F-313 C Mecânica Geral
Lista de Exercícios #3

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Os problemas sugeridos estão destacados em vermelho

1. **Teorema de Equipartição**
Neste problema queremos calcular a média temporal de uma grandeza $f(t)$ que é periódica no tempo, com período T. Definimos essa média como sendo:

$$\langle f(t) \rangle \equiv \frac{1}{T} \int_{t_0}^{t_0+T} dt \ f(t) \ ,$$

onde t_0 é um tempo de referência arbitrário (o resultado não poderá depender de t_0).

No caso do oscilador harmônico (OH) $1-\text{dim}$, encontramos a solução geral abaixo:

$$x(t) = A \ \cos(\omega_0 t + \theta) \ ,$$

onde ω_0 é a frequência do OH. A amplitude A e a fase θ dependem das condições iniciais (x_0, v_0).

(a) Para o OH, calcule a média temporal de energia cinética K. Expresse seu resultado em termos da energia total E.

(b) Calcule também a média temporal de energia potencial

$$V(x) = \frac{1}{2} m\omega_0^2 \ x^2 \ ,$$

expressando também o resultado em termos da energia total E.

(c) Discuta a física dos resultados de **a)** e **b)** fazendo os gráficos em função de t, de K e V (basta graficar um período). O resultado obtido é chamado **Teorema de Equipartição**. □

2. **Ressonância**

Considere o problema do oscilador forçado como tratado em aula, na situação sub-amortecida

$$\omega_0^2 \gg \alpha^2 \ .$$

(a) Encontre o máximo valor da amplitude em termos de ω, a frequência de oscilação da força. Mostre que o máximo diverge quando o amortecimento é muito fraco ($\alpha \to 0$). Isso representa o fenômeno da **Ressonância**.

(b) Encontre a defasagem ϕ entre a força e a coordenada, na situação de ressonância. Explique.

(c) Encontre também a defasagem ϕ para baixas e altas frequências. Explique.

(d) No caso do oscilador forçado, interessa calcular a média temporal do trabalho por unidade de tempo executado pela força impulsora sobre o sistema oscilante (potência transferida). Calcule então

$$\langle P \rangle = \langle F(t)\ \dot{x}(t) \rangle \ ,$$

onde os parênteses $\langle ... \rangle$ significam média temporal (como definida no problema # 1). Mostre que $\langle P \rangle$ depende de maneira crítica da defasagem ϕ, sendo máxima na ressonância. □

5. Prove the following inequalities. Give a geometric and an algebraic proof (in terms of components) for each:

a) $$|\boldsymbol{A}+\boldsymbol{B}| \leq |\boldsymbol{A}|+|\boldsymbol{B}|.$$

b) $$|\boldsymbol{A}\cdot\boldsymbol{B}| \leq |\boldsymbol{A}|\,|\boldsymbol{B}|.$$

c) $$|\boldsymbol{A}\times\boldsymbol{B}| \leq |\boldsymbol{A}|\,|\boldsymbol{B}|.$$

6. a) Obtain a formula analogous to Eq. (3.40) for the magnitude of the sum of three forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$, in terms of F_1, F_2, F_3, and the angles $\theta_{12}, \theta_{23}, \theta_{31}$ between pairs of forces. [Use the suggestions following Eq. (3.40).]

b) Obtain a formula in the same terms for the angle α_1, between the total force and the component force $\boldsymbol{F}_1$.

7. Prove Eqs. (3.54) and (3.55) from the definition (3.52) of vector differentiation.

8. Prove Eqs. (3.56) and (3.57) from the algebraic definition (3.53) of vector differentiation.

9. Give suitable definitions, analogous to Eqs. (3.52) and (3.53), for the integral of a vector function $\boldsymbol{A}(t)$ with respect to a scalar t:

$$\int_{t_1}^{t_2} \boldsymbol{A}(t)\,dt.$$

Write a set of equations like Eqs. (3.54)–(3.57) expressing the algebraic properties you would expect such an integral to have. Prove that on the basis of either definition

$$\frac{d}{dt}\int_0^t \boldsymbol{A}(t)\,dt = \boldsymbol{A}(t).$$

10. A 45° isosceles right triangle ABC has a hypotenuse AB of length $4a$. A particle is acted on by a force attracting it toward a point O on the hypotenuse a distance a from the point A. The force is equal in magnitude to k/r^2, where r is the distance of the particle from the point O. Calculate the work done by this force when the particle moves from A to C to B along the two legs of the triangle. Make the calculation by both methods, that based on Eq. (3.61) and that based on Eq. (3.63).

