i_1, i_2 around the two meshes as shown, we obtain

$$(L+L_1)\ddot{q}_1+(R+R_1)\dot{q}_1+\left(\frac{1}{C}+\frac{1}{C_1}\right)q_1+L\ddot{q}_2+R\dot{q}_2+\frac{1}{C}q_2=0, \tag{4.190}$$

and

$$(L+L_2)\ddot{q}_2+(R+R_2)\dot{q}_2+\left(\frac{1}{C}+\frac{1}{C_2}\right)q_2+L\ddot{q}_1+R\dot{q}_1+\frac{1}{C}q_1=0, \tag{4.191}$$

where q_1 and q_2 are the charges built up on C_1 and C_2 by the mesh currents i_1 and i_2. These equations have the same form as Eqs. (4.188), (4.189), and can be solved by similar methods. In electrical circuits, the damping is often fairly large, and finding the solution becomes a formidable task.

The discussion of this section can be extended to the case of any number of coupled mechanical or electrical harmonic oscillators, with analogous results. The algebraic details become almost prohibitive, however, unless we make use of more advanced mathematical techniques. We therefore postpone further discussion of this problem to Chapter 12.

All mechanical and electrical vibration problems reduce in the limiting case of small amplitudes of vibration to problems involving one or several coupled harmonic oscillators. Problems involving vibrations of strings, membranes, elastic solids, and electrical and acoustical vibrations in transmission lines, pipes, or cavities, can be reduced to problems of coupled oscillators, and exhibit similar normal modes of vibration. The treatment of the behavior of an atom or molecule according to quantum mechanics results in a mathematical problem identical with the problem of coupled harmonic oscillators, in which the energy levels play the role of oscillators, and external perturbing influences play the role of the coupling mechanism.

PROBLEMS

1. Formulate and prove a conservation law for the angular momentum about the origin of a system of particles confined to a plane.

2. Water is poured into a barrel at the rate of 120 lb per minute from a height of 16 ft. The barrel weighs 25 lb, and rests on a scale. Find the scale reading after the water has been pouring into the barrel for one minute.

3. A ballistic pendulum to be used to measure the speed of a bullet is constructed by suspending a block of wood of mass M by a cord of length l. The pendulum initially hangs vertically at rest. A bullet of mass m is fired into the block and becomes imbedded in it. The pendulum then begins to swing and rises until the cord makes a maximum angle θ with the vertical. Find the initial speed of the bullet in terms of M, m, l, and θ by applying appropriate conservation laws.

4. A box of mass m falls on a conveyor belt moving with constant speed v_0. The coefficient of sliding friction between the box and the belt is μ. How far does the box slide along the belt before it is moving with the same speed as the belt? What force F must be applied to the belt to keep it moving at constant speed after the box falls on it, and for how long? Calculate the impulse delivered by this force and check that momentum is conserved between the time before the box falls on the belt and the time when the box is moving with the belt. Calculate the work done by the force F in pulling the belt. Calculate the work dissipated in friction between the box and the belt. Check that the energy delivered to the belt by the force F is just equal to the kinetic energy increase of the box plus the energy dissipated in friction.

5. A scoop of mass m_1 is attached to an arm of length l and negligible weight. The arm is pivoted so that the scoop is free to swing in a vertical arc of radius l. At a distance l directly below the pivot is a pile of sand. The scoop is lifted until the arm is at a 45° angle with the vertical, and released. It swings down and scoops up a mass m_2 of sand. To what angle with the vertical does the arm of the scoop rise after picking up the sand? This problem is to be solved by considering carefully which conservation laws are applicable to each part of the swing of the scoop. Friction is to be neglected, except that required to keep the sand in the scoop.

6. a) A spherical satellite of mass m, radius a, moves with speed $\boldsymbol{v}$ through a tenuous atmosphere of density ρ. Find the frictional force on it, assuming that the speed of the air molecules can be neglected in comparison with $\boldsymbol{v}$, and that each molecule which is struck becomes embedded in the skin of the satellite. Do you think these assumptions are valid?

b) If the orbit is a circle 400 km above the earth (radius 6360 km), where $\rho = 10^{-11}\text{kg/m}^{-3}$, and if $a = 1$ m, $m = 100$ kg, find the change in altitude and the change in period of revolution in one week.

7. A lunar landing craft approaches the moon's surface. Assume that one-third of its weight is fuel, that the exhaust velocity from its rocket engine is 1500 m/sec, and that the acceleration of gravity at the lunar surface is one-sixth of that at the earth's surface. How long can the craft hover over the moon's surface before it runs out of fuel?

