36. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = 2ax(z^3 + y^3)$, $F_y = 2ay(z^3 + y^3) + 3ay^2(x^2 + y^2)$, $F_z = 3az^2(x^2 + y^2)$.
b) $F_\rho = a\rho^2 \cos\varphi$, $F_\varphi = a\rho^2 \sin\varphi$, $F_z = 2az^2$.
c) $F_r = -2ar \sin\theta \cos\varphi$, $F_\theta = -ar \cos\theta \cos\varphi$, $F_\varphi = ar \sin\theta \sin\varphi$.

37. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = axe^{-R}$, $F_y = bye^{-R}$, $F_z = cze^{-R}$, where $R = ax^2 + by^2 + cz^2$.
b) $\boldsymbol{F} = \boldsymbol{A} f(\boldsymbol{A} \cdot \boldsymbol{r})$, where $\boldsymbol{A}$ is a constant vector and $f(s)$ is any suitable function of $s = \boldsymbol{A} \cdot \boldsymbol{r}$.
c) $\boldsymbol{F} = (\boldsymbol{r} \times \boldsymbol{A}) f(\boldsymbol{A} \cdot \boldsymbol{r})$.

38. A particle is attracted toward the z-axis by a force $\boldsymbol{F}$ proportional to the square of its distance from the xy-plane and inversely proportional to its distance from the z-axis. Add an additional force perpendicular to $\boldsymbol{F}$ in such a way as to make the total force conservative, and find the potential energy. Be sure to write expressions for the forces and potential energy which are dimensionally consistent.

39. Show that $\boldsymbol{F} = \hat{\boldsymbol{r}} F(r)$ is a conservative force by showing by direct calculation that the integral

$$\int_{r_1}^{r_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

along any path between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ depends only on r_1 and r_2. [*Hint:* Express $\boldsymbol{F}$ and $d\boldsymbol{r}$ in spherical coordinates.]

40. Find the components of force for the following potential-energy functions:

a) $V = axy^2z^3$.
b) $V = \frac{1}{2}kr^2$.
c) $V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$.

41. Find the force on the electron in the hydrogen molecule ion for which the potential is

$$V = -\frac{e^2}{r_1} - \frac{e^2}{r_2},$$

where r_1 is the distance from the electron to the point $y = z = 0$, $x = -a$, and r_2 is the distance from the electron to the point $y = z = 0$, $x = a$.

42. Devise a potential-energy function which vanishes as $r \to \infty$, and which yields a force $\boldsymbol{F} \doteq -k\boldsymbol{r}$ when $r \to 0$. Find the force. Verify by doing the appropriate line integrals that the work done by this force on a particle going from $\boldsymbol{r} = 0$ to $\boldsymbol{r} = \boldsymbol{r}_0$ is the same if the particle travels in a straight line as it is if it follows the path shown in Fig. 3.32.

43. The potential energy for an isotropic harmonic oscillator is

$$V = \tfrac{1}{2}kr^2.$$

Plot the effective potential energy for the r-motion when a particle of mass m moves with this potential energy and with angular momentum L about the origin. Discuss the types of motion that are possible, giving as complete a description as is possible without carrying out the solution. Find the frequency of revolution for circular motion and the frequency of small radial oscillations about this circular motion. Hence describe the nature of the orbits which differ slightly from circular orbits.

44. Find the frequency of small radial oscillations about steady circular motion for the effective potential given by Eq. (3.232) for an attractive inverse square law force, and show that it is equal to the frequency of revolution.

45. Find $r(t)$, $\theta(t)$ for the orbit of the particle in Problem 43. Compare with the orbits found in Section 3.10 for the three-dimensional harmonic oscillator.

46. A particle of mass m moves under the action of a central force whose potential is

$$V(r) = Kr^4, \qquad K > 0.$$

For what energy and angular momentum will the orbit be a circle of radius a about the origin? What is the period of this circular motion? If the particle is slightly disturbed from this circular motion, what will be the period of small radial oscillations about $r = a$?

47. According to Yukawa's theory of nuclear forces, the attractive force between a neutron and a proton has the potential

$$V(r) = \frac{Ke^{-\alpha r}}{r}, \qquad K < 0.$$

a) Find the force, and compare it with an inverse square law of force.
b) Discuss the types of motion which can occur if a particle of mass m moves under such a force.
c) Discuss how the motions will be expected to differ from the corresponding types of motion for an inverse square law of force.
d) Find L and E for motion in a circle of radius a.
e) Find the period of circular motion and the period of small radial oscillations.
f) Show that the nearly circular orbits are almost closed when a is very small.