11. A particle moves around a semicircle of radius R, from one end A of a diameter to the other B. It is attracted toward its starting point A by a force proportional to its distance from A. When the particle is at B, the force toward A is F_0. Calculate the work done against this force when the particle moves around the semicircle from A to B.

12. A particle is acted on by a force whose components are

$$F_x = ax^3 + bxy^2 + cz,$$

$$F_y = ay^3 + bx^2y,$$

$$F_z = cx.$$

Calculate the work done by this force when the particle moves along a straight line from the origin to the point (x_0, y_0, z_0).

13. a) A particle in the xy-plane is attracted toward the origin by a force $F = k/y$, inversely proportional to its distance from the x-axis. Calculate the work done by the force when the particle moves from the point $x = 0$, $y = a$ to the point $x = 2a$, $y = 0$ along a path which follows the sides of a rectangle consisting of a segment parallel to the x-axis from $x = 0$, $y = a$ to $x = 2a$, $y = a$, and a vertical segment from the latter point to the x-axis.

b) Calculate the work done by the same force when the particle moves along an ellipse of semiaxes a, $2a$. [*Hint*: Set $x = 2a \sin \theta$, $y = a \cos \theta$.]

14. Find the r- and θ-components of $d\boldsymbol{a}/dt$ in plane polar coordinates, where $\boldsymbol{a}$ is the acceleration of a particle.

15. Find the components of $d^2\boldsymbol{A}/dt^2$ in cylindrical polar coordinates, where the vector $\boldsymbol{A}$ is a function of t and is located at a moving point.

16. Find the components of $d^3\boldsymbol{r}/dt^3$ in spherical coordinates.

*17. a) Plane parabolic coordinates f, h are defined in terms of cartesian coordinates x, y by the equations

$$x = f - h, \qquad y = 2(fh)^{1/2},$$

where f and h are never negative. Find f and h in terms of x and y. Let unit vectors $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{h}}$ be defined in the directions of increasing f and h respectively. That is, $\hat{\boldsymbol{f}}$ is a unit vector in the direction in which a point would move if its f-coordinate increases slightly while its h-coordinate remains constant. Show that $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{h}}$ are perpendicular at every point. [*Hint*: $\hat{\boldsymbol{f}} = (\hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy)[(dx)^2 + (dy)^2]^{-1/2}$, when $df > 0$, $dh = 0$. Why?]

b) Show that $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{h}}$ are functions of f, h, and find their derivatives with respect to f and h. Show that $\boldsymbol{r} = f^{1/2}(f+h)^{1/2}\hat{\boldsymbol{f}} + h^{1/2}(f+h)^{1/2}\hat{\boldsymbol{h}}$. Find the components of velocity and acceleration in parabolic coordinates.

18. A particle moves along the parabola

$$y^2 = 4f_0^2 - 4f_0 x,$$

where f_0 is a constant. Its speed v is constant. Find its velocity and acceleration components in rectangular and in polar coordinates. Show that the equation of the parabola in polar coordinates is

$$r \cos^2 \frac{\theta}{2} = f_0.$$

What is the equation of this parabola in parabolic coordinates (Problem 17)?

19. A particle moves with varying speed along an arbitrary curve lying in the xy-plane. The position of the particle is to be specified by the distance s the particle has traveled along the curve from some fixed point on the curve. Let $\hat{\boldsymbol{\tau}}(s)$ be a unit vector tangent to the curve at the point s in the direction of increasing s. Show that

$$\frac{d\hat{\boldsymbol{\tau}}}{ds} = \frac{\hat{\boldsymbol{v}}}{r},$$

where $\hat{\nu}(s)$ is a unit vector normal to the curve at the point s, and $r(s)$ is the radius of curvature at the point s, defined as the distance from the curve to the point of intersection of two nearby normals. Hence derive the following formulas for the velocity and acceleration of the particle:

$$\boldsymbol{v} = \dot{s}\hat{\boldsymbol{\tau}}, \qquad \boldsymbol{a} = \ddot{s}\hat{\boldsymbol{\tau}} + \frac{\dot{s}^2}{r}\hat{\boldsymbol{\nu}}.$$

