The Principles of Quantum Dynamics

If the Hamiltonian operator is known, an initial wave function $\psi(r, 0)$ develops in time into $\psi(r, t)$ according to the time-dependent Schrödinger equation (Chapters 2 and 3). This algorithm for calculating the future behavior of a wave packet from its past history was used in Chapter 13 for scattering calculations. We now extend the general principles of quantum mechanics (Chapters 9 and 10) to the laws governing the time evolution of quantum systems, utilizing several equivalent dynamical pictures (Schrödinger’s, Heisenberg’s, Dirac’s). The canonical quantization of systems with classical analogues is discussed and applied to the forced harmonic oscillator, which is the prototype of a system in interaction with its environment.


We now add the time parameter to the description of quantum states and generalize the fundamental postulate of quantum mechanics (Section 9.1) by asserting that:

The maximum information about the outcome of physical measurements on a system at time $t$ is contained in the probability amplitudes $\langle K | \Psi(t) \rangle$, which correspond to a complete set of observables $K$ for the system.

The only new feature here is that we now recognize formally that the state $\Psi$ is a function of time.

For the simple system of a particle with coordinates $x, y, z$ as observables, the amplitude is

$$\psi(r, t) = \langle x, y, z | \Psi(t) \rangle$$ (14.1)

which is the time-dependent wave function in the coordinate representation. The same state is represented in the momentum representation as

$$\phi(p, t) = \langle p_x, p_y, p_z | \Psi(t) \rangle$$ (14.2)

The basic question of quantum dynamics is this: Given an initial state $| \Psi(t_0) \rangle$ of the system, how is the state $| \Psi(t) \rangle$ determined from this, if indeed it is so determined? Or, in terms of the amplitudes that specify the state, how do the amplitudes $\langle L_j | \Psi(t) \rangle$ evolve in time from the initial amplitudes $\langle K_i | \Psi(t_0) \rangle$?

The assertion that $| \Psi(t_0) \rangle$ determines $| \Psi(t) \rangle$ is the quantum mechanical form of the principle of causality, and we shall assume it. The dynamical law that connects the initial and final amplitudes is contained in the further assumption that the composition rule (9.8) can be generalized to the time-dependent form

$$\langle L_j | \Psi(t) \rangle = \sum_i \langle L_j | T(t, t_0) | K_i \rangle \langle K_i | \Psi(t_0) \rangle$$ (14.3)

where the coefficients

$$\langle L_j | T(t, t_0) | K_i \rangle$$ (14.4)

315
are independent of the state \( |\Psi(t_0)\rangle \). They have a direct and simple interpretation: The expression (14.4) signifies the probability amplitude for finding the system at time \( t \) in the eigenstate of \( |L_j\rangle \) of the observables symbolized by \( L \), if at time \( t_0 \) it was known to be in the eigenstate \( |K_i\rangle \) of the observables \( K \). This quantity is called a \textit{transition amplitude}. With forethought it has been written in the form of a matrix element of an operator \( T(t, t_0) \), because from (14.3), which is valid for any state \( |\Psi(t_0)\rangle \), we can derive the transformation equation

\[
\langle L_j | T(t, t_0) | K_i \rangle = \sum_p \sum_q \langle L_j | N_p \rangle \langle N_p | T(t, t_0) | M_j \rangle \langle M_j | K_i \rangle \tag{14.5}
\]

which shows that (14.4) defines a representation-independent linear operator \( T(t, t_0) \). Equation (14.3) is consistent with the composition rule (9.8) if we require that

\[
\langle L_j | T(t, t) | K_i \rangle = \langle L_j | K_i \rangle \quad \text{for all times } t
\]

or

\[
T(t, t) = 1 \tag{14.6}
\]

**Exercise 14.1.** Prove the relation (14.5) from (14.3) without assuming that (14.4) is the matrix element of an operator.

It now follows from the composition rule (14.3) that the \textit{time development or evolution operator} \( T(t, t_0) \) relates the initial state \( |\Psi(t_0)\rangle \) to the final state \( |\Psi(t)\rangle \) according to

\[
|\Psi(t)\rangle = T(t, t_0) |\Psi(t_0)\rangle
\]

Since \( T(t, t_0) \) does not depend on \( |\Psi(t_0)\rangle \), the \textit{principle of superposition} applies to the time development of states. This means that if \( |\Psi_a(t_0)\rangle \) and \( |\Psi_b(t_0)\rangle \) separately evolve into \( |\Psi_a(t)\rangle \) and \( |\Psi_b(t)\rangle \), then a superposition \( c_a |\Psi_a(t_0)\rangle + c_b |\Psi_b(t_0)\rangle \) develops into \( c_a |\Psi_a(t)\rangle + c_b |\Psi_b(t)\rangle \), i.e., each component of the state moves independently of all the others, expressing the fundamental \textit{linearity} of quantum dynamics.

From (14.7) it follows that

\[
|\Psi(t_2)\rangle = T(t_2, t_1) |\Psi(t_1)\rangle = T(t_2, t_1) T(t_1, t_0) |\Psi(t_0)\rangle = T(t_2, t_0) |\Psi(t_0)\rangle
\]

Hence, the time development operator has the property

\[
T(t_2, t_0) = T(t_2, t_1) T(t_1, t_0) \tag{14.8}
\]

From (14.6) and (14.8) we infer that

\[
T(t, t_0) T(t_0, t) = T(t_0, t) T(t, t_0) = 1
\]

or

\[
[T(t, t_0)]^{-1} = T(t_0, t) \tag{14.9}
\]

For small \( \varepsilon \) we may write

\[
T(t + \varepsilon, t) = 1 - \frac{i}{\hbar} \varepsilon H(t) \tag{14.10}
\]
defining an operator $H(t)$. (The reason for introducing the factor $i/\hbar$, apparently capriciously, will become evident forthwith.) Since, by (14.8),

$$T(t + \varepsilon, t_0) = T(t + \varepsilon, t)T(t, t_0)$$

we have with (14.10) the differential equation for $T$,

$$\frac{dT(t, t_0)}{dt} = \lim_{\varepsilon \to 0} \frac{T(t + \varepsilon, t_0) - T(t, t_0)}{\varepsilon} = -\frac{i}{\hbar} H(t)T(t, t_0)$$

or

$$\frac{i\hbar}{dt} T(t, t_0) = H(t)T(t, t_0)$$

(14.11)

with the initial condition $T(t_0, t_0) = 1$.

The linear operator $H(t)$ is characteristic of the physical system under consideration. We will see that it is analogous to the Hamiltonian function in classical mechanics. This analogy has led to the name Hamiltonian operator for $H(t)$, even when the system has no classical counterpart.

We also have

$$|\Psi(t + \varepsilon)\rangle = T(t + \varepsilon, t)|\Psi(t)\rangle$$

or, to first order in $\varepsilon$,

$$|\Psi(t)\rangle + \varepsilon \frac{d}{dt} |\Psi(t)\rangle = \left[ 1 - \frac{i}{\hbar} \varepsilon H(t) \right] |\Psi(t)\rangle$$

Hence,

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H(t)|\Psi(t)\rangle$$

(14.12)

Its bra form is

$$-i\hbar \frac{d}{dt} \langle \Psi(t) | = \langle \Psi(t) | H^\dagger(t)$$

(14.13)

Equation (14.12) is the equation of motion for the state vector, giving the general law of motion for any Hamiltonian system.

To specialize to a particular system, we must select an appropriate Hamiltonian operator. The form of (14.12) is reminiscent of the time-dependent Schrödinger equation (3.42). This is no accident, for we have merely reformulated in abstract language those fundamental assumptions that have already proved their worth in wave mechanics. Of course, (14.12) is an equation for the state vector rather than for the wave function (14.1), but the distinction is one of generality only. It now becomes clear why Planck's constant was introduced in (14.10). In Chapter 15, we will see how the laws of wave mechanics derive from the general theory.

In wave mechanics, the differential operator $H$ was generally Hermitian, as it must be whenever the Hamiltonian corresponds to the energy operator of the system. Generally, we will assume that $H$ is Hermitian. A non-Hermitian Hamiltonian operator, like a complex index of refraction in optics, can be useful for describing the
Chapter 14  The Principles of Quantum Dynamics

dynamics of dissipative systems which, through absorption or decay, exchange energy with their environment.

If \( H(t) \) is Hermitian, the equation adjoint to (14.11) becomes

\[
-\frac{i\hbar}{dt} \frac{dT(t, t_0)}{dt} = T(t, t_0)H(t)
\]

(14.14)

and (14.13) becomes

\[
-\frac{i\hbar}{dt} \frac{d}{dt} \langle \Psi(t) \rangle = \langle \Psi(t) | H(t) \rangle
\]

(14.15)

By multiplying (14.11) on the left by \( T(t, t_0) \) and (14.14) on the right by \( T(t, t_0) \) and subtracting the two equations, we get

\[
\frac{d}{dt} [T(t, t_0)T(t, t_0)] = 0
\]

Since, by (14.6), the product of the two operators equals the identity at \( t = t_0 \), it follows that the time development operator is unitary:

\[
T(t, t_0)T(t, t_0) = 1
\]

(14.16)

Hence, the norm of any state vector remains unaltered during the motion. If \( \langle \Psi(t_0) \rangle \) is normalized to unity, such that \( \langle \Psi(t_0) | \Psi(t_0) \rangle = 1 \), then the normalization will be preserved in the course of time, and we have from (14.7) and (14.16) that

\[
\langle \Psi(t) | \Psi(t) \rangle = 1 \text{ for all times } t
\]

consistent with the assumption that \( |\langle L_j | \Psi(t) \rangle|^2 \) is the probability of finding the observable \( L \) to have the value \( L_j \) at time \( t \).

