ON ANGULAR MOMENTUM

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The commutation relations of an arbitrary angular momentum vector can be reduced to those of the harmonic oscillator. This provides a powerful method for constructing and developing the properties of angular momentum eigenvectors. In this paper many known theorems are derived in this way, and some new results obtained. Among the topics treated are the properties of the rotation matrices; the addition of two, three, and four angular momenta; and the theory of tensor operators.

1. INTRODUCTION

One of the methods of treating a general angular momentum in quantum mechanics is to regard it as the superposition of a number of elementary "spins," or angular momenta with \( j = \frac{1}{2} \). Such a spin assembly, considered as a Bose-Einstein system, can be usefully discussed by the method of second quantization. We shall see that this procedure unites the compact symbolism of the group theoretical approach with the explicit operator techniques of quantum mechanics.

We introduce spin creation and annihilation operators associated with a given spatial reference system, \( a_\zeta^+ = (a_\zeta^+, a_\zeta^-) \) and \( a_\zeta = (a_+, a_-) \), which satisfy

\[
[a_\zeta, a_\zeta^+] = 0, \quad [a_\zeta^+, a_\zeta^+] = 0,
\]

\[
[a_\zeta, a_\zeta^+] = \delta_{\zeta\zeta'}.
\]

(1.1)

The number of spins and the resultant angular momentum are then given by

\[
n = \sum a_\zeta^+ a_\zeta = n_+ + n_-,
\]

\[
J = \sum \zeta \zeta' \langle \zeta | \sigma | \zeta' \rangle a_\zeta^+ a_\zeta.
\]

(1.2)

With the conventional matrix representation for \( \sigma \), the components of \( J \) appear as

\[
J_+ = J_1 + iJ_2 = a_\zeta^+ a_\zeta, \quad J_- = J_1 - iJ_2 = a_\zeta^- a_\zeta,
\]

\[
J_3 = \frac{1}{2}(a_\zeta^+ a_\zeta - a_\zeta^- a_\zeta) = \frac{1}{2}(n_+ - n_-).
\]

(1.3)

Of course, this realization of the angular momentum commutation properties in terms of those of harmonic oscillators can be introduced without explicit reference to the composition of spins.

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To evaluate the square of the total angular momentum
\[ J^2 = \sum_{\zeta<\zeta'} a^+_{\zeta} a^-_{\zeta'} (\langle \zeta | \sigma | \zeta' \rangle \cdot \langle \zeta' | \sigma | \zeta'' \rangle), \]  
we employ the matrix elements of the spin permutation operator
\[ P^{(12)} = \frac{1}{2}(1 + \sigma^{(1)} \cdot \sigma^{(2)}). \]

Thus
\[ (\zeta | \sigma | \zeta') \cdot (\zeta' | \sigma | \zeta'') = 2\delta_{\zeta\zeta'} \delta_{\zeta''\zeta'} - \delta_{\zeta\zeta''}, \]

and
\[ J^2 = \frac{1}{2} \sum_{\zeta<\zeta'} a^+_{\zeta} a^-_{\zeta} a^+_{\zeta'} a^-_{\zeta'} - \frac{1}{2} n^2. \]

According to the commutation relations (1.1),
\[ \sum_{\zeta<\zeta'} a^+_{\zeta} a^-_{\zeta} a^+_{\zeta'} a^-_{\zeta'} = \sum_{\zeta} a^+_{\zeta} (n + 2)a^-_{\zeta} = n(n + 1), \]

whence
\[ J^2 = \frac{1}{2}n(n + 1); \]

a given number of spins, \( n = 0, 1, 2, \ldots \), possesses a definite angular momentum quantum number,
\[ j = \frac{1}{2}n = 0, \frac{1}{2}, 1, \ldots. \]

We further note that, according to (1.3), a state with a fixed number of positive and negative spins also has a definite magnetic quantum number,
\[ m = \frac{1}{2}(n_+ - n_-), \quad j = \frac{1}{2}(n_+ + n_-). \]

Therefore, from the eigenvector of a state with prescribed occupation numbers,
\[ \Psi(n_+ n_-) = \frac{(a^+)^{n_+}}{(n_+!)^{1/2}} \frac{(a^-)^{n_-}}{(n_-!)^{1/2}} \Psi_0, \]

we obtain the angular momentum eigenvector
\[ a_\pm \Psi_0 = 0, \]

\[ \Psi(jm) = \frac{(a^+)^{j+m}(a^-)^{j-m}}{[(j + m)!(j - m)!]^{1/2}} \Psi_0. \]

Familiar as a symbolic expression of the transformation properties of angular momentum eigenvectors\(^2\), this form is here a precise operator construction of the eigenvector.

\(^1\) A direct proof is given in Appendix A.
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On multiplying (1.13) with an analogous monomial constructed from the
components of the arbitrary spinor \( x_\pm = (x_+, x_-) \)

\[
\phi_{j_0}(x) \frac{x_+^{j_+} x_-^{j_-}}{[(j + m)! (j - m)!]^{1/2}}
\]

(1.14)

we obtain, after summation with respect to \( m \), and then with respect to \( j \),

\[
\sum_{m=-j}^{j} \phi_{jm}(x) \mathbf{Y}(jm) = (xa^+)^{2j} \frac{(2j)!}{(2j)!} \mathbf{Y}_0,
\]

(1.15)

and

\[
\sum_{jm} \phi_{jm}(x) \mathbf{Y}(jm) = e^{(xa^+)_m} \mathbf{Y}_0,
\]

(1.16)
in which we have written

\[
(xa^+) = \sum_{x_\pm} x_{\pm} a^\dagger_{\pm}.
\]

(1.17)

To illustrate the utility of (1.16), conceived of as an eigenvector generating
function, we shall verify the orthogonality and normalization of the eigenvectors (1.13). Consider, then,

\[
(e^{(xa^+)} \mathbf{Y}_0, e^{(xa^+)} \mathbf{Y}_0) = \sum_{jm} \phi_{jm}(x^*)(\mathbf{Y}(jm), \mathbf{Y}(jm')) \phi_{jm'}(y)
\]

\[
= (\mathbf{Y}_0, e^{(xa^+)} e^{(xa^+)} \mathbf{Y}_0).
\]

(1.18)

According to the commutation relations (1.1), and \( a_x \mathbf{Y}_0 = 0 \), we have

\[
a_x f(a^+) \mathbf{Y}_0 = \left( \frac{\partial f(a^+)}{\partial a_x^\dagger} \right) \mathbf{Y}_0,
\]

(1.19)

whence

\[
(\mathbf{Y}_0, e^{(xa^+)} e^{(xa^+)} \mathbf{Y}_0) = e^{(xa^+)}(e^{(xa^+)} \mathbf{Y}_0, \mathbf{Y}_0) = e^{(xa^+)}
\]

\[
= \sum_{jm} \phi_{jm}(x^*) \phi_{jm}(y).
\]

(1.20)

We have thus proved that

\[
(\mathbf{Y}(jm), \mathbf{Y}(jm')) = \delta_{jj'} \delta_{mm'}.
\]

(1.21)

As a second elementary example, we shall obtain the matrix elements of powers of \( J_\pm \) by considering the effect of the operators \( e^{\mathbf{J}_\pm} \) on (1.16). We have

\[
\sum_{jm} \phi_{jm}(x) e^{\mathbf{J}_+} \mathbf{Y}(jm) = e^{(xa^+ + x^- \mathbf{J}_+ \mathbf{J}_-)} \mathbf{Y}_0 = e^{(xa^+ + x^- \mathbf{J}_+ \mathbf{J}_-)} \mathbf{Y}_0
\]

\[
= e^{(xa^+ + x^- \mathbf{J}_+ \mathbf{J}_-)} \mathbf{Y}_0
\]

\[
= \sum_{jm} \phi_{jm}(x_+ + \lambda x_-, x_-) \mathbf{Y}(jm),
\]

(1.22)
and therefore
\[
\sum_{j} (jm|e^{\lambda x_{-}}|j'm') \phi_{jm}(x) = \phi_{jm}(x_{+} + \lambda x_{-}, x_{-}),
\]
(1.23)

which, on expansion, yields the nonvanishing matrix element
\[
(jm|J^{m-m'}|jm') = \left[ \frac{(j + m)!}{(j + m')!(j - m')!} \right]^{|m'|/2}, \quad m - m' > 0.
\]
(1.24)

Similarly
\[
\sum_{j} (jm|e^{\lambda x_{-}}|j'm') \phi_{jm}(x) = \phi_{jm}(x_{+}, x_{-} + \lambda x_{+}),
\]
(1.25)

and
\[
(jm|J^{m-m'}|jm') = \left[ \frac{(j + m)!}{(j + m')!(j - m')!} \right]^{|m'|/2}, \quad m' - m > 0.
\]
(1.26)

A particular consequence of (1.24) and (1.26) is
\[
\Psi'(jm) = \left[ \frac{1}{(2j)! (j + m)!} \right]^{|m'|/2} \cdot J^{m-m'}_{+} \Psi(j, -j)
= \left[ \frac{1}{(2j)! (j - m)!} \right]^{|m'|/2} \cdot J^{m-m'}_{-} \Psi(jj),
\]
(1.27)

which details the construction of an arbitrary eigenvector from those possessing the maximum values of $|m|$ compatible with a given $j$.

It is also possible to exhibit an operator which permits the construction of an arbitrary eigenvector from that possessing the minimum value of $j$ compatible with a given $m$. Indeed, (1.13), written in the form
\[
\Psi'(jm) = \frac{(a_{+} a_{+})^{j-|m|}}{(j + |m|)(j - |m|)!} \left( a_{+}^{\dagger} a_{+}^{\dagger} \right)^{|m|} a_{-},
\]
(1.28)

states that
\[
\Psi'(jm) = \left[ \frac{(2|m|)!}{(j + |m|)(j - |m|)!} \right]^{|m'|/2} K^{j-|m|}_{+} \Psi(|m|, m),
\]
(1.29)

where $K_{+}$ and two associated operators are defined by
\[
K_{+} = a_{+} a_{+}^{\dagger}, \quad K_{-} = a_{+} a_{-},
\]
(1.30)

\[
K_{3} = \frac{s}{2} (n_{+} + n_{-} + 1).
\]

It is easily seen that
\[
[J_{3}, K_{+}] = [J_{3}, K_{-}] = 0,
\]
(1.31)

and that
\[
[K_{3}, K_{+}] = K_{+}, \quad [K_{3}, K_{-}] = -K_{-},
\]

\[
[K_{+}, K_{-}] = -2K_{3}.
\]
(1.32)
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The latter are analogous to the commutation properties of $J$, save for the
algebraic sign of the commutator $[K_+, K_-]$. In keeping with this qualified
analogy we also have

$$J_z^2 - \frac{j}{4} = K_z(K_z - 1) - K_+K_- = K_z(K_z + 1) - K_-K_+ \tag{1.33}$$

as compared with

$$J_z^2 = J_z(J_z - 1) + J_+J_- = J_z(J_z + 1) + J_-J_+ \tag{1.34}$$

Noting that the eigenvalue of $K_z$ is $j + \frac{1}{2}$, we see that the roles of $j$ and $m$
are essentially interchanged in $K$. The hyperbolic nature of the space in which
the latter operates is thus related to the restriction $|m| \leq j$.

If (1.29) is multiplied by a similar numerical quantity, and then summed
with respect to $j$, one obtains

$$\sum_{j=|m|}^{\infty} \left[ \frac{(2|m|)!}{(j + |m|)!(j - |m|)!} \right]^{1/2} \lambda^{j-|m|} \Psi(jm) = F_{z|m|}(\lambda K_+) \Psi(|m|, m), \tag{1.35}$$

where

$$F_\tau(z) = rz^{-\tau/2} L_\tau(2z^{1/2}) = \sum_{n=0}^{\infty} \frac{r^l}{n!(\tau + n)!} z^n, \tag{1.36}$$

and $L_\tau$ is the cylinder function of imaginary argument. A simpler generating
function is given by

$$\sum_j \left[ \frac{1}{(2|m|)!} \right]^{1/2} \lambda^{j-|m|} \Psi(jm) = e^{\lambda_+} \Psi(|m|, m). \tag{1.37}$$

2. ROTATIONS

A significant interpretation is obtained for (1.15) by introducing the operators

$$a_+ = (xa^+), \quad a_- = (x^a), \quad a_+ = [x^a], \quad a_- = [xa], \tag{2.1}$$

where

$$[xy] = x_+y_- - x_-y_+. \tag{2.2}$$

With the restriction

$$(x^a x) = 1, \tag{2.3}$$

these operators also obey the commutation relations (1.1), and must therefore
constitute spin creation and annihilation operators associated with an altered
spatial reference system. Accordingly, (1.15) can be viewed as the expression
of the state $m = j$, in a rotated coordinate system, as a linear combination of the eigenvectors in a fixed coordinate system,

$$
\Psi'(j') = \frac{(a_j^+)^{j'}(a_j^+)^{\dagger}}{(2j)!^{1/2}} \Psi_0 = (2j)!^{1/2} \sum_{m=-j}^{j} \phi_{jm}(x) \Psi(jm).
$$

(2.4)

The unitary nature of this transformation is here easily verified,

$$
(2j)! \sum_{m} \phi_{jm}(x^*) \phi_{jm}(x) = \langle x^* x \rangle^{2j} = 1.
$$

(2.5)

In general

$$
\Psi'(jm') = \phi_{jm}(a_j^+) \Psi_0 = \sum_{m} \Psi(jm) U_{jm'}^{(j)},
$$

(2.6)

where the coefficients are to be inferred from

$$
\sum_{m} \phi_{jm}(a_j^+) U_{jm'}^{(j)} = \phi_{jm}(x^* x^+ + x^* x^- - x^* a_j^+ + x^* a_j^+).
$$

(2.7)

It is useful to introduce the unitary operator that generates $\Psi'(jm')$ from $\Psi(jm)$,

$$
\Psi'(jm') = U \Psi(jm),
$$

(2.8)

which permits an alternative construction of the coefficients in (2.6),

$$
U_{jm'}^{(j)} = (jm | U | jm').
$$

(2.9)

In terms of the successive rotations characterized by Eulerian angles, $\phi, \theta, \psi$, $U$ is given explicitly by

$$
U = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\psi J_1},
$$

(2.10)

where

$$
J^*_1 = e^{-i\phi J_3} J e^{i\phi J_3},
$$

$$
J^*_2 = e^{-i\theta J_2} J e^{i\theta J_2},
$$

(2.11)

are the operators appropriate to the coordinate systems produced by the previous rotations. The resulting expression for $U(\phi \theta \psi)$ is

$$
U = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\psi J_1}, \quad U^{-1} = e^{i\theta J_2} e^{i\psi J_1} e^{i\phi J_3}.
$$

(2.12)

The angular momentum operators associated with the new coordinate system,

$$
J' = U J U^{-1},
$$

(2.13)

can be constructed from the transformed creation and annihilation operators,

$$
a^+_j = U a^+_j U^{-1} = e^{-i(\phi + \psi)} \cos \frac{1}{2} \theta a^+_j + e^{-i(\phi - \psi)} \sin \frac{1}{2} \theta a^+_j
$$

$$
a^+_j = U a^+_j U^{-1} = e^{-i(\phi - \psi)} \sin \frac{1}{2} \theta a^+_j + e^{i(\phi + \psi)} \cos \frac{1}{2} \theta a^+_j.
$$

(2.14)
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In evaluating (2.14), we have made use of the relations

\[ e^{-i\phi J_z} a_\pm e^{i\phi J_z} = e^{\mp i(1/2)\phi} a_\pm, \]
\[ e^{-i\theta J_z} a_\mp e^{i\theta J_z} = \cos \frac{1}{2} \theta a_\pm \pm \sin \frac{1}{2} \theta a_\mp, \]

(2.15)

of which the former follows immediately from the significance of \( a_\pm \) as a positive (negative) spin creation operator, while the latter may be verified by differentiation with respect to \( \theta \), in conjunction with the commutation relations

\[ [a_\pm, J_z] = \mp (1/2) a_\pm. \]

(2.16)

The form of (2.14) is in agreement with (2.1) and (2.3), where

\[ x_+ = e^{-(i/2)(\phi + \psi)} \cos \frac{1}{2} \theta, \quad x_- = e^{-(i/2)(\phi - \psi)} \sin \frac{1}{2} \theta. \]

(2.17)

To construct the matrix of \( U \), we consider

\[ (e^{(x') a^+})^* U e^{(x a^+)} = \sum_{jm} \phi_{jm}(x^*) U^{(j)}_{mm'} \phi_{jm}(y) \]
\[ = (x a^+), \quad e^{(x a^+)} = (x a^+), \]

(2.18)

in which the \( a'^+ \) are the operators (2.14). On writing

\[ (y a^+') = (a^+ uy) \]

(2.19)

where \( u \) is the matrix

\[ u = \begin{pmatrix} e^{-(i/2)(\phi + \psi)} \cos \frac{1}{2} \theta, & -e^{-(i/2)(\phi - \psi)} \sin \frac{1}{2} \theta \\ e^{-(i/2)(\phi - \psi)} \sin \frac{1}{2} \theta, & e^{(i/2)(\phi + \psi)} \cos \frac{1}{2} \theta \end{pmatrix} \]

(2.20)

we immediately obtain

\[ \sum_{jm} \phi_{jm}(x^*) U^{(j)}_{mm'} \phi_{jm}(y) = e^{(x^* uy)}. \]

(2.21)

Since (2.12) implies that

\[ U^{(j)}_{mm'} (\phi \theta) \psi = e^{-i m \phi} U^{(j)}_{mm'} (\theta) e^{-i m \phi}, \]

(2.22)

where

\[ U^{(j)}_{mm'} (\theta) = (jm \mid e^{-i \theta J_z} \mid jm'), \]

(2.23)

we may simplify (2.21) by placing \( \phi = \psi = 0 \), thereby obtaining

\[ \sum_{jm} \phi_{jm}(x^*) U^{(j)}_{mm'} (\theta) \phi_{jm}(y) = \exp \{ \cos \frac{1}{2} \theta (x^* y) - \sin \frac{1}{2} \theta [x^* y] \}. \]

(2.24)

The matrix \( u \) is unitary and unimodular, that is, possesses a unit determinant. Its representation in terms of spin matrices has, as it must, the form of (2.12),

\[ u = e^{-(i/2)\phi \sigma z} e^{-(i/2)\psi \sigma z} e^{-(i/2)\theta \sigma z}. \]

(2.25)
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Any such unitary matrix can be presented as

\[ u = e^{-i\mathcal{H}} \]  

(2.26)

where \( \mathcal{H} \) is a Hermitian matrix. Since

\[ \det u = e^{-i \text{tr} \mathcal{H}}, \]  

(2.27)

\( \mathcal{H} \) must be a traceless Hermitian matrix and, accordingly, is a linear combination of the spin matrices, with real coefficients. Hence \( u \) can be written as

\[ u = e^{-i(\gamma/2)}n \cdot s \]  

(2.28)

where \( n \) is a unit vector, specified by two angles, \( \alpha \) and \( \beta \). The fact that (2.28) is the matrix describing a rotation through the angle \( \gamma \) about the axis \( n \) affirms the well-known equivalence between an arbitrary rotation and a simple rotation about a suitably chosen axis. The rotation angle \( \gamma \) is easily obtained by comparing the trace of \( u \), in its two versions,

\[ \frac{1}{2} \text{tr} u = \cos \frac{1}{2} \gamma = \cos \frac{1}{2} \theta \cos \frac{1}{2} (\phi + \psi). \]  

(2.29)

More generally, the trace of \( U \) for a given \( j \) depends only upon the rotation angle \( \gamma \). We define

\[ \chi^{(j)} = \sum_{m=-j}^{j} U_{nm}^{(j)} = \text{tr} P_j U, \]  

(2.30)

in which \( P_j \) is the projection operator for the states with quantum number \( j \). If we remark that \( U \) must also have the form of (2.28),

\[ U = e^{-i\alpha \cdot \tau} \]  

(2.31)

we immediately obtain

\[ \chi^{(j)} = \sum_{m=-j}^{j} \frac{1}{\sin \frac{1}{2} \gamma} \frac{e^{-i\alpha \gamma}}{\sin \frac{1}{2} \gamma}. \]  

(2.32)

However, we can also derive this directly from the generating function (2.21).