20. Using the properties of the vector symbol ∇, derive the vector identities:

$$\text{curl}\,(\text{curl}\,\boldsymbol{A}) = \text{grad}\,(\text{div}\,\boldsymbol{A}) - \nabla^2\boldsymbol{A},$$

$$u\,\text{grad}\,v = \text{grad}\,(uv) - v\,\text{grad}\,u.$$

Then write out the x-components of each side of these equations and prove by direct calculation that they are equal in each case. (One must be very careful, in using the first identity in curvilinear coordinates, to take proper account of the dependence of the unit vectors on the coordinates.)

21. Calculate curl $\boldsymbol{A}$ in cylindrical coordinates.

22. If the particle in Problem 12 moves with a constant velocity $\boldsymbol{v}$, what is the impulse delivered to it by the given force?

23. a) Given that the particle in Problem 11 moves with a constant speed v around the semicircle, find the rectangular components $F_x(t)$, $F_y(t)$ of the additional force which must act on it besides the force given in Problem 11. Take the x-axis along the diameter AB.

b) Calculate the impulse delivered by this additional force.

24. A particle of mass m moves with constant speed v around a circle of radius r, starting at $t = 0$ from a point P on the circle. Find the angular momentum about the point P at any time t, the force, and the torque about P, and verify that the angular momentum theorem (3.140) is satisfied.

25. A particle of mass m moves according to the equations

$$x = x_0 + at^2,$$

$$y = bt^3,$$

$$z = ct.$$

Find the angular momentum $\boldsymbol{L}$ at any time t. Find the force $\boldsymbol{F}$ and from it the torque $\boldsymbol{N}$ acting on the particle. Verify that the angular momentum theorem (3.144) is satisfied.

26. Give a suitable definition of the angular momentum of a particle about an axis in space. Taking the specified axis as the z-axis, express the angular momentum in terms of cylindrical coordinates. If the force acting on the particle has cylindrical components F_z, F_ρ, F_φ, prove that the time rate of change of angular momentum about the z-axis is equal to the torque about that axis.

27. A moving particle of mass m is located by spherical coordinates $r(t)$, $\theta(t)$, $\varphi(t)$. The force acting on it has spherical components F_r, F_θ, F_φ. Calculate the spherical components of the angular momentum vector and of the torque vector about the origin, and verify by direct calculation that the equation

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

follows from Newton's equation of motion.

28. The solutions plotted in Fig. 3.28 correspond to the first two of Eqs. (3.151). If $\theta_x = 0$, estimate θ_y for the case $\omega_x = 2\omega_y$ as drawn. Sketch the corresponding figure for the case $\theta_x = \theta_y$. Sketch a typical figure for the case $4\omega_x = 3\omega_y$.

29. Find a lowest order correction to Eq. (3.179) by putting $x_m = (mv_{x0}/b)(1-\delta)$ and solving Eq. (3.175) for δ, assuming $\delta \ll 1$ and $bv_{z0}/mg \gg 1$. [*Hint*: The algebra is not difficult, but you must think carefully about which are the most important terms in this limiting case.]

30. Find the maximum height $z_{\max}$ reached by a projectile whose equation of motion is Eq. (3.169). Expand your result in a power series in b, keeping terms in $z_{\max}$ up to first order in b, and check the lowest order term against Eq. (3.167).

31. A projectile is fired from the origin with initial velocity $\mathbf{v}_0 = (v_{x0}, v_{y0}, v_{z0})$. The wind velocity is $\mathbf{v}_w = w\hat{\mathbf{y}}$. Solve the equations of motion (3.180) for x, y, z as functions of t. Find the point x_1, y_1 at which the projectile will return to the horizontal plane, keeping only first-order terms in b. Show that if air resistance and wind velocity are neglected in aiming the gun, air resistance alone will cause the projectile to fall short of its target a fraction $4bv_{z0}/3mg$ of the target distance, and that the wind causes an additional miss in the y-coordinate of amount $2bwv_{z0}^2/(mg^2)$.

