The Liénard formula

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I. THE LIÉNARD FORMULA

We want to show

$$P=\frac{e^{2}}{4\pi c}\int d\Omega_{\hat{\mathbf{n}}}\frac{\left|\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^{2}}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{5}}=\frac{2e^{2}}{3c}\gamma^{6}\left[\left|\dot{\boldsymbol{\beta}}\right|^{2}-\left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^{2}\right], \tag{1}$$

where the integral is over all orientations of $\hat{\mathbf{n}}$, or

$$I=\int d\Omega_{\hat{\mathbf{n}}}\frac{\left|\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^{2}}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{5}}=\frac{8\pi}{3}\gamma^{6}\left[\left|\dot{\boldsymbol{\beta}}\right|^{2}-\left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^{2}\right]. \tag{2}$$

It is useful to note that

$$\gamma^{6}\left[\left|\dot{\boldsymbol{\beta}}\right|^{2}-\left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^{2}\right]=\gamma^{4}\left[\left|\dot{\boldsymbol{\beta}}\right|^{2}+\gamma^{2}\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^{2}\right], \tag{3}$$

so

$$I=\int d\Omega_{\hat{\mathbf{n}}}\frac{\left|\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^{2}}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{5}}=\frac{8\pi}{3}\gamma^{4}\left[\left|\dot{\boldsymbol{\beta}}\right|^{2}+\gamma^{2}\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^{2}\right]. \tag{4}$$

We can develop the square and write the integral as

$$I=\int d\Omega_{\hat{\mathbf{n}}}\left[-\frac{\left(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}}\right)^{2}}{\gamma^{2}\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{5}}+2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\frac{\left(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}}\right)}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{4}}+\frac{\left|\dot{\boldsymbol{\beta}}\right|^{2}}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{3}}\right]. \tag{5}$$

The 3rd term is easy. Setting $\boldsymbol{\beta}=\beta\hat{\mathbf{z}}$ and $\hat{\mathbf{n}}=\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}$,

$$\begin{aligned}I_{3}&=\left|\dot{\boldsymbol{\beta}}\right|^{2}\int d\Omega_{\hat{\mathbf{n}}}\frac{1}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^{3}}\\&=\left|\dot{\boldsymbol{\beta}}\right|^{2}\int_{0}^{2\pi}d\phi\int_{0}^{\pi}\sin\theta d\theta\frac{1}{\left(1-\beta\cos\theta\right)^{3}}\\&=2\pi\left|\dot{\boldsymbol{\beta}}\right|^{2}\int_{-1}^{1}d\mu\frac{1}{\left(1-\beta\mu\right)^{3}}\\&=2\pi\left|\dot{\boldsymbol{\beta}}\right|^{2}\frac{2}{\left(1-\beta^{2}\right)^{2}}\\&=4\pi\gamma^{4}\left|\dot{\boldsymbol{\beta}}\right|^{2},\end{aligned} \tag{6}$$

where we used

$$\begin{aligned}J_{1}\left(\beta\right)=\int_{-1}^{1}d\mu\frac{1}{\left(1-\beta\mu\right)^{3}}&=\frac{1}{2\beta}\left.\frac{1}{\left(1-\beta\mu\right)^{2}}\right|_{\mu=-1}^{\mu=1}\\&=\frac{1}{2\beta}\left[\frac{1}{\left(1-\beta\right)^{2}}-\frac{1}{\left(1+\beta\right)^{2}}\right]\\&=\frac{1}{2\beta}\left[\frac{4\beta}{\left(1-\beta^{2}\right)^{2}}\right]=\frac{2}{\left(1-\beta^{2}\right)^{2}}\end{aligned} \tag{7}$$

The 2nd term is not hard either

$$
\begin{aligned}
I_2 &= 2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\int d\Omega_{\hat{\mathbf{n}}}\frac{\left(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}}\right)}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^4} \\
&= 2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\frac{\dot{\beta}_z\cos\theta+\dot{\beta}_x\sin\theta\cos\phi+\dot{\beta}_y\sin\theta\sin\phi}{\left(1-\beta\cos\theta\right)^4} \\
&= 4\pi\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\int_0^{\pi}\sin\theta d\theta\frac{\dot{\beta}_z\cos\theta}{\left(1-\beta\cos\theta\right)^4} \\
&= 4\pi\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\dot{\beta}_z\int_{-1}^{1}d\mu\frac{\mu}{\left(1-\beta\mu\right)^4} \\
&= 4\pi\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\dot{\beta}_z\frac{8\beta}{3\left(1-\beta^2\right)^3} \\
&= \frac{32\pi}{3}\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2,
\end{aligned}
\tag{8}
$$