48. Solve the orbital equation (3.222) for the case $F = 0$. Show that your solution agrees with Newton's first law.

49. It will be shown in Chapter 6 (Problem 7) that the effect of a uniform distribution of dust of density ρ about the sun is to add to the gravitational attraction of the sun on a planet of mass m an additional attractive central force

$$F' = -mkr,$$

where

$$k = \frac{4\pi}{3}\rho G.$$

a) If the mass of the sun is M, find the angular velocity of revolution of the planet in a circular orbit of radius r_0, and find the angular frequency of small radial oscillations. Hence show that if F' is much less than the attraction due to the sun, a nearly circular orbit will be approximately an ellipse whose major axis precesses slowly with angular velocity

$$\omega_p = 2\pi\rho\left(\frac{r_0^3 G}{M}\right)^{1/2}.$$

b) Does the axis precess in the same or in the opposite direction to the orbital angular velocity? Look up M and the radius of the orbit of Mercury, and calculate the density of dust required to cause a precession of 41 seconds of arc per century.

50. a) Discuss by the method of the effective potential the types of motion to be expected for an attractive central force inversely proportional to the cube of the radius:

$$F(r) = -\frac{K}{r^3}, \qquad K > 0.$$

b) Find the ranges of energy and angular momentum for each type of motion.

c) Solve the orbital equation (3.222), and show that the solution is one of the forms:

$$\frac{1}{r} = A\cos[\beta(\theta-\theta_0)], \tag{1}$$

$$\frac{1}{r} = A\cosh[\beta(\theta-\theta_0)], \tag{2}$$

$$\frac{1}{r} = A\sinh[\beta(\theta-\theta_0)], \tag{3}$$

$$\frac{1}{r} = A(\theta-\theta_0), \tag{4}$$

$$\frac{1}{r} = \frac{1}{r_0}e^{\pm\beta\theta}. \tag{5}$$

d) For what values of L and E does each of the above types of motion occur? Express the constants A and β in terms of E and L for each case.

e) Sketch a typical orbit of each type.

51. (a) Discuss the types of motion that can occur for a central force

$$F(r) = -\frac{K}{r^2} + \frac{K'}{r^3}.$$

Assume that $K > 0$, and consider both signs for K'.

b) Solve the orbital equation, and show that the bounded orbits have the form (if $L^2 > -mK'$)

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\alpha\theta}.$$

c) Show that this is a precessing ellipse, determine the angular velocity of precession, and state whether the precession is in the same or in the opposite direction to the orbital angular velocity.

52. Sputnik I had a perigee (point of closest approach to the earth) 227 km above the earth's surface, at which point its speed was 28,710 km/hr. Find its apogee (maximum) distance from the earth's surface and its period of revolution. (Assume the earth is a sphere, and neglect air resistance. You need only look up g and the earth's radius to do this problem.)

53. Explorer I had a perigee 360 km and an apogee 2,549 km above the earth's surface. Find its distance above the earth's surface when it passed over a point 90° around the earth from its perigee.

54. A comet is observed a distance of 1.00×10^8 km from the sun, traveling toward the sun with a velocity of 51.6 km per second at an angle of 45° with the radius from the sun. Work out an equation for the orbit of the comet in polar coordinates with origin at the sun and x-axis through the observed position of the comet. (The mass of the sun is 2.00×10^{30} kg.)

55. It can be shown (Chapter 6, Problems 17 and 21) that the correction to the potential energy of a mass m in the earth's gravitational field, due to the oblate shape of the earth, is approximately, in spherical coordinates, relative to the polar axis of the earth,

$$V' = -\frac{\eta m M G R^2}{5r^3}(1-3\cos^2\theta),$$

where M is the mass of the earth and $2R$, $2R(1-\eta)$ are the equatorial and polar diameters of the earth. Calculate the rate of precession of the perigee (point of closest approach) of an earth satellite moving in a nearly circular orbit in the equatorial plane. Look up the equatorial and polar diameters of the earth, and estimate the rate of precession in degrees per revolution for a satellite 400 miles above the earth.

***56.** Calculate the torque on an earth satellite due to the oblateness potential energy correction given in Problem 55. A satellite moves in a circular orbit of radius r whose plane is inclined so that its normal makes an angle α with the polar axis. Assume that the orbit is very little affected in one revolution, and calculate the average torque during a revolution. Show that the effect of such a torque is to make the normal to the orbit precess in a cone of half angle α about the polar axis, and find a formula for the rate of precession in degrees per revolution. Calculate the rate for a satellite 400 miles above the earth, using suitable values for M, η, and R.