32. Solve for the next term beyond those given in Eqs. (3.176) and (3.178).

33. A projectile is to be fired from the origin in the xz-plane (z-axis vertical) with muzzle velocity v_0 to hit a target at the point $x = x_0$, $z = 0$. (a) Neglecting air resistance, find the correct angle of elevation of the gun. Show that, in general, there are two such angles unless the target is at or beyond the maximum range.

b) Find the first-order correction to the angle of elevation due to air resistance.

34. Show that the forces in Problems 11 and 12 are conservative, find the potential energy, and use it to find the work done in each case.

35. Determine which of the following forces are conservative, and find the potential energy for those which are:

a) $F_x = 6abz^3y - 20bx^3y^2$, $F_y = 6abxz^3 - 10bx^4y$, $F_z = 18abxz^2y$.
b) $F_x = 18abyz^3 - 20bx^3y^2$, $F_y = 18abxz^3 - 10bx^4y$, $F_z = 6abxyz^2$.
c) $\mathbf{F} = \hat{\mathbf{x}}F_x(x) + \hat{\mathbf{y}}F_y(y) + \hat{\mathbf{z}}F_z(z)$.

36. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = 2ax(z^3+y^3)$, $F_y = 2ay(z^3+y^3)+3ay^2(x^2+y^2)$, $F_z = 3az^2(x^2+y^2)$.
b) $F_\rho = a\rho^2 \cos\varphi$, $F_\varphi = a\rho^2 \sin\varphi$, $F_z = 2az^2$.
c) $F_r = -2ar \sin\theta \cos\varphi$, $F_\theta = -ar\cos\theta\cos\varphi$, $F_\varphi = ar\sin\theta\sin\varphi$.

37. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = axe^{-R}$, $F_y = bye^{-R}$, $F_z = cze^{-R}$, where $R = ax^2+by^2+cz^2$.
b) $\boldsymbol{F} = \boldsymbol{A}f(\boldsymbol{A}\cdot\boldsymbol{r})$, where $\boldsymbol{A}$ is a constant vector and $f(s)$ is any suitable function of $s = \boldsymbol{A}\cdot\boldsymbol{r}$.
c) $\boldsymbol{F} = (\boldsymbol{r}\times\boldsymbol{A})f(\boldsymbol{A}\cdot\boldsymbol{r})$.

38. A particle is attracted toward the z-axis by a force $\boldsymbol{F}$ proportional to the square of its distance from the xy-plane and inversely proportional to its distance from the z-axis. Add an additional force perpendicular to $\boldsymbol{F}$ in such a way as to make the total force conservative, and find the potential energy. Be sure to write expressions for the forces and potential energy which are dimensionally consistent.

39. Show that $\boldsymbol{F} = \hat{\boldsymbol{r}}F(r)$ is a conservative force by showing by direct calculation that the integral

$$\int_{r_1}^{r_2} \boldsymbol{F}\cdot d\boldsymbol{r}$$

along any path between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ depends only on r_1 and r_2. [*Hint:* Express $\boldsymbol{F}$ and $d\boldsymbol{r}$ in spherical coordinates.]

40. Find the components of force for the following potential-energy functions:

a) $V = axy^2z^3$.
b) $V = \frac{1}{2}kr^2$.
c) $V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$.

41. Find the force on the electron in the hydrogen molecule ion for which the potential is

$$V = -\frac{e^2}{r_1} - \frac{e^2}{r_2},$$

where r_1 is the distance from the electron to the point $y = z = 0$, $x = -a$, and r_2 is the distance from the electron to the point $y = z = 0$, $x = a$.

42. Devise a potential-energy function which vanishes as $r \to \infty$, and which yields a force $\boldsymbol{F} \doteq -k\boldsymbol{r}$ when $r \to 0$. Find the force. Verify by doing the appropriate line integrals that the work done by this force on a particle going from $\boldsymbol{r} = 0$ to $\boldsymbol{r} = \boldsymbol{r}_0$ is the same if the particle travels in a straight line as it is if it follows the path shown in Fig. 3.32.

43. The potential energy for an isotropic harmonic oscillator is

$$V = \tfrac{1}{2}kr^2.$$

Plot the effective potential energy for the r-motion when a particle of mass m moves with this potential energy and with angular momentum L about the origin. Discuss the types of motion