Often \( H \) does not depend on the time, and then \( T \) can be obtained for finite time intervals by applying the rule (14.8) repeatedly to \( n \) intervals, each of length \( \varepsilon = (t - t_0)/n \). Hence, by (14.10) we have, with the initial condition \( T(t_0, t_0) = I \),

\[
T(t, t_0) = \lim_{n \to \infty} \left( 1 - \frac{i}{\hbar} e^{\varepsilon H} \right)^n = \lim_{n \to \infty} \left[ I - \frac{i}{\hbar} \frac{(t - t_0)}{n} H\right]^n
\]

In the limit we get by the definition of the exponential function,

\[
T(t, t_0) = \exp \left[ -\frac{i}{\hbar} (t - t_0)H \right]
\]

(14.17)

It is obvious that \( T \) is unitary if \( H \) is Hermitian.

Quantum dynamics is a general framework and contains no unambiguous prescription for the construction of the operator \( H \) whose existence it asserts. The Hamiltonian operator must be found on the basis of experience, using the clues provided by the classical description, if one is available. Physical insight is required to make a judicious choice of operators to be used in the description of the system (such as coordinates, momenta, and spin variables) and to construct the Hamiltonian in terms of these variables.

Contact with measurable quantities and classical concepts can be established if we calculate the time derivative of the expectation value of an operator \( A \), which may itself vary with time:

\[
\frac{d}{dt} \langle \Psi(t) | A | \Psi(t) \rangle = -\langle \Psi(t) | HA | \Psi(t) \rangle + \langle \Psi(t) | AH | \Psi(t) \rangle + i\hbar \langle \Psi(t) | \frac{\delta A}{\delta t} | \Psi(t) \rangle
\]
\[ i\hbar \frac{d}{dt} \langle A \rangle = \langle AH - HA \rangle + i\hbar \left( \frac{\partial A}{\partial t} \right) = \langle [A, H] \rangle + i\hbar \left( \frac{\partial A}{\partial t} \right) \tag{14.18} \]

where, as usual, the brackets \( \langle \ldots \rangle \) signify expectation values of the operators enclosed. We see that the commutation relations of \( H \) with observables play an important role in the theory. If \( A \) is independent of time and commutes with \( H \), the expectation value of \( A \) is constant, and \( A \) is said to be a constant of the motion.

A special example of a time-dependent operator is the density operator for the state \( |\Psi(t)\rangle \).

\[ \rho = |\Psi(t)\rangle\langle\Psi(t)| \tag{14.19} \]

From the equations of motion (14.12) and (14.15),

\[ i\hbar \frac{\partial \rho}{\partial t} = H|\Psi(t)\rangle\langle\Psi(t)| - |\Psi(t)\rangle\langle\Psi(t)|H = [H, \rho] \tag{14.20} \]

Hence, (14.18) gives the simple result

\[ \frac{d}{dt} \langle \rho \rangle = 0 \tag{14.21} \]

which is not surprising in view of the conservation of probability. The definition (14.19) implies that

\[ \langle [H, \rho] \rangle = \langle \Psi(t) \rangle [H, |\Psi(t)\rangle\langle\Psi(t)|] |\Psi(t)\rangle = 0 \tag{14.22} \]

which, by (14.20), leads to the conclusion:

\[ \left( \frac{\partial \rho}{\partial t} \right) = 0 \tag{14.23} \]

These results have sometimes led to the oxymoronic proposition that the density operator is a "time-dependent constant of the motion"!

The formal relations derived in this section are all rooted in what we have called the quantum mechanical principle of causality, which states that the probability amplitude for finding the value of \( K \), of the generic observables \( K \) at time \( t \) can be written as the inner product \( \langle K | \Psi(t) \rangle \). The observables \( K \) and their eigenvectors are regarded as constant. Physically, this implies that a system represented by the same state vector at two different times has the same statistical distribution at the two times with regard to all observables of the system. In other words, \( |\Psi(t)\rangle \) completely characterizes the state of a system at time \( t \), which was the fundamental assumption made in Chapter 9. We will now discuss other, equivalent, formulations of quantum dynamics.

2. The Pictures of Quantum Dynamics. The mathematical formulation of quantum dynamics given in the last section is not unique. There we showed that the state vector of a system alone may be made responsible for the time evolution, but this is not the only way of dealing with dynamics in the framework of vector spaces. State vectors themselves are not observable or measurable quantities. Rather, the eigenvalues of Hermitian operators and probability amplitudes such as (14.3) and (14.4) are the physically significant objects of quantum mechanics. Comparison with
observation is made in terms of the eigenvalues of observables and of expansion
coefficients (probability amplitudes), which are inner products in the abstract state
vector space. Measuring an observable $L$ at time $t$ means finding one of its eigen­
values $L_j$, the probability of the particular result being given by $|\langle L_j | \Psi(t) \rangle|^2$ if $|\Psi(t)\rangle$
denotes the state of the system at time $t$.

It follows that two vector space formulations, or pictures, are equivalent and
equally acceptable provided that

(a) the operators corresponding to the observables maintain their eigenvalue
spectra in the two formulations, and

(b) the inner products of physical states with eigenvectors for the same eigen­
values are equal in the two pictures.

It follows from (a) and (b) that all expectation values remain the same from one
picture to the other.

Starting from the Schrödinger picture of quantum dynamics, a new picture is
obtained, satisfying conditions (a) and (b), by applying a time-dependent unitary
transformation $U(t)$ to all states and operators. All the state vectors change from
$|\Psi(t)\rangle$ to $U(t)|\Psi(t)\rangle$, and every operator $A$ is transformed into $U(t)AU^\dagger(t)$. Owing
to the unitary property of $U(t)$, all eigenvalues and all inner products remain invari­
ant, but the eigenvectors of an observable change from $|A\rangle$ to $U(t)|A\rangle$. Expectation
values remain unchanged.

The simplest choice for the transformation $U(t)$ is

$$U(t) = T(0, t) = T^\dagger(t, 0) \quad (14.24)$$

Since

$$U(t)|\Psi(t)\rangle = T(0, t)|\Psi(t)\rangle = |\Psi(0)\rangle \quad (14.25)$$

this transformation has the effect of referring all state vectors back to their values
at the initial time 0. In this formulation, called the Heisenberg picture, the state
vectors are constant in time, and we denote them as

$$|\Psi(0)\rangle = |\overline{\Psi}\rangle \quad (14.26)$$

The observables, which in the Schrödinger picture were represented by operators
fixed in time (unless they happened to have an explicit time dependence), are rep­
resented in the Heisenberg picture by time-dependent operators,

$$\overline{L}(t) = U(t)LU^\dagger(t) = T^\dagger(t, 0)L T(t, 0) \quad (14.27)$$

At the initial time $t = 0$,

$$\overline{L}(0) = L \quad (14.28)$$

These equations exhibit the evolution of the Heisenberg operators. To identify the
Heisenberg picture, the bar notation, as in $|\overline{\Psi}\rangle$ and $\overline{L}(t)$, will be used only in this
chapter. Elsewhere in this book, when there is no ambiguity, we will simply use
$|\Psi\rangle$ and $L(t)$ for Heisenberg states and operators.

As a special case, note that if $H$ is constant and energy is conserved, then

$$T(t, 0) = \exp\left( -\frac{i}{\hbar} H t \right) \quad (14.29)$$
and \( \bar{H}(t) = H \). If \( H \) is constant, the Hamiltonian operator does not change in time, even in the Heisenberg picture.

By differentiating (14.27) with respect to \( t \) and using the equation of motion (14.11) for the time development operator, we obtain the Heisenberg equation of motion for an observable:

\[
\frac{\delta \bar{L}(t)}{\delta t} = \left[ \bar{L}(t), \bar{H}(t) \right] + \frac{i\hbar}{\delta t} \frac{\delta \bar{L}(t)}{\delta t}
\]

(14.30)

The last term arises from the definition

\[
\frac{\delta \bar{L}(t)}{\delta t} = \frac{\delta \bar{L}(t)}{\delta t} = \frac{\delta \bar{L}(t)}{\delta t}
\]

(14.31)

in the event that \( L \) is explicitly time-dependent. To emphasize that \( \frac{\delta \bar{L}(t)}{\delta t} \) is the Heisenberg form of the derivative of the explicitly time-dependent operator \( L \), strictly speaking we should write this operator as

\[
\frac{\delta \bar{L}(t)}{\delta t} = \frac{\delta \bar{L}(t)}{\delta t}
\]

(14.32)

The expectation value of the operator equation (14.30) in the Heisenberg picture, where the states are constant in time, is the counterpart of Eq. (14.18) in the Schrödinger picture.

Eigenvalues of observables, corresponding to the same eigenvalues, differ in the Schrödinger and Heisenberg pictures by the unitary transformation \( U(t) \):

\[
|L_j, t\rangle = T(t, 0)|L_j\rangle
\]

(14.33)

or

\[
|L_j\rangle = T(t, 0)|L_j, t\rangle
\]

(14.34)

Differentiating this last equation with respect to \( t \), we obtain

\[
\frac{dT(t, 0)}{dt} |L_j, t\rangle + T(t, 0) \frac{d}{dt} |L_j, t\rangle = 0
\]

or, using (14.11),

\[
\frac{i\hbar}{dt} |L_j, t\rangle = -\hbar |L_j, t\rangle
\]

(14.35)

which is very similar to the equation of motion (14.2) in the Schrödinger picture, except for the all-important minus sign. Its appearance shows that if in the Schrödinger picture we regard the state vectors as "rotating" in a certain direction in abstract vector space and the operators with their eigenvectors as fixed, then in the Heisenberg picture the state vectors stand still and the operators with the eigenvectors "rotate" in the opposite direction. But the mutual relation between state vectors and operators is the same in the two pictures. They are related to each other in much the same way as the two kinematic descriptions of the rotation of a rigid body with respect to a reference frame. We can consider the body moving in a fixed frame, or the body as being at rest, with the reference frame rotating backward. Since the two pictures are connected by a unitary transformation, the probability amplitudes (inner products) are equal:

\[
\langle L_j | \Psi(t) \rangle = \langle L_j, t | \Psi \rangle
\]

(14.36)
Chapter 14 The Principles of Quantum Dynamics

Exercise 14.2. Find the equation of motion for the Heisenberg bra \((I_j, t)\).