For simplicity we shall assume the reference system to be so chosen that \( u \) is a diagonal matrix, with eigenvalues \( e^{\pm i(\gamma/2)v} \). We replace \( x^*_i \) with \( t(\partial/\partial y_i) \) and evaluate the derivatives at \( y_t = 0 \). According to

\[ \phi_{jm}(\partial/\partial y_j)\phi_{im}(y)|_{y_t=0} = \delta_{m,m'}, \]  

(2.33)

we then have

\[ \sum_j t^{2j}\chi^{(j)} = \exp \left( t e^{-i(\gamma/2)\tau} \frac{\partial}{\partial y_+} y_+ \right) \exp \left( t e^{i(\gamma/2)\tau} \frac{\partial}{\partial y_-} y_- \right) \bigg|_{y_t=0} \]  

(2.34)

in which the notation reflects the necessity of placing the derivatives to the left of the powers of \( y_t \). Now

\[ \text{This trace is the character of group theory.} \]
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\[
\exp \left( \lambda \frac{\partial}{\partial y}; y \right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left( \frac{\partial}{\partial y} \right)^n y^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = \frac{1}{1 - \lambda},
\]

and therefore

\[
\sum_{j} t^{2j} \chi^{(j)}(\lambda) = \frac{1}{1 - t \cdot \exp(-\frac{i}{2\gamma})} \frac{1}{1 - t \cdot \exp(\frac{i}{2\gamma})} = \frac{1}{1 - 2t \cdot \cos \frac{1}{2}\gamma + t^2},
\]

which is a generating function for the \( \chi^{(j)} \). On writing

\[
\frac{1}{1 - t \cdot \exp(-\frac{i}{2\gamma})} \frac{1}{1 - t \cdot \exp(\frac{i}{2\gamma})} = \frac{1}{2it \cdot \sin \frac{1}{2}\gamma} \left[ \frac{1}{1 - t \cdot \exp(-\frac{i}{2\gamma})} - \frac{1}{1 - t \cdot \exp(\frac{i}{2\gamma})} \right],
\]

and expanding in powers of \( t \), one obtains

\[
\chi^{(j)}(\gamma) = \frac{\sin(j + \frac{1}{2})\gamma}{\sin \frac{1}{2}\gamma}.
\]

Symmetry properties of \( U^{(j)}_{mn}(\phi \theta \psi) \) are easily inferred from (2.21). According to the invariance of \( (x^x, y, y) \) under the substitutions \( \phi \rightarrow \phi + \pi, \ x^x \rightarrow y, \ \text{and} \ \phi \rightarrow \phi - \pi, \ \theta \rightarrow \pi - \theta, \ \psi \rightarrow -\psi \), we have

\[
U^{(j)}_{mn}(\phi \theta \psi) = U^{(j)}_{m-n}(\psi + \pi, \phi - \pi, \theta - \pi) = t^{2j} U^{(j)}_{m-n}(\phi - \pi, \pi - \theta, \psi).
\]

Among the additional equivalent forms produced by successive application of these transformations are

\[
\begin{align*}
\text{i}^{2j} U^{(j)}_{m-n}(\phi - \pi, \phi + \pi, -\theta, -\psi) &= U^{(j)}_{m-n}(\pi - \phi, \theta, -\pi - \psi) \\
&= U^{(j)}_{m-n}(\psi, \psi, -\phi).
\end{align*}
\]

We also note that

\[
U^{(j)}_{m-n}(\phi \theta \psi) = U^{(j)}_{m-n}(\phi + \pi, \theta - \pi, \psi).
\]

On removing the angles \( \phi \) and \( \psi \) with the aid of (2.22), we find that the content of (2.39) and (2.40) is

\[
U^{(j)}_{m-n}(\theta) = (-1)^{j - m} U^{(j)}_{m-n}(\pi - \theta) = (-1)^{j - m} U^{(j)}_{m-n}(\pi - \theta)
\]

\[
= (-1)^{j - m} U^{(j)}_{m-n}(\theta) = U^{(j)}_{m-n}(\theta) = U^{(j)}_{m-n}(\theta).
\]
In view of these relations, it is sufficient to exhibit $U_{mn}^{(j)}(\theta)$ for non-negative values of $m$ and $m'$.

On expanding the generating function (2.24) in terms of $\phi_{jm}(x^*)$, or of $\phi_{jm}(y)$, we obtain the equivalent expressions

$$
\sum_m U_{mn}^{(j)}(\theta) \phi_{jm}(y) = \phi_{jm}(\cos \frac{1}{2} \theta y_+ - \sin \frac{1}{2} \theta y_- , \sin \frac{1}{2} \theta y_+ + \cos \frac{1}{2} \theta y_-) ,
$$

(2.43a)

$$
\sum_m \phi_{jm}(x^*) U_{mn}^{(j)}(\theta) = \phi_{jm}(\cos \frac{1}{2} \theta x^*_+ + \sin \frac{1}{2} \theta x^*_- , -\sin \frac{1}{2} \theta x^*_+ + \cos \frac{1}{2} \theta x^*_- ) ,
$$

(2.43b)

of which the latter is the counterpart of (2.7). As a convenient means of constructing $U_{mn}^{(j)}(\theta)$, we place

$$
x^*_+ = \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta , \quad x^*_- = t - \cos^2 \frac{1}{2} \theta ,
$$

so that (2.43b) reads

$$
\sum_m (\sin \frac{1}{2} \theta \cos \frac{1}{2} \theta)_j^{j+m} \frac{(t - \cos^2 \frac{1}{2} \theta)^{j-m}}{((j+m)!(j-m)!)^{1/2}} U_{mn}^{(j)}(\theta)
$$

$$
= (-1)^{j-m} \left[ \frac{(\sin \frac{1}{2} \theta)^{j+m}(\cos \frac{1}{2} \theta)^{j-m}}{((j+m)!(j-m)!)^{1/2}} \right] t^{j+m}(1-t)^{j-m} .
$$

(2.44)

Thus

$$
U_{mn}^{(j)}(\theta) = (-1)^{j-m} \left[ \frac{(j+m)!}{(j-m)!((j+m)!(j-m)!)^{1/2}} \right]^{-1/2} \cdot (\sin \frac{1}{2} \theta)^{-m+m'} (\cos \frac{1}{2} \theta)^{-m-m'}
$$

$$
\cdot \left[ (\frac{d}{dt})^{j-m} t^{j+m}(1-t)^{j-m'} \right] t = \cos^2 \frac{1}{2} \theta .
$$

(2.45)

The structure of the right side will be recognized as that of the Jacobi polynomial,

$$
F_n(a,b;t) = F(-n,a+n,b;t) = \frac{(b-1)!}{(b+n-1)!} t^{1-b}(1-t)^{b-n}
$$

$$
\cdot \left( \frac{d}{dt} \right)^n t^{b+n-1}(1-t)^{a-b+n},
$$

(2.46)

whence

$$
U_{mn}^{(j)}(\theta) = (-1)^{j-m} \left[ \frac{(j+m)!}{(j-m)!((j+m)!(j-m)!)^{1/2}} \right]^{-1/2} \cdot (\sin \frac{1}{2} \theta)^{m-m'} (\cos \frac{1}{2} \theta)^{m+m'}
$$

$$
\cdot F_{j-m}(2m+1,m+m'+1;\cos^2 \frac{1}{2} \theta) .
$$

(2.47)

This is equivalent to the result obtained by P. Güttinger, Z. Phys. 73, 169 (1931).
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Other forms can be obtained from (2.43), corresponding to the variety of transformations permissible to hypergeometric functions. Thus the known relation

\[ F(a, b, c; x) = (1 - x)^{-a} F\left( a, c - b, c; \frac{x}{1-x} \right) \]

applied to (2.47), gives

\[ U_{nm}^{(j)}(\theta) = \frac{(-1)^{j-m'}}{(m + m')!} \left[ \frac{(j + m)!}{(j - m')!} \right]^{1/2} (\sin \frac{1}{2} \theta)^{2j} (\cot \frac{1}{2} \theta)^{m + m'} F(m - j, m' - j, m + m' + 1; -\cot^2 \frac{1}{2} \theta). \]  

(2.49)

Another aspect of reference system transformation is best discussed in terms of

\[ U_{nm}^{(j)*}(\phi \theta \psi) = e^{i m \phi} U_{nm}^{(j)}(\theta) e^{i m' \psi} = (jm') U^{-1} |jm\rangle. \]

(2.50)

This quantity is the transformation function

\[ (\Psi'(jm'), \Psi(jm)) = (\omega, jm'|jm), \]

(2.51)

in which we have used \( \omega \) to designate collectively the angles \( \phi \theta \psi \), relating the new reference system to the fixed one. We shall be interested in the differential characterization of this transformation function, in its dependence upon the Eulerian angles. Now

\[ \frac{1}{i} \frac{\partial}{\partial \phi} U^{-1} = U^{-1} J_3 \]

\[ \frac{1}{i} \frac{\partial}{\partial \phi} U^{-1} = J_3 U^{-1} = U^{-1} J_3^* \]

(2.52)

\[ \frac{1}{i} \frac{\partial}{\partial \theta} U^{-1} = U^{-1} e^{-i \phi} J_3 e^{i \phi} J_3 = U^{-1} J_\theta, \]

where

\[ J_3 = J_3 \cos \theta + \frac{i}{2} \sin \theta (J_+ e^{-i \phi} + J_- e^{i \phi}), \]

\[ J_\theta = \frac{1}{2i} (J_+ e^{-i \phi} - J_- e^{i \phi}), \]

(2.53)

and, therefore

\[ \frac{1}{i} \frac{\partial}{\partial \phi} (\omega| \omega) = (\omega|J_3| \omega) \]

\[ e^{i \phi} \left[ \left( \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \left( \frac{1}{i} \frac{\partial}{\partial \phi} - \cos \theta \frac{1}{i} \frac{\partial}{\partial \phi} \right) \right) \right] (\omega| \omega) = (\omega|J_+| \omega) \]

(2.54)

\[ e^{-i \phi} \left[ \left( \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \left( \frac{1}{i} \frac{\partial}{\partial \phi} - \cos \theta \frac{1}{i} \frac{\partial}{\partial \phi} \right) \right) \right] (\omega| \omega) = (\omega|J_-| \omega). \]
This is a differential operator representation of an arbitrary angular momentum vector. The familiar differential operators associated with an orbital angular momentum emerge if the transformation function is independent of \( \psi \). Since this corresponds to \( n' = 0 \), the quantum number \( j \) must then be an integer.\(^5\)

The differential operators (2.54) are well-known in connection with angular momentum of a rigid body, and, accordingly, the eigenvalue equation for \( J^2 \) in this representation will be identical with the symmetrical top wave equation. To construct this equation directly, we remark that

\[
J^2 = J^2_3 + \left( \frac{1}{2} J^+, e^{-i\theta} + \frac{1}{2} J^- e^{i\theta} \right)^2 - \left( \frac{1}{2} J^+_+ e^{-i\theta} - \frac{1}{2} J^- e^{i\theta} \right)^2
\]

\[
= J^2_3 + \left[ \frac{J^3_3 - J^3_3 \cos \theta}{\sin \theta} \right]^2 + J^2_\theta
\]

\[
= \frac{J^2_3 - 2J^2_3 \cos \theta + J^2_3}{\sin^2 \theta} + J^2_\theta + \cot \theta \frac{1}{i} J_\theta,
\]

since

\[
[J'_z, J_3] = \sin \theta \frac{1}{i} J_\theta.
\]  

On referring to (2.52), we immediately obtain

\[
- \left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} - 2 \cos \theta \frac{\partial}{\partial \psi} \frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial \phi^2 \phi} \right) \right] U^{-1} = U^{-1} J^2,
\]  

(2.57)

and the analogous differential equation for \( (\omega_1) \), including the eigenvalue equation

\[
\left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + j(j + 1) \right] \frac{m^2 - 2mm' \cos \theta + m'^2}{\sin^2 \theta} \right] (\omega, j'm', jm) = 0. \tag{2.58}
\]

An integral theorem concerning the angular dependence of \( U \), or \( U^{-1} \), is stated by

\[
\int U \, d\omega = P_0,
\]

(2.59)

where \( P_0 \) is the projection operator for the state \( j = 0 \), and

\[
d\omega = \frac{1}{2} \sin \theta \, d\theta \cdot \frac{1}{4\pi} \, d\phi \cdot \frac{1}{4\pi} \, d\psi,
\]

(2.60)

\[
\int d\omega = 1.
\]

\(^5\) The fact that the general differential operators (2.54) admit half-integral values of \( j \) has been noticed by F. Bopp and R. Haag, Z. Naturforsch. 5a, 644 (1950).
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The integration domain is here understood to be

\[ 0 \leq \phi < 4\pi, \quad 0 \leq \psi < 4\pi, \quad 0 \leq \theta \leq \pi. \]

(2.61)

To prove this theorem we subject (2.57) to the angular integrations contained in \( d\omega \). In virtue of the periodicity possessed by \( U^{-1} \) over \( 4\pi \) intervals of \( \phi \) and \( \psi \), we obtain

\[ \int U^{-1} d\omega J^2 = -\frac{1}{2} \left[ \sin \theta \frac{\partial}{\partial \theta} \int U^{-1} d\phi \frac{d\psi}{4\pi} \right]_{\theta=0} = 0. \]

(2.62)

This result asserts the vanishing of \( \int U^{-1} d\omega \), and the Hermitian conjugate \( \int U d\omega \), except for the state with \( j = 0 \). The fact that the rotation operator \( U \) reduces to unity for this spherically symmetrical state completes the proof of (3.59). We shall defer application of this theorem to the next section.

3. ADDITION OF TWO ANGULAR MOMENTA

Two kinematically independent angular momenta, \( J_1 \) and \( J_2 \), can be expressed by

\[ J_1 = \sum_{\xi} a^+_{\xi} (\xi | \xi \rangle \langle \xi |) a_\xi, \]

\[ J_2 = \sum_{\xi} b^+_{\xi} (\xi | \xi \rangle \langle \xi |) b_\xi, \]

(3.1)

where the \( a \) and \( b \) operators individually obey (1.1), but are mutually commutative. In studying the eigenvectors of the total angular momentum,

\[ J = J_1 + J_2, \]

(3.2)

the following scalar operators play an important role:

\[ J_+ = (a^+ b), \quad J_- = (b^+ a), \]

\[ J_3 = \frac{1}{2} [(a^+ a) - (b^+ b)] = \frac{1}{2} (n_1 - n_2), \]

(3.3)

and

\[ \mathcal{K}_+ = [a^+ b^+], \quad \mathcal{K}_- = [a b], \]

\[ \mathcal{K}_3 = \frac{1}{2} [(a^+ a) + (b^+ b)] + 1 = \frac{1}{2} n + 1. \]

(3.4)

As one can easily verify by direct calculation, the operators \( J \) and \( \mathcal{K} \) commute with each other (as well as with \( J \)), and obey

\[ [J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2 J_3, \]

\[ [\mathcal{K}_3, \mathcal{K}_\pm] = \pm \mathcal{K}_\pm, \quad [\mathcal{K}_+, \mathcal{K}_-] = -2 \mathcal{K}_3. \]

(3.5)

It will be noted that the commutation properties of the \( J \) operators are those of a conventional angular momentum, while the \( \mathcal{K} \) operators are analogous.

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to the hyperbolic angular momentum $K$, which was discussed in the first section. We shall denote the eigenvalues of $J_3$ and $N_3$ by $\mu$ and $\nu$, respectively. These quantities have the following significance,

$$\mu = j_1 - j_2, \quad \nu = j_1 + j_2 + 1.$$  \hfill (3.6)

In evaluating the square of the resultant angular momentum, we encounter

$$2J_1 \cdot J_2 = \sum_{\alpha \beta} a_\alpha^* a_\beta b_\alpha^* b_\beta \cdot (\xi^* | \xi^\prime \rangle \langle \xi^\prime | \xi^\prime \rangle),$$

$$= \sum_{\alpha \beta} a_\alpha^* a_\beta b_\alpha^* b_\beta - \frac{i}{2} n_1 n_2. \hfill (3.7)$$

This can be expressed either in terms of the $J$ operators, or of the $N$ operators, since

$$J_- J_+ = \sum \frac{b_\alpha^* a_\beta a_\beta^* b_\alpha}{\xi} = n_2 + \sum \frac{a_\alpha^* a_\beta b_\alpha^* b_\beta}{\xi}, \hfill (3.8)$$

and

$$N_+ N_- = \sum \frac{a_\alpha^* b_\alpha (a_\beta b_\beta - a_\beta^* b_\beta^*)}{\xi} = n_1 n_2 - \sum \frac{a_\alpha^* a_\beta b_\alpha^* b_\beta}{\xi}. \hfill (3.9)$$

Indeed,

$$J^2 = J_3 (J_3 + 1) - J_- J_+ = J_3 (J_3 - 1) + J_+ J_-, \hfill (3.10)$$

and

$$J^2 = N_3 (N_3 - 1) - N_+ N_- = N_3 (N_3 + 1) - N_+ N_+. \hfill (3.11)$$

From the first, conventional, representation of $J^2$ in terms of the angular momentum $J$, we infer that

$$j \geq |\mu|, \hfill (3.12)$$

or

$$j \geq |j_1 - j_2|, \hfill (3.13)$$

while the hyperbolic representation implies that

$$\nu - 1 \geq j, \hfill (3.14)$$

or

$$j_1 + j_2 \geq j. \hfill (3.15)$$

We have thus arrived at

$$j_1 + j_2 \geq j \geq |j_1 - j_2|, \hfill (3.16)$$

the familiar restriction on the composition of two angular momenta.

