the car will come to a stop, but if $b \leq b_c$, the car will rebound and roll back down the track. (Note that the car is not fastened to the spring. As long as it pushes on the spring, it moves according to the harmonic oscillator equation, but instead of pulling on the spring, it will simply roll back down the track.)

39. A mass m subject to a linear restoring force $-kx$ and damping $-b\dot{x}$ is displaced a distance x_0 from equilibrium and released with zero initial velocity. Find the motion in the underdamped, critically damped, and overdamped cases.

40. Solve Problem 39 for the case when the mass starts from its equilibrium position with an initial velocity v_0. Sketch the motion for the three cases.

41. Solve Problem 39 for the case when the mass has an initial displacement x_0 and an initial velocity v_0 directed back toward the equilibrium point. Show that if $|v_0| > |\gamma_1 x_0|$, the mass will overshoot the equilibrium in the critically damped and overdamped cases so that the remarks at the end of Section 2.9 do not apply. Sketch the motion in these cases.

42. It is desired to design a bathroom scale with a platform deflection of one inch under a 200-lb man. If the motion is to be critically damped, find the required spring constant k and the damping constant b. Show that the motion will then be overdamped for a lighter person. If a 200-lb man steps on the scale, what is the maximum upward force exerted by the scale platform against his feet while the platform is coming to rest?

43. A mass of 1000 kg drops from a height of 10 m on a platform of negligible mass. It is desired to design a spring and dashpot on which to mount the platform so that the platform will settle to a new equilibrium position 0.2 m below its original position as quickly as possible after the impact *without overshooting*.

a) Find the spring constant k and the damping constant b of the dashpot. Be sure to examine your proposed solution $x(t)$ to make sure that it satisfies the correct initial conditions and does not overshoot.

b) Find, to two significant figures, the time required for the platform to settle within 1 mm of its final position.

44. A force $F_0 e^{-at}$ acts on a harmonic oscillator of mass m, spring constant k, and damping constant b. Find a particular solution of the equation of motion by starting from the guess that there should be a solution with the same time dependence as the applied force.

45. a) Find the motion of a damped harmonic oscillator subject to a constant applied force F_0, by guessing a "steady-state" solution of the inhomogeneous equation (2.91) and adding a solution of the homogeneous equation.

b) Solve the same problem by making the substitution $x' = x - a$, and choosing the constant a so as to reduce the equation in x' to the homogeneous equation (2.90). Hence show that the effect of the application of a constant force is merely to shift the equilibrium position without affecting the nature of the oscillations.

46. An underdamped harmonic oscillator is subject to an applied force

$$F = F_0 e^{-at} \cos(\omega t + \theta).$$

Find a particular solution by expressing F as the real part of a complex exponential function and looking for a solution for x having the same exponential time dependence.

47. An undamped harmonic oscillator ($b = 0$), initially at rest, is subject beginning at $t = 0$ to an applied force $F_0 \sin \omega t$. Find the motion $x(t)$.

48. An undamped harmonic oscillator ($b = 0$) is subject to an applied force $F_0 \cos \omega t$. Show that if $\omega = \omega_0$, there is no steady-state solution. Find a particular solution by starting with a solution for $\omega = \omega_0 + \varepsilon$, and passing to the limit $\varepsilon \to 0$. [*Hint:* If you start with the steady-state solution and let $\varepsilon \to 0$, it will blow up. Try starting with a solution which fits the initial condition $x_0 = 0$, so that it cannot blow up at $t = 0$.]

49. A critically damped harmonic oscillator with mass m and spring constant k, is subject to an applied force $F_0 \cos \omega t$. If, at $t = 0$, $x = x_0$ and $v = v_0$, what is $x(t)$?

50. A force $F_0 \cos(\omega t + \theta_0)$ acts on a damped harmonic oscillator beginning at $t = 0$.

a) What must be the initial values of x and v in order that there be no transient?

b) If instead $x_0 = v_0 = 0$, find the amplitude A and phase θ of the transient in terms of F_0, θ_0.

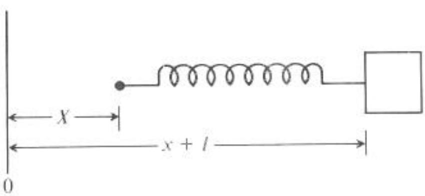

Fig. 2.11

51. A mass m is attached to a spring with force constant k, relaxed length l, as shown in Fig. 2.11. The left end of the spring is not fixed, but is instead made to oscillate with amplitude a, frequency ω, so that $X = a \sin \omega t$, where X is measured from a fixed reference point 0. Write the equation of motion, and show that it is equivalent to Eq. (2.148) with an applied force $ka \sin \omega t$, if the friction is given by Eq. (2.31). Show that, if the friction comes instead from a dashpot connected between the ends of the spring, so that the frictional force is $-b(\dot{x} - \dot{X})$, then the equation of motion has an additional applied force $\omega ba \cos \omega t$.