Exercise 14.3. Show that in the Heisenberg picture the density operator for state \(|\Psi\rangle\),

\[
\bar{\rho} = |\Psi(0)\rangle\langle\Psi(0)| = |\bar{\Psi}\rangle\langle\bar{\Psi}|
\] (14.37)

satisfies the equations

\[
\frac{i\hbar}{\partial t} \frac{\partial \bar{\rho}}{\partial t} = i\hbar T^*(t, 0) \frac{\partial \rho}{\partial t} T(t, 0) = [\bar{H}, \bar{\rho}] \quad \text{and} \quad \frac{d\bar{\rho}}{dt} = 0 \] (14.38)

and that the expectation value of \(\bar{\rho}\) is constant as in (14.21).

Instead of attributing the time evolution to either the state vectors or the operators, as in the Schrödinger and Heisenberg pictures, it is obviously possible to partition the time development operator in infinitely many different ways. We can arrange to let both the state vectors and the observables carry a complementary share of the time development. If the system and its Hamiltonian are complicated, it is often sensible to choose for \(U(t)\) the adjoint of the time development operator for a suitably chosen model Hamiltonian \(H_0\), which is the solution of

\[
\frac{i\hbar}{dt} dU(t) = -U(t)H_0
\] (14.39)

subject to the initial condition \(U(0) = 1\). Generally, \(H_0\) may be time-dependent.

The observables are then transformed into new operators

\[
\tilde{I}(t) = U(t)LU^*(t)
\] (14.40)

and these satisfy the equation of motion

\[
\frac{i\hbar}{dt} \tilde{I}(t) = [\tilde{I}(t), \tilde{H}_0] + i\hbar \frac{\partial \tilde{I}(t)}{\partial t}
\] (14.41)

More important is the equation for the transformed state vector,

\[
|\bar{\Psi}(t)\rangle = U(t)|\Psi(t)\rangle
\] (14.42)

By differentiating this equation with respect to \(t\), and using (14.12), we obtain

\[
\frac{i\hbar}{dt} |\Psi(t)\rangle = -\tilde{H}_0|\Psi(t)\rangle + U(t)HU^*(t)|\Psi(t)\rangle = U(t)(H - H_0)U^{-1}(t)|\Psi(t)\rangle
\]

We define an interaction term \(V\) as the difference between the Hamiltonian \(H\) and the model Hamiltonian \(H_0\) by

\[
H = H_0 + V
\] (14.43)

giving us the simple-looking formula

\[
\frac{i\hbar}{dt} |\bar{\Psi}(t)\rangle = \bar{V}(t)|\Psi(t)\rangle
\] (14.44)

where

\[
\bar{V}(t) = U(t)VU^*(t)
\] (14.45)
Quantization Postulates for a Particle

is the transformed version of $V$. The resulting formulation of quantum dynamics is known as the interaction (or Dirac) picture. Note that if the model Hamiltonian $H_0$ is time-independent and thus conservative,

$$U(t) = \exp\left(\frac{i}{\hbar} H_0 t\right)$$  \hspace{1cm} (14.46)

and

$$\tilde{H}_0(t) = H_0$$  \hspace{1cm} (14.47)

If $H_0$ is chosen to be the full Hamiltonian, $H_0 = H$, the interaction picture coalesces with the Heisenberg picture. If $H_0 = 0$, the Schrödinger picture is recovered. The interaction picture will be found useful when we consider time-dependent perturbation theory in Chapter 19.

To demonstrate the total equivalence of the various pictures of quantum dynamics, let us suppose that at time $t_1$ the system has the definite sharp value $A'$ for the observable $A$. We ask: "What is the probability that at time $t_2$ the system will have value $B''$ if the observable $B$ is measured?" The answer is that the required probability is the square of the absolute value of the amplitude

$$\langle B'', t_2 \mid \tilde{T}(t_2, t_1) \mid A', t_1 \rangle$$

where $\tilde{T}(t_2, t_1)$ is the time development operator for the state vector in the interaction (or really a generic) picture as defined by

$$|\tilde{\Psi}(t)\rangle = \tilde{T}(t, t_0) |\tilde{\Psi}(t_0)\rangle$$  \hspace{1cm} (14.48)

From this it follows, with (14.42), that

$$\tilde{T}(t_2, t_1) = U(t_2)T(t_2, t_1)U(t_1)$$  \hspace{1cm} (14.49)

Since we infer from (14.42) that

$$|\tilde{A}', t\rangle = U(t)|A'\rangle$$ and $\langle B'', t| = \langle B''| U^\dagger(t)$  \hspace{1cm} (14.50)

we see that the transition amplitude can be expressed equivalently in all pictures as

$$\langle B'', t_2 \mid \tilde{T}(t_2, t_1) \mid A', t_1 \rangle = \langle B'' | T(t_2, t_1) | A' \rangle = \langle B'' , t_2 \mid | A', t_1 \rangle = \langle B'', t_2 | A', t_1 \rangle$$

(14.51)

The distinctiveness of the Schrödinger and Heisenberg pictures is manifested by the important fact that the Hamiltonian $H$ (energy) is the same in both pictures.

**Exercise 14.4.** Show that the expression

$$\langle B'', t_2 \mid \tilde{T}(t_2, t_1) \mid A', t_1 \rangle$$

for the transition amplitude is quite general and gives the correct answer if the Schrödinger ($H_0 = 0$) or Heisenberg ($H_0 = H$) pictures are employed.

3. **Quantization Postulates for a Particle.** Let us now apply the general equations of quantum dynamics to the special case of a point particle with a mass $m$. We are concerned with the quantum behavior of this system, but it does have a classical analogue—Newtonian mechanics, or its more sophisticated Lagrangian or Hamil-
tonian forms. The position operators \(x, y, z\) are assumed to form a complete set of commuting observables for this physical system.

For the purposes of this chapter it is worthwhile to distinguish in the notation between the classical observables, \(x, y, z\), which represent numbers, and the corresponding quantum observables, \(x, y, z\), which stand for operators.

Setting \(A = x\) in (14.18), we obtain

\[
\frac{d\langle x \rangle}{dt} = \frac{\langle xH - Hx \rangle}{i\hbar}
\]

(14.52)

On the left-hand side there is a velocity, but if we wish to compare this equation with the classical equation for velocities we cannot simply let the operators go over into their classical analogues, because classical observables commute and we would have zero on the right-hand side. Hence, we must let \(\hbar \to 0\) at the same time. Thus, we formulate a heuristic version of the correspondence principle as follows:

*If a quantum system has a classical analogue, expectation values of operators behave, in the limit \(\hbar \to 0\), like the corresponding classical quantities.*

This principle provides us with a test that the quantum theory of a system with a classical analogue must meet, but it does not give us an unambiguous prescription of how to construct the quantum form of any given classical theory. Certainly, we cannot expect that every valid classical equation can be turned into a correct quantum equation merely by replacing classical variables by expectation values of operators. For example, \(xp = mx(dx/dt) = (1/2)m(dx^2/dt)\) is a valid classical equation if not a particularly useful one; yet, for operators \(\langle xp \rangle = (1/2)m(dx/dt)(x^2)\) is generally wrong, although

\[
\langle x \rangle \langle p_x \rangle = m \frac{d\langle x \rangle}{dt} \quad \text{and} \quad \left\langle \frac{1}{2} (xp + px) \right\rangle = \frac{1}{2} m \frac{d}{dt} \langle x^2 \rangle
\]

are both correct. The trouble comes from the noncommunitivity of \(x\) and \(p_x\).

To make the conversion from classical to quantum mechanics, the correspondence principle must be supplemented by a set of quantization rules. These rules have to be consistent with the correspondence principle, but their ultimate test lies in a comparison between the theoretical predictions and the experimental data.

We expect that (14.52) is the quantum analogue of one of Hamilton's equations,

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p_x}
\]

(14.53)

where \(H\) is the classical Hamiltonian function of \(x, y, z, p_x, p_y, p_z\) which characterize the system. The correspondence principle requires that

\[
\lim_{\hbar \to 0} \frac{\langle xH - Hx \rangle}{i\hbar} = \frac{\partial H}{\partial p_x}
\]

(14.54)

Similarly, for \(A = p_x\), we have in quantum mechanics

\[
\frac{d\langle p_x \rangle}{dt} = \frac{\langle p_x H - Hp_x \rangle}{i\hbar}
\]

(14.55)

and classically,

\[
\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}
\]

(14.56)
The correspondence principle requires that
\[
\lim_{\hbar \to 0} \left( \frac{p_x H - H p_x}{\hbar} \right) = -\frac{\partial H}{\partial x}
\]  
(14.57)

Similar equations follow for \( y \) and \( z \) and their conjugate momenta.

All these conditions can be satisfied if we do the following:

1. **Let \( H \) be a Hermitian operator identical in form with \( H \) but replace all coordinates and momenta by their corresponding operators.**

2. **Postulate the fundamental commutation relations between the Hermitian operators representing coordinates and momenta:**

\[
[x, p_x] = [y, p_y] = [z, p_z] = i\hbar I \quad (14.58)
\]

\[
[x, p_y] = [x, p_z] = [y, p_z] = [z, p_y] = 0 \quad (14.59)
\]

The coordinates, \( x, y, z \), commute with each other; the three momenta \( p_x, p_y, p_z \), also commute.