An eigenvector of $J^2$ is conveniently labelled by the eigenvalues of $J_3$, $J_3$, and $N_3$. In virtue of (3.6), the resulting eigenvector $\Psi(jm\nu)$ is equivalently designated as $\Psi(j_1 j_2 jm)$. In particular, the state with $\nu = j + 1$ corresponds to $j_1 + j_2 = j$, and $2j_1 = j + \mu$, $2j_2 = j - \mu$. The special state
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of this type with \( m = j \) can be realized in only one way, since \( m = j_1 + j_2 \) requires that \( m_1 = j_1, m_2 = j_2 \). Thus

\[
\Psi(jj + 1) = \frac{(a_+^j)^{j+\mu} (b_+^j)^{j-\mu}}{[(j + \mu)! (j - \mu)!]^{1/2}} \Psi_0. \tag{3.17}
\]

With an arbitrary reference system, this result becomes

\[
((2j)!)^{1/2} \sum_{m=-j}^j \phi_{jm}(x) \Psi(jm + 1) = \frac{(xa^j)^{j+\mu}(xb^j)^{j-\mu}}{[(j + \mu)! (j - \mu)!]^{1/2}} \Psi_0, \tag{3.18}
\]

according to (2.4). We multiply this \( j = \) analogue of (1.13) with \( \phi_{jm}(\xi) \), and sum with respect to \( \mu \),

\[
((2j)!)^{1/2} \sum_{m\mu} \phi_{jm}(x) \phi_{jm}(\xi) \Psi(jm + 1) = \frac{\xi_+ (xa^j) + \xi_- (xb^j)}{(2j)!} \Psi_0. \tag{3.19}
\]

Further summation with respect to \( j \) then yields

\[
\sum_{j\mu} ((2j)!)^{1/2} \phi_{jm}(x) \phi_{jm}(\xi) \Psi(jm + 1) = e^{\xi_+(xa^j) + \xi_- (xb^j)} \Psi_0. \tag{3.20}
\]

To complete the determination of the eigenvector \( \Psi(jm_\mu) \), we need the analogue of (1.29), specifying the eigenvector with arbitrary \( \nu \) in terms of that with the minimum value, \( j + 1 \). For this purpose, we examine the operator\(^6\)

\[
V = \ell^2 X_3 = 1 \tag{3.21}
\]

which has the following significant properties,

\[
t \frac{\partial}{\partial t} V = (2X_3 - 1)V, \quad \left( i \frac{\partial}{\partial t} \right)^2 V = (2X_3 - 1)^2 V, \tag{3.22}
\]

and

\[
V^{-1} \mathcal{X}_-_V = \ell^2 \mathcal{X}_-, \quad \mathcal{X}_-_V = \ell^2 V \mathcal{X}_-. \tag{3.23}
\]

In conjunction with

\[
4 \ell^2 + 1 = (2X_3 - 1)^2 - 4 \mathcal{X}_+ \mathcal{X}_-, \tag{3.24}
\]

we obtain

\[
\left( \frac{\partial^2}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t} - \frac{4 \ell^2 + 1}{\ell^2} \right) V - 4 \mathcal{X}_+ V \mathcal{X}_- = 0, \tag{3.25}
\]

an ordered operator form of Bessel's equation. The solution is

\[
V = t^{(4\ell^2 + 1)^{1/2}} F_{(4\ell^2 + 1)^{1/2}}(t^2 \mathcal{X}_+ + P; \mathcal{X}_-), \tag{3.26}
\]

where \( P \) is an integration constant, and the notation is intended to indicate that \( P \) is inserted between the powers of \( \mathcal{X}_+ \) and \( \mathcal{X}_- \) in the ordered operator

\(^6\) Our procedure here is based upon the general method of Appendix A.
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expansion of the function $F$ defined in (1.36). The second solution of the Bessel equation has been rejected in order to conform with the fact that $\rho_2 \mathcal{K}_2 - 1$ must vanish as $t \to 0$, in view of the non-negative character of $\mathcal{K}_2 - 1$. The operator (3.26) can also be written as

$$
V = \sum_j \epsilon_{j+1}^j F_{j+1}(\rho_2 \mathcal{K}_2^{-1}; P_{j+1}; \mathcal{K}_2^{-1})
$$

$$
= \sum_{j=0}^{\infty} \epsilon_{j+1}^j \sum_{n-j+1}^{\infty} P_{j+1},
$$

(3.27)

where $P_{j+1}$ is the projection operator for the state with the indicated eigenvalues. According to the well-known Bessel function power series we then have

$$
P_{j+1} = \omega_{j+1}(\mathcal{K}_2^{-1}) P_{j+1} \omega_{j+1}(\mathcal{K}_2^{-1}),
$$

(3.28)

where

$$
\omega_{j+1}(\lambda) = \left[ \frac{(2j+1)!}{(\nu+j)! (\nu-j)!} \right]^{1/2} \lambda^{j+1/2}.
$$

(3.29)

This yields the desired eigenvector relation,

$$
\mathcal{P}(j \nu \mu \nu) = \omega_{j+1}(\mathcal{K}_2^{-1}) \mathcal{P}(j \nu \mu \nu + 1).
$$

(3.30)

It will be noted that, with respect to $j$ and $\nu$, Eq. (3.30) is converted into (1.29) by the substitutions

$$
\begin{align*}
&j \to |m| - \frac{1}{2}, & \nu \to j + \frac{1}{2},
\end{align*}
$$

(3.31)

which are in accord with the significance of $K$. Corresponding, then, to the generating functions (1.35) and (1.37), we have

$$
\sum_{j=0}^{\infty} \omega_{j+1}(\lambda) \mathcal{P}(j \mu \nu) = F_{j+1}(\lambda \mathcal{K}_2^{-1}) \mathcal{P}(j \mu \nu + 1),
$$

(3.32)

and

$$
\frac{(2j+1)!}{(\nu+j)! (\nu-j)!} \sum_{j=0}^{\infty} \chi_{\lambda j}(\nu) \mu \nu) = e^{\lambda \mathcal{K}_2^{-1}} \mathcal{P}(j \mu \nu + 1),
$$

(3.33)

in which

$$
\chi_{\lambda j}(\nu) = \left[ \frac{(\nu+j)!}{(\nu-j)!} \right]^{1/2} \lambda^{j+1/2}.
$$

(3.34)

The application of the operator $e^{\lambda \mathcal{K}_2^{-1}}$ to (3.20) thus produces

$$
\sum (2j+1)^{-1/2} \phi_{j}(x) \phi_j(\xi) \chi_{\lambda j}(\nu) \mathcal{P}(j \mu \nu) = e^{(\lambda \mathcal{K}_2^{-1})} \mathcal{P}(j \mu \nu + 1),
$$

(3.35)

The eigenvectors are exhibited somewhat more explicitly in the result

\footnote{The normalization constant does not automatically appear in the corresponding group theory formula. B. L. van der Waerden, "Die gruppentheoretische Methode in der Quantenmechanik" (Berlin, 1932).}
obtained by applying $\omega_\mu(\mathcal{X}^\gamma_+)$ to (3.18),

$$
\sum_{m=-j}^{j} \phi_{jm}(x) \Psi(j_1,j_2,jm) = \left[ \frac{2j+1}{(j_1+j_2+j+1)!} \right]^{1/2} \left[ \frac{(a+b^+)^{j_1+j_2-j} \cdot (ab^+)^{j_1-j_2} \cdot (ab)^{j_2+j-j_1}}{(j_1+j_2-j)!(j_1-j_2)!(j_2+j-j_1)!} \right]^{1/2} \Psi_0.
$$

(3.36)

in which we have employed $j_1$ and $j_2$, rather than $\mu$ and $\nu$. For the purpose of converting (3.36) into a convenient expression for the transformation function

$$(j_1,j_2,jm|j_1,m_1,j_2,m_2) = (\Psi(j_1,j_2,jm), \Psi(j_1,m_1,j_2,m_2)),$$

(3.37)

we make the replacement $x_+ \rightarrow x^+\, , \, x_- \rightarrow -x^+$, and take the scalar product with the generating function of the $\Psi(j_1,m_1,j_2,m_2),$

$$
\sum_{j_1m_1j_2m_2} \phi_{j_1m_1}(x) \phi_{j_2m_2}(y) \Psi(j_1,m_1,j_2,m_2) = e^{-\mu x^+ + \nu y^+} \Psi_0.
$$

(3.38)

The ensuing formula can be written

$$
\sum_{m_1m_2} \phi_{j_1m_1}(x) \phi_{j_2m_2}(y) \phi_{j_3m_3}(z) X(j_1,j_2,j_3; m_1,m_2,m_3)
$$

$$
= [(j_1+j_2+j_3+1)!]^{-1/2} \cdot \left[ \frac{(yz)^{j_3-j_1} \cdot (zx)^{j_3-j_2} \cdot (xy)^{j_3-j_1}}{(j_2+j_3-j_1)!(j_3+j_1-j_2)(j_1+j_2-j_3)!} \right]^{1/2},
$$

(3.39)

in virtue of the definition$^8$

$$(j_1,j_2,jm|j_1,m_1,j_2,m_2) = (2j+1)^{1/2}(-1)^{j_1-j_2+m} X(j_1,j_2,j_3; m_1,m_2,m_3).
$$

(3.40)

Multiplication with

$$
\Phi_{j_1j_2j_3}(x^\beta y) = [(J+1)!]^{1/2} \cdot \frac{\alpha^{j_1} \beta^{j_2} \gamma^{j_3}}{(J-2j_1)!(J-2j_2)!(J-2j_3)!}^{1/2}
$$

$$
J = j_1 + j_2 + j_3,
$$

(3.41)

and summation with respect to $j_1$, $j_2$, and $j_3$, then yields the generating function

$$
\sum_{j_1j_2j_3} \phi_{j_1m_1}(x) \phi_{j_2m_2}(y) \phi_{j_3m_3}(z) \Phi_{j_1j_2j_3}(x^\beta y) X(j_1,j_2,j_3; m_1,m_2,m_3) = e^{\mu y^+ + \nu x^+ + \gamma x y}.
$$

(3.42)

$^8$ This $X$ coefficient is related to the $V$ coefficient of G. Racah, Phys. Rev. 62, 438 (1942), by $X = (-1)^{j_3-j_1-j_2} V$. We have introduced the $X$ coefficient by virtue of its greater symmetry; compare Eqs. (3.44), (3.45) with Eq. (19a) of Racah's paper (henceforth referred to as R). Editors' note: This $X$ coefficient is identical to the Wigner 3-$j$ symbol; i.e., $X(j_1,j_2,j_3; m_1,m_2,m_3) = \left( \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ \hline m_1 \\ m_2 \\ m_3 \end{array} \right)$. 

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Symmetry properties of the $X$ coefficients can be easily inferred from the invariance of the generating function to particular substitutions. Thus, the null effect of multiplying $x_+, y_+, z_+$ by $e^{i(\pi/2)p}$, and $x_-, y_-, z_-$ by $e^{-i(\pi/2)p}$, indicates that $X$ vanishes unless

$$m_1 + m_2 + m_3 = 0.$$  \hspace{1cm} (3.43)

The invariance of the generating function for simultaneous cyclic permutations of $x, y, z$ and $\alpha, \beta, \gamma$ implies the corresponding property for $X$:

$$X(j_1j_2j_3; m_1m_2m_3) = X(j_3j_1j_2; m_3m_1m_2) = X(j_3j_2j_1; m_3m_1m_2).$$  \hspace{1cm} (3.44)

The interchange of $x$ and $y$, combined with the substitutions $\alpha \leftrightarrow -\beta$, $\gamma \leftrightarrow -\gamma$, discloses the behavior of the $X$ coefficients with respect to non-cyclic permutations,

$$X(j_2j_1j_3; m_1m_2m_3) = X(j_1j_3j_2; m_1m_2m_3) = X(j_3j_2j_1; m_2m_1m_3)$$

$$= (-1)^r X(j_1j_2j_3; m_1m_2m_3),$$  \hspace{1cm} (3.45)

while the exchange of $x_+, y_+, z_+$ with $x_-, y_-, z_-$, in conjunction with sign reversals for $\alpha, \beta, \gamma$, leads to

$$X(j_1j_2j_3; -m_1-m_2-m_3) = (-1)^r X(j_1j_2j_3; m_1m_2m_3).$$  \hspace{1cm} (3.46)

Among the implied properties of the transformation function (3.37) are

$$(j_3j_1jm|j_2m_2jm_1m_3) = (j_1j_2j - m_1j_2 - m_3)$$

$$= (-1)^{j_1+j_2-j_3} (j_1j_2jm|j_1m_1j_2m_3).$$  \hspace{1cm} (3.47)

The expression for $X(j_1j_2j_3; m_1m_2m_3)$, obtained by expanding (3.39), is

$$X(j; m) = [(J + 1)!]^{-1/2} \sum_n (-1)^n \prod_{i=1}^2 \frac{[(j_i + m_i)!(j_i - m_i)!]}{(J - 2j_i - n_i)!}$$

in which

$$n = n_1 + n_2 + n_3,$$  \hspace{1cm} (3.48)

and the summation is to be extended over all $n_i$ subject to

$$J - 2j_i \geq n_i \geq 0,$$  \hspace{1cm} (3.49)

and

$$n_2 - n_3 = m_1 - j_2 + j_3, n_3 - n_1 = m_2 - j_3 + j_1, n_1 - n_2 = m_3 - j_1 + j_2.$$  \hspace{1cm} (3.50)

The latter conditions can also be written as

$$J - 2j_1 - n_1 = j_2 + m_2 - n_3 = j_3 - m_3 - n_2$$

$$J - 2j_2 - n_2 = j_3 + m_3 - n_1 = j_1 - m_1 - n_3$$

$$J - 2j_3 - n_3 = j_1 + m_1 - n_2 = j_2 - m_2 - n_1.$$  \hspace{1cm} (3.51)
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It follows from the non-negative character of these quantities that the \( n_i \) are uniquely determined if one of the nine integers \( J - 2j_i, j_i + m_i, j_i - m_i \) is equal to zero. In general, the number of terms in the sum (3.48) exceeds by unity the smallest of these nine integers. It is a matter of convenience which of the \( n_i \) is chosen as the summation parameter.

The \( X \) coefficient can also be exhibited in closed form whenever the \( |m_i| \) have the minimum values compatible with the given \( j_i \). The simplest illustration of this is provided by \( X(j_1, j_2, j_3; 000) \) corresponding to integral values of \( j_1, j_2, \) and \( j_3 \). Note that this quantity vanishes, according to (3.46), if \( \frac{1}{2} J \) is not an integer. Our procedure here is to place \( x_- = \partial/\partial x_+ \), with analogous substitutions for \( y_- \) and \( z_- \), and to evaluate the derivatives at \( x_+ = y_+ = z_+ = 0 \). Since

\[
[(j_1 + m_1)(j_1 - m_1)]^{-1/2} (\partial/\partial x_+)^{j_1 - m_1} x_+^{j_1 + m_1} x_+ = 0 = \delta_{m_1, 0},
\]

this effectively isolates the \( m = 0 \) terms in (3.42). The reduction of the generating function can be performed with the aid of the following theorem concerning ordered operators, which will be proved in Appendix B. If \( a \) and \( a^+ \) are two operators satisfying \([a, a^+] = 1\), and \( f(a^+) \) is an arbitrary function, we have

\[
e^{az; a^+} f(a^+) = \frac{1}{1 - z} f \left( \frac{a^+}{1 - z} \right) e^{az; a^+} f.
\]

The differential operator realization of this, with \( a = \partial/\partial a^+ \), is the form actually employed.

The result of the calculation is

\[
\sum_j \Phi_{j_1, j_2, j_3} (a^+ \gamma) X(j_1, j_2, j_3; 000) = (1 + a^2 + \beta^2 + \gamma^2)^{-1},
\]

which is a generating function for \( X(j; 0) \). On writing

\[
(1 + a^2 + \beta^2 + \gamma^2)^{-1} = \sum_{J=0,2, \ldots} (-1)^J (a^2 + \beta^2 + \gamma^2)^J \]

\[
= \sum_J (-1)^{J} \frac{(\frac{1}{2} J)!}{(\frac{1}{2} J - j_1)! (\frac{1}{2} J - j_2)! (\frac{1}{2} J - j_3)!},
\]

we obtain the explicit formula\(^8\)

\[
X(j; 0) = (-1)^{J} \frac{(\frac{1}{2} J)!}{[(J + 1)!]^{1/2}} \prod_{i=1}^{3} \frac{[(J - 2j_i)!]^{1/2}}{(\frac{1}{2} J - j_i)!}.
\]

We extend this argument by making the substitutions \( x_- \to \partial/\partial x_+ \), \( y_- \to \partial/\partial y_+ \), \( z_- \to \partial/\partial z_+ \), and evaluating the derivatives for arbitrary \( x_+, y_+ \).

\(^8\) This result is contained in R. Eq. (22').
and \( z_+ \). In view of

\[
\phi_{j_1 m_1}(x) \to \left[ (j_1 + m_1)! \right]^{1/2} \frac{x_{+}^{2m_1}}{(2m_1)!}, \quad m_1 \geq 0
\]

\[
0, \quad m_1 < 0,
\]  

(3.58)

and

\[
\phi_{j_3 m_3}(x) \to \left[ (j_3 + m_3)! \right]^{1/2} \frac{x_{-}^{2m_3}}{(2m_3)!}, \quad m_3 \leq 0
\]

\[
0, \quad m_3 > 0,
\]  

(3.59)

we shall thereby obtain the \( X \) coefficient for \( m_1 \geq 0, m_2 \geq 0, m_3 = m_1 + m_2 \). The values of \( X \) when two of the \( m_i \) are negative can then be inferred from (3.46). The generating function now becomes

\[
e^{a[x_1] + \beta[x_2] + \gamma[x_3]} \to (1 + \alpha^2 + \beta^2 + \gamma^2)^{-1}
\]

\[
\exp \left\{ \frac{x_{-}}{1 + \alpha^2 + \beta^2 + \gamma^2} \left[ (ax_1 - \beta x_2 + (\beta y - \alpha y_1) \right] \right\} \]

(3.60)

and, on expanding in powers of \( x_+, y_+ \), and \( z_- \), we find that

\[
\sum_f \Phi_{j_1 j_2 j_3} (\alpha \beta \gamma) \prod_i \left[ \frac{(j_i + |m_i|)!}{(j_i - |m_i|)!} \right]^{1/2} X(j; m)
\]

\[
= (2|m_3|)! \frac{(ax_1 - \beta x_2 + (\beta y - \alpha y_1)^{2m_3}}{(1 + \alpha^2 + \beta^2 + \gamma^2)^{2m_3}} \]

(3.61)

The result attained by further expansion of (3.61) is

\[
\left[ (J + 1)! \prod_i \frac{(j_i + |m_i|)!}{(j_i - |m_i|)!} (J - 2j_i)! \right]^{1/2} X(j; m)
\]

\[
= \sum_{n_1 n_2} (-1)^{J_1 + 2|m_1|} \frac{(j_1)!}{(j_1 - j_1 - |m_1|)!} \frac{(2m_1)!}{(2n_1)!} \frac{(2m_2)!}{(2n_2)!} \]  

(3.62)

where

\[
J_1 = J + n_1 - n_2, \quad J_2 = J - n_1 + n_2, \quad J_3 = J + n_1 + n_2.
\]  

(3.63)

The double summation is to be extended over such non-negative integers that satisfy

\[
J - 2j_1 - n_2 \geq 2m_1 - n_1 \geq 0
\]

\[
J - 2j_2 - n_1 \geq 2m_2 - n_2 \geq 0
\]

\[
J - 2j_3 \geq 2|m_3| - n_1 - n_2 \geq 0.
\]  

(3.64)
and for which \( J + n_1 + n_2 \) is an even integer. The sum consists of a single term if one of the \( J - 2j_i \) vanishes, or if \( m_1 = m_2 = 0 \). This simplification may also result from the evenness requirement on \( J \). Thus

\[
\left[ \frac{(j_1 + \frac{1}{2})(j_2 + \frac{1}{2})(J + 1)!}{(J - 2j_i)!} \right] X(j_1 j_2 j_3; \frac{1}{2}0 - \frac{1}{2}) = (-1)^{J-1} \frac{(4J)!}{(4J - j_1 - \frac{1}{2})(4J - j_2 - \frac{1}{2})(4J - j_3 - \frac{1}{2})!} \quad J \text{ even}
\]

\[
= (-1)^{J+1} \frac{(4J + \frac{1}{2})!}{(4J - j_1)(4J - j_2 - \frac{1}{2})(4J - j_3)!} \quad J \text{ odd, (3.65)}
\]

which are the \( X \) coefficients with the minimum \( |m_1| \) corresponding to half-integer values for two of the \( j_i \).