8. A toy rocket consists of a plastic bottle partly filled with water containing also air at a high pressure p. The water is ejected through a small nozzle of area A. Calculate the exhaust velocity v by assuming that frictional losses of energy are negligible, so that the kinetic energy of the escaping water is equal to the work done by the gas pressure in pushing it out. Show that the thrust of this rocket engine is then $2\,pA$. (Assume that the water leaves the nozzle of area A with velocity v.) If the empty rocket weighs 500 g, if it contains initially 500 g of water, and if $A = 5\text{ mm}^2$, what pressure is required in order that the rocket can just support itself against gravity? If it is then released so that it accelerates upward, what maximum velocity will it reach? Approximately how high will it go? What effects are neglected in the calculation, and how would each of them affect the final result?

***9.** A two-stage rocket is to be built capable of accelerating a 100-kg payload to a velocity of 6000 m/sec in free flight in empty space (no gravitational field). (In a two-stage rocket, the first stage is detached after exhausting its fuel, before the second stage is fired.) Assume that the fuel used can reach an exhaust velocity of 1500 m/sec, and that structural requirements imply that an empty rocket (without fuel or payload) will weigh 10% as much as the fuel it can carry. Find the optimum choice of masses for the two stages so that the total take-off weight is a minimum. Show that it is impossible to build a single-stage rocket which will do the job.

10. A rocket is to be fired vertically upward. The initial mass is M_0, the exhaust velocity $-u$ is constant, and the rate of exhaust $-(dM/dt) = A$ is constant. After a total mass ΔM is exhausted, the rocket engine runs out of fuel.

a) Neglecting air resistance and assuming that the acceleration g of gravity is constant, set up and solve the equation of motion.

*b) Show that if M_0, u, and ΔM are fixed, then the larger the rate of exhaust A, that is, the faster it uses up its fuel, the greater the maximum altitude reached by the rocket.

11. Assume that essentially all of the mass M of the gyroscope in Fig. 4.1 is concentrated in the rim of the wheel of radius R, and that the center of mass lies on the axis at a distance l from the pivot point Q. If the gyroscope rotates rapidly with angular velocity ω, show that the angular velocity of precession of its axis in a cone making an angle α with the vertical is approximately

$$\omega_p = gl/(R^2\omega^2).$$

12. A diver executing a $2\frac{1}{2}$ flip doubles up with his knees in his arms in order to increase his angular velocity. Estimate the ratio by which he thus increases his angular velocity relative to his angular velocity when stretched out straight with his arms over his head. Explain your reasoning.

13. A uniform spherical planet of radius a revolves about the sun in a circular orbit of radius r_0, and rotates about its axis with angular velocity ω_0, normal to the plane of the orbit. Due to tides raised on the planet by the sun, its angular velocity of rotation is decreasing. Find a formula expressing the orbit radius r as a function of angular velocity ω of rotation at any later or earlier time. [You will need formulas (5.9) and (5.91) from Chapter 5.] Apply your formula to the earth, neglecting the effect of the moon, and estimate how much farther the earth will be from the sun when the day has become equal to the present year. If the effect of the moon were taken into account, would the distance be greater or less?

***14.** A mass m of gas and debris surrounds a star of mass M. The radius of the star is negligible in comparison with the distances to the particles of gas and debris. The material surrounding the star has initially a total angular momentum $\boldsymbol{L}$, and a total kinetic and potential energy E. Assume that $m \ll M$, so that the gravitational fields due to the mass m are negligible in comparison with that of the star. Due to internal friction, the surrounding material continually loses mechanical energy. Show that there is a maximum energy ΔE which can be lost in this way, and that when this energy has been lost, the material must all lie on a circular ring around the star (but not necessarily uniformly distributed). Find ΔE and the radius of the ring. (You will need to use the method of Lagrange multipliers.)

15. A particle of mass m_1, energy T_{1I} collides elastically with a particle of mass m_2, at rest. If the mass m_2 leaves the collision at an angle ϑ_2 with the original direction of motion of m_1, what is the energy T_{2F} delivered to particle m_2? Show that T_{2F} is a maximum for a head-on collision, and that in this case the energy lost by the incident particle in the collision is

$$T_{1I} - T_{1F} = \frac{4m_1 m_2}{(m_1+m_2)^2} T_{1I}.$$

16. A cloud-chamber picture shows the track of an incident particle which makes a collision and is scattered through an angle ϑ_1. The track of the target particle makes an angle ϑ_2 with the direction of the incident particle. Assuming that the collision was elastic and that the target particle was initially at rest, find the ratio m_1/m_2 of the two masses. (Assume small velocities so that the classical expressions for energy and momentum may be used.)

17. A proton of mass m_1 collides elastically with an unknown nucleus in a bubble chamber and is scattered through an angle ϑ_1. The ratio p_{1F}/p_{1I} is determined from the curvature of its initial and final tracks. Find the mass m_2 of the target nucleus. How might it be possible to determine whether the collision was indeed elastic?

18. In an experiment in which particles of mass m_1 collide elastically with stationary particles of mass m_2, it is desired to place a counter in a position where it will count particles which have lost half their initial momentum. At what angle ϑ_1 with the incident beam should the counter be placed? For what range of mass ratios m_1/m_2 does this problem have an answer?