52. An automobile weighing one ton (2000 lb, including passengers but excluding wheels and everything else below the springs) settles one inch closer to the road for every 200 lb of passengers. It is driven at 20 mph over a washboard road with sinusoidal undulations having a distance between bumps of 1 ft and an amplitude of 2 in (height of bumps and depth of holes from mean road level). Find the amplitude of oscillation of the automobile, assuming it moves vertically as a simple harmonic oscillator without damping (no shock absorbers). (Neglect the mass of wheels and springs.) If shock absorbers are added to provide damping, is the ride better or worse? (Use the result of Problem 51.)

53. An undamped harmonic oscillator of mass m, natural frequency ω_0, is initially at rest and is subject at $t = 0$ to a blow so that it starts from $x_0 = 0$ with initial velocity v_0 and oscillates freely until $t = 3\pi/2\omega_0$. From this time on, a force $F = B \cos(\omega t + \theta)$ is applied. Find the motion.

54. Find the motion of a mass m subject to a restoring force $-kx$, and to a damping force $(\pm)\mu mg$ due to dry sliding friction. Show that the oscillations are isochronous (period independent of amplitude) with the amplitude of oscillation decreasing by $2\mu g/\omega_0^2$ during each half-cycle until the mass comes to a stop. [*Hint:* Use the result of Problem 45. When the force has a different algebraic form at different times during the motion, as here, where the sign of the damping force must be chosen so that the force is always opposed to the velocity, it is necessary to solve the equation of motion separately for each interval of time during which a particular expression for the force is to be used, and to choose as initial conditions for each time interval the final position and velocity of the preceding time interval.]

55. An undamped harmonic oscillator ($\gamma = 0$), initially at rest, is subject to a force given by Eq. (2.191).

a) Find $x(t)$.

b) For a fixed p_0, for what value of δt is the final amplitude of oscillation greatest?

c) Show that as $\delta t \to 0$, your solution approaches that given by Eq. (2.190).

56. Find the solution analogous to Eq. (2.190) for a critically damped harmonic oscillator subject to an impulse p_0 delivered at $t = t_0$.

57. a) Find, using the principle of superposition, the motion of an underdamped oscillator $[\gamma = (1/3)\omega_0]$ initially at rest and subject, after $t = 0$, to a force

$$F = A \sin \omega_0 t + B \sin 3\omega_0 t,$$

where ω_0 is the natural frequency of the oscillator.

b) What ratio of B to A is required in order for the forced oscillation at frequency $3\omega_0$ to have the same amplitude as that at frequency ω_0?

58. A force $F_0(1 - e^{-at})$ acts on a harmonic oscillator which is at rest at $t = 0$. The mass is m, the spring constant $k = 4ma^2$, and $b = ma$. Find the motion. Sketch $x(t)$.

***59.** Solve Problem 58 for the case $k = ma^2$, $b = 2ma$.

60. Find, by the Fourier-series method, the steady-state solution for the damped harmonic oscillator subject to a force

$$F(t) = \begin{cases} 0, & \text{if } nT < t \le (n+\frac{1}{2})T, \\ F_0, & \text{if } (n+\frac{1}{2})T < t \le (n+1)T, \end{cases}$$

where n is any integer, and $T = 6\pi/\omega_0$, where ω_0 is the resonance frequency of the oscillator. Show that if $\gamma \ll \omega_0$, the motion is nearly sinusoidal with period $T/3$.

*An asterisk is used, as explained in the Preface, to indicate problems which may be particularly difficult.

61. Find, by the Fourier-series method, the steady-state solution for an undamped harmonic oscillator subject to a force having the form of a rectified sine-wave:

$$F(t) = F_0 \left|\sin \omega_0 t\right|,$$

where ω_0 is the natural frequency of the oscillator.

62. Solve Problem 58 by using Green's solution (2.210).

63. An underdamped oscillator initially at rest is acted upon, beginning at $t = 0$, by a force given by Eq. (2.191). Find its motion by using Green's solution (2.210).

64. Using the result of Problem 56, find by Green's method the motion of a critically damped oscillator initially at rest and subject to a force $F(t)$.