The orthogonality and normalization of the eigenvectors \( \Psi(jm \mu \nu) \) can be verified, with the aid of (3.39), by an extension of the procedure leading to (1.21). According to Eq. (C7) of Appendix C, we have

\[
\exp \left\{ \lambda (a^+ b^+) + \xi_+ (xa^+) + \xi_- (xb^+) \right\} \Psi_0, \quad \exp \left\{ \kappa (a^+ b^+) + \eta_+ (ya^+) + \eta_- (yb^+) \right\} \Psi_0 = \frac{1}{(1 - \lambda^* \kappa)^2} \exp \left\{ \frac{(\xi^* \eta)(x^* y)}{1 - \lambda^* \kappa} \right\}
\]

(3.66)

and the expansion

\[
\frac{1}{(1 - \lambda^* \kappa)^2} \exp \left\{ \frac{(\xi^* \eta)(x^* y)}{1 - \lambda^* \kappa} \right\}
\]

\[
= \sum_{j \mu \nu} \frac{1}{2j + 1} \phi_{jm}(x^*) \phi_{jm}(y) \phi_{j \mu}(\xi^*) \phi_{j \mu}(\eta) X_{jm}(\lambda^*) X_{j \mu}(\kappa)
\]

(3.67)

establishes that

\[
(\Psi(jm \mu \nu), \Psi(j'm' \mu' \nu')) = \delta_{jj'} \delta_{mm'} \delta_{\mu \mu'} \delta_{\nu \nu'}.
\]

(3.68)

The unitary nature of the transformation \( \Psi(j_1 j_2 j_3 m_1 m_2 m_3) \to \Psi(j_1 j_2 j_3 m_1 m_2 m_3) \), and of its inverse, imposes the following conditions upon the \( X \) coefficients,

\[
\sum_{m_1 m_2 m_3} X(j_1 j_2 j_3, m_1 m_2 m_3) X(j_1 j_2 j_3; m_1 m_2 m_3) = \frac{1}{2j_3 + 1} \delta_{j_3 j_3} \delta_{m_3 m_3}
\]

(3.69)

and

\[
\sum_{j \mu \nu} (2j + 1) X(j_1 j_2 j_3; m_1 m_2 m_3) X(j_1 j_2 j_3; m_1 m_2 m_3) = \delta_{m_1 m_1}' \delta_{m_3 m_3}'.
\]

(3.70)

As a particular consequence of (3.69), we have

\[
\sum_m [X(j; m)]^2 = 1.
\]

(3.71)
The Rotation Matrices

The results of this section can be applied in developing further the properties of the matrices $U^{(j)}_{m_1 m_2}(\phi \theta \psi)$, which were introduced in Section 2. If $U$ is the operator generating a reference system rotation for the composite system with angular momentum $J = J_1 + J_2$, while $U_1$ and $U_2$ are the corresponding operators for the individual angular momenta, we have

$$U = U_1 U_2,$$

(3.72)

according to the exponential form (2.31). In particular, theorem (2.69) states that

$$\int U_1 U_2 d\omega = P_0,$$

(3.73)

where $P_0$ is the projection operator for the $J = 0$ state of the resultant angular momentum. On taking matrix elements of the latter equation, we find

$$\int U^{(j_1)}_{m_1 m_2} (\omega) U^{(j_2)}_{m_2 m_3} (\omega) d\omega = (j_1 m_1 j_2 m_2) P_0 (j_1 m_1' j_2 m_2')$$

$$= (j_1 m_1 j_2 m_2) (j_1 j_2 00)(j_1 j_2 00) (j_1 m_1' j_2 m_2')$$

$$= \frac{1}{2j_1 + 1} \delta_{j_1 j_1} \delta_{-m_1 m_2} \delta_{-m_1 m_2'} (-1)^{m_1 - m_2'}$$

(3.74)

since

$$(j_1 j_2 00 | j_1 m_1 j_2 m_2) = (2j_1 + 1)^{-1/2} (-1)^{j_1 - m_1} \delta_{j_1 j_2} \delta_{-m_1 m_2}.$$  

(3.75)

In view of (2.41), it is also possible to write (3.74) as

$$\int U^{(j_1)}_{m_1 m_2} (\omega) U^{(j_2)}_{m_2 m_3} (\omega) d\omega = \frac{1}{2j_1 + 1} \delta_{j_1 j_1} \delta_{m_1 m_2} \delta_{-m_1 m_2'},$$

(3.76)

which expresses the orthogonality properties of the rotation matrices, in their dependence upon the rotation parameters.

The orthogonality relation of the trace $\chi^{(j)}$, derived from (3.76), is

$$\int \chi^{(j_1)} (\omega)^* \chi^{(j_2)} (\omega) d\omega = \delta_{j_1 j_2}.$$  

(3.77)

This integral can be simplified, since the $\chi^{(j)}$ depend only upon the rotation angle $\gamma$. We write

$$d\omega = \int_0^{2\pi} d\gamma \sin \frac{\gamma}{2} \delta \left( \cos \frac{\gamma}{2} - \cos \frac{\phi + \phi'}{2} \right) d\omega,$$

(3.78)

and, after first performing the $d\omega$ integration, obtain

$$\int_0^{2\pi} \chi^{(j_1)}(\gamma)^* \chi^{(j_2)}(\gamma) \frac{1}{\pi} \sin^2 \frac{\gamma}{2} d\gamma = \delta_{j_1 j_2},$$

(3.79)

which can be verified directly.
ON ANGULAR MOMENTUM

We return to (3.72) and observe that its matrix element is

\[ U_{m_1m_1'}^{(j)}(\omega)U_{m_2m_2'}^{(j)}(\omega) = \sum_{j_{mm'}} (j_1m_1j_2m_2j_{j_1j_2}) U_{nmn}^{(j)}(\omega)(j_{j_1j_2}m_1m_2) U_{nmn'}^{(j)}(\omega)(j_{j_1j_2}m_1m_2') \]

\[ = \sum_{j_{mm'}} (2j + 1)X(j_{j_1j_2}; m_1m_2; m) (-1)^m' U_{nmn}^{(j)}(\omega)X(j_{j_1j_2}; m_1m_2; m'), \]  

(3.80)

or

\[ U_{m_1m_1'}^{(j)}(\omega)U_{m_2m_2'}^{(j)}(\omega) \]

\[ = \sum_{j_{mm'}} (2j + 1)X(j_{j_1j_2}; m_1m_2m) U_{nmn}^{(j)}(\omega)X(j_{j_1j_2}; m_1m_2; m'). \]  

(3.81)

With the use of the orthogonality relation (3.76), this can be presented in the symmetrical form

\[ \int U_{m_1m_1'}^{(j)}U_{m_2m_2'}^{(j)}\ d\omega = X(j_{j_1j_2}; m_1m_2m) X(j_{j_1j_2}; m_1m_2; m'). \]  

(3.82)

Specializations of this integral are provided by

\[ U_{00}^{(j)} = \left( \frac{4\pi}{2l + 1} \right)^{1/2} Y_{lm}^*(\theta\phi), \]  

(3.83)

and

\[ U_{00}^{(0)} = P_l(\cos\theta), \]  

(3.84)

where \( Y_{lm} \) is the spherical harmonic associated with integral \( l \), and \( P_l(\cos\theta) \) is the Legendre polynomial.

Thus

\[ \int Y_{l_1m_1} Y_{l_2m_2} Y_{l_3m_3} \sin\theta \ d\theta \ d\phi \frac{1}{2\pi} \]  

\[ = \left[ \prod_{i} \left( \frac{2l_i + 1}{4\pi} \right) \right]^{1/2} X(l; 0) X(l; m), \]  

(3.85)

and

\[ \int_{0}^{\pi} P_{l_1}(\cos\theta) P_{l_2}(\cos\theta) P_{l_3}(\cos\theta) \sin\theta \ d\theta = [X(l; 0)]^2. \]

(3.86)

The multiplication property of the trace, as derived from (3.80) is

\[ \chi^{(j_1)}(y)\chi^{(j_2)}(y) = \sum_{j = |j_1 - j_2|}^{j_1 + j_2} \chi^{(j)}(y), \]  

(3.87)

which can also be expressed in the form

\[ \int_{0}^{2\pi} \chi^{(j_1)}(y)\chi^{(j_2)}(y) \sin\theta \ d\theta = \begin{cases} 1, & J - 2j_i \geq 0 \\ 0, & \text{otherwise} \end{cases} \]

(3.88)

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One can regard this as a realization of the projection operator statement of the angular momentum composition law,

\[ P_{j_1} P_{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} P_j, \]

(3.89)

since (3.87) is the trace of the equation obtained by multiplying (3.89) with \( U_1 U_2 = U \).

We shall conclude this discussion by deriving the completeness relations for the functions \( \chi^{(j)}(\omega) \) and \( U^{(j)}_{\text{min}}(\omega) \). Referring to (2.36), the generating function of the \( \chi^{(j)} \), we replace \( t \) therein with \( t e^{i(\frac{\gamma}{2} \omega')} \) and obtain

\[ \sum_j t^{2j} \chi^{(j)}(\omega) e^{(j+1)\gamma'} = \frac{1}{(1 + t^2) \cos \frac{\gamma'}{2} - 2t \cos \frac{\gamma'}{2} - i \sin \frac{\gamma'}{2} (1 - t^2)}, \]

(3.90)

the imaginary part of which can be written

\[ \sum_j t^{2j} \chi^{(j)}(\omega) \chi^{(j)}(\omega') = \frac{(1 - t^2)}{(1 - t^2)^2 \left( 1 - \frac{4t}{1 + t^2} \cos \frac{\gamma}{2} \cos \frac{\gamma'}{2} \right) + 4t^2 \left( \cos \frac{\gamma}{2} - \cos \frac{\gamma'}{2} \right)^2}. \]

(3.91)

We now consider the limit \( t \to 1 \), and infer from the known result

\[ \lim_{\varepsilon \to 0} \frac{\ln \pi}{\varepsilon} = \delta(x), \]

(3.92)

that

\[ \sum_j \chi^{(j)}(\omega) \chi^{(j)}(\omega') = \frac{\pi}{2 \sin(\gamma/2)} \delta \left( \cos \frac{\gamma}{2} - \cos \frac{\gamma'}{2} \right). \]

(3.93)

However

\[ \int \frac{\pi}{2 \sin \gamma/2} \delta \left( \cos \frac{\gamma}{2} - \cos \frac{\gamma'}{2} \right) d\omega = \int_0^{2\pi} \delta \left( \cos \frac{\gamma}{2} - \cos \frac{\gamma'}{2} \right) \sin \frac{\gamma}{2} \frac{1}{2} d\gamma = 1, \]

(3.94)

so that (3.93) can be written

\[ \sum_j \chi^{(j)}(\omega) \chi^{(j)}(\omega') = \delta(\omega - \omega'), \]

(3.95)

which is the completeness relation of the \( \chi^{(j)} \). As a specialization of (3.95), we place \( \gamma' = 0 \) and find

\[ \sum_j (2j + 1) \chi^{(j)}(\omega) = \delta(\omega). \]

(3.96)
ON ANGULAR MOMENTUM

An operator expression for the composition of successive rotations is given by

\[ U(\omega) U^{-1}(\omega') = U(\omega - \omega'). \]  \hspace{1cm} (3.97)

We take the trace of this equation for the states with quantum number \( j \), and, in virtue of the unitary property of \( U \), obtain

\[ \sum_{mm'} U^{(j)}_{mm}(\omega) U^{(j)}_{mm'}(\omega')^* = \chi^{(j)}(\omega - \omega'), \]  \hspace{1cm} (3.98)

which is in the nature of an addition theorem. The completeness relation for the \( U^{(j)}_{mm}(\omega) \) is reached on multiplying (3.98) with \( 2j + 1 \) and summing with respect to \( j \). In view of (3.96), we have

\[ \sum_{jmm'} (2j + 1) U^{(j)}_{mm}(\omega) U^{(j)}_{mm'}(\omega')^* = \delta(\omega - \omega'). \]  \hspace{1cm} (3.99)

On integration of (3.98) and (3.99) with respect to the Eulerian angle \( \psi \), there emerges the addition theorem and the completeness relation of the spherical harmonics.

4. THREE AND FOUR ANGULAR MOMENTA

Eigenveectors for the resultant of three angular momenta can be built up in several ways, as symbolized by

\[ J = J_1 + (J_2 + J_3) = J_2 + (J_3 + J_1) = J_3 + (J_1 + J_2). \]  \hspace{1cm} (4.1)

Thus, according to the first procedure, we construct \( \Psi(j_1m_1j_2j_3j_4m_4) \) and then \( \Psi(j_1j_2j_3j_4m_1m_2m_3m_4) \), while the last method yields \( \Psi(j_3j_2j_1j_4j_3j_4j_2j_1m_1m_2m_3m_4) \). The notation \([j_1j_2j_3]\), for example, is intended to indicate that these angular momenta are not involved explicitly in the composition of \( j_1 \) and \( j_2 \) to form \( j \). Similarly, four angular momenta can be combined in various pairs,

\[ J = (J_1 + J_2) + (J_3 + J_4) = (J_2 + J_3) + (J_4 + J_1) = (J_1 + J_2) + (J_3 + J_4), \]  \hspace{1cm} (4.2)

in which the first method, say, yields \( \Psi(j_1j_2j_3j_4j_1j_2j_3j_4j_3j_4j_2j_1m_1m_2m_3m_4m_5m_6) \) through the intermediary of \( \Psi(j_1j_2j_3j_4j_1j_2j_3j_4j_3j_4j_2j_1m_1m_2m_3m_4m_5m_6) \). Our problem in this section is the evaluation of the transformation function connecting two such schemes of adding four angular momenta. The analogous question for three angular momenta can be regarded as a specialization of this more symmetrical problem.

To facilitate the addition of angular momenta in pairs, we observe that the generating function (3.35), written as

\[ \sum_{j_1j_2j_3j_4} (2j + 1)^{-1/2} \phi_{jm}(x) \Phi_{j_1j_2}(\alpha_1, \alpha_2, \alpha_3) \Phi_{j_3j_4}(\alpha_4, \alpha_5) \Psi(j_1j_2j_3j_4) \]

\[ = \exp(\alpha_1^2a^{+}a) + \alpha_2^2(xa^{+}) + \alpha_4(xb^{+}) \Psi_{0}, \]  \hspace{1cm} (4.3)

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can be obtained from
\[
\sum_{j_1m_1j_2m_2} \phi_{j_1m_1}(t_1) \phi_{j_2m_2}(t_2) \Psi(j_1m_1j_2m_2) = \exp((t_1a^+ + (t_2b^+))\Psi_0
\]
by the application of the differential operator
\[
\exp\left(\alpha_0 \left[ \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \right] + \alpha_2 \left( x \frac{\partial}{\partial t_2} \right) + \alpha_1 \left( z \frac{\partial}{\partial t_2} \right) \right)
\]
with the understanding that the derivatives are to be evaluated at \(t_1 = t_2 = 0\).

Accordingly, if we apply (4.5) and
\[
\exp\left(\beta_3 \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] + \beta_2 \left( y \frac{\partial}{\partial y} \right) + \beta_1 \left( y \frac{\partial}{\partial y} \right) \right)
\]
to the generating function of the \(\Psi(j_1m_1j_2m_2j_3m_3j_4m_4)\), namely,
\[
\exp((t_1a^+ + (t_2b^+ + (t_3c^+ + (t_4d^+))\Psi_0,
\]
we shall obtain a function generating \(\Psi(j_1j_2j_3j_4m_1m_2j_3j_4m_4)\). The further application of the operator
\[
\exp\left(\gamma_3 \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] + \gamma_2 \left( z \frac{\partial}{\partial y} \right) + \gamma_1 \left( z \frac{\partial}{\partial y} \right) \right)
\]
then produces
\[
\sum_{j_1j_2j_3j_4j_5j_6j_7} \left[ (2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1) \right]^{-1/2} \phi_{j_1j_2j_3j_4j_5j_6j_7}(z) \Phi_{j_1j_2j_3j_4j_5j_6j_7} = e^{Q} \Psi_0,
\]
in which
\[
Q = \alpha_0[a^+b^+] + \beta_0[c^+d^+] + \gamma_3 \alpha_1 \beta_1[b^+c^+] + \gamma_3 \alpha_1 \beta_1[b^+d^+] + \gamma_3 \alpha_1 \beta_1[a^+d^+]
\]
\[
+ \gamma_3 \alpha_1 \beta_1[a^+c^+] + \gamma_3 \alpha_1 \beta_1[a^+d^+] + \gamma_2 \alpha_1 \beta_1[b^+c^+] + \gamma_2 \alpha_1 \beta_1[b^+d^+] + \gamma_2 \alpha_1 \beta_1[a^+d^+]
\]
\[
+ \gamma_2 \alpha_1 \beta_1[a^+c^+] + \gamma_1 \beta_2 [a^+c^+] + \gamma_1 \beta_2 [a^+d^+] + \gamma_1 \beta_2 [a^+d^+].
\]
As an important specialization of (4.9), yielding the eigenvectors with \(j = 0\), we place \(\gamma_3 = 0\), and \(\gamma_3 = 1\), with the result
\[
\sum_{j_1j_2j_3j_4j_5j_6j_7} (2j_1 + 1)^{-1/2} \phi_{j_1j_2j_3j_4j_5j_6j_7}(z) \Phi_{j_1j_2j_3j_4j_5j_6j_7} = e^{R} \Psi_0,
\]
\[
R = \alpha_0[a^+b^+] + \beta_0[c^+d^+] + \alpha_1 \beta_1[b^+c^+] + \alpha_1 \beta_1[b^+d^+] + \alpha_1 \beta_1[a^+d^+]
\]
\[
+ \alpha_2 \beta_2 [a^+c^+],
\]
where \(j_1 = j_2 = j_3\). An analogous equation for a different mode of addition is
\[
\sum_{j_1j_2j_3j_4j_5j_6j_7} (2j_1 + 1)^{-1/2} \phi_{j_1j_2j_3j_4j_5j_6j_7}(z) \Phi_{j_1j_2j_3j_4j_5j_6j_7} = e^{R'} \Psi_0,
\]
\[
R' = \alpha_0[a^+c^+] + \beta_0[b^+d^+] + \alpha_1 \beta_1[c^+d^+] + \alpha_1 \beta_1[b^+c^+] + \alpha_1 \beta_1[a^+d^+]
\]
\[
+ \alpha_2 \beta_2 [a^+b^+].
\]
ON ANGULAR MOMENTUM

The transformation function connecting the two schemes is determined by

\[ \sum \frac{(-1)^{j^*+j^*+j^*} \Phi_{j,i,j^*}(\alpha)\Phi_{j^*,j^*}(\beta^*)\Phi_{j^*,i,j^*}(\alpha^*)\Phi_{j^*,j^*}(\beta^*)}{j_{i,j^*} j_{j^*} j_{j^*}} W(j_{i,j^*}, j_{j^*}, j_{j^*}) = \omega \frac{\Psi_0}{e^{R\Psi_0}}, \quad (4.13) \]

in which we have written

\[ (j_{i,j^*}j_{j^*}j_{j^*}) (j_{j^*}j_{j^*}j_{j^*}) = (-1)^{j^*+j^*+j^*} \quad (2j^*+1)(2j^*+1)^{1/2} W(j_{i,j^*}, j_{j^*}, j_{j^*}^*). \quad (4.14) \]

We now employ the theorem (Equation (C28))

\[ \exp(\frac{1}{4} \sum_{\mu=1}^{4} \lambda_{\mu} A_{\mu}^+ A_{\mu} + \frac{1}{4} \sum_{\mu=1}^{4} \kappa_{\mu} A_{\mu}^+ A_{\mu}) \Psi_0 \] = \[ [A_{\mu}, A_{\mu}^+] = \delta_{\mu,\nu} \delta_{\mu,\nu}, \quad (4.16) \]

and \(|\lambda|, |\kappa|\) are the determinants of the antisymmetrical matrices \(\lambda_{\mu}\) and \(\kappa_{\mu}\).