Prescription (1) must be applied with care if \( H \) contains terms such as \( x p_x \), because \( x \) and \( p_x \) are noncommuting and would upon translation into operator language give rise to a non-Hermitian \( H \). The symmetrized operator \((x p_x + p_x x)/2\) can then be used instead. It is Hermitian and leads to the correct classical limit. Sometimes there may be several different ways of symmetrizing a term to make it Hermitian. Thus the Hermitian operators \( x^2 p_x^2 + p_x^2 x^2 \) and \((x p_x + p_x x)^2/2\) both have the classical limit \( 2x^2 p_x^2 \), but they are not identical. In practice, it is usually possible to avoid such ambiguities.

**Exercise 14.5.** Show that the operators \( x^2 p_x^2 + p_x^2 x^2 \) and \((x p_x + p_x x)^2/2\) differ only by terms of order \( \hbar^2 \).

The consistency of conditions (1) and (2) and their agreement with (14.54) and (14.57) can be verified for any \( H \) that can be expanded in powers of the coordinates and momenta. For instance, the commutation relation \([x, p_x] = i\hbar I\) agrees with (14.54) and (14.57), as can be seen if we choose \( H = p_x \), and \( H = x \), respectively. The consistency proof can be continued by letting \( H = x^n \). Then

\[
\frac{1}{i\hbar} [p_x, x^n] = \frac{x^{n-1}}{i\hbar} [p_x, x] + \frac{1}{i\hbar} [p_x, x^{n-1}] x = -nx^{n-1}
\]
by virtue of repeated invocation of the quantum conditions (14.58). This is in agreement with the classical limit (14.57) because for \( H = x^n \),

\[
-\frac{\partial x^n}{\partial x} = -nx^{n-1}
\]

More generally, we can continue this type of reasoning to prove that for any two functions, \( F \) and \( G \), of the coordinates and momenta, which can be expanded in a power series, the relation

\[
\lim_{\hbar \to 0} \frac{\langle GF - FG \rangle}{i\hbar} = \frac{\delta G \delta F}{\delta x \delta p_x} - \frac{\delta F \delta G}{\delta x \delta p_x} + \frac{\delta G \delta F}{\delta y \delta p_y} - \frac{\delta F \delta G}{\delta y \delta p_y}
\]

\[
- \frac{\delta F \delta G}{\delta z \delta p_z} + \frac{\delta G \delta F}{\delta z \delta p_z}
\]

(14.60)
holds, where F and G are the same functions of the ordinary variables as \( F \) and \( G \) are of the corresponding operators. Equation (14.60) is assumed to be valid for any smooth functions of coordinates and momenta, even if a power series expansion cannot be made. Equations (14.54) and (14.57) are special cases of (14.60).

In classical mechanics, the expression on the right-hand side of (14.60) is abbreviated as \([G, F]_{\text{p.B.}}\) and is known as the Poisson bracket\(^1\) (P.B.) of F and G. Dirac discovered that this is the classical analogue of the commutator \([G, F]/i\hbar\).

**Exercise 14.6.** Illustrate the validity of Eq. (14.60) by letting \( G = x^2 \) and \( F = p_x \), and evaluating both the operator expression on the left, in the limit \( \hbar \to 0 \), and the corresponding Poisson bracket on the right.

All of these arguments can be summarized in the proposition that the classical equation of motion for an arbitrary function \( A(x, p_x, t) \),

\[
\frac{dA}{dt} = [A, H]_{\text{P.B.}} + \frac{\partial A}{\partial t}
\]

(14.61)

is the correspondence limit of the quantum equation of motion (14.18),

\[
\frac{d\langle A \rangle}{dt} = \frac{\langle [A, H] \rangle}{i\hbar} + \frac{\langle \frac{\partial A}{\partial t} \rangle}{i\hbar}
\]

(14.62)

for the operator \( A(x, p_x, t) \), which is the same function of its arguments as \( A(x, p_x, t) \).

4. **Canonical Quantization and Constants of the Motion.** So far we have considered only descriptions of the physical system in terms of Cartesian coordinates for a point particle. Yet, the connection between classical and quantum mechanics was established by the use of Hamilton's equations of classical mechanics, which are by no means restricted to Cartesian coordinates. Rather, these equations are well known to have the same general form for a large class of variables called canonical coordinates and momenta, and denoted by the symbols \( q \) and \( p \). Since in the Hamiltonian form of mechanics the Cartesian coordinates do not occupy a unique position, we ask whether the quantization procedure of Section 14.3 could not equally well have been applied to more general canonical variables. Could we replace \( x \) by \( q \) and \( p_x \) by \( p \) (assuming for convenience only one degree of freedom), satisfying the more general commutation relations

\[
qp - pq = i\hbar
\]

(14.63)

instead of (14.58), and could we still apply the same quantization rules?

To show that we are indeed at liberty to use canonical variables other than the Cartesian ones, we must prove that the same form of quantum mechanics results whether we use \( x, p_x \), or \( q, p \) to make the transition to quantum mechanics. To prove that we can pass from the upper left to the lower right corner of Figure 14.1 equivalently by routes 1, 2 or 3, 4 we first consider an infinitesimal canonical transfor-

\(^1\)Goldstein (1980), p. 397.
Canonical Quantization and Constants of the Motion

**Figure 14.1.** Classical canonical transformations and quantum mechanical unitary transformations.

Canonical transformation (step 1 in the figure), i.e., a transformation that is generated by an infinitesimal function $eG(x, px)$ from the relations

$$q = x + \epsilon \frac{\partial G}{\partial p_x}, \quad p = p_x - \epsilon \frac{\partial G}{\partial x}$$

(14.64)

The new Hamiltonian is

$$H'(q, p) = H(x, p_x) = H(q, p) - \epsilon \frac{\partial H}{\partial x} \frac{\partial G}{\partial p_x} + \epsilon \frac{\partial H}{\partial p_x} \frac{\partial G}{\partial x} = H(q, p) + \epsilon [G, H]_{PB}. \quad (14.65)$$

This canonical transformation is paralleled in quantum theory by step 4. Agreement with (14.64) and (14.65) in the correspondence limit is assured if we define the Hermitian operators

$$q = x + \frac{\epsilon}{i\hbar} [x, G], \quad p = p_x + \frac{\epsilon}{i\hbar} [p_x, G]$$

(14.66)

More generally, for an arbitrary function $F(x, p)$ we find to first order,

$$F(q, p) = F(x, p_x) + \epsilon \frac{\partial F}{\partial x} \frac{\partial G}{\partial p_x} - \epsilon \frac{\partial F}{\partial p_x} \frac{\partial G}{\partial x} = F(x, p_x) + \epsilon [F, G]_{PB}. \quad (14.67)$$

The corresponding operators satisfy the equation

$$F(q, p) = F(x, p_x) + \frac{\epsilon}{i\hbar} [F, G]$$

(14.68)

again to first order in $\epsilon$. In quantum mechanics, the new Hamiltonian is constructed by correspondence with (14.65) as

$$H'(q, p) = H(x, p_x) = H(q, p) + \frac{\epsilon}{i\hbar} [G, H]$$

(14.69)

The Hermitian operator $G(x, p_x)$ is constructed from $G(x, p_x)$ by letting $x$ and $p_x$ become operators. The commutators are evaluated by applying the quantization rules of the last section for Cartesian coordinates.
To first order in $\epsilon$, (14.68) may be written as

$$F(q, p) = (1 + i\frac{\epsilon}{\hbar} G)F(x, p)\left(1 - i\frac{\epsilon}{\hbar} G\right)$$  \hspace{1cm} (14.70)

showing that the new operators are obtained from the old Cartesian ones by an infinitesimal unitary transformation (step 4):

$$F(q, p) = U_\epsilon F(x, p)U_\epsilon^\dagger$$ \hspace{1cm} (14.71)

with

$$U_\epsilon = 1 + i\frac{\epsilon}{\hbar} G$$ \hspace{1cm} (14.72)

The Hermitian operator $G$ is the generator of this infinitesimal transformation.

In terms of the new variables, the quantum analogue of the classical Hamiltonian, (14.69), becomes

$$H'(q, p) = H(x, p) = H(U_\epsilon^* q U_\epsilon, U_\epsilon^* p U_\epsilon) = U_\epsilon^* H(q, p) U_\epsilon$$ \hspace{1cm} (14.73)

[See Exercise 9.16 for the conditions under which the last equality in (14.73) holds.]

The commutation relations are invariant under unitary transformations because

$$[q, p] = U_\epsilon x U_\epsilon^* U_\epsilon p U_\epsilon^* - U_\epsilon p U_\epsilon^* U_\epsilon x U_\epsilon^* = U_\epsilon [x, p] U_\epsilon^* = i\hbar I$$ \hspace{1cm} (14.74)

and we have arrived at (14.63). This completes the proof that the quantization rules of 14.3 can be extended to new canonical variables that are infinitesimally close to Cartesian.

The quantization procedure based on rules (1) and (2) of Section 14.3 can now be immediately generalized to all those canonical variables that can be obtained from the Cartesian ones by a succession of infinitesimal canonical transformations. This is true because two classical canonical transformations made in succession can be replaced by a single direct one. Similarly, in quantum mechanical transformations, successive application of unitary operators is equivalent to the application of a single unitary operator. If we let $\epsilon = \lambda/N$ (where $\lambda$ is a finite parameter and $N$ is an integer), and apply the same unitary operator $N$ times, we obtain the in limit $N \to \infty$ the unitary operator,

$$U = \lim_{N \to \infty} \left[1 + i\frac{\lambda}{N\hbar} G\right]^N = \exp\left(i\frac{\lambda}{\hbar} G\right)$$ \hspace{1cm} (14.75)

This finite unitary transformation changes the Cartesian variables into

$$q = U_\lambda x U_\lambda^* \hspace{1cm} p = U_\lambda p U_\lambda^*$$ \hspace{1cm} (14.76)

The commutation relations are also invariant under the finite transformations.

