $\sigma_a(z) = \varepsilon_o E_{2\rho}(\rho \to a, z)$, $\sigma_b(z) = -\varepsilon_o E_{2\rho}(\rho \to b, z)$ and $\sigma_c(z) = \varepsilon_o E_{4\rho}(\rho \to c, z)$, where the subscripts 2ρ and 4ρ mean the radial component of $\vec{E}$ in the regions $a < \rho < b$ and $c < \rho$, respectively. This means that:

$$\sigma_a=\frac{\varepsilon_o}{a}\frac{1}{\ln(b/a)}\left[\frac{\phi_A-\phi_D+\phi_C-\phi_B}{\ell}z+\frac{\phi_A+\phi_D-\phi_C-\phi_B}{2}\right], \tag{12}$$

$$\sigma_b(z)=-\frac{a}{b}\sigma_a(z), \tag{13}$$

$$\sigma_c(z)=\frac{\varepsilon_o}{c}\frac{1}{\ln(\ell/c)}\left[\frac{\phi_B-\phi_C}{\ell}z+\frac{\phi_C+\phi_B}{2}\right]. \tag{14}$$

Jefimenko obtained only Eqs. (12) and (13), but not (14). This last one was calculated here for the first time.

An alternative way of obtaining ϕ and $\vec{E}$ is to begin with the surface charges as given by Eqs. (12) to (14). We then calculate the electric potential ϕ (and $\vec{E} = -\nabla\phi$) through

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{j=1}^{3}\iint_{S_j}\frac{\sigma(\vec{r}_j)\mathrm{d}a_j}{\left|\vec{r}-\vec{r}_j\right|} . \tag{15}$$

Here the sum goes over the three surfaces. We checked our results with this procedure.

We can calculate the vector potential utilizing

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\iiint\frac{\vec{J}(\vec{r}\,')\mathrm{d}V'}{|\vec{r}-\vec{r}\,'|} , \tag{16}$$

where $\mu_o = 4\pi\times 10^{-7}$ kg m C^{-2} and dV' is a volume element. With the approximation above we obtain:

$$\vec{A}(\rho \le a,\varphi,z) = \frac{\mu_o I}{2\pi}\left[\frac{\rho^2}{2a^2} - \frac{c^2\ln(c/a) - b^2\ln(b/a)}{c^2-b^2}\right]\hat{z} , \tag{17}$$

$$\vec{A}(a \le \rho \le b,\varphi,z) = \frac{\mu_o I}{2\pi}\left[\frac{c^2\ln(c/\rho) - b^2\ln(b/\rho)}{c^2-b^2} - \frac{1}{2}\right]\hat{z} , \tag{18}$$

$$\vec{A}(b \le \rho \le c,\varphi,z) = \frac{\mu_o I}{2\pi}\left[\frac{c^2\ln(c/\rho)}{c^2-b^2} - \frac{c^2-\rho^2}{2(c^2-b^2)}\right]\hat{z} , \tag{19}$$

$$\vec{A}(c \le \rho,\varphi,z) = 0 . \tag{20}$$

The magnetic field can be obtained either through the magnetic circuital law $\oint\vec{B}\cdot\mathrm{d}\ell = \mu_o I$, or through $\vec{B} = \nabla\times A$. Both approaches yield the same result, namely:

$$\vec{B}(\rho \le a,\varphi,z) = \frac{\mu_o I\rho}{2\pi a^2}\hat{\varphi} , \tag{21}$$

$$\vec{B}(a \le \rho \le b,\varphi,z) = \frac{\mu_o I\hat{\varphi}}{2\pi\rho} , \tag{22}$$

$$\vec{B}(b \le \rho \le c,\varphi,z) = \frac{\mu_o I}{2\pi}\frac{c^2-\rho^2}{c^2-b^2}\frac{\hat{\varphi}}{\rho} , \tag{23}$$

$$\vec{B}(c \le \rho,\varphi,z) = 0 . \tag{24}$$

This completes the solution of this problem.