For the application in question,

\[ |\lambda|^{1/2} |\kappa|^{1/2} = -\alpha_0 \beta_0 \alpha_0 \beta_0, \]

\[ \frac{1}{4} \sum_{\mu,\nu} \lambda_{\mu} \kappa_{\nu} = \alpha_0 \alpha_0 \beta_0 \beta_0 + \beta_0 \alpha_0 \beta_0 \beta_0 + \alpha_0 \alpha_0 \beta_0 \beta_0 - \alpha_0 \beta_0 \alpha_0 \beta_0 + \alpha_0 \beta_0 \alpha_0 \beta_0 + \alpha_0 \beta_0 \alpha_0 \beta_0. \quad (4.17) \]

On changing the signs of \(\alpha_0\) and \(\beta_0\), we obtain for the generating function of the \(W\) coefficients,

\[ \Phi_{j,i,j^*}(\alpha)\Phi_{j^*,j^*}(\beta)\Phi_{j^*,i,j^*}(\alpha^*)\Phi_{j^*,j^*}(\beta^*) \]

\[ = \frac{(-1)^{j^*+j^*+j^*}}{(2j^*+1)(2j^*+1)^{1/2}} W(j_{i,j^*}, j_{j^*}, j_{j^*}^*). \quad (4.18) \]

The symmetry properties expressed by

\[ W(j_{i,j^*}, j_{j^*}, j_{j^*}) = W(j_{j^*}, j_{j^*}, j_{j^*}) = W(j_{i,j^*}, j_{j^*}, j_{j^*}^*) \]

\[ = W(j_{j^*}, j_{j^*}, j_{j^*}^*). \quad (4.19) \]

follow from the invariance of (4.18) under the respective substitutions:

\( \alpha_0 \leftrightarrow \beta_0, \beta_0 \leftrightarrow \alpha_0, \alpha \leftrightarrow \beta, \beta_0 \leftrightarrow \beta_0, \alpha \leftrightarrow \beta, \beta \leftrightarrow \beta \), while the more complicated transformation \((\alpha_0 \alpha_0 \beta_0 \beta_0) \rightarrow (-\alpha_0 \alpha_0 \beta_0 \beta_0), \alpha_0 \beta_0 \alpha_0 \beta_0 \rightarrow (\beta_0 \beta_0 \alpha_0 \beta_0), \beta_0 \beta_0 \alpha_0 \beta_0 \rightarrow (\beta_0 \beta_0 \beta_0 \beta_0) \)

\( \beta_0 \beta_0 \beta_0 \beta_0 \rightarrow (\beta_0 \beta_0 \beta_0 \beta_0) \) yields

\[ W(j_{i,j^*}, j_{j^*}, j_{j^*}^*) = (-1)^{j^*+j^*+j^*} W(j_{j^*}, j_{j^*}, j_{j^*}^*). \quad (4.20) \]

\[ \text{For simplicity we have assumed that the parameters } \alpha, \beta \text{ are real. The generating function (4.18) is valid without this restriction.} \]

\[ \text{The } W \text{ coefficient thereby defined is the same as that discussed in } R. \]

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Twenty-four equivalent forms for $W$ are obtained by repeated use of (4.19) and (4.20).

Further characteristics of $W$ follow from the composition properties of the transformation function (4.14), which we shall temporarily indicate by $(12, 34j'|13, 24j^*)$. Thus

$$\sum_{j'} (12, 34j'|13, 24j^*)(13, 24j^*|12, 34j^*) = \delta_{j'j^*}$$  \hspace{1cm} (4.21)

and

$$\sum_{j'} (12, 34j'|13, 24j^*)(13, 24j^*|14, 23j^*) = (12, 34j'|14, 23j^*)$$.  \hspace{1cm} (4.22)

All of these quantities can be expressed in terms of $W$. The interchange of 2 and 4, and of 3 and 4 in (4.14) yields, with the aid of (3.47),

$$(13, 24j^*|14, 23j^*) = (-1)^{j_3+j_4-j_1}(2j^*+1)(2j^*+1)^{1/2}W(j_1, j_2, j_3; j^*, j^*)$$,

and

$$(12, 34j^*|14, 23j^*) = (-1)^{j^*+j_3-j_1}(2j^*+1)(2j^*+1)^{1/2}W(j_1, j_2, j_3; j^*, j^*)$$.  \hspace{1cm} (4.23)

Therefore

$$\sum_{j'} (2j^*+1)W(j_1, j_2, j_3; j^*, j')W(j_1, j_2, j_3; j^*, j'^*) = \frac{1}{2j^*+1} \delta_{j'j'^*}$$  \hspace{1cm} (4.25)

and

$$\sum_{j'} (-1)^{j^*+j'^*+j_3+j_4}(2j^*+1)W(j_1, j_2, j_3; j^*, j')W(j_1, j_2, j_3; j^*, j'^*)$$

$$= W(j_1, j_2, j_3; j^*, j'^*)$$.  \hspace{1cm} (4.26)

These formulae can be combined by placing $j_2 = j_1$, $j' = j^*$ in (4.26) and, after multiplication with $2j^*+1$, performing the summation with respect to $j'$ by means of (4.25). We obtain

$$\sum_{j'} (2j^*+1)W(j_1, j_2, j_3; j^*, j') = \sum_{j'} (-1)^{j^*+j_3+j_4}$$

in which the values assumed by $j^*$ are those compatible with the existence of $W(j_1, j_2, j_3; j^*, j'^*)$, namely, $j^* \geq |j_1 - j_3|$, $j^* \leq j_1 + j_3$, $2j_2$. Accordingly,

$$\sum_{j'} (2j^*+1)W(j_1, j_2, j_3; j^*, j') = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$  \hspace{1cm} (4.27)

where $k$ is the smaller of the two integers $j_1 + j_3 - |j_1 - j_3|$, $2j_2 - |j_1 - j_3|$. One of the consequences of (4.28),

$$W(j_1, j_2, j_3; j_2j_3) = \frac{1}{2j_2+1}, \quad j_1 \leq 2j_2$$  \hspace{1cm} (4.29)
is a particular example of

\[ W(j_1j_2j_3; j_2j_3) = [(2j_2 + 1)(2j_3 + 1)]^{-1/2}, \quad |j_2 - j_3| \leq j_1 \leq j_2 + j_3, \quad (4.30) \]

which follows from (4.14) on remarking that, with \( j_4 = 0 \), the interchange of \( j_2 \) and \( j_3 \) simply multiplies the eigenvector with \((-1)^{j_2 + j_3 - j_1}\).

The relation between the \( W \) and \( X \) coefficients can be inferred from (4.14) by writing

\[
\Psi(j_1, j_2, j_3; j_1'j_2'j_3') = \sum_{m_{12}} (2j_1' + 1)^{-1/2}(-1)^{j_2' - m_{12}} \Psi(j_1, j_2, j_3; m_{12}j_2j_3j_1', m_{12}) \\
= (2j_1' + 1)^{1/2}(-1)^{j_1' + j_2' - j_3} \sum_{m_{12}} X(j_1, j_2, j_3; m_1m_2 - m_{12})(-1)^{j_2' - m_{12}} \\
\cdot X(j_2', j_3'; m_3m_4m_{12}) \Psi(j_1, j_2, j_3; m_1m_2m_3m_4m_1), \quad (4.31)
\]

which, with the similar representation of \( \Psi((j_1, j_2, j_3'; j_2', j_3', j_1')0) \), yields

\[
\sum_{m_1} X(j_1, j_2, j_3; m_1m_2 - m_{12})(-1)^{j_2 - m_{12}} X(j_3, j_2, j_1; m_3m_4m_{12}) \\
\cdot X(j_2', j_3'; m_3m_4 - m_{13})(-1)^{j_3 - m_{13}} X(j_1', j_2', j_3; m_3m_4m_{13}) \\
= (-1)^{j_1 + j_2' + j_3' - j_2 - j_3'} W(j_1, j_2, j_3; j_1', j_2', j_3'). \quad (4.32)
\]

The general expression obtained for \( W \) by expanding the generating function (4.18) can be cast into the form

\[
W(j_1, j_2, j_3; j_1', j_2', j_3') = \prod_{r} \left[ \frac{1}{(n + p + 1, n_r, p_r)} \right]^{-1/2} \prod_{r,s} \left[ \frac{1}{(n_r + p_r)} \right]^{1/2} \\
\cdot \sum (-1)^{p_3} \left( n + p + 1 \right)! \prod_{r,s} n_r!p_r! , \quad (4.33)
\]

where

\[
n = \sum_{r=1}^{4} n_r, \quad p = \sum_{s=1}^{3} p_s, \quad (4.34)
\]

and the summation is to be extended over the non-negative integers, \( n_r, p_r \), for which

\[
\begin{align*}
  j_2 + j_3 - j_1 - p_1 &= j_4 + j_3' - j_2 - p_2 = j_2 + j_4 - j_3' - p_3 = n_1, \\
  j_3 + j_4 - j_1 - p_1 &= j_1 + j_3' - j_2 - p_2 = j_1 + j_3 - j_3' - p_3 = n_2, \\
  j_4 + j_1 - j_3 - p_2 &= j_1 + j_4 - j_2 - p_2 = j_1 + j_4 - j_2 - p_2 = n_3, \\
  j_4 + j_2 - j_3 - p_1 &= j_1 + j_4 - j_2 - p_1 = j_1 + j_2 - j_3' - p_3 = n_4.
\end{align*}
\quad (4.35)
\]

The number of terms in the sum exceeds by unity the smallest of the twelve quantum number combinations, \( j_2 + j_3' - j_1 \), etc.; the sum reduces to a single
term if one such combination vanishes. The choice of summation parameter is a matter of convenience.

We now return to the general problem, that of evaluating the transformation function

\[
(\{j_1,j_2\}i_{12}J_{12}j_{12}J_{12}j_{12}M_{12}m_{12}m_{12}m_{12}m_{12}m_{12}m_{12}) = (-1)^{j_{13}+j_{34}+j_{14}} \\
\cdot[(2j_{13}+1)(2j_{34}+1)(2j_{14}+1)]^{1/2}S(j_1,j_2,j_3,j_4; j_{12},j_{34},j_{14},j_{34}; j_{14}). \tag{4.36}
\]

A generating function for the $S$ coefficient is given by\(^\text{12}\)

\[
\sum \Phi(j_1,j_2,j_3,j_4; \beta_{12} \beta_{34} \beta_{14} \beta_{34}) \Phi(j_1,j_2,j_3,j_4; \gamma_{12} \gamma_{34} \gamma_{14} \gamma_{34}) \\
\cdot \Phi(j_1,j_2,j_3,j_4; \beta_{12} \beta_{34} \gamma_{12} \gamma_{34}) \Phi(j_1,j_2,j_3,j_4; \beta_{12} \gamma_{34} \beta_{14} \gamma_{34}) \\
= [1 + \alpha_2 \beta_2 \gamma_2 \beta_1 - \gamma_2 (\alpha_2 \alpha_3 \beta_2 + \beta_2 \alpha_2 \beta_1) - \gamma_3 (\alpha_2 \alpha_3 \beta_2 + \beta_2 \alpha_2 \beta_1) \\
- \gamma_2 \gamma_3 (\alpha_2 \alpha_3 \beta_2 + \beta_2 \alpha_2 \beta_1) - \gamma_3 \gamma_4 (\alpha_2 \alpha_3 \beta_2 + \beta_2 \alpha_2 \beta_1) - \gamma_1 \gamma_4 (\alpha_2 \alpha_3 \beta_2 + \beta_2 \alpha_2 \beta_1)]^{-1}, \tag{4.37}
\]

where the sum is over all $j$'s. The connection with the $X$ coefficients is contained in

\[
\sum X(j_1,j_2,j_{12}; m_{12}m_{12}m_{12}m_{12}m_{12}m_{12}) X(j_3,j_4,j_{34}; m_{34}m_{34}m_{34}m_{34}m_{34}m_{34}) \\
\cdot X(j_1,j_2,j_{12}; m_{12}m_{12}m_{12}m_{12}m_{12}m_{12}) X(j_3,j_4,j_{34}; m_{34}m_{34}m_{34}m_{34}m_{34}m_{34}) \\
= (-1)^{j_{14}+j_{13}+j_{34}+j_{34}} \cdot S(j_1,j_2,j_3,j_4; j_{12},j_{34},j_{14},j_{34}; j_{14}). \tag{4.38}
\]

(the sum is over all $m$'s) and the $W$ coefficient appears as a special example,

\[
S(j_1,j_2,j_3,j_4; j_{14}j_{34}j_{14}j_{34}; 0) = [(2j_{14}+1)(2j_{34}+1)]^{-1/2} W(j_1,j_2,j_3,j_4; j_{14}j_{34}). \tag{4.39}
\]

In view of the complexity of the $S$ coefficient we shall be content to record here only those cases that can be expressed in terms of $W$. This occurs whenever one of the nine quantum numbers involved in the $S$ coefficient equals zero, which is a consequence of (4.39) and the fact that the symmetry of $S$ is such that any of the other quantum numbers can appear in the position of $j$. Thus, it follows from either (4.37) or (4.38) that

\[
S(j_1,j_2,j_3,j_4; j_{12},j_{34},j_{14},j_{34}; j_{14}) \\
= (-1)^{j_{14}+j_{34}+j_{13}+j_{13}} S(j_1,j_2,j_3,j_4; j_{12},j_{34},j_{14},j_{34}; j_{14}) \\
= (-1)^{j_{14}+j_{34}+j_{13}+j_{34}} S(j_{12},j_{34},j_{14},j_{34}; j_{14}j_{34}). \tag{4.40}
\]

which are representative of the eight permutations of this type. We obtain from (4.39) that

\[
S(j_1,j_2,j_3,j_4; j_{12},j_{34},j_{14},j_{34}; j_{14}) \\
= (-1)^{j_{13}+j_{14}j_{12},j_{34}+1}(2j_{13}+1) S(j_{12},j_{34},j_{13},j_{14},j_{34}). \tag{4.41}
\]

12 This is obtained with the aid of Eq. (3.30).
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and

\[ S(j_1j_2j_3\lambda; j_4j_5j_6j_7j_8; j) = (-1)^{j_8-j_7-j_6} \tilde{\eta}(2j_6 + 1)(2j_8 + 1)^{-1/2} W(j_1; j_2j_3j_4j_5; j_6j_7j_8). \] (4.42)

The latter result contains the solution to the problem of three angular momenta. Expressed in terms of a transformation function, without explicit reference to the angular momentum with zero quantum number, (4.42) states that

\[ ([j_1j_2]_1j_3j_4j_m)[j_1j_2][j_3j_4j_m] \]
\[ = (-1)^{j_1+2j_3-j_2-j_4} \tilde{\eta}(2j_2 + 1)(2j_4 + 1)^{1/2} W(j_1; j_2j_3j_4; j_5j_6j_7). \] (4.43)

A slightly simpler form is obtained on permuting the indices 1 and 2, together with a change in sense of addition for \( j_1 \) and \( j_2 \),

\[ ([j_1j_2]_1j_3j_4j_m)[j_1j_2][j_3j_4j_m] \]
\[ = (2j_1 + 1)(2j_2 + 1)^{1/2} W(j_1; j_2j_3j_4; j_5j_6j_7). \] (4.44)

As a particular consequence of this result, note that, according to (4.30),

\[ ([j_1j_2]_1j_3j_4j_m)[j_1j_2][j_3j_4j_m] = 1, \] (4.45)

that is, the eigenvector for the null resultant of three angular momenta is independent of the mode of addition, provided that the order of the angular momenta is preserved. As one representation of this eigenvector we have

\[ \Psi(j_1j_2j_3\lambda) = \sum_{m_3} [(2j_3 + 1)^{-1/2}(-1)^j_1j_3m_3]\Psi((j_1j_2)j_3j_4j_5j_6j_7j_8), \] (4.46)

and therefore

\[ (j_1j_2j_3j_4j_5j_6j_7j_8) = (2j_3 + 1)^{-1/2}(-1)^j_1j_3m_3(j_1j_2j_3j_4j_5j_6j_7j_8) \]
\[ = (-1)^{j_1+j_2-j_3-j_4} X(j_1j_2j_3; m_1m_2m_3). \] (4.47)

in virtue of (3.40). Thus, the \( X \) coefficient, originally defined in terms of the addition of two angular momenta, now appears as characterizing three angular momenta with a null resultant.

This possibility, of replacing \( J_1 + J_2 = J \) with \( J_1 + J_2 + J_3 = 0 \), depends upon the circumstance that the negative of an angular momentum operator is, in a certain sense, also an angular momentum operator. The commutation relations

\[ J \times J = iJ \] (4.48)

imply that

\[ (-J) \times (-J) = -i(-J), \] (4.49)

\[ ^{13} \text{G. Racah, Phys. Rev. 63, 367 (1943).} \]
which reassemble the form (4.48) on changing the sign of $i$ (complex, not Hermitian conjugation). Therefore

$$J' = -J^*$$  \hspace{1cm} (4.50)

is an angular momentum operator. To find the eigenvectors of $J'$, we notice that a rotation operator $U$ is a function of $iJ$ and real angles. Therefore

$$U' = U^*$$  \hspace{1cm} (4.51)

is the same function of $J'$ that $U$ is of $J$. On taking the complex conjugate of the equation

$$U\Psi(jm') = \sum_m \Psi(jm) U_{mm'}^{(j)}$$  \hspace{1cm} (4.52)

we obtain

$$U'\Psi^*(jm') = \sum_m \Psi^*(jm)(-1)^{m-m'} U_{mm'}^{(j)}$$  \hspace{1cm} (4.53)

with the aid of (2.41). Hence

$$\Psi'(jm) = (-1)^{j+m}\Psi^*(j-m)$$  \hspace{1cm} (4.54)

are the eigenvectors associated with $J'$.

