FI-001 Mecânica Quântica I
Lista # 3

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1 **Partícula de spin 1/2 em um campo magnético**

Considere um elétron de spin 1/2 na presença de um campo magnético. Sabemos que o Hamiltoniano do sistema é dado por:

$$\mathcal{H} = -\gamma \left(\vec{B} \cdot \vec{S}\right) = -\gamma \left(B_x S_x + B_y S_y + B_z S_z\right) \qquad ,$$

onde

$$\gamma = \frac{e}{mc}$$

é a *razão giromagnética* e $\vec{S}$ é o operador do spin. Na representação que diagonaliza S_z, as três componentes do spin têm matrizes:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

i) Calcule $\mathcal{H}^2$ e as energias possíveis do sistema, para uma orientação arbitrária do campo magnético.

ii) Encontre explicitamente os autoestados de $\mathcal{H}$ na base que digonaliza S_z.

iii) Suponha agora que o sistema está inicialmente em um dos autoestados de S_z. Encontre as probabilidades de medir as diferentes energias possíveis (estude o problema para as duas autofunções de S_z).

iv) Calcule a matriz do operador de evolução temporal

$$U(t, 0)$$

na mesma base que diagonaliza S_z.

v) Suponha agora que no tempo $t = 0$, o sistema está em um autoestado de S_z, com projeção do spin bem definida ($\sigma =\uparrow$ou $\downarrow$). Encontre o estado do sistema para um instante posterior t arbitrário, supondo que nenhuma medição foi feita sobre ele.

vi) Usando os resultados do ponto anterior, determine as probabilidades de encontrar os distintos valores de S_z no tempo t. Quando é que achamos os mesmos valores para essas probabilidades em t e em 0 ?

□

2 Versão de Dirac da MQ

Considere o caso em que o operador *Hamiltoniano* pode ser separado na forma

$$\mathcal{H} = \mathcal{H}_0 + V \ ,$$

com

$$[\mathcal{H}_0, V] \neq 0 \ .$$

Definimos uma transformação unitária por

$$U_D(t) \equiv \exp\left(-\frac{i}{\hbar}\mathcal{H}_0 \ t\right)$$

que define a chamada versão de *Dirac* ou *Representação de Interação* da Mecânica Quântica. Transformamos *kets-estados* e operadores dinâmicos a partir da versão de Schrödinger por:

$$|\alpha, t_0; t >_D = U_D^\dagger(t) \ |\alpha, t_0; t >_S$$

$$A^D = \ U_D^\dagger(t) \cdot A^S \cdot U_D(t)$$

Encontre as equações dinâmicas que são satisfeitas por *kets-estados* e operadores na versão de Dirac.

□

3 Estados Coerentes do Oscilador Harmônico Simples (OHS).

Uma das maneiras possíveis de definir um estado coerente do OHS é como sendo um autoestado do operador de destruição:

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle \quad ,$$

onde λ é, em geral, um número complexo.

i) Mostre que o estado

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(\lambda a^\dagger\right)|0\rangle$$

é um estado coerente normalizado.

ii) Mostre que este estado minimiza a relação de incerteza para todo tempo, isto é:

$$< (\Delta x)^2 >< (\Delta p_x)^2 >= \frac{\hbar^2}{4} \ .$$

O estado coerente é portanto o estado do OHS que está mais próximo do limite clássico. O estado fundamental é um caso particular de um estado coerente, com $\lambda = 0$.

iii) Escreva o estado como

$$|\lambda\rangle = \sum_n \ f(n) \ \ |n\rangle$$

e mostre que a distribuição $|f(n)|^2$ como função do número quântico n, é de tipo Poisson. Encontre o valor mais provável de n e portanto da energia. Esse valor da energia coincide com seu valor médio.

iv) Mostre que obtemos um estado coerente aplicando o operador de traslação espacial sobre o estado fundamental do oscilador harmônico. Mostre também que o estado coerente mais geral é obtido pela ação do operador *deslocamento* dado abaixo:

$$D(\lambda) = \exp\left(\lambda a^\dagger - \lambda^* a\right) \ .$$

v) Na versão de Schrödinger, mostre que o Teorema de Ehrenfest (onde as médias são tomadas para estados coerentes) fornece a equação clássica de movimento.