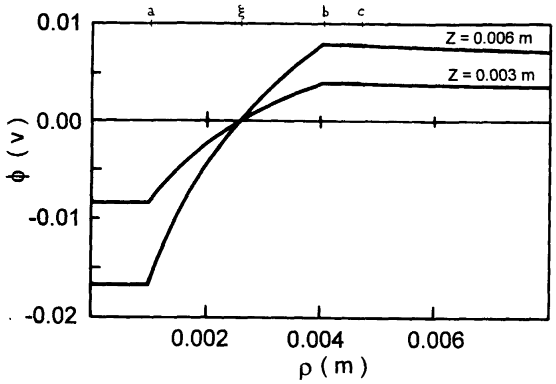

Figure 2. Electric potential as a function of ρ.

2. The Symmetrical Case

We now consider two equal batteries symmetrically located on both ends, such that $\phi_d = -\phi_A = \phi_1$ and $\phi_B = -\phi_C = \phi_3$. In this case the potential is simply proportional to z without any additive constant. We can then write it as $\phi(\rho, \varphi, z) = R(\rho)z$, where $R(\rho)$ in terms of the currents and conductivities is given by

$$R(\rho \le a) = -\frac{1}{\pi g_1 a^2}, \tag{25}$$

$$R(a \le \rho \le b) = -\frac{I}{\pi}\frac{1}{\ln(b/a)}\left[\frac{\ln(b/\rho)}{g_1 a^2} - \frac{\ln(\rho/a)}{g_3(c^2-b^2)}\right], \tag{26}$$

$$R(b \le \rho \le c) = \frac{I}{\pi g_3(c^2-b^2)}, \tag{27}$$

$$R(c \le \rho) = \frac{I}{\pi}\frac{\ln(\ell/\rho)}{\ln(\ell/c)}\frac{1}{g_3(c^2-b^2)}. \tag{28}$$

A plot of $\phi(\rho) = R(\rho)z$ versus ρ is given in Figure 2. In order to obtain this plot we utilized the following data: $a = 0.0010$ m, $b = 0.0040$ m, $c = 0.0047$ m, $I = 50$ A, $g_1 = 5.7 \times 10^6$ m^{-1} Ω^{-1}, $g_3 = 2 \times 10^6$ m^{-1} Ω^{-1} and l = 1 m. There are two curves, one for $z = 0.003$ m and another for $z = 0.006$ m. We see that the potential is constant for $0 \le \rho \le a$, increases between a and b, is constant between b and c, decreasing for $\rho > c$. As $\vec{E} = -\nabla\phi$, the z component of $\vec{E}$ is given by $E_z = -R(\rho)$, so that its behaviour is the same as that of $\phi(\rho)/z$ with an overall change of sign. The point where $\phi(\rho) = R(\rho) = 0$ is $\rho = \xi$, where

$$\xi = \exp\frac{g_1 a^2 \ln(a) + g_3(c^2-b^2)\ln(b)}{g_1 a^2 + g_3(c^2-b^2)}. \tag{29}$$

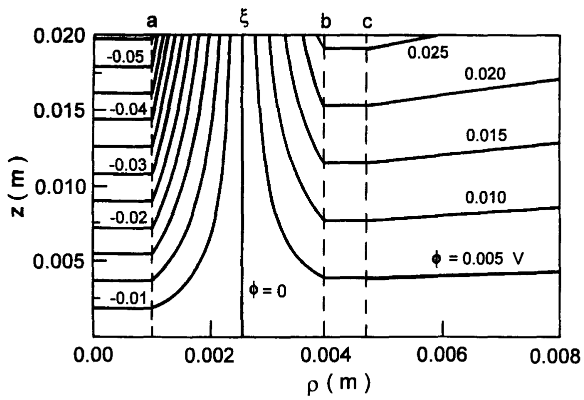

Figure 3. Equipotentials.

