# FI 008 – Eletrodinâmica I

1º Semestre de 2021

20/04/2021

Aula 10

# Ondas eletromagnéticas

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= 0\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} &= 0;\\ \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} &= 0.\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)};\\ \mathbf{B}(\mathbf{x},t) &= \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}.\end{aligned}$$

$$\mathbf{E}_0,\mathbf{B}_0\in\mathbb{C}$$

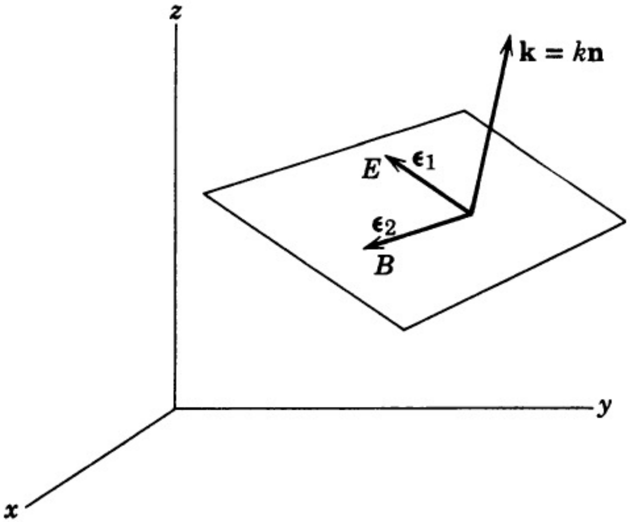
$$E_x^{phys}(\mathbf{x},t)=\mathrm{Re}\left[|E_{0x}|\,e^{i\delta_x}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]=|E_{0x}|\cos\left(\mathbf{k}\cdot\mathbf{x}-\omega t+\delta_x\right),\ldots$$

$$\boxed{\omega=c\,|\mathbf{k}|=ck\in\mathbb{R}}$$

$$\boxed{\mathbf{k}=k\hat{\mathbf{n}}\in\mathbb{R}}$$

$$\boxed{\begin{aligned}\hat{\mathbf{n}}\cdot\mathbf{E}_0 &= 0;\\ \hat{\mathbf{n}}\cdot\mathbf{B}_0 &= 0;\\ \hat{\mathbf{n}}\times\mathbf{E}_0 &= c\mathbf{B}_0;\\ \hat{\mathbf{n}}\times\mathbf{B}_0 &= -\frac{1}{c}\mathbf{E}_0.\end{aligned}}$$

# Tríade ortonormal




$$\hat{\boldsymbol{\epsilon}}_1 \times \hat{\boldsymbol{\epsilon}}_2 = \hat{\mathbf{n}}$$

$$E_0 \in \mathbb{C}$$

$\hat{\epsilon}_1, \hat{\epsilon}_2 \in \mathbb{R}$

$$\mathbf{E}_0 = E_0 \hat{\boldsymbol{\epsilon}}_1;$$
$$\mathbf{B}_0 = \frac{E_0}{c} \hat{\boldsymbol{\epsilon}}_2.$$

Outra solução linearmente independente é:

$E_0' \in \mathbb{C}$

$$\mathbf{E}_0 = E_0' \hat{\boldsymbol{\epsilon}}_2;$$
$$\mathbf{B}_0 = -\frac{E_0'}{c} \hat{\boldsymbol{\epsilon}}_1.$$

# O vetor de Poynting e a densidade de energia

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \qquad u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

Teorema: $A(t) = A_0 e^{i\omega t},\ A_0 \in \mathbb{C}$

$$B(t) = B_0 e^{i\omega t},\ B_0 \in \mathbb{C}$$

então:

$$\left\langle A^{phys}(t)\, B^{phys}(t)\right\rangle \equiv \frac{\omega}{2\pi}\int_0^{2\pi/\omega} A^{phys}(t)\, B^{phys}(t)\, dt = \frac{1}{2}\mathrm{Re}\left[A_0^* B_0\right] = \frac{1}{2}\mathrm{Re}\left[A_0 B_0^*\right]$$

$$\left\langle \mathbf{S}^{phys}(t)\right\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left[\mathbf{E}_0^*\times\mathbf{B}_0\right] = \frac{1}{2\mu_0 c}\left|\mathbf{E}_0\right|^2\hat{\mathbf{n}}$$

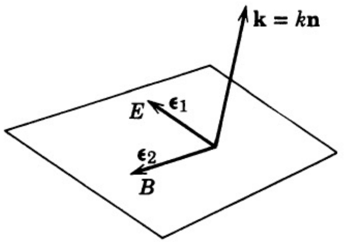
$$\left\langle u^{phys}(t)\right\rangle = \frac{\epsilon_0}{4}\mathrm{Re}\left[\mathbf{E}_0^*\cdot\mathbf{E}_0\right] + \frac{1}{4\mu_0}\mathrm{Re}\left[\mathbf{B}_0^*\cdot\mathbf{B}_0\right] = \frac{\epsilon_0}{2}\left|\mathbf{E}_0\right|^2$$

$$\left\langle \mathbf{S}^{phys}(t)\right\rangle = c\left\langle u^{phys}(t)\right\rangle\hat{\mathbf{n}}$$

# Polarização de ondas planas no vácuo

Base completa para **k=*k*n**

$$\mathbf{E}_0 = E_0\hat{\boldsymbol{\epsilon}}_1; \quad \mathbf{E}_0 = E_0\hat{\boldsymbol{\epsilon}}_2;$$
$$\mathbf{B}_0 = \frac{E_0}{c}\hat{\boldsymbol{\epsilon}}_2. \quad \mathbf{B}_0 = -\frac{E_0}{c}\hat{\boldsymbol{\epsilon}}_1.$$



