FI-001 Mecânica Quântica I
Lista # 4

Prof. G. Cabrera

1. **Ação Clássica em 1-dimensão**

   Avalie a Ação das trajetórias clássicas para a partícula livre e para uma partícula na presença de um potencial de oscilador harmônico. Suponha que os pontos inicial e final da trajetória são respectivamente $(x_0, t_0 = 0)$ e (x, t). A Ação precisa ser escrita em termos desses pontos.

   Resposta :

$$
\begin{aligned}
S_{livre} &= \frac{m(x-x_0)^2}{2t} \quad , \\
S_{osc} &= \frac{m\omega}{2\sin\omega t}\left[\left(x^2+x_0^2\right)\cos\omega t - 2xx_0\right] \quad .
\end{aligned}
$$

   ■

2. **Uma maneira alternativa para calcular o Propagador sem integrais de trajetória**

   Neste ponto, o objetivo é obter uma equação diferencial para o Propagador que seja de fácil integração. Usamos a notação seguinte para o Propagador:

$$
K(x', t;\ x_0, t_0 = 0) = < x' | \exp\{\frac{i}{\hbar}\mathcal{H}t\} | x_0 > \quad .
$$

   Mostre que temos a identidade:

$$
< x' | x_H(-t) \exp\{-\frac{i}{\hbar}\mathcal{H}t\} | x_0 >= x_0 K(x', t;\ x_0, t_0 = 0) \quad ,
$$

   onde $x_H(-t)$ é o operador posição na versão de Heisenberg retrasado no tempo. Este último operador pode ser obtido a partir da integração das equações de Heisenberg. Mostre que depois da feito isso obtemos uma equação diferencial para o Propagador de primeira ordem na coordenada. Integre esta equação para os casos da partícula livre e do oscilador harmônico. Lembre que para estados estacionários reais, o Propagador é simétrico nas variáveis (x', x_0). Para obter o pre-fator do propagador, use a propriedade de **composição**, isto é:

$$
K(x_2, t_2; x_1, t_1) = \int dx' \ \ K(x_2, t_2; x', t')\ K(x', t'; x_1, t_1) \quad . \qquad ■
$$

3. **Propagador em 1-dimensão**

Mostre que o Propagador, para potenciais quadráticos

$$V(x) = a + bx + \frac{1}{2}x^2 \ ,$$

que inclui os casos da partícula livre e do oscilador harmônico, é obtido apenas pela contribuição da trajetória clássica. O resultado é obtido expandindo a Ação em torno da trajetória clássica. Mostre que em todos casos o Propagador pode ser escrito como:

$$K(x,t;x_0,t_0) = F(t-t_0)\exp\left[i\frac{S_{class}}{\hbar}\right] \ ,$$

onde S_{class} é a Ação da trajetória clássica calculada no problema 1. O prefator $F(t-t_0)$ é função apenas do intervalo de tempo e foi calculado no problema # 2.

Resposta :

$$F_{livre} = \sqrt{\frac{m}{2\pi\hbar(t-t_0)}} \ ,$$
$$F_{osc} = \left[\frac{m\omega}{2\pi i\hbar\sin\omega(t-t_0)}\right]^{1/2} \ .$$

■

4. **Versão escalar do Efeito de Aharonov-Bohm**

Considere a experiência de Young das duas fendas, mas onde as fendas são substituídas por longos tubos condutores (ver Fig.1).

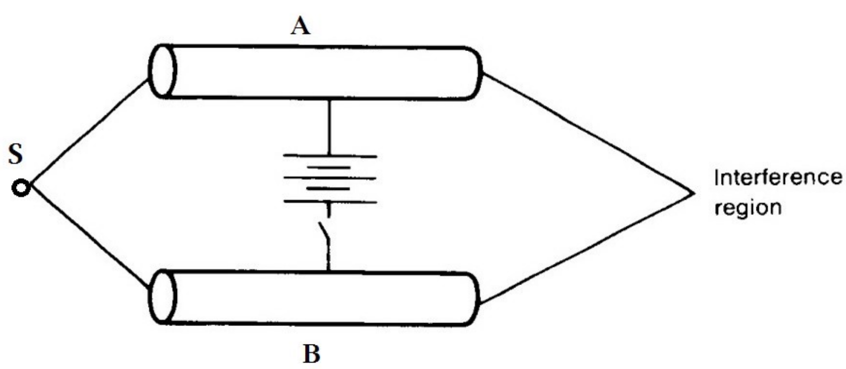


Fig 1 Aharonov-Bohm escalar

A fonte emisora S emite partículas carregadas na forma de pacotes de ondas bem definidos, de maneira que, para tubos suficientemente longos, podemos ter certeza que durante um tempo Δt a partícula está dentro do tubo. Durante este tempo, uma diferença de potencial V_A é aplicada no tubo A e V_B no tubo B. O potencial é espacialmente uniforme em cada tubo e portanto a partícula não experimenta nenhuma força dentro, pois o campo elétrico é nulo. No restante do tempo não existe nenhuma tensão nos tubos. Descreva como o diagrama de interferência na tela é afetado pelas tensões V_A e V_B.