57. It can be shown that the orbit given by the special theory of relativity for a particle of mass m moving under a potential energy $V(r)$ is the same as the orbit which the particle would follow according to Newtonian mechanics if the potential energy were

$$V(r) - \frac{[E - V(r)]^2}{2mc^2},$$

where E is the energy (kinetic plus potential), and c is the speed of light. Discuss the nature of the orbits for an inverse square law of force according to the theory of relativity. Show by comparing the orbital angular velocity with the frequency of radial oscillations for nearly circular motion that the nearly circular orbits, when the relativistic correction is small, are precessing ellipses, and calculate the angular velocity of precession. [See Eq. (14.101).]

58. Mars has a perihelion (closest) distance from the sun of 2.06×10^8 km, and an aphelion (maximum) distance of 2.485×10^8 km. Assume that the earth moves in the same plane in a circle of radius 1.49×10^8 km with a period of one year. From this data alone, find the speed of Mars at perihelion. Assume that a Mariner space probe is launched so that its perihelion is at the earth's orbit and its aphelion at the perihelion of Mars. Find the velocity of the Mariner relative to Mars at the point where they meet. Which has the higher velocity? Which has the higher average angular velocity during the period of the flight?

59. Mariner 4 left the earth on an orbit whose perihelion distance from the sun was approximately the distance of the earth (1.49×10^8 km), and whose aphelion distance was approximately the distance of Mars from the sun (2.2×10^8 km). With what velocity did it leave relative to the earth? With what velocity must it leave the earth (relative to the earth) in order to escape altogether from the sun's gravitational pull? (You need no further data to answer this problem except the length of the year, if you assume the earth moves in a circle.)

60. a) A satellite is to be launched from the surface of the earth. Assume the earth is a sphere of radius R, and neglect friction with the atmosphere. The satellite is to be launched at an angle α with the vertical, with a velocity v_0, so as to coast without power until its velocity is horizontal at an altitude h_1 above the earth's surface. A horizontal thrust is then applied by the last stage rocket so as to add an additional velocity Δv_1 to the velocity of the satellite. The final orbit is to be an ellipse with perigee h_1 (point of closest approach) and apogee h_2 (point farthest away) measured from the earth's surface. Find the required initial velocity v_0 and additional velocity Δv_1, in terms of R, α, h_1, h_2, and g, the acceleration of gravity at the earth's surface.

b) Write a formula for the change δh_1 in perigee height due to a small error $\delta\beta$ in the final thrust direction, to order $(\delta\beta)^2$.

61. Two planets move in the same plane in circles of radii r_1, r_2 about the sun. A space probe is to be launched from planet 1 with velocity v_1 relative to the planet, so as to reach the orbit of planet 2. (The velocity v_1 is the relative velocity after the probe has escaped from the gravitational field of the planet.) Show that v_1 is a minimum for an elliptical orbit whose perihelion and aphelion are r_1 and r_2. In that case, find v_1, and the relative velocity v_2 between the space probe and planet 2 if the probe arrives at radius r_2 at the proper time to intercept planet 2. Express your results in terms of r_1, r_2, and the length of the year Y_1 of planet 1. Look up the appropriate values of r_1 and r_2, and estimate v_1 for trips to Venus and Mars from the earth.

62. A rocket is in an elliptical orbit around the earth, perigee r_1, apogee r_2, measured from the center of the earth. At a certain point in its orbit, its engine is fired for a short time so as to give a velocity increment Δv in order to put the rocket on an orbit which escapes from the earth with a final velocity v_0 relative to the earth. (Neglect any effects due to the sun and moon.) Show that Δv is a minimum if the thrust is applied at perigee, parallel to the orbital velocity. Find Δv in that case in terms of the elliptical orbit parameters ε, a, the acceleration g at a distance R from the earth's center, and the final velocity v_0. Can you explain physically why Δv is smaller for larger ε?

63. A satellite moves around the earth in an orbit which passes across the poles. The time at which it crosses each parallel of latitude is measured so that the function $\theta(t)$ is known. Show how to find the perigee, the semimajor axis, and the eccentricity of its orbit in terms of $\theta(t)$, and the value of g at the surface of the earth. Assume the earth is a sphere of radius R.