Now observe that the following dyadic, formed from the eigenvectors of a single angular momentum,

$$(2j + 1)^{-1/2}\sum_m \Psi(jm)\Psi^*(jm),$$  \hspace{1cm} (4.55)

is unchanged by a rotation of the reference system, since

$$\sum_{mm'} \Psi(jm') (jm'|U|jm)(jm|U^{-1}|jm^*) \Psi^*(jm^*) = \sum_m \Psi(jm)\Psi^*(jm).$$  \hspace{1cm} (4.56)

Therefore, on employing (4.54) we infer that the vector

$$(2j + 1)^{-1/2}\sum_m \Psi(jm)(-1)^{j-m}\Psi'(j-m)$$  \hspace{1cm} (4.57)

describes the spherically symmetrical state of two angular momenta, which is in agreement with (3.75). This is the basic example of the relationship involved in (4.47).

5. Tensor Operators

An irreducible tensor operator of rank $j(=0, \frac{1}{2}, 1, \ldots)$ is a set of $2j + 1$ operators, $T(jm)$, which transforms in the following manner under a change in coordinate system,

$$UT(jm')U^{-1} = \sum_{m=-j}^{j} T(jm) U_{mm'}^{(j)}.$$  \hspace{1cm} (5.1)
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On taking the Hermitian conjugate of this equation and employing (2.41), we find that $i^{2m} T(j - m)^\dagger$ transforms in the same manner as $T(jm)$. We therefore define the Hermitian conjugate tensor $T^\dagger$ according to

$$T^\dagger(jm) = i^{2m}(T(j - m))^\dagger.$$  \hspace{1cm} (5.2)

The tensor that is conjugate to $T^\dagger$ is then described by

$$T^{\dagger\dagger}(jm) = i^{2m}(T^\dagger(j - m))^\dagger = i^{2m}(i^{-2m} T(jm))^\dagger = (-1)^{2m} T(jm),$$  \hspace{1cm} (5.3)

or

$$T^{\dagger\dagger} = (-1)^{2j} T.$$

This shows that Hermitian tensors, $T^\dagger = T$, exist only for integral $j$,\(^{14}\) and satisfy

$$T(jm) = (-1)^{m} T(j - m)^\dagger.$$  \hspace{1cm} (5.5)

The product of two tensor operators transforms under coordinate system rotations according to

$$UT_1(j_1m_1')T_2(j_2m_2')U^{-1} = (UT_1(j_1m_1')U^{-1})(UT_2(j_2m_2')U^{-1}) = \sum_{m_{12}} T_1(j_1m_1)T_2(j_2m_2)U^{(j_1)}_{m_{11}}U^{(j_2)}_{m_{22}}.$$

It follows from (3.80) that

$$\sum_{m_{12}} T_1(j_1m_1)T_2(j_2m_2)(j_1m_1,j_2m_2; j_1,j_2, jm) = T(j_1j_2jm)$$
\hspace{1cm} (5.7)

obeys

$$UT(j_1j_2jm')U^{-1} = \sum_{m} T(j_1j_2jm)U^{(j_m)}_{mm'},$$
\hspace{1cm} (5.8)

and is therefore an irreducible tensor of rank $j$.

For a tensor operator applied to an angular momentum eigenvector we have, analogously,

$$U(T(j_1m_1')\Psi(j_2m_2')) = (UT(j_1m_1')U^{-1})(U\Psi(j_2m_2')) = \sum_{m_{12}} T(j_1m_1)\Psi(j_2m_2)U^{(j_1)}_{m_{11}}U^{(j_2)}_{m_{22}},$$
\hspace{1cm} (5.9)

so that

$$\sum_{m_{12}} T(j_1m_1)\Psi(j_2m_2)(j_1m_1,j_2m_2; j_1,j_2, jm) = \Phi(j_1j_2jm)$$
\hspace{1cm} (5.10)

obeys

$$U\Phi(j_1j_2jm') = \sum_{m} \Phi(j_1j_2jm)U^{(j_m)}_{mm'}.$$  \hspace{1cm} (5.11)

\(^{14}\) It is similarly impossible to identify the $\Psi'(jm)$ of (4.54) with $\Psi(jm)$, for all $m$, if $j$ is half-integral.
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and is therefore an angular momentum eigenvector with quantum numbers \( j \) and \( m \).

The magnetic quantum number dependence of tensor operator matrix elements is contained in the last statement. On introducing explicitly the additional quantum numbers necessary to form a complete set, we are led to write

\[
\sum_{\gamma m} T(kq|\gamma j' m') = \sum_{\gamma} \Psi(\gamma jm)(2j + 1)^{-1/2}[\gamma j|T^{(k)}|\gamma j'],
\]

where we have employed different letters for the tensor operator indices in order to simplify the notation. It follows from (5.12) that\(^{15}\)

\[
(\gamma jm|T(kq)|\gamma j' m') = (2j + 1)^{-1/2}[\gamma j|T^{(k)}|\gamma j'](kqjm|kqjm') \nonumber \]

\[= (-1)^{k-j+m}[\gamma j|T^{(k)}|\gamma j']X(jk'; -mqm').
\]

(5.13)

As an alternative derivation of the latter result,\(^{16}\) we remark that

\[
(\gamma jm|T(kq)|\gamma j' m') = (U\Psi(\gamma jm), UT(kq)U^{-1}U\Psi(\gamma j' m'))
\]

\[= \sum_{m'm'} (\gamma jm'|T(kq'),\gamma j'm')(\gamma j'm'\gamma jm)(-1)^{m'-m'U^{(k)}_{m'^{j}}U^{(j)}_{m'}).
\]

(5.14)

An integration with respect to \( \omega \) then yields, according to (3.82),

\[
(\gamma jm|T(kq)|\gamma j' m') = \sum_{m'm'} (-1)^{m-m'}X(jk'; -mqm')(\gamma jm|T(kq)|\gamma j'm'),
\]

(5.15)

which is (5.13), with

\[\{\gamma j|T^{(k)}|\gamma j'\} = \sum_{mqm'} (-1)^{k'-m}X(jk'; -mqm')(\gamma jm|T(kq)|\gamma j'm').
\]

(5.16)

According to the definition of the Hermitian conjugate tensor, we have

\[
(\gamma jm|T^{*}(kq)|\gamma j' m') = \varepsilon^{2}(\gamma j'm'|T(k - q)|\gamma jm)^{*}
\]

\[= \varepsilon^{2}(-1)^{k-j+m}[\gamma j'|T^{(k)}|\gamma j']^*X(jk'; -mqm'),
\]

(5.17)

or

\[\{\gamma j|T^{(k)}|\gamma j'\} = \varepsilon^{2}(-1)^{k'-j-m}[\gamma j'|T^{(k)}|\gamma j']^*,
\]

in which use has been made of the \( X \) coefficient properties contained in (3.45) and (3.46). For a Hermitian tensor, this result reads

\[
\{\gamma j|T^{(k)}|\gamma j'\} = (-1)^{k-j}[\gamma j'|T^{(k)}|\gamma j']^*.
\]

(5.18)

\[\frac{1}{2}(\gamma j|T^{(k)}|\gamma j')^* = (-1)^{k-j} T^{(k)}(\gamma j'|\gamma j').
\]

The relation between the rectangular bracket symbol and the analogous quantity defined in \( R \) is

\[
[\gamma j|T^{(k)}|\gamma j'] = (-1)^{k-j}[\gamma j|T^{(k)}|\gamma j']^*.
\]

This is the method employed by E. Wigner, "Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren" (Braunschweig, 1931), p. 263.
If the tensor operators $T_1$ and $T_2$ of (5.7) refer to the same dynamical variables, we may write

$$\langle \gamma j m | T(k_1, k_2, k_3) | \gamma' j' m' \rangle = (-1)^{k'-j'+m} \langle \gamma j | T^{(k)}(k_1, k_2, k_3) | \gamma' j' m' \rangle X(jk'; -mqm'),$$  \hspace{1cm} (5.20)

where in view of (5.16),

$$\langle \gamma j | T^{(k)}(k_1, k_2, k_3) | \gamma' j' m' \rangle = \sum_{q_1 q_2} (-1)^{q_1 - j - m} X(jk'; -mqm') \langle k_1 q_1, k_2 q_2, k_3 | \gamma j' m' \rangle.$$  \hspace{1cm} (5.21)

The resulting magnetic quantum number summation, involving four $X$ coefficients, can be identified with a $W$ coefficient,

$$\langle \gamma j | T^{(k)}(k_1, k_2, k_3) | \gamma' j' \rangle = (2k + 1)^4 \sum_{j' j''} W(k_1, k_2, j'; k) \langle \gamma j | T^{(k_1)}(k_1 q_1) | \gamma' j' m' \rangle \langle \gamma j' j'' j'' \rangle \langle \gamma' j' | T^{(k_2)}(k_2 q_2) | \gamma' j'' m'' \rangle.$$  \hspace{1cm} (5.22)

When $T_1$ and $T_2$ are tensor operators associated with different dynamical variables, so that

$$[T_1, J_2] = [T_2, J_1] = 0,$$  \hspace{1cm} (5.23)

we have

$$\langle \gamma j_1 j_2 j_3 j_4 | T(k_1, k_2, k_3) | \gamma' j_1' j_2' j_3' j_4' m' \rangle = (-1)^{k' - j' + m} \langle \gamma j_1 j_2 j_3 j_4 | T^{(k)}(k_1, k_2, k_3) | \gamma' j_1' j_2' j_3' j_4' m' \rangle X(jk'; -mqm').$$  \hspace{1cm} (5.24)

Here

$$\langle \gamma j_1 j_2 j_3 j_4 | T^{(k)}(k_1, k_2, k_3) | \gamma' j_1' j_2' j_3' j_4' m' \rangle = \sum_{j_5 j_6 j_7 j_8} (-1)^{j_5 - j - m} X(jk'; -mqm')$$

$$\cdot \langle k_1 q_1, k_2 q_2, k_3 | \gamma j_1 j_2 j_3 j_4 m_1 m_2 \rangle \langle j_1 j_2 j_3 j_4 m_1 m_2 | \gamma' j_1' j_2' j_3' j_4' m_1' m_2' \rangle$$

$$\cdot \langle \gamma j_5 j_6 j_7 j_8 | T^{(k_4)}(k_4 q_4) | \gamma' j_1' j_2' j_3' j_4' \rangle \langle \gamma' j_5 j_6 j_7 j_8 | T^{(k_5)}(k_5 q_5) | \gamma' j_1' j_2' j_3' j_4' \rangle.$$  \hspace{1cm} (5.25)

where the sum is over all $m$ variables. This magnetic quantum number summation, involving six $X$ coefficients, can be identified with an $S$ coefficient,

$$\langle \gamma j_1 j_2 j_3 j_4 | T^{(k)}(k_1, k_2, k_3) | \gamma' j_1' j_2' j_3' j_4' \rangle = \sum_{j_5 j_6 j_7 j_8} \langle \gamma j_1 j_2 j_3 j_4 | T^{(k_4)}(k_4 q_4) | \gamma' j_1' j_2' j_3' j_4' \rangle \langle \gamma' j_5 j_6 j_7 j_8 | T^{(k_5)}(k_5 q_5) | \gamma' j_1' j_2' j_3' j_4' \rangle.$$  \hspace{1cm} (5.26)

Special examples which require only the $W$ coefficient are

$$\langle \gamma j_1 j_2 j_3 | T^{(0)}(k_1, k_2, k_3) | \gamma' j_1' j_2' j_3' \rangle = \left[ \frac{2j_1 + 1}{2k_1 + 1} \right]^{1/2} (-1)^{j_1 + j_2 - k_1} W(j_1 j_2 j_3 j_4; j_1 j_2 j_3; j_1 j_2)$$

$$\cdot \sum_{j'_j} \langle \gamma j_1 j_2 j_3 | T^{(k)}(k_1 q_1) | \gamma' j_1' j_2' j_3' \rangle \langle \gamma j_1 j_2 j_3 | T^{(k_2)}(k_2 q_2) | \gamma' j_1' j_2' j_3' \rangle.$$  \hspace{1cm} (5.27)
\[ [\gamma_{j_1j_3} T^{(k)}_{1} \gamma'_{j_1'j_3'}] \]
\[ = [(2j + 1)(2j' + 1)]^{1/2} (-1)^{j_1 + j_3 - j_1' - j_3'} W(j_1, j_3; j_1', j_3'; j_2 k) [\gamma_{j_1'} T^{(k)}_{1} \gamma'_{j_3'}]. \] (5.28)

and

\[ [\gamma_{j_2j_3} T^{(k)}_{2} \gamma'_{j_2'j_3'}] \]
\[ = [(2j + 1)(2j' + 1)]^{1/2} (-1)^{j_1 + j_3 - j_1' - j_3'} W(j_2, j_3; j_2', j_3'; j_1 k) [\gamma_{j_2'} T^{(k)}_{2} \gamma'_{j_3'}]. \] (5.29)

Further relations connecting the \( S \) and \( W \) coefficients can be deduced from these results. We shall illustrate this for the simpler situation in which only \( W \) is involved. We multiply the two scalar operators\(^{17}\)
\[ T^{(0)}(k_1, k_1) = \sum_{k_1} (2k_1 + 1)^{-1/2} T_1(k_1 g_1) (-1)^{s_1 - s_{1_1}} T_2(k_1 - g_1), \] (5.30)

and
\[ T^{(0)}(k_2, k_2) = \sum_{k_2} (2k_2 + 1)^{-1/2} T_1(k_2 g_2) (-1)^{s_2 - s_{1_2}} T_2(k_2 - g_2), \] (5.31)

to obtain
\[ T^{(0)}(k_1, k_1) T^{(0)}(k_2, k_2) = \sum_{k_1, k_2} [(2k_1 + 1)(2k_2 + 1)]^{-1/2} T_1(k_1 g_1) T_1(k_2 g_2) \]
\[ \cdot (-1)^{s_1 + s_2 - s_{1_1} - s_{1_2}} T_2(k_1 - g_1) T_2(k_2 - g_2). \] (5.32)

On writing
\[ T_1(k_1 g_1) T_1(k_2 g_2) = \sum_{k_1, k_2} T_1(k_1, k_2) k(k_1, k_2) g_1 g_2, \] (5.33)

and
\[ T_2(k_1 - g_1) T_2(k_2 - g_2) = \sum_{k_1, k_2} T_2(k_1, k_2) k - g_1 (-1)^{s_1 + s_2 - s}(k_1, k_2) g_1 g_2, \] (5.34)

this becomes
\[ T^{(0)}(k_1, k_1) T^{(0)}(k_2, k_2) \]
\[ = \sum_{k_1, k_2} [(2k_1 + 1)(2k_2 + 1)]^{-1/2} T_1(k_1, k_2) g_1 g_2 \]
\[ = \sum_{k} \left( \frac{2k + 1}{(2k_1 + 1)(2k_2 + 1)} \right)^{1/2} T^{(0)}([k], [k_1, k_2], k). \] (5.35)

A matrix element of this equation, when evaluated with the aid of (5.22) and (5.27), yields the information that
\[ W(j_1 j_2 j_3 j_4; j k) W(j_1 j_2 j_3 j_4; j k) \]
\[ = (-1)^{s_1 + s_2 - s} \sum_{k} (2k + 1)(-1)^{s_1 + s_2 - s} W(j_1 j_2 j_3 j_4; j k) \]
\[ \cdot W(k_1, k_2 j_3; j k) W(k_1, k_2 j_4; j k). \] (5.36)

\(^{17}\) Here \( T_1 \) and \( T_2 \) are functions of different dynamical variables.
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Tensor operators can be constructed from the spin creation and annihilation operators. Thus, consider the operator

$$e^{\zeta (za^+) + \zeta (za)} = \sum_{k,q} \phi_{kq}(z) \phi_{kq}(\zeta) t(kqa), \quad (5.37)$$

formed from the commuting quantities \((za^+)\) and \([za]\). On subjecting this to a unitary transformation, we find

$$e^{\zeta (za^+) + \zeta (za)} = \sum_{k,q} \phi_{kq}(z) \phi_{kq}(\zeta) U(kqa) U^{-1}, \quad (5.38)$$

where the transformed creation and annihilation operators are described by (2.14). Now, according to (2.19), we have

$$(za^+ = (z'a^+, \quad [za] = [z'a], \quad z' = uz, \quad (5.39)$$

in which the second statement stems from the fact that \(a_-\) and \(a_+\) transform in the same way as \(a^+\) and \(-a^-\). Therefore,

$$e^{\zeta (za^+) + \zeta (za)} = \sum_{k,q} \phi_{kq}(uz) \phi_{kq}(\zeta) t(kqa)$$

$$= \sum_{k,q} U^{(k)}_{q'} \phi_{kq}(z) \phi_{kq}(\zeta) t(kqa), \quad (5.40)$$

on employing (2.21). We have thereby shown that

$$Ut(kq' \alpha) U^{-1} = \sum_q t(kqa) U^{(k)}_{q'}. \quad (5.41)$$

On taking the Hermitian conjugate of (5.37) and making the substitution

$$z^*_{-} \rightarrow z_-, \ z^*_{+} \rightarrow -z_+, \ \zeta^*_{+} \rightarrow \zeta_+, \ \zeta^*_{-} \rightarrow -\zeta_-, \quad (5.42)$$

which restores this generating operator to its original form, we find that

$$t(kqa) = (-1)^{q + q'(k - q - a)^t. \quad (5.43)$$

Accordingly, the adjoint tensor is given by

$$t^t(kqa) = t^{2a} t(kq - a). \quad (5.44)$$

The significance of \(a\) can be appreciated from

$$\frac{(za^+)^k [za]^{k - a}}{[(k + a)(k - a)]^{1/2}} = \sum_k \phi_{kq}(z) t(kqa), \quad (5.45)$$

namely, \(2a\) is the excess of creation with respect to annihilation operators. Therefore, if \(t(kqa)\) is applied to an angular momentum eigenvector with quantum number \(j'\), it will produce an eigenvector with quantum number \(j\), such that

$$\alpha = j - j'. \quad (5.46)$$
To evaluate the matrix elements of $t(k\alpha)$, we examine
\[ \langle e^{i\alpha \pi} \gamma_0 | e^{i\alpha \pi} + \frac{\alpha \cdot \alpha}{2} | e^{i\alpha \pi} \gamma_0 \rangle = \sum_{jm/m'k\alpha} \phi_{jm}(x^\alpha) (jm|m(k\alpha)|j'm') \phi_{j'm'}(y) \phi_{k\alpha}(z) \phi_{k\alpha}(\zeta) = e^{i\alpha \pi} + \frac{\alpha \cdot \alpha}{2} + \frac{\alpha \cdot \alpha}{2}. \] (5.46)