■

5. **Potenciais puramente de "Gauge"**

   Considere uma partícula com carga e na presença dos potenciais eletromagnéticos abaixo

$$\vec{A}(\vec{r},t) = -\nabla\Lambda(\vec{r},t) \ , \qquad \varphi(\vec{r},t) = \frac{1}{c}\,\partial_t\Lambda(\vec{r},t) \ ,$$

   onde $\Lambda(\vec{r},t)$ é uma função escalar arbitrária. Fala-se que estes potenciais são *puramente de gauge*.

   a) Quais são os campos eletromagnéticos descritos pelos potenciais acima?
   b) Calcule o Propagador da partícula na presença dos potenciais acima. Expresse seu resultado em termos do Propagador na ausência dos potenciais.

■

6. **Propagador do Oscilador Harmônico,** $1-\mathrm{dim}$

   O chamado 'Kernel de Mehler' é uma função $M(x,y;\rho)$ que pode ser expandida em série de polinômios de Hermite, na forma

$$\begin{aligned} M(x,y;\rho) &= \frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{\rho^2\left(x^2+y^2\right)-2\rho xy}{\left(1-\rho^2\right)}\right\} = \\ &= \sum_{n=0}^{\infty}\frac{(\rho/2)^n}{n!}H_n\left(x\right)H_n\left(y\right) \ , \end{aligned} \tag{1}$$

   onde os H_n são os polinômios de Hermite, como definidos em Mecânica Quântica.

   Calcule o propagador do Oscilador Harmônico em forma fechada, usando a identidade (1), compare com a solução do problema # 1 da presente lista e discuta o resultado. ■

7. **Exercícios do livro do Sakurai.**

   Problemas # 20-37 do Cap.2 do livro *MQM* do Sakurai, Revised Edition (ver anexo abaixo). ■

17. Show for the one-dimensional simple harmonic oscillator

$$\langle 0|e^{ikx}|0\rangle = \exp\left[-k^2\langle 0|x^2|0\rangle/2\right],$$

where x is the position *operator*.

18. A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle,$$

where λ is, in general, a complex number.

a. Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$$

is a normalized coherent state.

b. Prove the minimum uncertainty relation for such a state.

c. Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, hence of E.

d. Show that a coherent state can also be obtained by applying the translation (finite-displacement) operator $e^{-ipl/\hbar}$ (where p is the momentum operator, and l is the displacement distance) to the ground state. (See also Gottfried 1966, 262–64.)

19. Let

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \qquad J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \qquad N = a_+^\dagger a_+ + a_-^\dagger a_-$$

where $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of two *independent* simple harmonic oscillator satisfying the usual simple harmonic oscillator commutation relations. Prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [\mathbf{J}^2, J_z] = 0, \qquad \mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right)N\left[\left(\frac{N}{2}\right)+1\right].$$

20. Consider a particle of mass m subject to a one-dimensional potential of the following form:

$$V = \begin{cases} \frac{1}{2}kx^2 & \text{for } x > 0 \\ \infty & \text{for } x < 0. \end{cases}$$

a. What is the ground-state energy?

b. What is the expectation value $\langle x^2\rangle$ for the ground state?

21. A particle in one dimension is trapped between two rigid walls:

$$V(x)=\begin{cases}0, & \text{for } 0<x<L\\ \infty, & \text{for } x<0, \quad x>L.\end{cases}$$

At $t=0$ it is known to be exactly at $x=L/2$ with certainty. What are the *relative* probabilities for the particle to be found in various energy eigenstates? Write down the wave function for $t \geq 0$. (You need not worry about absolute normalization, convergence, and other mathematical subtleties.)

22. Consider a particle in one dimension bound to a fixed center by a δ-function potential of the form

$$V(x)=-v_0\delta(x), \quad (v_0 \text{ real and positive}).$$

Find the wave function and the binding energy of the ground state. Are there excited bound states?

23. A particle of mass m in one dimension is bound to a fixed center by an attractive δ-function potential:

$$V(x)=-\lambda\delta(x), \quad (\lambda>0).$$

At $t=0$, the potential is suddenly switched off (that is, $V=0$ for $t>0$). Find the wave function for $t>0$. (Be quantitative! But you need not attempt to evaluate an integral that may appear.)

24. A particle in one dimension $(-\infty<x<\infty)$ is subjected to a constant force derivable from

$$V=\lambda x, \quad (\lambda>0).$$

a. Is the energy spectrum continuous or discrete? Write down an approximate expression for the energy eigenfunction specified by E. Also sketch it crudely.

b. Discuss briefly what changes are needed if V is replaced by

$$V=\lambda|x|.$$

25. Consider an electron confined to the *interior* of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho=\rho_a$ and ρ_b, and also at the top and bottom, $z=0$ and L.

a. Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn}=\left(\frac{\hbar^2}{2m_e}\right)\left[k_{mn}^2+\left(\frac{l\pi}{L}\right)^2\right] \quad (l=1,2,3,\ldots,\ m=0,1,2,\ldots),$$

where k_{mn} is the nth root of the transcendental equation

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a)-N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a)=0.$$

b. Repeat the same problem when there is a uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ for $0<\rho<\rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

c. Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \quad (N = 0, \pm 1, \pm 2, \ldots).$$