Sommerfeld or Griffiths's solutions are recovered taking $g_3(c^2 - b^2) \to \infty$, such that $\xi \to b$, $\sigma_c(z) \to 0$, $\vec{E}(\rho > b) \to 0$ and $\phi(\rho \geq b) \to 0$ for any z. The opposite solution when the current flows in an inner conductor of infinite conductivity, returning in an outer conductor of finite area and finite conductivity is also easily obtained from above, yielding $\xi \to a$, $\vec{E}(\rho < a) \to 0$ and $\phi(\rho \leq a) \to 0$ for any z.

In Figure 3 we plotted the equipotentials with the same data as above, in SI units. The values of the surface charges at $z = 0.001$ m obtained from Eqs. (12) to (14) are: $\sigma_a = -6.54174 \times 10^{-12}$ C m^{-2}, $\sigma_b = 2.61670 \times 10^{-11}$ C m^{-2} and $\sigma_c = 4.49027 \times 10^{-13}$ C m^{-2}. As the surface charges vary linearly with z, it is easy to find their values at any other distances from the center of the cable.

3. Discussion

The distribution of charges given by Eqs. (12) to (14) shows that the facing surfaces $\rho = a$ and $\rho = b$ work as a set of capacitors. That is, the charge at the position $\rho = a$, z, in a length dz, d$q_a(z) = 2\pi\, a$ dz $\sigma_a(z)$, is equal and opposite to the charge at the position $\rho = b$, z, in the same length dz: d$q_b(z) = 2\pi b$ d$z\sigma_b(z) = -$d$q_a(z)$. The field outside the coaxial cable depends then only on the surface charges at the external wall of the return conductor, $\sigma_c(z)$:

$$\phi(c \leq \rho, \varphi, z) = \frac{c}{\varepsilon_o} \sigma_c(z) \ln \frac{\ell}{\rho} . \tag{30}$$

The flux of energy from Poynting vector $\vec{S} = \vec{E} \times \vec{B} / \mu_o$ is also represented in Figure 3. That is, the lines of Poynting flux lie in the equipotential surfaces, as had been pointed out by [17] and [18]. The classical view is that the energy comes from the batteries (not represented in Figure 1). In Figure 3 it would come from the top of the graph moving downwards towards decreasing values of z, along the equipotential lines. It would then enter the conductors and move radially in them. In the inner conductor it would dissipate as heat while moving radially from $\rho = a$ to $\rho = 0$, while in the outer conductor it also moves radially from $\rho = b$ to $\rho = c$, being completely dissipated as heat along this trajectory. The only region where the lines of Poyting flux do not follow the equipotential surfaces is for $\rho > c$. In this region there is no magnetic field. Although we have obtained an electric field and equipotential lines here, the Poynting vector goes to zero.

Beyond the generalizations of the previous works, the main nontrivial conclusion of this analysis are Eqs. (10), (20) and (24). They show that although there is no vector potential nor magnetic field outside a coaxial cable, the electric field won't be zero when there is a finite resistivity in the outer conductor. As the previous works quoted above considered only the case of a return conductor with zero resistivity, this aspect did not appear. The existence of the tangential component E_z of $\vec{E}$ outside the coaxial cable might be guessed from Maxwell's equations. That is, as there is a resisitivity in the outer conductor of finite area and finite g, there must be an electric field at $b \le \rho \le c$ balancing Ohm's resistance in a dc current. As the tangential component of the electric field is continuous in any boundary, this means that E_z must also exist outside the external conductor. Although this may seem trivial, it is almost never mentioned for the case of a coaxial cable. More important than this is that we have shown that there will also exist a radial component of $\vec{E}$, $E\rho$ given by Eq. (10). Although it is inversely proportional to ρ, it will have a reasonable value close to the cable and in principle might be measured in the laboratory. To our knowledge the first to mention this external electrical field outside a resistive coaxial cable was Russell in his important paper of 1983, [24]. Our work presents a clear analytical calculation of this field, which Russell could only estimate. Our paper might be considered the quantitative implementation of his insights.