The substitution $x^\alpha_+ \rightarrow x_-, x^\alpha_- \rightarrow -x_+$ places this in the form
\[ \sum_{jm/m'k\alpha} (-1)^{j'+m} \phi_{j'-m}(x) (jm|m(k\alpha)|j'm') \phi_{j'm'}(y) \phi_{k\alpha}(z) \phi_{k\alpha}(\zeta) = e^{i\alpha \pi} + \frac{\alpha \cdot \alpha}{2} - \frac{\alpha \cdot \alpha}{2}. \] (5.47)

and comparison with (3.42) shows that
\[ e^{i\alpha \pi} + \frac{\alpha \cdot \alpha}{2} - \frac{\alpha \cdot \alpha}{2} = \sum_{jm/m'k\alpha} \phi_{j'-m}(x) \phi_{k\alpha}(z) \phi_{j'm'}(y) \phi_{k\alpha}(\zeta) \left( \frac{(j' + j + k + 1)!}{(j' - k)!} \right)^{1/2} X(jk') - m\alpha m' \] (5.48)

Therefore
\[ (jm|m(k\alpha)|j'm') = \delta_{\alpha, j-j'} (-1)^{k+1-j} \left[ \frac{(j' + k + m + 1)!}{(j' - k)!} \right]^{1/2} X(jk') - m\alpha m'. \] (5.49)

or
\[ [j|(d^k)(\alpha)|j'] = \delta_{\alpha, j-j'} \left[ \frac{(j' + k + m + 1)!}{(j' - k)!} \right]^{1/2} . \] (5.50)

Of particular interest are the operators with $\alpha = 0$ ($k$ integral),
\[ \frac{(za)_k[z\alpha]}{k!} = \sum_{\gamma} \phi_{k\gamma}(z) t(k\gamma 0). \] (5.51)

Indeed
\[ -(za_+)[za] = a_+ \cdot J, \] (5.52)

where $a_+$ is a null vector,
\[ a_+ \cdot a = 0. \] (5.53)

with the components
\[ a_1 = -x_+^2 + x_-^2, \quad a_2 = -i(x_+^2 - x_-^2), \quad a_3 = 2x_+^2. \] (5.54)

It is well known that if $r$ is a position vector, $(a \cdot r)_k$ is a spherical harmonic of order $k$,
\[ \frac{(a \cdot r)_k}{2^k k!} = \left[ \frac{4\pi}{2k + 1} \right]^{1/2} \sum_{\gamma} \phi_{k\gamma}(z) Y_{k\gamma}(r), \] (5.55)

where $Y_{k\gamma}(r)$, which usually designates a surface spherical harmonic, here includes the factor $r^k$. Accordingly, we write
\[ \frac{(a \cdot J)_k}{2^k k!} = \left[ \frac{4\pi}{2k + 1} \right]^{1/2} \sum_{\gamma} \phi_{k\gamma}(z) Y_{k\gamma}(J). \] (5.56)
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in which \( Y_{kq}(J) \) differs from the analogous \( Y_{kq}(r) \) only in that the order of factors is significant. With this notation, we have

\[
t(kq0) = \left[ \frac{4\pi}{4k + 1} \right]^{1/2} (-2)^k Y_{kq}(J),
\]

and

\[
[j[Y^{(k)}(J)|j] = \left[ \frac{2k + 1}{4\pi} \right]^{1/2} (2j + k + 1)! \left( \frac{2j - k}{2j} \right)^{1/2}.
\]

Notice also that the tensor \( t(kq0) \) is Hermitian, according to (5.43), so that the operator harmonics satisfy

\[
Y_{kq}(J)^* = (-1)^q Y_{k-q}(J).
\]

The matrix elements of the tensor operator

\[
Y(k_1k_2kq) = \sum_{q_{12}} Y_{k_1q_{11}}(J) Y_{k_2q_{22}}(J) Y_{kq_{12}}(J) (k_1g_{11}k_2g_{22}k_1k_2kq)
\]

are described by

\[
[j[Y^{(k)}(k_1k_2)|j] = (2k + 1)^{1/2} W(k_1k_2;j,j; k_1|j[Y^{(k)}(J)|j] Y^{(k)}(J)|j],
\]

in view of (5.22). With respect to their effect on an eigenvector with quantum number \( j \), one can assert that

\[
Y(k_1k_2kq) = Y_{kq}(J)|j[Y^{(k)}(k_1k_2)|j]/[j[Y^{(k)}(J)|j],
\]

which becomes a generally valid operator equation on replacing \( j(j+1) \) with \( J^2 \). Hence

\[
\sum_{q_{12}} Y_{k_1q_{11}}(J) Y_{k_2q_{22}}(J) (k_1g_{11}k_2g_{22}k_1k_2kq)
\]

\[
= Y_{kq}(J)(2k + 1)^{1/2} W(k_1k_2;j,j; k_1|j[Y^{(k)}(J)|j] Y^{(k)}(J)|j]/[j[Y^{(k)}(J)|j].
\]

The example of this result for \( k = 0 \) can be written

\[
\sum_{q} Y_{kq}(J) Y_{kq}(J)^* = \frac{1}{2j + 1} [j[Y^{(k)}(J)|j] = \frac{2k + 1}{4\pi} \frac{1}{4j^2} (2j + k + 1)! (2j + 1)(2j - k)!.
\]

in which we have employed

\[
W(kkjj; 0j) = (-1)^k[(2j + 1)(2k + 1)]^{-1/2}.
\]

One can easily exhibit the right side of (5.64) as a function of \( j(j+1) \), and thus obtain the operator equation

\[
\sum_{q} Y_{kq}(J) Y_{kq}(J)^* = \frac{2k + 1}{4\pi} [J^2]^k,
\]
\[ \{J^2\}^k = \prod_{n=0}^{k-1} \left[ J^2 - \frac{n(n+1)}{2} \right]. \]  

(5.66)

The structure of the operator \(\{J^2\}^k\) can also be inferred from the two requirements that it annihilate any eigenvector with \(j < \frac{k}{2}\), and that it simplify to the \(k\)th power of \(J^2\) as \(j\) becomes very large.

We return to (5.63), displayed in the form

\[ \langle k_{1g} | Y_{k_{1g}}(J) \rangle \langle k_{2g} | Y_{k_{2g}}(J) \rangle = \left[ \frac{(2k_1 + 1)(2k_2 + 1)}{4\pi} \right]^{1/2} \sum_{k_q} Y_{k_q}(J) \langle k_{1g} k_{2g} | k_q \rangle \langle k_q k_{1g} k_{2g} \rangle, \]  

(5.67)

where

\[ \left[ \frac{(J^2)^k}{\left[ J^2 \right]^{1/2} \left[ (J^2)^k \right]^{1/2}} \right]^{1/2} \sum_{k_{1g} k_{2g}} Y_{k_{1g} k_{2g}}(J) = (2j + 1)^{1/2} W(k_1, k_2; j; k). \]  

(5.68)

The analogous equation for \(\langle k_{1g} | Y_{k_{1g}}(J) \rangle \langle k_{2g} | Y_{k_{2g}}(J) \rangle\) differs from (5.67) only by the inclusion of the factor \((-1)^{k_1 + k_2 - k}\), as follows from (3.47). The addition and subtraction of these two equations then yields

\[ \{ Y_{k_{1g}}, Y_{k_{2g}} \} = \left[ \frac{(2k_1 + 1)(2k_2 + 1)}{4\pi} \right]^{1/2} \sum_{\text{even, } k_q} Y_{k_q}(J) \langle k_{1g} k_{2g} | k_q \rangle \langle k_q k_{1g} k_{2g} \rangle, \]  

(5.69)

and

\[ [ Y_{k_{1g}}, Y_{k_{2g}} ] = \left[ \frac{(2k_1 + 1)(2k_2 + 1)}{4\pi} \right]^{1/2} \sum_{\text{odd, } k_q} Y_{k_q}(J) \langle k_{1g} k_{2g} | k_q \rangle \langle k_q k_{1g} k_{2g} \rangle. \]  

(5.70)

where the parity referred to is that of \(k_1 + k_2 - k\). In the latter equation we have the commutation properties of these operator functions of \(J\).

As an elementary application of (5.70), we take its trace for the states with quantum number \(j\). In view of the null trace possessed by a commutator, we infer that the trace of \(Y_{k_q}^k(J)\) vanishes for every \(k\) that can occur in (5.70). Since these \(k\) values are \(|k_1 - k_2| + 1, |k_1 - k_2| + 3, \ldots, k_1 + k_2 - 1\), we obtain\(^{18}\)

\[ \text{tr}^{(j)} Y_{k_q}^k(J) = 0, \quad k > 0, \]  

(5.71)

\(^{18}\) This theorem is easily proved for an arbitrary tensor operator by taking the trace of (5.1) for states with a given \(j\), and integrating with respect to \(\omega\)

\[ \sum_{m=-j}^{j} \langle \gamma jm | T(k) | \gamma jm \rangle = 0, \quad k > 0. \]

Of course, \(k\) must be integral if the individual matrix elements are not to vanish.
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\[ \frac{1}{2j+1} \text{tr}^{(j)} Y_{kq}(J) = \delta_{40}. \]  (5.72)

With the aid of this result, the trace of (5.67) is evaluated as

\[ \frac{1}{2j+1} \text{tr}^{(j)} Y_{k12}(J)^{i} Y_{k21}(J) = \left[ \prod_{i} \frac{2k_i + 1}{4\pi} \right]^{1/2} \cdot X(k; q)(-1)^{j1+k2}(2j + 1)^{1/2}W(k_1 k_2 jj; k_3 k_3). \]  (5.73)

which expresses the orthogonality of the operator harmonics. The multiplication of (5.67) with \(Y_{k21}(J)\) then yields

\[ \frac{1}{2j+1} \text{tr}^{(j)} Y_{k12}(J) Y_{k21}(J) Y_{k2k}(J) = \left[ \prod_{i} \frac{2k_i + 1}{4\pi} \right]^{1/2} \cdot X(k; q)(-1)^{j1+k2}(2j + 1)^{1/2}W(k_1 k_2 jj; k_3 k_3). \]  (5.74)

A comparison with (3.85) shows that, in the limit of large \(j\),

\[ (-1)^{k1+k2}(2j + 1)^{1/2}W(k_1 k_2 jj; k_3 k_3) \rightarrow X(k_1 k_2 k_3; 000). \]  (5.75)

Turning to tensor operators formed from two angular momenta, we remark that, for matrix elements diagonal in \(j\),

\[ \sum_{k1k2} Y_{k12}(J_1) Y_{k21}(J_2)(k_1 k_2 jj; j k) \]

\[ = Y_{kq}(J)(2j + 1)(2k + 1)^{1/2}(-1)^{j1+j2-k2}S(j_1 j_2 j_2 j_2; j j k_2 k_2) \]

\[ \cdot \frac{[j_1 | Y^{(k2)} j_1 ] [j_2 | Y^{(k2)} j_2 ]}{[j | Y^{(k2)} j]}. \]  (5.76)

No such restriction is required for the special example

\[ \sum_{k} Y_{kq}(J_1)^{i} Y_{kq}(J_2) = (-1)^{j_1+j_2-j}W(j_1 j_2 j_2 j_2; j j k)[j_1 | Y^{(k2)} j_1 ][j_2 | Y^{(k2)} j_2 ]]. \]  (5.77)

In terms of the Legendre polynomial operator defined by

\[ \sum_{k} Y_{kq}(J_1)^{i} Y_{kq}(J_2) = \frac{2k + 1}{4\pi} P_{k}(J_1, J_2), \]  (5.78)

the latter equation can be written

\[ [(J_1^2 + J_2^2)]^{1/2}P_{k}(J_1, J_2) = (-1)^{j_1+j_2-j}(2j_1 + 1)^{1/2}(2j_2 + 1)^{1/2}W(j_1 j_2 j_2 j_2; j j k). \]  (5.79)

which indicates the limiting form of the right side for large \(j_1, j_2, \) and \(j\). The simple result obtained for \(j = 0\) can be expressed as

\[ P_{k}(J, -J) = (-1)^{k}[j^2]^{k}. \]  (5.80)
A multiplication theorem for the Legendre operator is obtained from the observation that
\[
\frac{2k_1+1}{4\pi} \frac{2k_2+1}{4\pi} P_{k_1}(J_1, J_2) P_{k_2}(J_1, J_2) = \sum_{k_3} (Y_{k_3}(J_1) Y_{k_3}(J_1))^j Y_{k_3}(J_2) Y_{k_3}(J_2),
\]
(5.81)
namely,\(^{19}\)
\[
P_{k_1}(J_1, J_2) P_{k_2}(J_1, J_2) = \sum_k (2k+1) P_k(J_1, J_2) (-1)^{k_1+k_2-k} f_{k_1k_2k}(J_1^2) f_{k_1k_2k}(J_2^2).
\]
(5.82)
On placing \(k_3 = 1\), we obtain a simple recurrence relation from which the Legendre operators can be constructed successively, starting with
\[
P_0(J_1, J_2) = 1.
\]
(5.83)
The coefficients in the recurrence relation can be computed from
\[
W(k_1 jj; k + 1 jj) = -W(k + 1 1 jj; kj)
\]
\[
= \left[ \frac{4j(j+1) - k^2 - 2k}{4j(j+1)(2j+1)} \right]^{1/2} \left( \frac{k+1}{2k+1(2k+3)} \right)^{1/2},
\]
(5.84)
and
\[
\sum_k (2k+1)(2j+1) |W(k_1 jj; kj)|^2 = 1.
\]
(5.85)
Thus
\[
f_{k1k+1}(J^2) = \left( \frac{k+1}{(2k+1)(2k+3)} \right)^{1/2},
\]
\[
f_{k1k-1}(J^2) = -\left( \frac{k}{(2k-1)(2k+1)} \right)^{1/2} \left( J^2 - \frac{k^2 - 1}{4} \right),
\]
\[
(f_{k1k}(J^2))^2 = \frac{k(k+1)}{2k+1},
\]
(5.86)
and therefore,
\[
\left( J_1 \cdot J_2 + \frac{k(k+1)}{4} \right) P_k(J_1, J_2) = \frac{k+1}{2k+1} P_{k+1}(J_1, J_2)
\]
\[+ \frac{k}{2k+1} \left( J_1^2 - \frac{k^2 - 1}{4} \right) \left( J_2^2 - \frac{k^2 - 1}{4} \right) P_{k-1}(J_1, J_2).
\]
(5.87)
\(^{19}\) This is a particular example of the theorem on the product of two \(W\) coefficients [Eq. (5.36)].
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As the first few Legendre operators, obtained in succession from (5.87) with
\( k = 0, 1, 2, \) we have

\[
P_1(J_1, J_2) = J_1 \cdot J_2,
\]

\[
P_2(J_1, J_2) = \frac{3}{2} J_1 \cdot J_2 (J_1 \cdot J_2 + \frac{1}{2}) - \frac{1}{2} J_1^2 J_2^2,
\]

\[
P_3(J_1, J_2) = \frac{5}{2} J_1 \cdot J_2 (J_1 \cdot J_2 + \frac{1}{2}) (J_1 \cdot J_2 + \frac{3}{2}) - \frac{5}{2} (J_1 \cdot J_2 + \frac{1}{2}) J_1^2 J_2^2
\]

\[- \frac{3}{2} J_1 \cdot J_2 (J_1^2 - \frac{1}{2}) (J_2^2 - \frac{1}{2}).
\]

(5.88)

A useful check upon these results is afforded by (5.80).

A statement analogous to (5.62) can be made for an arbitrary tensor
operator; as far as matrix elements diagonal in \( j \) are concerned,

\[
T(kq) = Y_{kj}(J) \frac{[j] T^{(k)}|j]}{[j] Y^{(k)}| j].
\]

(5.89)

The coefficient in this relation can be expressed in other ways. Thus, we have

\[
\sum_q Y_{kj}(J)^{\dagger} T(kq) = \sum_q Y_{kj}(J)^{\dagger} Y_{kj}(J) \frac{[j] T^{(k)}|j]}{[j] Y^{(k)}| j].
\]

(5.90)

which leads to the projection rule

\[
T(kq) \rightarrow \frac{4\pi}{2k+1} Y_{kj}(J) \frac{[j] T^{(k)}|j]}{[j] Y^{(k)}| j].
\]

(5.91)

for isolating the part of a tensor operator that contributes to matrix elements
diagonal in \( j \). Alternatively, we consider the particular matrix element

\[
(jj|[j] T^{(k)}|jj) = (jj| Y_{k0}(J)|jj) \frac{[j] T^{(k)}|j]}{[j] Y^{(k)}| j].
\]

(5.92)

Now

\[
(jj| Y_{k0}(J)|jj) = (-1)^k [j] Y^{(k)}|j] X(kj; -j0j) = \left[ \frac{2k + 1}{4\pi} \right]^{1/2} \frac{1}{2^k (2j)!} (2j)!.
\]

(5.93)

so that, for matrix elements diagonal in \( j \),

\[
T(kq) \rightarrow \left[ \frac{4\pi}{2k+1} \right]^{1/2} Y_{kj}(J) \frac{2^k (2j-k)!}{(2j)!} (jj|[j] T^{(k)}|jj).
\]

(5.94)

APPENDIX A

We shall describe a method which produces simultaneously the eigenvalues and eigenvectors of the angular momentum operators. Consider for
this purpose the unitary operator

\[
V = \exp(i\chi \frac{1}{2} a + i\phi J_3),
\]

(A1)
which has the eigenvalues $\exp(i \gamma X + i m \phi)$. The operator $V$ can be interpreted as

$$V = \sum_j \left[ \exp(i \gamma X + i m \phi) \right] P jm,$$

(A2)

where $P jm$, the projection operator for the state with the indicated eigenvalues, is represented in terms of the corresponding eigenvector by the dyadic

$$P jm = \Psi jm \Psi jm^*.$$

(A3)

Accordingly, if $V$ can be constructed and displayed in the form (A2), we shall have achieved our goal.