64. A particle of mass m moves in an elliptical orbit of major axis $2a$, eccentricity ε, in such a way that the radius to the particle from the center of the ellipse sweeps out area at a constant rate

$$\frac{dS}{dt} = C,$$

and with period τ independent of a and ε. (a) Write out the equation of the ellipse in polar coordinates with origin at the center of the ellipse.
b) Show that the force on the particle is a central force, and find $F(r)$ in terms of m, τ.

65. Show that the Rutherford cross-section formula (3.276) holds also when one of the charges is negative.

66. A particle is reflected from the surface of a hard sphere of radius R in such a way that the incident and reflected lines of travel lie in a common plane with the radius to the point of impact and make equal angles with the radius. Find the cross-section $d\sigma$ for scattering through an angle between Θ and $\Theta + d\Theta$. Integrate $d\sigma$ over all angles and show that the total cross-section has the expected value πR^2.

67. Exploit the analogy u, $\theta \leftrightarrow x$, t between Eqs. (3.222) and (2.39) in order to develop a solution of Eq. (3.222) analogous to the solution (2.46) of Eq. (2.39). Use your solution to show that the scattering angle Θ (Fig. 3.42) for a particle subject to a central force $F(r)$ is given by

$$\Theta = \left|\pi - 2s \int_0^{u_0} \left[1 - s^2u^2 - V(u^{-1})/(\tfrac{1}{2}mv_0^2)\right]^{-1/2} du\right|,$$

where $V(r = u^{-1})$ is the potential energy,

$$V(r) = \int_r^\infty F(r)\, dr,$$

s is the impact parameter, and u_0 is the value of u at which the quantity in square brackets vanishes. [This problem is not difficult if you keep clearly in mind the physical and geometrical significance of the various quantities involved at each step in the solution.]

68. Show that a hard sphere as defined in Problem 66 can be represented as a limiting case of a central force where

$$V(r) = \begin{matrix} 0, \text{ if } r > R, \\ \infty, \text{ if } r < R, \end{matrix}$$

that is, show that such a potential gives the same law of reflection as specified in Problem 66. Hence use the result of Problem 67 to solve Problem 66.

69. Use the result of Problem 67 to derive the Rutherford cross-section formula (3.276).

70. A rocket moves with initial velocity v_0 toward the moon of mass M, radius r_0. Find the cross-section σ for striking the moon. Take the moon to be at rest, and ignore all other bodies.

71. Show that for a repulsive central force inversely proportional to the cube of the radius,

$$F(r) = \frac{K}{r^3}, \qquad K > 0,$$

the orbits are of the form (1) given in Problem 50, and express β in terms of K, E, L, and the mass m of the incident particle. Show that the cross-section for scattering through an angle between Θ and $\Theta + d\Theta$ for a particle subject to this force is

$$d\sigma = \frac{2\pi^3 K}{mv_0^2} \frac{\pi - \Theta}{\Theta^2(2\pi - \Theta)^2} d\Theta.$$

72. A particle of charge q, mass m at rest in a constant, uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ is subject, beginning at $t = 0$, to an oscillating electric field

$$\boldsymbol{E} = E_0\hat{\boldsymbol{x}} \sin \omega t.$$

Find its motion.

73. Solve Problem 72 for the case $\omega = qB_0/mc$.

74. A charged particle moves in a constant, uniform electric and magnetic field. Show that if we introduce a new variable

$$\boldsymbol{r}' = \boldsymbol{r} - \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} ct,$$

the equation of motion for $\boldsymbol{r}'$ is the same as that for $\boldsymbol{r}$ except that the component of $\boldsymbol{E}$ perpendicular to $\boldsymbol{B}$ has been eliminated.

75. A particle of charge q in a cylindrical magnetron moves in a uniform magnetic field

$$\boldsymbol{B} = B\hat{\boldsymbol{z}},$$

and an electric field, directed radially outward or inward from a central wire along the z-axis,

$$\boldsymbol{E} = \frac{a}{\rho}\hat{\boldsymbol{\rho}},$$

where ρ is the distance from the z-axis, and $\hat{\boldsymbol{\rho}}$ is a unit vector directed radially outward from the z-axis. The constants a and B may be either positive or negative.

a) Set up the equations of motion in cylindrical coordinates.

b) Show that the quantity

$$m\rho^2\dot{\varphi} + \frac{qB}{2c}\rho^2 = K$$

is a constant of the motion.

c) Using this result, give a qualitative discussion, based on the energy integral, of the types of motion that can occur. Consider all cases, including all values of a, B, K, and E.

d) Under what conditions can circular motion about the axis occur?

e) What is the frequency of small radial oscillations about this circular motion?