We note that if (14.76) holds, the eigenvalue spectra of $x$ and $q$ are the same, as

$$U_\lambda x U_\lambda^* |x\rangle = q U_\lambda |x\rangle = x' U_\lambda |x\rangle$$

Hence,

$$q' = x' \hspace{1cm} |q\rangle' = U_\lambda |x\rangle$$

We see that the quantization of the system can be carried through by the use of the general commutation relations (14.63) for any pair of canonical variables that
can be obtained from \( x, p_x \) by a continuous succession of infinitesimal transformations. For more general canonical transformations than these, the standard quantization procedure may or may not be valid. Clearly, it will be valid whenever the new operators can be obtained from the old ones by a unitary transformation.

A simple example of a failure of the standard quantization is provided by the transition to spherical polar coordinates, \( r, \varphi, \theta \). The transformation to these from Cartesian coordinates is canonical, but it cannot be generated by a succession of infinitesimal transformations, because of the discontinuity of the spherical polar coordinates. Nor does a unitary transformation between \( x, y, z \) and \( r, \varphi, \theta \) exist, for if it did the eigenvalues of the latter operators would have to be the same as those of \( x, y, z \), and range from \( -\infty \) to \( +\infty \), contrary to the definition of \( r, \varphi, \theta \). The general procedure for expressing the Hamiltonian operator and the Schrödinger equation in terms of curvilinear coordinates will be given in the Appendix, Section 3.

Because of its close connection with the classical canonical formalism, the quantization procedure described here is referred to as canonical quantization. The correspondence between canonical transformations and unitary operators has led to the frequent designation of unitary operators as quantum mechanical canonical transformations. This terminology has asserted itself, even though some unitary transformations have no classical analogue, and vice versa.

**Exercise 14.7.** Show that the transformation

\[
U a x U^\dagger = a x \cos \Theta + b p_x \sin \Theta
\]

\[
U b p_x U^\dagger = -a x \sin \Theta + b p_x \cos \Theta
\]

is canonical, if \( a \) and \( b \) are real-valued constants, and \( \Theta \) is a real-valued angle parameter. Construct the unitary operator \( U \) that effects this transformation. For the special case \( \Theta = \pi/2 \), calculate the matrix elements of \( U \) in the coordinate representation. Noting that this transformation leaves the operator \( a^2 x^2 + b^2 p_x^2 \) invariant, rederive the result of Exercises 3.8 and 3.21.

**Exercise 14.8.** Show that the reflection operator, defined by the relation \( U|\mathbf{x}'\rangle = |-\mathbf{x}'\rangle \), gives rise to a unitary transformation which takes \( x \) into \( -x \) and \( p_x \) into \( -p_x \).

An important application of canonical transformations concerns the finding of constants of the motion, which are observables that commute with the Hamiltonian \( H \) (see Section 14.1).

A useful way of obtaining constants of the motion for a time-independent Hamiltonian operator \( H(q, p) \) consists in noting that if a (finite or infinitesimal) canonical transformation to new variables \( q', p' \) is made, the new Hamiltonian \( H' \) is related to the old one by the equation

\[
H(q, p) = H'(q', p') = U H'(q, p) U^\dagger
\]

which is just an extension of (14.73) to finite canonical transformations. If the canonical transformation leaves the Hamiltonian invariant, so that the new Hamiltonian \( H' \) is the same function of the canonical variables as the old one,

\[
H'(q, p) = H(q, p)
\]

then

\[
U H - H U = [U, H] = 0
\]
Hence,

$$\frac{d}{dt} \langle U \rangle = \frac{1}{i\hbar} \langle UH - HU \rangle = 0$$

and thus the unitary operator $U$ is a constant of the motion if the transformation leaves $H$ invariant. If, in particular, $H$ is invariant under an infinitesimal transformation

$$U_e = 1 + \frac{ie}{\hbar} G$$

then the (Hermitian) generator $G$ of the transformation commutes with $H$,

$$\{G, H\} = 0 \quad (14.80)$$

and thus $G$ is a constant of the motion. In this way, physical observables are obtained which are constants of the motion.

As an example, consider a free particle whose Hamiltonian is

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) \quad (14.81)$$

According to (14.66), the generator of infinitesimal translations $e^\varepsilon G$

$$e^G = -\varepsilon_p x - \varepsilon_y p_y - \varepsilon_z p_z = -\varepsilon \cdot p$$

produces no change in the momenta

$$p' = p \quad (14.82)$$

but, owing to the fundamental commutation relations, changes the coordinates to

$$r' = r - \frac{1}{i\hbar} [r, \varepsilon \cdot p] = r - \varepsilon \quad (14.83)$$

Thus, the transformation describes a coordinate translation, as expected from the connection between momentum and displacement operators.

Evidently, any $H$ that does not depend on the coordinates is invariant under the transformation (14.82), (14.83). Hence, $e^G = -\varepsilon \cdot p$ (for arbitrary $\varepsilon$) and $p$ itself are constants of the motion, and we conclude that linear momentum is conserved under these conditions.

Similarly, for infinitesimal rotations we recall the generator

$$e^G = -\delta \phi \hat{\mathbf{n}} \cdot \mathbf{L} \quad (14.84)$$

from Section 11.1. For a vector operator $A$, such a rotation induces the change

$$\delta A = \frac{\varepsilon}{i\hbar} [A, G] = -\frac{\delta \phi}{i\hbar} [A, \hat{\mathbf{n}} \cdot \mathbf{L}] = -\delta \phi \hat{\mathbf{n}} \times A \quad (14.85)$$

by (11.19). The operator $A \cdot A = A^2$ is a rotationally invariant scalar operator, as was shown in Section 11.1.

The Hamiltonian for a particle in a central-force field,

$$H = \frac{p^2}{2m} + V(r) \quad (14.86)$$

is invariant under rotations, since $p^2$ and $r$ are scalar operators. Hence, $e^G = -\delta \phi \hat{\mathbf{n}} \cdot \mathbf{L}$ (for arbitrary vectors $\hat{\mathbf{n}}$ and rotation angles $\delta \phi$) and $L$ itself are
constants of the motion, and orbital angular momentum is conserved for a system with spherical symmetry.

**Exercise 14.9.** Show that if both $A$ and $B$ are constants of the motion, they either commute or the commutator $i[A, B]$ is also a constant of the motion. Prove that if the entire spectrum of $H$ is nondegenerate, then $A$ and $B$ must commute. If the constants of the motion $A$ and $B$ do not commute, there must be degenerate energy eigenvalues. Illustrate this theorem by constructing an example Hamiltonian for which $A = L_x$ and $B = L_y$ are constants of the motion.

5. **Canonical Quantization in the Heisenberg Picture.** The canonical quantization procedure can be formulated in any of the pictures of quantum dynamics, since they are all related to the Schrödinger picture by a generally time-dependent, unitary transformation $U(t)$. Such a transformation leaves every algebraic relation between operators

$$f(A, B, C, \ldots) = 0$$

formally unchanged (see Exercise 9.16):

$$f(\tilde{A}, \tilde{B}, \tilde{C}, \ldots) = f(UA^U, UBU^U, UCU^U, \ldots) = Uf(A, B, C, \ldots)U^U = 0$$

Hence, the canonical commutation relations for conjugate variables

$$[q, p] = qp - pq = i\hbar$$

become the same in any picture:

$$[\tilde{q}(t), \tilde{p}(t)] = (UqU^U)(UpU^U) - (UpU^U)(UqU^U) = i\hbar$$

The dynamical law,

$$i\hbar \frac{d(\tilde{A})}{dt} = \langle [\tilde{A}, \hat{H}] \rangle$$

for any operator $A$ that does not depend on time explicitly also has the same form in all pictures.

**Exercise 14.10.** Using Eqs. (14.41) and (14.44), verify that the equation of motion (14.87) for expectation values of observables holds in any picture of quantum dynamics.

The equations of motion for the canonical variables are derived from (14.41):

$$i\hbar \frac{d\tilde{q}(t)}{dt} = [\tilde{q}(t), H_0]$$

$$i\hbar \frac{d\tilde{p}(t)}{dt} = [\tilde{p}(t), H_0]$$

and depend on the choice of $H_0$, the model Hamiltonian for the particular dynamical picture.

In the Heisenberg picture \( (H_0 = H) \), the equation of motion can be expressed for any dynamical variable

\[
\dot{F}(q, p, t) = F(q(t), p(t), t)
\]

according to (14.30) in summary form as

\[
\frac{d}{dt} \bar{F}(q, p, t) = \frac{1}{i\hbar} \left[ \bar{F}(q, p, t), H \right] + \frac{\partial}{\partial t} \bar{F}(q, p, t)
\]

(14.89)

By (14.61), this is the quantum analogue of the classical equation of motion,

\[
\frac{d}{dt} F(q, p, t) = \left[ F(q, p, t), H \right]_{PB} + \frac{\partial}{\partial t} F(q, p, t)
\]

(14.90)

The formal similarity between the classical equation of motion for canonical variables and the quantum equations for the corresponding operators, and not merely their expectation values, confers a measure of distinction on the Heisenberg picture of quantum dynamics, which otherwise is just one of infinitely many (unitarily) equivalent pictures. In the Heisenberg picture, the transition from classical to quantum theory for a system that has a classical analogue is made simply by replacing the canonical variables by operators that can change in time, subject to the commutation relations

\[
[q_k(t), \pi_l(t)] = i\hbar \delta_{kl}
\]

\[
[q_k(t), \pi_l(0)] = [p_k(t), \pi_l(t)] = 0
\]

(14.91) (14.92)

and by postulating that if the classical equations of motion are expressed in terms of Poisson brackets, the correspondence

\[
[F, G]_{PB} \rightarrow \frac{i}{\hbar} [\bar{F}, \bar{G}]
\]

(14.93)

is to be made—barring, as usual, complications that may arise from ambiguities in the ordering of operators.