vi) Um estado obtido como a combinação linear

$$|\psi\rangle = A\left(|\lambda\rangle + |-\lambda\rangle\right) \ ,$$

onde $|\lambda\rangle$ é um estado coerente, é chamado de "*Gato de Schrödinger Par*". Calcule a constante de normalização A, notando que os estados coerentes acima não são ortogonais. Calcule a média do Operador Número e mostre que o *Gato* apresenta oscilações na distribuição de número, sendo nula a probabilidade de observar um estado com um número ímpar de quanta (interferência destrutiva) e que para um número par temos interferência construtiva (dai o nome de *Gato Par*). □

4 Oscilador Harmônico na representação de momentum

Para o oscilador harmônico $1-\dim$, gere todas as funções de onda na representação de momentum a partir da função de onda do estado fundamental. Proceda de maneira análoga a como feito na representação de coordenadas. □

5 Paradoxo

Para matrizes A e B de dimensão finita, sabemos que $Tr\left([A,B]\right) = 0$. Pareceria então, que tomando o traço da relação fundamental de comutação

$$[\mathbf{x}, \mathbf{p}] = i\hbar \ ,$$

deveriamos ter necessariamente que $\hbar = 0$. Para discutir este paradoxo, use as matrizes de dimensão infinita de $\mathbf{x}$ e $\mathbf{p}$ no caso do oscilador harmônico. Calcule as matrizes de $\mathbf{xp}$ e $\mathbf{px}$ e explique porque a conclusão de $\hbar = 0$ não é válida (problema **6.3** do livro *Quantum Mechanics* de Leslie Ballentine). □

6 Livro do Sakurai

Problemas # 1-19 do Cap.2 do livro *MQM* do Sakurai, Revised Edition (ver anexo abaixo).

□

PROBLEMS

1. Consider the spin-precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z,$$

write the Heisenberg equations of motion for the time-dependent operators $S_x(t)$, $S_y(t)$, and $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as functions of time.

2. Look again at the Hamiltonian of Chapter 1, Problem 11. Suppose the typist made an error and wrote H as

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (You may assume $H_{11} = H_{22} = 0$ for simplicity.)

3. An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-axis.
   a. Obtain the probability for finding the electron in the $s_x = \hbar/2$ state as a function of time.
   b. Find the expectation value of S_x as a function of time.
   c. For your own peace of mind show that your answers make good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

4. Let $x(t)$ be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate

$$[x(t), x(0)].$$

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*Empirically the equality in magnitude between the electron charge and the proton charge is established to an accuracy of four parts in 10^{19}.

5. Consider a particle in one dimension whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x).$$

By calculating $[[H, x], x]$ prove

$$\sum_{a'} |\langle a''|x|a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$

where $|a'\rangle$ is an energy eigenket with eigenvalue $E_{a'}$.

6. Consider a particle in three dimensions whose Hamiltonian is given by

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}).$$

By calculating $[\mathbf{x}\cdot\mathbf{p}, H]$ obtain

$$\frac{d}{dt}\langle \mathbf{x}\cdot\mathbf{p}\rangle = \left\langle \frac{\mathbf{p}^2}{m} \right\rangle - \langle \mathbf{x}\cdot\nabla V\rangle.$$

To identify the preceding relation with the quantum-mechanical analogue of the virial theorem it is essential that the left-hand side vanish. Under what condition would this happen?

7. Consider a free-particle wave packet in one dimension. At $t = 0$ it satisfies the minimum uncertainty relation

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4} \quad (t = 0).$$

In addition, we know

$$\langle x\rangle = \langle p\rangle = 0 \quad (t = 0).$$

Using the Heisenberg picture, obtain $\langle(\Delta x)^2\rangle_t$ as a function of $t (t \geq 0)$ when $\langle(\Delta x)^2\rangle_{t=0}$ is given. (*Hint:* Take advantage of the property of the minimum-uncertainty wave packet you worked out in Chapter 1, Problem 18).

8. Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a' and a'', respectively $(a' \neq a'')$. The Hamiltonian operator is given by

$$H = |a'\rangle\delta\langle a''| + |a''\rangle\delta\langle a'|,$$

where δ is just a real number.

a. Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?

b. Suppose the system is known to be in state $|a'\rangle$ at $t = 0$. Write down the state vector in the Schrödinger picture for $t > 0$.

c. What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in state $|a'\rangle$ at $t = 0$?

d. Can you think of a physical situation corresponding to this problem?