19. Show that an elastic collision corresponds to a coefficient of restitution $e = 1$, that is, show that for a head-on elastic collision between two particles, Eq. (4.85) holds with $e = 1$.

20. Calculate the energy loss $-Q$ for a head-on collision between a particle of mass m_1, velocity v_1 with a particle of mass m_2 at rest, if the coefficient of restitution is e.

21. A particle of mass m_1, momentum p_{1I} collides elastically with a particle of mass m_2, momentum p_{2I} going in the opposite direction. If m_1 leaves the collision at an angle ϑ_1 with its original course, find its final momentum.

22. Find the relativistic corrections to Eq. (4.81) when the incident particle m_1 and the emitted particle m_3 move with speeds near the speed of light. Assume that the recoil particle m_4 is moving slowly enough so that the classical relation between energy and momentum can be used for it.

23. A particle of mass m_1, momentum p_1 collides with a particle of mass m_2 at rest. A reaction occurs from which two particles of masses m_3 and m_4 result, which leave the collision at angles ϑ_3 and ϑ_4 with the original path of m_1. Find the energy Q produced in the reaction in terms of the masses, the angles, and p_1.

24. A nuclear reaction whose Q is known occurs in a photographic plate in which the tracks of the incident particle m_1 and the two product particles m_3 and m_4 can be seen. Find the energy of the incident particle in terms of m_1, m_3, m_4, Q, and the measured angles ϑ_3 and ϑ_4 between the incident track and the two final tracks. What happens if $Q = 0$?

25. A billiard ball sliding on a frictionless table strikes an identical stationary ball. The balls leave the collision at angles $\pm\vartheta$ with the original direction of motion. Show that after the collision the balls must have a rotational energy equal to $1-\frac{1}{2}\cos^{-2}\vartheta$ of the initial kinetic energy, assuming that no energy is dissipated in friction.

26. A neutral particle of unknown momentum and direction produces a reaction in a bubble chamber in which two charged particles of masses m_3, m_4 emerge with momenta p_3, p_4. The angle between their tracks is α. Find the direction and momentum of the incident particle. If the mass m_1 of the incident particle is known or suspected, find the energy Q released in the reaction. (Nonrelativistic velocities.)

27. The Compton scattering of x-rays can be interpreted as the result of elastic collisions between x-ray photons and free electrons. According to quantum theory, a photon of wavelength λ has a kinetic energy hc/λ, and a linear momentum of magnitude h/λ, where h is Planck's constant and c is the speed of light. In the Compton effect, an incident beam of x-rays of known wavelength λ_I in a known direction is scattered in passing through matter, and the scattered radiation at an angle ϑ_1 to the incident beam is found to have a longer wavelength λ_F, which is a function of the angle ϑ_1. Assuming an elastic collision between an incident photon and an electron of mass m at rest, set up the equations expressing conservation of energy and momentum. Use the relativistic expressions for the energy and momentum of the electron. Show that the change in x-ray wavelength is

$$\lambda_F - \lambda_I = \frac{h}{mc}(1-\cos\vartheta_1),$$

and that the ejected electron appears at an angle given by

$$\tan\vartheta_2 = \frac{\sin\vartheta_1}{[1+(h/\lambda_I mc)](1-\cos\vartheta_1)}.$$

28. Work out a correction to Eq. (3.267) which takes into account the motion of the central mass M under the influence of the revolving mass m. A pair of stars revolve about each other, so close together that they appear in the telescope as a single star. It is determined from spectroscopic observations that the two stars are of equal mass and that each revolves in a circle with speed v and period τ under the gravitational attraction of the other. Find the mass m of each star by using your formula.

29. A space ship of mass m, initial velocity $\boldsymbol{v}_0$ approaches the moon and passes by it. The distance of closest approach is R (measured from the center of the moon). The velocity $\boldsymbol{v}_0$ is perpendicular to the orbital velocity $\boldsymbol{V}$ of the moon. Show that if the space ship passes behind the moon, it will gain energy and calculate the increase in its kinetic energy as it leaves the vicinity of the moon. Assume that $M \gg m$, where M is the mass of the moon.

30. A star of mass m, initial speed v_0, approaches a second star of mass $2m$ at rest. The first star travels initially along a line which if continued would pass the second star at a distance s. Find the final speed and direction of motion of each star.

31. Show that if the incident particle is much heavier than the target particle ($m_1 \gg m_2$), the Rutherford scattering cross section [Eq. (3.276)] in laboratory coordinates is approximately

$$d\sigma \doteq \left(\frac{q_1 q_2}{2m_2 v_0^2}\right)^2 \frac{4\gamma^2}{[1-(1-\gamma^2\vartheta_1^2)^{1/2}]^2(1-\gamma^2\vartheta_1^2)^{1/2}} 2\pi \sin\vartheta_1 \, d\vartheta_1$$

if $\gamma\vartheta_1 < 1$, where $\gamma = m_1/m_2$. Otherwise, $d\sigma = 0$.