The simple commutation relations (14.91) and (14.92) are not valid if the operators are taken at two different times. Thus, generally, \( \bar{q}(t) \) and \( \bar{q}(0) \) do not commute, nor is the commutator \( [\bar{q}(0), \bar{p}(t)] \) equal to \( i\hbar I \). For example, if the system is a free particle in one dimension with \( H = \frac{p^2}{2m} \), we have

\[
\bar{p}(t) = \bar{p}(0) \quad \text{and} \quad \bar{q}(t) = \bar{q}(0) + \frac{\bar{p}(0)}{m} t
\]

(14.94)

hence,

\[
[q(t), \bar{q}(0)] = -\frac{i\hbar}{m} t
\]

(14.95)

If this commutation relation is applied to (10.54), we find the uncertainty relation

\[
(\Delta q)(\Delta q) = \Delta\bar{q}(t)\Delta\bar{q}(0) \leq \frac{\hbar}{2m} |t|
\]

(14.96)

which shows that if the particle moves freely, its wave packet in the coordinate representation must spread in the long run as \( |t| \to \infty \). In Eq. (14.96), the notation
on the left-hand side emphasizes that the variances, being expectation values, are
independent of the choice of the quantum dynamical picture. Except for denoting
the coordinate by \( q \) rather than \( x \), this inequality provides a precise formulation of
the statements about wave packets contracting long before the present time and
spreading in the distant future, that were made in Section 2.4. If the particle is free,
initially narrow wave packets spread more rapidly than those that initially are broad.

If the initial wave packet at \( t = 0 \) is the minimum uncertainty wave packet
\((10.66)\), with

\[
\Delta \bar{q}(0)\Delta \bar{p} = \frac{\hbar}{2}
\]

then, using \((14.96)\),

\[
\Delta \bar{q}(t) \geq \frac{|t|}{m} \Delta \bar{p}
\]

This inequality is consistent with the result of an explicit calculation of the time
dependence of the variance of \( q \) for a wave packet that has "minimum uncertainty"
at \( t = 0 \):

\[
(\Delta q)^2 = (\Delta q_0)^2 + \frac{(\Delta p_0)^2t^2}{m^2}
\]  \((14.97)\)

**Exercise 14.11.** For a free particle in one dimension and an arbitrary initial
wave packet, calculate the time development of \((\Delta q)^2\) and show (as in Problem 2 in
Chapter 3) that

\[
(\Delta q)^2 = (\Delta q_0)^2 + \frac{t}{m} [(qp + p(q) - 2(q)p)] + \frac{(\Delta p)^2t^2}{m^2}
\]  \((14.98)\)

Verify that for the minimum uncertainty wave packet this result agrees with \((14.97)\).
Also compare with the value of the variance \((\Delta q)^2\) as a function of time for a beam
of freely moving classical particles whose initial positions and momenta have dis­
tributions with variances \((\Delta q_0)^2\) and \((\Delta p)^2\).

If the close correspondence between the classical theory and quantum dynamics
gives the Heisenberg picture a certain preferred status, the Schrödinger picture is
perhaps a more intuitive form of quantum mechanics. The Schrödinger picture is
particularly suitable for a discussion of scattering processes, which are more natu­
really described by moving wave packets, albeit complex-valued ones, than by op­
ners changing in time. From our present general point of view, however, it is a
matter of taste and convenience whether we navigate in the Heisenberg or the Schrö­
dinger picture, or any other picture. Once again, the harmonic oscillator offers the
simplest nontrivial illustration.

In the Heisenberg picture, the oscillator Hamiltonian may, similar to the Schrö­
dinger picture form, be written as

\[
H = \hbar \omega \left( a^\dagger a \right) + \frac{1}{2}
\]  \((14.99)\)
Applying (14.30), we obtain the equation of motion for the lowering (annihilation) operator $\bar{a}(t)$,

$$i\hbar \frac{d\bar{a}(t)}{dt} = [\bar{a}(t), \hat{H}] = \hbar \omega [\bar{a}(t), \bar{a}(t)'] \bar{a}(t)$$

Using the commutation relation (10.72), transcribed into the Heisenberg picture, we find the simple differential equation

$$\frac{d\bar{a}(t)}{dt} + i\omega \bar{a}(t) = 0$$

(14.100)

Although the operator equations of motion are difficult to solve directly in most problems, necessitating passage to a representation in which these equations become conventional systems of linear differential and integral equations, the present example is an important exception. Equation (14.100) can be solved immediately, even though $\bar{a}(t)$ is an operator:

$$\bar{a}(t) = \bar{a}(t_0) e^{-i\omega (t-t_0)}$$

(14.101)

Similarly,

$$\bar{a}^+(t) = \bar{a}^+(t_0) e^{i\omega (t-t_0)}$$

(14.102)

If we choose $t_0 = 0$, the initial values of the operators $\bar{a}(0) = a$ and $\bar{a}^+(0) = a^+$ are two mutually adjoint operators that satisfy the commutation relation (10.72). They are the raising (creation) and lowering (annihilation) operators in the Schrödinger picture, which coalesces with the Heisenberg picture at $t = 0$; they serve as the constants of integration of the dynamical problem. The canonical variables $p$ and $q$, and any function of these, can be expressed in terms of $a$ and $a^+$. Thus, in principle the equations of motion have all been integrated.

**Exercise 14.12.** Work out directly from (14.27),

$$\bar{a}(t) = \exp\left(\frac{i}{\hbar} \hat{H} t\right) a \exp\left(-\frac{i}{\hbar} \hat{H} t\right)$$

as an application of the identity (3.59). Then determine $\bar{q}(t)$ and $\bar{p}(t)$.

We now choose a basis and introduce a fixed representation. The most convenient one is the same as in Section 10.6: A basis supported by the eigenvectors of $\hat{H}$. This may be called the energy representation. The matrices representing $\bar{a}(t)$ and $\bar{a}^+(t)$ are obtained by multiplying the matrices (10.90a) and (10.90b) representing $a$ and $a^+$ by $e^{-i\omega t}$ and $e^{i\omega t}$, respectively. In the energy representation of the Heisenberg picture, the coordinate operator of the harmonic oscillator is explicitly represented by the matrix:

$$\bar{q}(t) = \sqrt{\frac{\hbar}{2m\omega}} (\bar{a}(t) + \bar{a}^+(t))$$

(14.103)
This matrix is evidently Hermitian, as it should be. Its elements are harmonic functions of time—a general property of the matrix representing any Heisenberg operator in an energy representation that is supported by the fixed eigenvectors of $H$:

$$H|E\rangle = E|E\rangle$$

Indeed, for any operator, which is not explicitly time-dependent,

$$\tilde{A}(t) = \exp\left(\frac{i}{\hbar} Ht\right) A \exp\left(-\frac{i}{\hbar} Ht\right)$$

giving us the matrix element

$$\langle E'|\tilde{A}(t)|E''\rangle = \langle E'|A|E''\rangle \exp\left[\frac{i}{\hbar} (E'-E)t\right]$$

(14.104)

The special feature of the harmonic oscillator as a perfect periodic system is that all matrix elements oscillate with integral multiples of the same frequency, $\omega$.

**Exercise 14.13.** In either the Heisenberg or Schrödinger picture, show that if at $t=0$ a linear harmonic oscillator is in a coherent state, with eigenvalue $\alpha$, it will remain in a coherent state, with eigenvalue $\alpha e^{-i\omega t}$, at time $t$.

6. The Forced Harmonic Oscillator. For many applications, especially in many-body and field theory, it is desirable to consider the dynamical effects produced by the addition of a time-dependent interaction that is linear in $q$ to the Hamiltonian of the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - qQ(t)$$

where $Q(t)$ is a real-valued function of $t$. This perturbation corresponds to an external time-dependent force that does not depend on the coordinate $q$ (dipole interaction). With no additional effort, we may generalize the Hamiltonian even further by introducing a velocity-dependent term:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - qQ(t) - pP(t)$$

(14.105)

where $P(t)$ is also a real function of $t$.

With the substitutions (10.69) and (10.71), the Hamiltonian (14.105) may be cast in the form

$$H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + f(t)a + f^*(t)a^\dagger$$

$$= \hbar \omega \left( \bar{a}^\dagger(t)\bar{a}(t) + \frac{1}{2} \right) + f(t)\bar{a}(t) + f^*(t)\bar{a}^\dagger(t)$$

(14.106)

in either the Schrödinger or Heisenberg picture, provided that we define the complex-valued function $f(t)$:

$$f(t) = -\frac{\hbar}{\sqrt{2m\omega}} Q(t) + i \frac{\hbar m \omega}{2} P(t)$$

(14.107)
In most applications, we are interested in the lasting rather than the transient changes produced by the time-dependent forces in an initially unperturbed harmonic oscillator. It is therefore reasonable to assume that the disturbance $f(t) \neq 0$ acts only during the finite time interval $T_1 < t < T_2$ and that before $T_1$ and after $T_2$ the Hamiltonian is that of a free oscillator. The time development of the system is conveniently studied in the Heisenberg picture, in which the state vector $|\Psi\rangle$ is constant, and the operators are subject to a unitary transformation as they change from the free oscillation regime before $T_1$ to a free oscillation regime after $T_2$.

Using the equal-time commutation relation,

$$[[\bar{a}(t), \bar{a}^\dagger(t)]] = 1$$

we derive the equation of motion

$$i\hbar \frac{d\bar{a}(t)}{dt} = [\bar{a}(t), H] = \hbar \omega \bar{a}(t) + f^*(t)$$

or

$$\frac{d\bar{a}(t)}{dt} + i\omega \bar{a}(t) = -\frac{i}{\hbar} f^*(t)$$

(14.109)

This inhomogeneous differential equation is easily solved by standard methods. For instance, it can be multiplied by $e^{i\omega t}$ and cast in the form

$$\frac{d}{dt} [\bar{a}(t)e^{i\omega t}] = -\frac{i}{\hbar} f^*(t)e^{i\omega t}$$

which can then be integrated to produce the general solution

$$\bar{a}(t) = \bar{a}(t_0)e^{-i\omega(t-t_0)} - \frac{i}{\hbar} \int_{t_0}^t e^{-i\omega(t-t')} f^*(t') \, dt'$$

(14.110)

If we choose $t_0 = 0$, this equation simplifies to

$$\bar{a}(t) = ae^{-i\omega t} - \frac{i}{\hbar} \int_0^t e^{-i\omega(t-t')} f^*(t') \, dt'$$

(14.111)

Although it calls for unnecessarily heavy artillery, it is instructive to review the solution of (14.109) by the use of Green’s functions to illustrate a method that has proved useful in many similar but more difficult problems.

