FI-001 Mecânica Quântica I
Lista # 2

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1. **Grupo Linear Geral em uma dimensão**

   Considere a transformação linear em uma dimensão com dois parâmetros reais (a, b):

   $$\mathbf{R}(a,b)\,|x'> \;\; = \;\; |ax'+b> \quad ,$$

   com $a \neq 0$.

   a) Encontre a ação de $\mathbf{R}$ sobre a função de onda $< x'|\alpha >= \psi_\alpha(x')$ e pesquise se o operador $\mathbf{R}$ é unitário.

   b) Mostre que as transformações $\{\mathbf{R}\}$ formam um grupo. Diga se ele é Abeliano (comutativo) ou não.

   c) Encontre a transformação infinitesimal e os correspondentes *geradores infinitesimais*. Quantos geradores temos?

   d) Encontre as relações de comutação entre os geradores.

   ■

2. **Pacote de ondas Gaussiano**

   Em 1-dim, considere a função de onda

   $$< x'|\alpha > \;\; = \;\; \psi_\alpha(x') = \left[\frac{1}{\pi^{1/4} d^{1/2}}\right] \exp\left[ikx' - \frac{x'^2}{2d^2}\right] .$$

   a) Verifique que a função de onda está normalizada. Em caso contrário, encontre o fator apropriado de normalização.

   b) Para a função de onda acima, encontre os valores médios de $x,\ x^2,\ p$ e p^2.

   c) Calcule as dispersões $< (\Delta x)^2 >$ e $< (\Delta p)^2 >$ e verifique que este pacote satisfaz a igualdade no Teorema sobre as Relações de Incerteza. Este cálculo será útil para responder a parte *c)* do problema **18**, do Cap. 1, do livro do Sakurai.

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3. **Transformada de Fourier**

   A função de onda dada no problema anterior (pacote Gaussiano), é a representação do ket $|\alpha >$ na base do operador de posição. Expresse o mesmo ket na base do operador momentum.

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4. **Representação de Coordenadas**

   Para dim-1, escreva os *auto kets* dos operadores de posição e momentum na representação de coordenadas e obtenha as correspondentes funções de onda.

   a) Verifique que as funções de onda são mutuamente ortonormais, no sentido da *delta* de Dirac.

   b) Considere o caso quando é feita uma medição da posição da partícula com resultado x_0. Qual é nesse momento sua função de onda? O que se espera em uma medição imediatamente posterior do momentum ?

   ■

5. **Identidade de Baker-Haussdorff**

   Sejam A e B dois observáveis **não** compatíveis. Mostre que se tem a identidade

$$\exp(A)\ B\ \exp(-A) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + ...$$

   Dica. Considere o operador

$$F(\lambda) \equiv \exp(\lambda A)\ B\ \exp(-\lambda A)\ ,$$

   função de um parâmetro λ e obtenha a série de Taylor de F com centro $\lambda = 0$, calculando as sucessivas derivadas de F em relação a λ.

   Discuta o caso de A e B serem compatíveis. Seja agora $B = x$, o operador de posição e $\mathcal{T}(a_0)$ o operador de uma translação finita em a_0 $(1 - \text{dim})$. Encontre

$$\mathcal{T}(a_0)\ B\ \mathcal{T}^{\dagger}(a_0)$$

   e interprete o resultado.

   ■

6. **Problemas de #17 a #33 do livro do Sakurai, Cap. 1 (ver anexo).** ■

17. Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0,$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0, \qquad [A_2, H] = 0.$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = \mathbf{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

18. a. The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

b. Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A|\alpha\rangle = \lambda\,\Delta B|\alpha\rangle$$

with λ purely *imaginary*.

c. Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}.$$

Prove that the requirement

$$\langle x'|\Delta x|\alpha\rangle = (\text{imaginary number})\langle x'|\Delta p|\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

19. a. Compute

$$\langle(\Delta S_x)^2\rangle \equiv \langle S_x^2\rangle - \langle S_x\rangle^2,$$

where the expectation value is taken for the $S_z +$ state. Using your result, check the generalized uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2,$$

with $A \to S_x$, $B \to S_y$.

b. Check the uncertainty relation with $A \to S_x$, $B \to S_y$ for the $S_x +$ state.

20. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the

uncertainty product

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

21. Evaluate the *x-p* uncertainty product $\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle$ for a one-dimensional particle confined between two rigid walls

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Do this for both the ground and excited states.

22. Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it *in seconds*.

23. Consider a three-dimensional ket space. If a certain set of orthonormal kets—say, $|1\rangle$, $|2\rangle$, and $|3\rangle$—are used as the base kets, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

a. Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

b. Show that A and B commute.

c. Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

24. a. Prove that $(1/\sqrt{2})(1 + i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

b. Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

25. Some authors define an *operator* to be real when every member of its matrix elements $\langle b'|A|b''\rangle$ is real in some representation ($\{|b'\rangle\}$ basis in this case). Is this concept representation independent, that is, do the

matrix elements remain real even if some basis other than $\{|b'\rangle\}$ is used? Check your assertion using familiar operators such as S_y and S_z (see Problem 24) or x and p_x.

26. Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|.$$

27. a. Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A|a'\rangle = a'|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.

    b. Using the continuum analogue of the result obtained in (a), evaluate

$$\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle.$$

    Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are *operators*.

28. a. Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{\text{classical}}.$$

    b. Let x and p_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right].$$

    c. Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \qquad (x|x'\rangle = x'|x'\rangle)$$

    is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?

29. a. On page 247, Gottfried (1966) states that

$$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \qquad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

    can be "easily derived" from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

    b. Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classical}}$.

30. The translation operator for a finite (spatial) displacement is given by

$$\mathscr{T}(\mathbf{l}) = \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right),$$

where **p** is the momentum *operator*.

a. Evaluate

$$[x_i, \mathscr{T}(\mathbf{l})].$$

b. Using (a) (or otherwise), demonstrate how the expectation value $\langle \mathbf{x} \rangle$ changes under translation.

31. In the main text we discussed the effect of $\mathscr{T}(d\mathbf{x}')$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle \mathbf{x} \rangle$ and $\langle \mathbf{p} \rangle$ under infinitesimal translation. Using (1.6.25), (1.6.45), and $|\alpha\rangle \to \mathscr{T}(d\mathbf{x}')|\alpha\rangle$ only, prove $\langle \mathbf{x} \rangle \to \langle \mathbf{x} \rangle + d\mathbf{x}', \langle \mathbf{p} \rangle \to \langle \mathbf{p} \rangle$ under infinitesimal translation.

32. a. Verify (1.7.39a) and (1.7.39b) for the expectation value of p and p^2 from the Gaussian wave packet (1.7.35).

b. Evaluate the expectation value of p and p^2 using the momentum-space wave function (1.7.42).

33. a. Prove the following:

$$\text{(i)} \qquad \langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle,$$

$$\text{(ii)} \qquad \langle \beta|x|\alpha\rangle = \int dp' \, \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p'),$$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

b. What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right),$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.