Acknowledgements

One of the authors (AKTA) wishes to thank the European Community and FAEP–UNICAMP for financial support. He thanks also Drs. Franco Selleri, Mark A. Heald, Álvaro Vannucci and Daniel Gardelli for discussions and suggestions.

References

[1] W. F. Edwards, C. S. Kenyon, and D. K. Lemon. Continuing investigation into possible electric fields arising from steady conduction currents. *Physical Review D*, 14:922–938, 1976.

[2] D. F. Bartlett and B. F. L. Ward. Is an electron's charge independent of its velocity?*Physical Review D*, 16:3453–3458, 1977.

[3] G. Bonnet. Electric field arising from a steady current passing through a superconductor. *Physics Letters A*, 82:465–467, 1981.

[4] J. C. Curé. A modified version of the Millikan oil drop experiment to test the probable existence of a new electrodynamic field. *Physics Letters B*, 116:158–160, 1982.

[5] D. F. Bartlett and S. Maglic. Test of an anomalous electromagnetic effect. *Review of Scientific Instruments*, 61:2637–2639, 1990.

[6] D. F. Bartlett and W. F. Edwards. Invariance of charge to Lorentz transformation. *Physics Letters A*, 151:259–262, 1990.

[7] C. S. Kenyon and W. F. Edwards. Test of current–dependent electric fields. *Physics Letters A*, 156:391–394, 1991.

[8] A. K. T. Assis. Can a steady current generate an electric field?*Physics Essays*, 4:109–114, 1991.

[9] A. K. T. Assis. *Weber*'s *Electrodynamics* . Kluwer Academic Publishers, Dordrecht, 1994. ISBN: 0–7923–3137–0.

[10] A. K. T. Assis. *Eletrodinamica de Weber – Teoria, Aplicacoes e Exercicios*. Editora da Universidade Estadual de Campinas – UNICAMP, Campinas, 1995. ISBN: 85–268–0358–1.

[11] T. Ivesic. The relativistic electric fields arising from steady conduction currents. *Physics Letters A*, 144:427–431, 1990.

[12] T. Ivesic. Electric fields from steady currents and unexplained electromagnetic experiments. *Physical Review A*, 44:2682–2685, 1991.

[13] T. Ivesic. The definitions of charge and the invariance of charge. *Physics Letters A*, 162:96–102, 1992.

[14] T. Ivesic. Reply to "Comments on 'Electric fields from steady currents and unexplained electromagnetic experiments'". *Physical Review E*, 44:4140–4142, 1993.

[15] F. Selleri. Noninvariant one–way velocity of light and particle collisions. *Foundations of Physics Letters*, 9:43–60, 1996.

[16] F. Selleri. Noninvariant one–way velocity of light. *Foundations of Physics*, 26:641–664, 1996.

[17] M. A. Heald. Electric fields and charges in elementary circuits. *American Journal of Physics*, 52:522–526, 1984.

[18] J. D. Jackson. Surface charges on circuit wires and resistors play three roles. *American Journal of Physics*, 64:855–870, 1996.

[19] A. Marcus. The electric field associated with a steady current in long cylindrical conductor. *American Journal of Physics*, 9:225–226, 1941.

[20] A. Sommerfeld. *Electrodynamics*. Academic Press, New York, 1964.

[21] D. J. Griffiths. *Introduction to Electrodynamics*. Prentice Hall, Englewood Cliffs, second edition, 1989.

[22] B. R. Russell. Surface charges on conductors carrying steady currents. *American Journal of Physics*, 36:527–529, 1968.

[23] O. D. Jefimenko. *Electricity and Magnetism*. Electret Scientific Company, Star City, 2nd edition, 1989.

[24] B. R. Russell. Surface charges and leaky shields. *American Journal of Physics*, 51:269–270, 1983.