A Green’s function appropriate to Eq. (14.109) is a solution of the equation

$$\frac{dG(t-t')}{dt} + i\omega G(t-t') = \delta(t-t')$$

(14.112)

because such a function permits us to write a particular solution of Eq. (14.109) as

$$\bar{a}(t) = -\frac{i}{\hbar} \int_{-\infty}^{t'} G(t-t')f^*(t') \, dt'$$

(14.113)

This is easily verified by substituting (14.113) into (14.109). Obviously, for $t \neq t'$ the Green’s function is proportional to $e^{-i\omega(t-t')}$, but at $t = t'$ there is a discontinuity. By integrating (14.112) over an infinitesimal interval from $t' - \varepsilon$ to $t' + \varepsilon$, we derive the condition

$$\lim_{\varepsilon \to 0} [G(t') - G(t')] = 1$$

(14.114)

for $\varepsilon > 0$. 
Two particular Green's functions are useful:

\[ G_R(t - t') = \eta(t - t')e^{-i\omega(t-t')} \]  

(14.115) and

\[ G_A(t - t') = -\eta(t' - t)e^{-i\omega(t-t')} \]  

(14.116)

where the Heaviside step function \( \eta(t) = 0 \) for \( t < 0 \) and \( \eta(t) = 1 \) for \( t > 0 \) (see Appendix, Section 1). These two particular solutions of (14.112) are known as retarded and advanced Green's functions, respectively.

We note that

\[ \tilde{a}(t) = -\frac{i}{\hbar} \int_{-\infty}^{t} G_R(t - t')f^*(t') \, dt' \]

is the particular solution of (14.109), which vanishes for \( t < T_1 \). Similarly,

\[ \tilde{a}(t) = -\frac{i}{\hbar} \int_{-\infty}^{t} G_A(t - t')f^*(t') \, dt' \]

is the particular solution of (14.109) which vanishes for \( t > T_2 \).

If we denote by \( a_{in}(t) \) and \( a_{out}(t) \) those solutions of the homogeneous equation

\[ \frac{d\tilde{a}(t)}{dt} + i\omega\tilde{a}(t) = 0 \]

which coincide with the solution \( \tilde{a}(t) \) of the inhomogeneous equation (14.109) for \( t < T_1 \) and \( t > T_2 \) respectively, and if we choose \( t_0 = 0 \), we can write

\[ \tilde{a}(t) = a_{in}(t) - \frac{i}{\hbar} \int_{-\infty}^{t} G_R(t - t')f^*(t') \, dt' \]  

(14.117)

or, alternatively,

\[ \tilde{a}(t) = a_{out}(t) - \frac{i}{\hbar} \int_{-\infty}^{t} G_A(t - t')f^*(t') \, dt' \]  

(14.118)

Both (14.117) and (14.118) are equivalent to solution (14.111). By equating the right-hand sides of (14.117) and (14.118), we obtain the relation

\[ a_{out} = a_{in} - \frac{i}{\hbar} g^*(\omega) \]  

(14.119)

where

\[ g(\omega) = \int_{T_1}^{T_2} e^{-i\omega t'} f(t') \, dt' = \int_{-\infty}^{+\infty} e^{-i\omega t'} f(t') \, dt' \]  

(14.120)

is the Fourier transform of the generalized force \( f(t) \).
The solution (14.117) or (14.118) in the Heisenberg picture can be used to answer all questions about the time development of the system. It is nevertheless instructive also to consider the dynamical problem from the point of view of other pictures.

To implement the interaction picture, we regard the Hamiltonian of the forced oscillator as the sum, $H = H_0 + V(t)$, of an unperturbed Hamiltonian

$$H_0 = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right)$$

(14.121)

and an explicitly time-dependent "interaction" term,

$$V(t) = f(t)a + f^*(t)a^\dagger$$

(14.122)

Time-dependent Hamiltonians require more careful treatment than time-independent ones, because generally the interaction operators at two different times do not commute.

We choose the unperturbed Hamiltonian operator $H_0$ as the model Hamiltonian to define the interaction picture. According to (14.45), the transformed interaction operator is

$$\tilde{V}(t) = e^{i\omega a^\dagger a} f(t)a + f^*(t)a^\dagger e^{-i\omega a^\dagger a}$$

(14.123)

The interaction operator can be evaluated by use of the identity (3.59), since $[a^\dagger a, a] = -a$ and $[a^\dagger a, a^\dagger] = a^\dagger$. We thus obtain

$$\tilde{V}(t) = f(t)ae^{-i\omega t} + f^*(t)a^\dagger e^{i\omega t}$$

(14.124)

The equation of motion for the state vector in the interaction picture is

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = (f(t)ae^{-i\omega t} + f^*(t)a^\dagger e^{i\omega t}) |\Psi(t)\rangle$$

(14.125)

Similarly, the time development operator $\hat{T}(t_2, t_1)$ in the interaction picture as defined in Eq. (14.49) satisfies the equation of motion:

$$i\hbar \frac{d}{dt} \hat{T}(t, t_1) = \tilde{V}(t)\hat{T}(t, t_1)$$

(14.126)

Integration of (14.126) over the time interval $(t_1, t)$ and use of the initial condition $\hat{T}(t_1, t_1) = I$ produce an integral equation for the time development operator:

$$\hat{T}(t, t_1) = I - i \int_{t_1}^{t} \tilde{V}(t')\hat{T}(t', t_1) dt'$$

(14.127)

A formal solution of this equation can be constructed by successive iteration:

$$\hat{T}(t, t_1) = I - i \int_{t_1}^{t} \tilde{V}(t') dt' + \left( \frac{i}{\hbar} \right)^2 \int_{t_1}^{t} \tilde{V}(t') dt' \int_{t_1}^{t_1} \tilde{V}(t'') dt'' + \ldots$$

(14.128)

It is sometimes convenient to write this series expansion in a more symmetric form by using the time-ordered product of operators. We define time ordering of two operators as

$$\text{T}[\tilde{V}(t')\tilde{V}(t'')] = \begin{cases} \tilde{V}(t')\tilde{V}(t'') & \text{if } t'' \leq t' \\ \tilde{V}(t'')\tilde{V}(t') & \text{if } t' \leq t'' \end{cases}$$

(14.129)
This convention is easily generalized to products of any number of time-dependent operators. With it we can prove that if \( t > t_1 \), the time development operator may be written in the form

\[
\hat{T}(t, t_1) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \right)^n \int_{t_1}^{t} \cdots \int_{t_1}^{t} \frac{dt'_1 \cdots dt'_n}{1} T[\hat{V}(t'_1) \hat{V}(t'_2) \cdots \hat{V}(t'_n)]
\]

or formally and compactly as

\[
\hat{T}(t, t_1) = T \exp \left[-\frac{i}{\hbar} \int_{t_1}^{t} \hat{V}(t') \, dt' \right]
\]

(14.130)

To prove that (14.130) is a solution of (14.126), it is sufficient to write to first order in \( \varepsilon \),

\[
\hat{T}(t + \varepsilon, t_1) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \right)^n \int_{t_1}^{t} \cdots \int_{t_1}^{t} \frac{dt'_1 \cdots dt'_n}{1} T[\hat{V}(t'_1) \hat{V}(t'_2) \cdots \hat{V}(t'_n)]
\]

\[
= \hat{T}(t, t_1) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \right)^n n\hat{V}(t) \int_{t_1}^{t} \cdots \int_{t_1}^{t} \frac{dt'_1 \cdots dt'_{n-1}}{1} T[\hat{V}(t'_1) \hat{V}(t'_2) \cdots \hat{V}(t'_{n-1})]
\]

or

\[
\frac{\hat{T}(t + \varepsilon, t_1) - \hat{T}(t, t_1)}{\varepsilon} = -\frac{i}{\hbar} \hat{V}(t) \hat{T}(t, t_1)
\]

which in the limit \( \varepsilon \rightarrow 0 \) reduces to (14.126).

If the formula (14.131) is applied to the forced linear harmonic oscillator, with the interaction potential (14.124), we obtain

\[
\hat{T}(t, t_1) = T \exp \left[-\frac{i}{\hbar} \int_{t_1}^{t} \left(f(t')a \hat{e}^{-i\omega t'} + f^*(t')a^* \hat{e}^{i\omega t'} \right) \, dt' \right]
\]

(14.132)

Although this is a compact expression for the time development operator, because of the presence of the unwieldy time ordering operator it is not yet in a form convenient for calculating transition amplitudes.