where the integral was obtained differentiating Eq. (7) with respect to β

$$
\begin{aligned}
J_2\left(\beta\right) = \int_{-1}^{1}d\mu\frac{\mu}{\left(1-\beta\mu\right)^4} &= \frac{1}{3}\frac{d}{d\beta}\left[\int_{-1}^{1}d\mu\frac{1}{\left(1-\beta\mu\right)^3}\right] \\
&= \frac{1}{3}\frac{dJ_1\left(\beta\right)}{d\beta} \qquad (9)\\
&= \frac{1}{3}\frac{d}{d\beta}\left[\frac{2}{\left(1-\beta^2\right)^2}\right] \\
&= \frac{2}{3}\frac{4\beta}{\left(1-\beta^2\right)^3} = \frac{8}{3}\frac{\beta}{\left(1-\beta^2\right)^3}. \qquad (10)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_1 &= -\frac{1}{\gamma^2}\int d\Omega_{\hat{\mathbf{n}}}\frac{\left(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}}\right)^2}{\left(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}\right)^5} \\
&= -\frac{1}{\gamma^2}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\frac{\left(\dot{\beta}_z\cos\theta+\dot{\beta}_x\sin\theta\cos\phi+\dot{\beta}_y\sin\theta\sin\phi\right)^2}{\left(1-\beta\cos\theta\right)^5} \\
&= -\frac{1}{\gamma^2}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\frac{\dot{\beta}_z^2\cos^2\theta+\dot{\beta}_x^2\sin^2\theta\cos^2\phi+\dot{\beta}_y^2\sin^2\theta\sin^2\phi}{\left(1-\beta\cos\theta\right)^5},
\end{aligned}
\tag{11}
$$

where the crossed terms vanish upon ϕ integration. Integration over ϕ now gives

$$
I_1 = -\frac{\pi}{\gamma^2}\int_0^{\pi}\sin\theta d\theta\frac{2\dot{\beta}_z^2\cos^2\theta+\left(\dot{\beta}_x^2+\dot{\beta}_y^2\right)\sin^2\theta}{\left(1-\beta\cos\theta\right)^5},
\tag{12}
$$

where we used

$$
\int_0^{2\pi}\left\{\begin{array}{c}\cos^2\phi\\ \sin^2\phi\end{array}\right\}d\phi = \pi.
\tag{13}
$$

Thus,

$$
I_1 = -\frac{\pi}{\gamma^2}\int_{-1}^{1}d\mu\frac{2\dot{\beta}_z^2\mu^2+\left(\dot{\beta}_x^2+\dot{\beta}_y^2\right)\left(1-\mu^2\right)}{\left(1-\beta\mu\right)^5}.
\tag{14}
$$

We now need

$$J_3 = \int_{-1}^{1} d\mu \frac{1}{(1-\beta\mu)^5} = \frac{1}{4\beta}\ \frac{1}{(1-\beta\mu)^4}\Bigg|_{\mu=-1}^{\mu=1} = 2\frac{1+\beta^2}{(1-\beta^2)^4} = 2\gamma^8\left(1+\beta^2\right), \tag{15}$$

$$J_4 = \int_{-1}^{1} d\mu \frac{\mu^2}{(1-\beta\mu)^5} = \frac{1}{4}\frac{dJ_2(\beta)}{d\beta} = \frac{2}{3}\frac{d}{d\beta}\left[\frac{\beta}{(1-\beta^2)^3}\right] = \frac{2}{3}\frac{1+5\beta^2}{(1-\beta^2)^4} = \frac{2}{3}\gamma^8\left(1+5\beta^2\right). \tag{16}$$

From these we get

$$\begin{aligned} I_1 &= -\frac{2\pi}{3}\gamma^6\left[\left(2\dot{\beta}_z^2 - \dot{\beta}_x^2 - \dot{\beta}_y^2\right)\left(1+5\beta^2\right) + 3\left(\dot{\beta}_x^2 + \dot{\beta}_y^2\right)\left(1+\beta^2\right)\right] \\ &= -\frac{4\pi}{3}\gamma^6\left[\left(\dot{\beta}_x^2 + \dot{\beta}_y^2 + \dot{\beta}_z^2\right) + \beta^2\left(5\dot{\beta}_z^2 - \dot{\beta}_x^2 - \dot{\beta}_y^2\right)\right] \\ &= -\frac{4\pi}{3}\gamma^6\left[\left(\dot{\beta}_x^2 + \dot{\beta}_y^2 + \dot{\beta}_z^2\right) - \beta^2\left(\dot{\beta}_z^2 + \dot{\beta}_x^2 + \dot{\beta}_y^2\right) + 6\beta^2\dot{\beta}_z^2\right] \\ &= -\frac{4\pi}{3}\gamma^6\left[\frac{1}{\gamma^2}\left|\dot{\boldsymbol{\beta}}\right|^2 + 6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right] = -\frac{4\pi}{3}\gamma^4\left[\left|\dot{\boldsymbol{\beta}}\right|^2 + 6\gamma^2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right]. \end{aligned} \tag{17}$$

Putting it all together

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= -\frac{4\pi}{3}\gamma^4\left[\left|\dot{\boldsymbol{\beta}}\right|^2 + 6\gamma^2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right] + \frac{32\pi}{3}\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 + 4\pi\gamma^4\left|\dot{\boldsymbol{\beta}}\right|^2 \\ &= \frac{8\pi}{3}\gamma^4\left|\dot{\boldsymbol{\beta}}\right|^2 + \frac{8\pi}{3}\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 = \frac{8\pi}{3}\gamma^4\left[\left|\dot{\boldsymbol{\beta}}\right|^2 + \gamma^2\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right]. \end{aligned} \tag{18}$$