26. Consider a particle moving in one dimension under the influence of a potential $V(x)$. Suppose its wave function can be written as $\exp[iS(x,t)/\hbar]$. Prove that $S(x,t)$ satisfies the classical Hamilton-Jacobi equation to the extent that $\hbar$ can be regarded as small in some sense. Show how one may obtain the correct wave function for a plane wave by starting with the solution of the classical Hamilton-Jacobi equation with $V(x)$ set equal to zero. Why do we get the exact wave function in this particular case?
27. Using spherical coordinates, obtain an expression for $\mathbf{j}$ for the ground and excited states of the hydrogen atom. Show, in particular, that for $m_l \neq 0$ states, there is a circulating flux in the sense that $\mathbf{j}$ is in the direction of increasing or decreasing ϕ, depending on whether m_l is positive or negative.
28. Derive (2.5.16) and obtain the three-dimensional generalization of (2.5.16).
29. Define the partition function as

$$Z = \int d^3x' K(\mathbf{x}', t; \mathbf{x}', 0)|_{\beta = it/\hbar},$$

as in (2.5.20)–(2.5.22). Show that the ground-state energy is obtained by taking

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta}, \quad (\beta \to \infty).$$

Illustrate this for a particle in a one-dimensional box.
30. The propagator in momentum space analogous to (2.5.26) is given by $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$. Derive an explicit expression for $\langle \mathbf{p}'', t|\mathbf{p}', t_0\rangle$ for the free-particle case.
31. a. Write down an expression for the classical action for a simple harmonic oscillator for a finite time interval.
    b. Construct $\langle x_n, t_n|x_{n-1}, t_{n-1}\rangle$ for a simple harmonic oscillator using Feynman's prescription for $t_n - t_{n-1} = \Delta t$ small. Keeping only terms up to order $(\Delta t)^2$, show that it is in complete agreement with the $t - t_0 \to 0$ limit of the propagator given by (2.5.26).
32. State the Schwinger action principle (see Finkelstein 1973, 155). Obtain the solution for $\langle x_2 t_2|x_1 t_1\rangle$ by integrating the Schwinger principle and compare it with the corresponding Feynman expression for $\langle x_2 t_2|x_1 t_1\rangle$. Describe the classical limits of these two expressions.

33. Show that the wave-mechanics approach to the gravity-induced problem discussed in Section 2.6 also leads to phase-difference expression (2.6.17).
34. a. Verify (2.6.25) and (2.6.27).
    b. Verify continuity equation (2.6.30) with **j** given by (2.6.31).
35. Consider the Hamiltonian of a spinless particle of charge e. In the presence of a static magnetic field, the interaction terms can be generated by

$$\mathbf{p}_{\text{operator}} \to \mathbf{p}_{\text{operator}} - \frac{e\mathbf{A}}{c},$$

where **A** is the appropriate vector potential. Suppose, for simplicity, that the magnetic field **B** is uniform in the positive z-direction. Prove that the above prescription indeed leads to the correct expression for the interaction of the orbital magnetic moment $(e/2mc)\mathbf{L}$ with the magnetic field **B**. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on its physical significance.
36. An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$).
    a. Evaluate

$$[\Pi_x, \Pi_y],$$

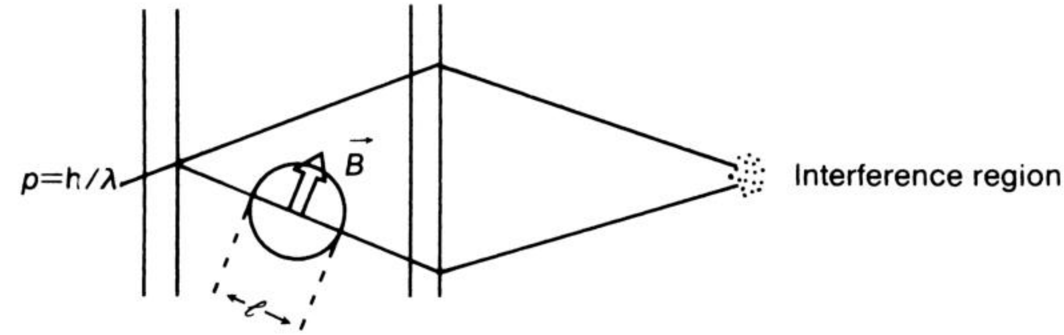
where

$$\Pi_x \equiv p_x - \frac{eA_x}{c}, \qquad \Pi_y \equiv p_y - \frac{eA_y}{c}.$$

    b. By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right),$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.
37. Consider the neutron interferometer.



Prove that the difference in the magnetic fields that produce two successive maxima in the counting rates is given by

$$\Delta B = \frac{4\pi\hbar c}{|e|g_n \bar{\lambda} l},$$

where g_n $(=-1.91)$ is the neutron magnetic moment in units of $-e\hbar/2m_n c$. [If you had solved this problem in 1967, you could have published it in *Physical Review Letters*!]