To derive a more manageable formula, we use the group property (14.8) and write

\[
\hat{T}(t, t_1) = \lim_{N \rightarrow \infty} e^{V_N} e^{V_{N-1}} \cdots e^{V_1}
\]

(14.133)

where, by definition,

\[
V_k = -\frac{i}{\hbar} \int_{t_1}^{t} \hat{V}(t') \, dt' \quad \text{and} \quad N_k = t - t_1
\]

(14.134)

Expression (14.133) is valid, even if the interaction operators at different times do not commute, because the time intervals of length \( \varepsilon \) are infinitesimally short and are not subject to internal time ordering. The expression (14.133) can be further reduced if the commutators [\( \hat{V}(t'), \hat{V}(t'') \)] are numbers for all \( t' \) and \( t'' \). This is indeed the case for the forced harmonic oscillator, since according to (14.124),

\[
[V(t'), V(t'')] = \int_{t_1}^{t_1} f^*(t') f(t'') \hat{e}^{-i\omega(t'-t')} - f^*(t') f(t'') \hat{e}^{i\omega(t'-t')}
\]

(14.135)
The identity (3.61) can be applied repeatedly to give

\[ \tilde{T}(t, t_1) = \lim_{N \to \infty} \exp\left\{ \sum_{k=1}^{N} \left( V_k + \frac{1}{2} \left[ V_k, \sum_{n=1}^{k} V_n \right] \right) \right\} \]

or, if the limit \( N \to \infty \) and \( \epsilon \to 0 \) is carried out,

\[ \tilde{T}(t, t_1) = \exp\left\{ -\frac{i}{\hbar} \int_{t_1}^{t} \tilde{V}(t') \, dt' \right\} \]

For the forced harmonic oscillator, inserting (14.124) and (14.135) into (14.136), we thus obtain the time development operator in the interaction picture in the desired form:

\[ \tilde{T}(t, t_1) = e^{i\hat{\beta}(t, t_1)} \exp\left\{ -\frac{i}{\hbar} \int_{t_1}^{t} \tilde{V}(t') \, dt' \right\} \]

where we have defined

\[ \hat{\beta}(t, t_1) = -\frac{i}{\hbar} \int_{t_1}^{t} \tilde{V}(t') \, dt' \]

This expression can be connected with the Fourier integral of the applied force given in (14.120):

\[ g(\omega) = -i \hbar \tilde{\chi}^{*}(T_2, T_1) = -i \hbar \tilde{\chi}^{*}(-\infty, +\infty) \]

In (14.137), the real phase \( \beta \) appears as the price we must pay for eliminating the time ordering operator, and it stands for:

\[ \beta(t, t_1) = \frac{i}{2\hbar^2} \int_{t_1}^{t} dt' \int_{t_1}^{t} dt'' \left( f(t') f^{*}(t'') e^{-i\omega(t'-t'')} - f^{*}(t') f(t'') e^{i\omega(t'-t'')} \right) \]

If the initial state at \( t = t_1 \) is a coherent oscillator state \( |\alpha\rangle \), as defined in Section 10.7, the state at time \( t \) is

\[ |\Psi(t)\rangle = \tilde{T}(t, t_1) |\alpha\rangle = e^{i\hat{\beta}(t, t_1)} e^{-\chi(C(t_1)) a^* + \chi(C(t)) a^*} e^{\gamma a^* - a^* a} |0\rangle = e^{i\gamma(t_1)} |\alpha + \gamma\rangle \]

where \( \gamma \), like \( \beta \), is a numerical phase. We arrive at the intriguing and important conclusion that, under the influence of the (dipole) interaction \( f(t)a^* + f^{*}(t)a \), a coherent state remains coherent at all times, because the time development operator (14.137) is a displacement operator for coherent states, like the operator \( D \) in (10.98).

Of particular interest is the limit of the operator \( \tilde{T}(t, t_1) \) as \( t_1 \to -\infty \) and \( t \to +\infty \). This limiting time development operator is known as the S (or scattering) operator and is defined formally as

\[ S = \tilde{T}(+\infty, -\infty) \]

For the forced harmonic oscillator with an interaction of finite duration during the interval \((T_1, T_2)\) the S operator is

\[ S = e^{i\theta} \exp\left\{ -\frac{i}{\hbar} g(\omega)a - \frac{i}{\hbar} g^{*}(\omega)a^* \right\} \]
where we have denoted

\[ \beta = \beta(+\infty, -\infty) \]  

(14.144)

Substituting the expression for \( g(\omega) \) defined in (14.120), we obtain

\[ S = e^{i\theta} \exp\left\{ -i \frac{\hbar}{\omega} \int_{-\infty}^{+\infty} \left[ f(t)a e^{-\omega t} + f^*(t)a^\dagger e^{\omega t} \right] dt \right\} \]  

(14.145)

As the link between the states of the system before the onset and after the cessation of the interaction, the scattering operator, or the \( S \) matrix representing it, was first illustrated in Section 6.3. We will encounter the same concepts again in Chapter 20.

If the oscillator is in the ground state before the start of the interaction, what is the probability that it will be found in the \( n \)th excited oscillator energy eigenstate after the interaction terminates? The interaction produces the state \( S|0\rangle \), which is a coherent state with eigenvalue \( \alpha = -(i\hbar)g^*(\omega) \). The transition probability of finding the oscillator in the \( n \)th eigenstate after the interaction is, according to (10.110), the Poisson distribution

\[ |\langle n|S|0\rangle|^2 = |\langle n|\alpha = -ig^*(\omega)\hbar\rangle|^2 = P_n(\alpha) \]  

(14.146)

with expectation value \( \langle n \rangle = |g(\omega)|^2/\hbar^2 \) for the oscillator quantum number. These results can be interpreted in terms of a system of quanta, \( n \) being the number of quanta present. The interaction term in the Hamiltonian is linear in \( \alpha^* \) and \( \alpha \) and creates or annihilates quanta. The strength of the interaction determines the average number \( \langle n \rangle \) of quanta present and characterizes the Poisson distribution, which represents the probability that a dipole interaction pulse incident on the vacuum state of our system of quanta leaves after its passage a net number of \( n \) quanta behind. These features of the dynamics of the forced or driven linear harmonic oscillator will help us understand the creation and annihilation of photons in Chapter 23.

Finally, we use the results from the interaction picture to deduce the time development operator in the Schrödinger picture. From Eq. (14.49) we infer that

\[ T(t_2, t_1) = e^{-i(\hbar/k)H_0(t_2 - t_1)} T(t_2, t_1) e^{i(\hbar/k)H_0 t_1} \]  

(14.147)

If we employ the oscillator Hamiltonian (14.121) for \( H_0 \) and the time development operator (14.137) in the interaction picture, we obtain

\[ T(t_2, t_1) = e^{iH(t_2 - t_1)} \exp\{-i*\xi(t_2, t_1)\omega e^{i\omega t_2} + \xi(t_2, t_1)\omega e^{-i\omega t_2} e^{-i(\hbar/k)H_0(t_2 - t_1)} \} \]  

(14.148)

**Exercise 14.14.** If \([A, B] = \gamma B \) (as in Exercise 3.15) prove that

\[ e^{\lambda A}f(B)e^{-\lambda A} = f(e^{\lambda}\gamma B) \]  

(14.149)

**Exercise 14.15.** Verify the expression (14.148) for the time development operator by applying (14.149) and (3.61).

**Exercise 14.16.** Show that

\[ S^t a S = a - \frac{i}{\hbar} g^*(\omega) = a + a_{\text{out}} - a_{\text{in}} \]  

(14.150)
where $S$ is the operator defined in Eq. (14.142) and $a_{in}$ and $a_{out}$ are related by (14.119). Noting that the operators $a$, $a_{in}$, and $a_{out}$ differ from each other by additive constants, and using the unitarity of $S$, deduce that

$$S^* a_{in} S = a_{out} \quad (14.151)$$

**Exercise 14.17.** For a forced harmonic oscillator with a transient perturbation (14.106), derive the change in the unperturbed energy, if $|\Psi\rangle$ is the initial state of the oscillator in the interaction picture, at asymptotically long times before the onset of the interaction. Show that

$$\Delta E = ig\omega(a) - ig^*(\omega)(a^\dagger) + \omega|g(\omega)|^2/\hbar \quad (14.152)$$

If $|\Psi\rangle$ is the ground state of the oscillator, verify that $\Delta E$ given by (14.152) agrees with a direct calculation based on the Poisson distribution formula (14.146).

**Problems**

1. A particle of charge $q$ moves in a uniform magnetic field $B$ which is directed along the $z$ axis. Using a gauge in which $A_z = 0$, show that $q = (cp_y - qA_y)/qB$ and $p = (cp_y - qA_y)/c$ may be used as suitable canonically conjugate coordinate and momentum together with the pair $z, p_z$. Derive the energy spectrum and the eigenfunctions in the $q$-representation. Discuss the remaining degeneracy. Propose alternative methods for solving this eigenvalue problem.

2. A linear harmonic oscillator is subjected to a spatially uniform external force $F(t) = C\eta(t)e^{-\lambda t}$ where $\lambda$ is a positive constant and $\eta(t)$ the Heaviside step function (A.23). If the oscillator is in the ground state at $t < 0$, calculate the probability of finding it at time $t$ in an oscillator eigenstate with quantum number $n$. Assuming $C = (\hbar m \lambda^2)^{1/2}$, examine the variation of the transition probabilities with $n$ and with the ratio $\lambda/\omega$, $\omega$ being the natural frequency of the harmonic oscillator.

3. If the term $V(t)$ in the Hamiltonian changes suddenly ("impulsively") between time $t$ and $t + \Delta t$, in a time $\Delta t$ short compared with all relevant periods, and assuming only that $[V(t'), V(t'')] = 0$ during the impulse, show that the time development operator is given by

$$T(t + \Delta t, t) = \exp \left[ -\frac{i}{\hbar} \int_t^{t + \Delta t} V(t') \, dt' \right]$$

Note especially that the state vector remains unchanged during a sudden change of $V$ by a finite amount.

4. A linear harmonic oscillator in its ground state is exposed to a spatially constant force which at $t = 0$ is suddenly removed. Compute the transition probabilities to the excited states of the oscillator. Use the generating function for Hermite polynomials to obtain a general formula. How much energy is transferred?

5. In the nuclear beta decay of a tritium atom ($^3$H) in its ground state, an electron is emitted and the nucleus changes into an $^3$He nucleus. Assume that the change is sudden, and compute the probability that the atom is found in the ground state of the helium ion after the emission. Compute the probability of atomic excitation to the $2S$ and $2P$ states of the helium ion. How probable is excitation to higher levels, including the continuum?