We write

$$V = \exp(\frac{i}{2} (\gamma_+ n_+ + \gamma_- n_-)),$$

$$\gamma_+ = \chi + \phi, \quad \gamma_- = \chi - \phi,$$

(A4)

and deduce the differential equations

$$\frac{\partial}{\partial \gamma_+} V = \frac{1}{2} i a^+_\zeta a^\zeta V$$

$$= \frac{1}{2} i [\exp(i \gamma_+)] a^+_\zeta V a^\zeta,$$

(A5)

with the aid of

$$V^{-1} a^\zeta V = \exp(\frac{i}{2} i \gamma_+) a^\zeta.$$

(A6)

The latter can be verified by differentiation,

$$\frac{\partial}{\partial \gamma_+} V^{-1} a^\zeta V = \frac{1}{2} i V^{-1} [a^\zeta, n_+] V = \frac{1}{2} i V^{-1} a^\zeta V,$$

(A7)

or from the general theorem

$$a^\zeta f(n_\zeta) = f(n_\zeta + 1) a^\zeta.$$

(A8)

In virtue of the operator ordering in (A5), the solution of these equations which reduces to unity for $\gamma_+ = 0$ is given by

$$V = \exp\{ (e^{i \gamma +} - 1) a^+_+; a_+ + (e^{i \gamma -} - 1) a^+_-; a_- \}$$

(A9)

where

$$\exp(\lambda a^+_+; a) = \sum_n \frac{\lambda^n}{n!} (a^+_+)^n (a)^n$$

(A10)

is a correspondingly ordered form of the exponential. We write this solution as

$$V = \exp(\sum_\zeta e^{i \gamma \zeta} a^\zeta_\zeta; P_0; a_\zeta).$$

(A11)
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which is intended to indicate that

\[ P_0 = \exp(-a^+_\pi; a_+ - a^-_\pi; a_-) = \exp(-(a^+; a)) \]  \hspace{1cm} (A12)

is to be inserted between the powers of \( a^+ \) and \( a^- \) in the ordered operator expansion of \( V \):

\[ V = \sum_{n_+, n_- = 0}^{\infty} \left\{ \exp\left[ \frac{i}{2} (n_+ \gamma_+ + n_- \gamma_-) \right] \right\} \frac{(a^+_\pi)^{n_+}(a^-)^{n_-}}{(n_+! n_-)!^{1/2}} P_0 \frac{(a^-)^{n_+}(a_\pi)^{n_-}}{(n_+! n_-)!^{1/2}}. \]  \hspace{1cm} (A13)

We have thus obtained the form (A2), with

\[ j = \frac{1}{2}(n_+ + n_-), \quad m = \frac{1}{2}(n_+ - n_-), \quad n_+, n_- = 0, 1, 2, \ldots, \]  \hspace{1cm} (A14)

and

\[ P(jm) = \phi_{jm}(a^+) P_0 \phi_{jm}(a), \]  \hspace{1cm} (A15)

in which we have employed the notation

\[ \phi_{jm}(a^+) = \frac{(a^+_\pi)^{j+m}(a^-)^{j-m}}{[(j+m)!(j-m)!]^{1/2}}. \]  \hspace{1cm} (A16)

In terms of the eigenvector \( \Psi_0 \), defined by

\[ P_0 = \Psi_0 \Psi_0^*, \]  \hspace{1cm} (A17)

the angular momentum eigenvectors are exhibited as

\[ \Psi(jm) = \phi_{jm}(a^+) \Psi_0. \]  \hspace{1cm} (A18)

The fundamental property of \( \Psi_0 = \Psi(00) \) is deduced from

\[ [a_\pi, P_0] = (\partial/\partial a^+_\pi) P_0 = -P_0 a_\pi, \]  \hspace{1cm} (A19)

or

\[ a^- P_0 = 0, \]  \hspace{1cm} (A20)

namely

\[ a_\pi \Psi_0 = 0. \]  \hspace{1cm} (A21)

The simple generating function for the eigenvectors, (1.16), can also be obtained by noting that

\[ \langle \Psi(jm), e^{i\alpha^+} \Psi_0 \rangle = \langle \Psi_0, \phi_{jm}(a)e^{i\alpha^+} \Psi_0 \rangle = \phi_{jm}(x). \]  \hspace{1cm} (A22)

Indeed,

\[ e^{i\alpha^+} \Psi_0 = \sum_{jm} \Psi(jm) \langle \Psi(jm), e^{i\alpha^+} \Psi_0 \rangle = \sum_{jm} \phi_{jm}(x) \Psi(jm). \]  \hspace{1cm} (A23)
The ordered operator
\[ A = \exp(za; a^+) \quad [a, a^+] = 1, \quad (B1) \]
satisfies
\[ [a, A] = (\partial/\partial a^+)A = zaA, \]
\[ [a^+, A] = -(\partial/\partial a)A = -Az a^+, \quad (B2) \]
or
\[ (1 - z)aA = Aa, \quad a^+A = (1 - z)Aa^+. \quad (B3) \]
Therefore
\[ \frac{\partial}{\partial z}A = aAa^+ = \frac{1}{1 - z}Aa + \frac{1}{1 - z}Aa^+a^+Aa \]
\[ = \frac{1}{1 - z}A + \frac{1}{(1 - z)^2}a^+Aa, \quad (B4) \]
the solution of which implies the ordered operator identity,
\[ \exp(za; a^+) = \frac{1}{1 - z} \exp\left(\frac{z}{1 - z} a^+; a\right) \quad (B5) \]
A particular consequence of this relation
\[ \exp(za; a^+) \cdot \Psi_0 = \frac{1}{1 - z} \Psi_0, \quad a \Psi_0 = 0, \quad (B6) \]
is derived directly in the text (Eq. (2.35)). The properties of \( A \) contained in (B3) are also displayed in the generalizations
\[ \exp(za; a^+) \cdot f(a^+) = \frac{1}{1 - z} f\left(\frac{a^+}{1 - z}\right) \exp\left(\frac{z}{1 - z} a^+; a\right), \]
\[ f(a) \exp(za; a^+) = \exp\left(\frac{z}{1 - z} a^+; a\right) \frac{1}{1 - z} f\left(\frac{a}{1 - z}\right). \quad (B7) \]
The particular examples of these identities provided by
\[ \exp(za; a^+) \cdot (a^+)^r = \frac{(a^+)^r}{(1 - z)^{r+1}} \exp\left(\frac{z}{1 - z} a^+; a\right) \quad (B8) \]
and
\[ a^r \exp(za; a^+) = \exp\left(\frac{z}{1 - z} a^+; a\right) \frac{a^r}{(1 - z)^{r+1}} \quad (B9) \]
are operator forms of the Laguerre polynomial generating functions. Thus, if
we place \( a^+ = x, a = \partial / \partial x \), and let both sides of \((B8)\) operate upon \( e^{-z} \), we obtain

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \frac{\partial}{\partial x} \right)^n x^{n+r} e^{-z} = \frac{x^r}{(1 - z)^{r+1}} \exp \left( \frac{z}{1 - z} \frac{\partial}{\partial x} \right) e^{-z} = \frac{x^r}{(1 - z)^{r+1}} \exp \left( - \frac{z}{1 - z} x \right) e^{-z}, \tag{B10}
\]

or

\[
\exp \left( - \frac{zx}{1 - z} \right) = \sum_{n=0}^{\infty} \frac{x^n I_n^{(r)}(x)}{n!}, \tag{B11}
\]

where

\[
L_n^{(r)}(x) = \frac{1}{n!} x^{-r} e^x \left( \frac{d}{dx} \right)^n (x^n e^{-z}). \tag{B12}
\]

A similar procedure applied to \((B9)\) yields

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \frac{\partial}{\partial x} \right)^n x^{n+r} e^{-z} = \frac{(-1)^r}{(1 - z)^{r+1}} \exp \left( - \frac{z}{1 - z} x \right) e^{-z}, \tag{B13}
\]

which proves the equivalence between \((B12)\) and

\[
L_n^{(r)}(x) = \frac{(-1)^r}{n!} e^x \left( \frac{d}{dx} \right)^n (x^n e^{-z}). \tag{B14}
\]

Another example of an ordered operator identity involves the cylinder function [Eq. \((1.36)\)]

\[
F_r(z) = rz^{-r/2}J_r(2z^{1/2}) = \frac{r}{2\pi i} \int dt \frac{\exp \left( t + \frac{z}{t} \right)}{t^{r+1}}. \tag{B15}
\]

We have

\[
a^r F_r(xa; a^+) = \frac{r}{2\pi i} \int dt \frac{e^t}{t^{r+1}} a^r \exp \left( \frac{z}{t} a; a^+ \right) = \frac{e^t}{2\pi i} \int dt \frac{e^{t-z}}{(t-z)^{r+1}} \exp \left( \frac{z}{t-z} a^+; a \right) a^r = e^r F_r(xa^+; a) a^r, \tag{B16}
\]

and similarly

\[
F_r(xa; a^+)(a^+) = e^{(a^+)r} F_r(xa^+; a). \tag{B17}
\]

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From these identities we obtain the Laguerre polynomial generating function

\[ \sum_{n=0}^{\infty} \frac{r^n}{(n+r)!} z^n L_n^r(x) = e^r F_r(-2x). \tag{B18} \]

Appendix C

It is our purpose in this section to evaluate a class of scalar products, the simplest illustration of which is

\[ T^{(3)} = (\exp(\lambda[a^+b^+]) + \xi_+(xa^+) + \xi_-(xb^+))\Psi_0, \]
\[ \exp(\kappa[a^+b^+] + \eta_+(ya^+) + \eta_-(yb^+))\Psi_0. \tag{C1} \]

Differentiation with respect to \( \xi_+^* \) yields

\[ (\partial/\partial \xi_+^*) T^{(3)} = (e\cdots\Psi_0, (x^*a)e\cdots\Psi_0) \]
\[ = \eta_+(x^*y)T^{(3)} + \kappa[(xb)e\cdots\Psi_0, e\cdots\Psi_0], \tag{C2} \]

or

\[ (1 - \lambda^*\kappa)(\partial/\partial \xi_+^*) T^{(3)} = \eta_+(x^*y)T^{(3)}. \tag{C3} \]

The solution of this, and analogous equations, is

\[ T^{(3)} = \exp\left(\frac{(\xi^*\eta)(x^*y)}{1 - \lambda^*\kappa}\right) T_0^{(3)}, \tag{C4} \]

where

\[ T_0^{(3)} = (\exp(\lambda[a^+b^+])\Psi_0, \exp(\kappa[a^+b^+])\Psi_0) \]
\[ = (\Psi_0, \exp(\lambda^*\kappa(a^+; a))\Psi_0) = \frac{1}{(1 - \lambda^*\kappa)^2}, \tag{C5} \]

in view of the simple generalization of (B6)

\[ \exp(z(a; a^+))\Psi_0 = \frac{1}{(1 - z)^2} \Psi_0. \tag{C6} \]

Therefore

\[ T^{(3)} = \frac{1}{(1 - \lambda^*\kappa)^2} \exp\left(\frac{(\xi^*\eta)(x^*y)}{1 - \lambda^*\kappa}\right). \tag{C7} \]

One can prove, in a similar manner, that

\[ (\exp(\lambda[a^+b^+]) + (xa^+) + (xb^+))\Psi_0, \exp(\kappa[a^+b^+] + (ya^+) + (yb^+))\Psi_0 \]
\[ = \frac{1}{(1 - \lambda^*\kappa)^2} \exp\left(\frac{1}{1 - \lambda^*\kappa} \{x^*_1 y_1 + x^*_2 y_2 + \kappa[x^*_1 x^*_2] + \lambda^*[y_1 y_2]\}\right). \tag{C8} \]
The general member of the class exemplified by \( (C1) \) is
\[
T^{(n)} = \left( \exp \left( \frac{1}{2} \sum_{\mu} \lambda_\mu [A^+_\mu A^+_{\mu}] + \sum_{\mu} \lambda_\mu (x A^+_\mu) \right) \right) \Psi_0,
\]
\[
\exp \left( \frac{1}{2} \sum_{\mu} \kappa_\mu [A^+_\mu A^+_{\mu}] + \sum_{\mu} \eta_\mu (y A^+_\mu) \right) \Psi_0, \tag{C9}
\]
where the \( A^\nu \) are \( n \) sets of two-component operators, obeying
\[
[A^\nu_{\nu}, A^\pm_{\nu'}] = \delta_{\nu \nu'} \delta_{\nu' \nu'}, \tag{C10}
\]
while \( \lambda_\mu \) and \( \kappa_\mu \) form antisymmetrical matrices. Following the same procedure, we evaluate
\[
(\partial/\partial \xi^*_{\nu}) T^{(n)} = (e \cdots \Psi_0, (x A^+_\nu) e \cdots \Psi_0)
\]
\[
= \eta_\nu (x A^+_\nu) T^{(n)} + \sum_{\nu} \kappa_\mu (x A^+_\nu) e \cdots \Psi_0, \tag{C11}
\]
whence
\[
(\partial/\partial \xi^*_{\nu}) T^{(n)} + \sum_{\nu} \kappa_\mu \eta_\nu (x A^+_\nu) T^{(n)} = \eta_\nu (x A^+_\nu) T^{(n)}. \tag{C12}
\]
The solution of this equation can be expressed in a matrix notation as
\[
T^{(n)} = \exp \left[ (x A^+_\nu) \sum_{\nu} \lambda^*_{\nu} \left( \frac{1}{1 + \kappa \lambda^*} \right) \right] \eta_\nu T^{(n)}_0, \tag{C13}
\]
where
\[
T^{(n)}_0 = (\exp(\frac{1}{2} \sum_{\mu} \lambda_\mu [A^+_\mu A^+_{\mu}]) \Psi_0, \exp(\frac{1}{2} \sum_{\mu} \kappa_\mu [A^+_\mu A^+_{\mu}]) \Psi_0)
\]
\[
= (\Psi_0^*, Q \Psi_0), \tag{C14}
\]
and
\[
Q = \exp(\frac{1}{2} \sum_{\mu} \lambda^*_{\mu} [A^\mu_{\mu}]) \cdot \exp(\frac{1}{2} \sum_{\mu} \kappa_\mu [A^+_\mu A^+_{\mu}]). \tag{C15}
\]
To evaluate \( T^{(n)}_0 \), we employ the following properties of \( Q \),
\[
(\partial/\partial \lambda_{\mu}) Q = [A^\mu_{\nu} A^\nu_{\mu}] Q, \tag{C16}
\]
and
\[
[x A^\mu_{\nu}] Q - Q [x A^\mu_{\nu}] = -Q \sum_{\nu} \kappa_\nu (x A^+_\nu), \tag{C17a}
\]
\[
Q (x A^+_\nu) - (x A^+_\nu) Q = \sum_{\nu} \lambda^*_{\nu} [x A^\nu_{\nu}] Q, \tag{C17b}
\]
in which \( x \) is an arbitrary constant spinor. One can combine \( (C17a, b) \) into
\[
\sum_{\nu} (1 + \kappa \lambda^*_{\nu}) [x A^\nu_{\nu}] Q = Q [x A^\nu_{\nu}] - \sum_{\nu} \kappa_\nu (x A^+_\nu) Q, \tag{C18}
\]
or
\[
[x A^\nu_{\nu}] Q = \sum_{\nu} \left( \frac{1}{1 + \kappa \lambda^*_{\nu}} \right) Q [x A^\nu_{\nu}] - \sum_{\nu} \left( \frac{1}{1 + \kappa \lambda^*_{\nu}} \kappa \right) (x A^+_\nu) Q. \tag{C19}
\]

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Therefore
\[
[A_\mu A_\nu]Q = \sum_{\beta} \left( \frac{1}{1 + \kappa \lambda^{*}} \right) [A_\mu Q A_\nu] - \sum_{\beta} \left( \frac{1}{1 + \kappa \lambda^{*}} \right) (A^{\beta}_{\mu} A_\nu)Q
- 2 \left( \frac{1}{1 + \kappa \lambda^{*}} \right) \kappa Q.
\] (C20)
from which we obtain \((A_{\mu} \Psi_0 = 0)\)
\[
(\partial/\partial \lambda^{*}_{\nu}) T_0^{(n)} = -2 \left( \frac{1}{1 + \kappa \lambda^{*}} \right) \kappa T_0^{(n)}.
\] (C21)
Thus, with respect to changes in the matrix \(\lambda^{*}\), we have
\[
\delta \log T_0^{(n)} = \frac{1}{2} \sum_{\mu \nu} \delta \lambda^{*}_{\mu \nu} (\partial/\partial \lambda^{*}_{\nu}) \log T_0^{(n)} = -\text{tr} \left( \frac{1}{1 + \kappa \lambda^{*}} \kappa \delta \lambda^{*} \right).
\] (C22)
On comparing this with the theorem on differentiation of a determinant,
\[
\delta \log |A| = \text{tr}(A^{-1} \delta A),
\] (C23)
we obtain the desired general evaluation,
\[
T_0^{(n)} = \frac{1}{1 + \kappa \lambda^{*}}.
\] (C24)
A recurrence relation for \(T_0^{(n)}\) can also be established with the aid of (C13). Thus, we have
\[
T_0^{(n)} = \left( \exp\left( \frac{1}{2} \sum_{\mu \nu=1}^{n-1} \lambda_{\mu \nu} [A^{\mu}_{\mu} A^{\nu}_{\nu}] + \sum_{\mu \nu=1}^{n-1} \lambda_{\mu \nu} [A^{\mu}_{\mu} A^{\nu}_{\nu}] \right) \Psi_0, \exp\left( \frac{1}{2} \sum_{\mu \nu=1}^{n-1} \kappa_{\mu \nu} [A^{\mu}_{\mu} A^{\nu}_{\nu}] + \sum_{\mu \nu=1}^{n-1} \kappa_{\mu \nu} [A^{\mu}_{\mu} A^{\nu}_{\nu}] \right) \Psi_0 \right) T_0^{(n-1)}
= \left( \Psi_0, \exp \left( \sum_{\mu \nu = 1}^{n-1} \lambda_{\mu \nu} \left( \frac{1}{1 + \kappa \lambda^{*}} \right) \kappa_{\nu \nu} \right) T_0^{(n-1)} \right)^2 T_0^{(n-1)}
= \left( 1 - \sum_{\mu \nu} \lambda^{*}_{\mu \nu} \left( \frac{1}{1 + \kappa \lambda^{*}} \right) \kappa_{\nu \nu} \right)^2 T_0^{(n-1)},
\] (C25)
in which \(\kappa'\) and \(\lambda'\) designate the matrices of dimensionality \(n - 1\).

The actual construction of the \(T_0^{(n)}\) can be performed without detailed calculations. It follows from (C24) and (C25) that \(T_0^{(n)}\) has the form of the inverse square of a power series in the components of \(\lambda^{*}\) and \(\kappa\), where the last terms of the series, \((-1)^{n} |\lambda^{*}|^{1/2} |\kappa|^{1/2}\), vanishes for \(n\) odd. Thus, beginning with
\[
T_0^{(3)} = [1 - \lambda_{12}^{*} \kappa_{12}]^{-2} = [1 - |\lambda^{*}|^{1/2} |\kappa|^{1/2}]^{-2},
\] (C26)
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We infer that $T^{(3)}_0$ has the same structure, suitably extended for the additional dimension,

$$T^{(3)}_0 = [1 - (\lambda^{a}_{12} \kappa_{12} + \lambda^{a}_{23} \kappa_{23} + \lambda^{a}_{24} \kappa_{24})]^{-2} = [1 - \frac{1}{3} \sum_{\mu \nu = 1}^{3} \lambda^{a}_{\mu \nu} \kappa_{\mu \nu}]^{-2}, \quad (C27)$$

and therefore

$$T^{(4)}_0 = [1 - \frac{4}{3} \sum_{\mu \nu = 1}^{4} \lambda^{a}_{\mu \nu} \kappa_{\mu \nu} + |\lambda|^1]^{-2}, \quad (C28)$$

where

$$|\lambda|^{1/2} = \lambda_{12} \lambda_{24} + \lambda_{23} \lambda_{14} + \lambda_{31} \lambda_{12}$$

$$= \frac{1}{2} \sum_{\mu \nu \tau \sigma = 1}^{4} \epsilon_{\mu \nu \tau \sigma} \lambda_{\mu \nu} \lambda_{\tau \sigma}, \quad (C29)$$

and $\epsilon$ is the completely antisymmetric symbol. For the last indication of this general procedure we remark that, as the extension of (C28), we have

$$T^{(5)}_0 = [1 - \frac{5}{3} \sum_{\mu \nu = 1}^{5} \lambda^{a}_{\mu \nu} \kappa_{\mu \nu} + \sum_{\mu = 1}^{5} (\lambda^{a}_{\mu})_{\mu} \kappa_{\mu}]^{-2}, \quad (C30)$$

in which

$$(\lambda)_{\mu} = \frac{1}{2} \sum_{\mu \nu \tau \sigma = 1}^{5} \epsilon_{\mu \nu \tau \sigma} \lambda_{\mu \nu} \lambda_{\tau \sigma}. \quad (C31)$$