9. A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle(|L\rangle)$, where we have neglected spatial variations within each half of the box. The most general state vector can then be written as

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle,$$

where $\langle R|\alpha\rangle$ and $\langle L|\alpha\rangle$ can be regarded as "wave functions." The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian

$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|),$$

where Δ is a real number with the dimension of energy.

a. Find the normalized energy eigenkets. What are the corresponding energy eigenvalues?

b. In the Schrödinger picture the base kets $|R\rangle$ and $|L\rangle$ are fixed, and the state vector moves with time. Suppose the system is represented by $|\alpha\rangle$ as given above at $t = 0$. Find the state vector $|\alpha, t_0 = 0;\ t\rangle$ for $t > 0$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.

c. Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

d. Write down the coupled Schrödinger equations for the wave functions $\langle R|\alpha, t_0 = 0; t\rangle$ and $\langle L|\alpha, t_0 = 0; t\rangle$. Show that the solutions to the coupled Schrödinger equations are just what you expect from (b).

e. Suppose the printer made an error and wrote H as

$$H = \Delta|L\rangle\langle R|.$$

By explicitly solving the most general time-evolution problem with this Hamiltonian, show that probability conservation is violated.

10. Using the one-dimensional simple harmonic oscillator as an example, illustrate the difference between the Heisenberg picture and the Schrödinger picture. Discuss in particular how (a) the dynamic variables x and p and (b) the most general state vector evolve with time in each of the two pictures.

11. Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by

$$\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle,$$

where p is the momentum operator and a is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$.

12. a. Write down the wave function (in coordinate space) for the state specified in Problem 11 at $t = 0$. You may use

$$\langle x'|0\rangle = \pi^{-1/4} x_0^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right], \quad \left(x_0 \equiv \left(\frac{\hbar}{m\omega}\right)^{1/2}\right).$$

b. Obtain a simple expression for the probability that the state is found in the ground state at $t = 0$. Does this probability change for $t > 0$?

13. Consider a one-dimensional simple harmonic oscillator.

a. Using

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \pm \frac{ip}{m\omega}\right), \quad \left.\begin{matrix} a|n\rangle \\ a^\dagger|n\rangle \end{matrix}\right\} = \left\{\begin{matrix} \sqrt{n}\,|n-1\rangle \\ \sqrt{n+1}\,|n+1\rangle, \end{matrix}\right.$$

evaluate $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|\{x, p\}|n\rangle$, $\langle m|x^2|n\rangle$, and $\langle m|p^2|n\rangle$.

b. Check that the virial theorem holds for the expectation values of the kinetic and the potential energy taken with respect to an energy eigenstate.

14. a. Using

$$\langle x'|p'\rangle = (2\pi\hbar)^{-1/2} e^{ip'x'/\hbar} \qquad \text{(one dimension)}$$

prove

$$\langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'}\langle p'|\alpha\rangle.$$

b. Consider a one-dimensional simple harmonic oscillator. Starting with the Schrödinger equation for the state vector, derive the Schrödinger equation for the *momentum-space* wave function. (Make sure to distinguish the operator p from the eigenvalue p'.) Can you guess the energy eigenfunctions in momentum space?

15. Consider a function, known as the **correlation function**, defined by

$$C(t) = \langle x(t)x(0)\rangle,$$

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one-dimensional simple harmonic oscillator.

16. Consider again a one-dimensional simple harmonic oscillator. Do the following algebraically, that is, without using wave functions.

a. Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.

b. Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$ using (i) the Schrödinger picture and (ii) the Heisenberg picture.

c. Evaluate $\langle(\Delta x)^2\rangle$ as a function of time using either picture.

17. Show for the one-dimensional simple harmonic oscillator

$$\langle 0|e^{ikx}|0\rangle = \exp\left[-k^2\langle 0|x^2|0\rangle/2\right],$$

where x is the position *operator*.

18. A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle,$$

where λ is, in general, a complex number.

a. Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$$

is a normalized coherent state.

b. Prove the minimum uncertainty relation for such a state.

c. Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, hence of E.

d. Show that a coherent state can also be obtained by applying the translation (finite-displacement) operator $e^{-ipl/\hbar}$ (where p is the momentum operator, and l is the displacement distance) to the ground state. (See also Gottfried 1966, 262–64.)

19. Let

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \qquad J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \qquad N = a_+^\dagger a_+ + a_-^\dagger a_-$$

where $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of two *independent* simple harmonic oscillator satisfying the usual simple harmonic oscillator commutation relations. Prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [\mathbf{J}^2, J_z] = 0, \qquad \mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right)N\left[\left(\frac{N}{2}\right)+1\right].$$