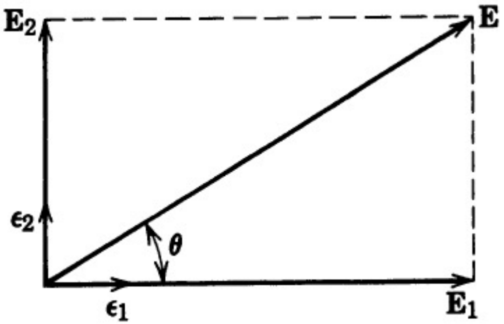

Solução geral: $\mathbf{E}_0 = E_1\hat{\boldsymbol{\epsilon}}_1 + E_2\hat{\boldsymbol{\epsilon}}_2 \quad \mathbf{B}_0 = \frac{1}{c}\hat{\mathbf{n}} \times \mathbf{E}_0$

$$\boxed{E_1, E_2 \in \mathbb{C} \qquad \hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, \hat{\mathbf{n}} \in \mathbb{R}}$$

## a) Polarização linear (ou luz plano-polarizada):

$$E_1 = |E_1|\, e^{i\delta} \quad E_2 = |E_2|\, e^{i\delta}$$

$$\mathbf{E}^{phys}(\mathbf{x}, t) = (|E_1|\,\hat{\boldsymbol{\epsilon}}_1 + |E_2|\,\hat{\boldsymbol{\epsilon}}_2)\cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta)$$



# Polarização de ondas planas no vácuo

b) Polarização circular (ou luz circularmente polarizada):

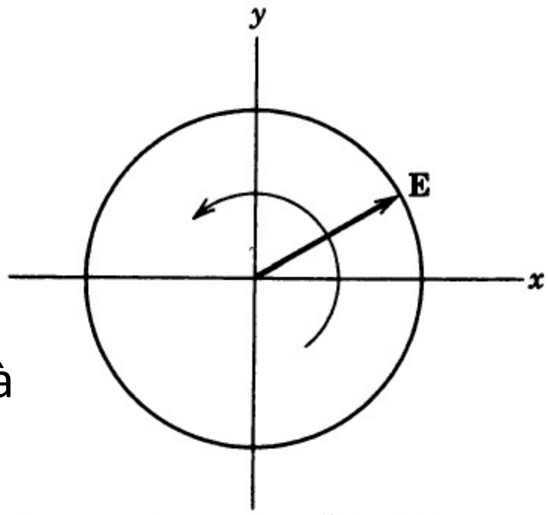
$$E_1 = |E_1|\, e^{i\delta} \quad E_2 = |E_1|\, e^{i(\delta \pm \pi/2)} = \pm i E_1$$

$$\mathbf{E}^{phys}(\mathbf{x}, t) \; = \; |E_1| \left[\hat{\boldsymbol{\epsilon}}_1 \cos\left(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta\right) \mp \hat{\boldsymbol{\epsilon}}_2 \sin\left(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta\right)\right]$$

$$\mathbf{E}^{phys}(0, t) \; = \; |E_1| \left[\hat{\boldsymbol{\epsilon}}_1 \cos\left(\omega t - \delta\right) \pm \hat{\boldsymbol{\epsilon}}_2 \sin\left(\omega t - \delta\right)\right]$$

Sinal superior: luz circularmente polarizada à esquerda ou de helicidade positiva.

Sinal inferior: luz circularmente polarizada à direita ou de helicidade negativa.




$\mathbf{E}(\mathbf{x}, t) = E_0(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$

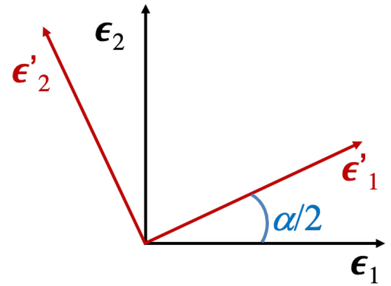
# Polarização de ondas planas no vácuo

c) Caso mais geral: polarização elíptica (ou luz elipticamente polarizada)

$$E_{\pm} = \frac{1}{\sqrt{2}}\left(E_1 \mp iE_2\right)$$

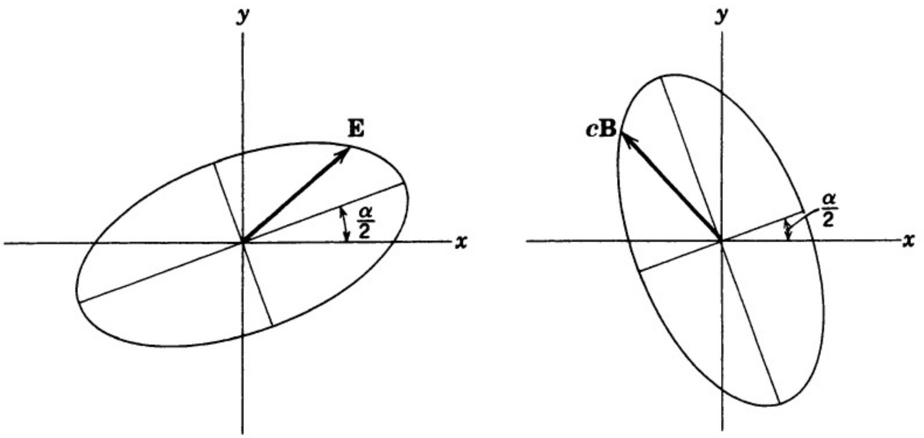
$$\frac{E_-}{E_+} = re^{i\alpha} \quad r, \alpha \in \mathbb{R}$$

$$= \frac{E_1 + iE_2}{E_1 - iE_2}$$




$$\mathbf{E}^{phys}(\mathbf{x},t) = \frac{|E_+|}{\sqrt{2}}\left[(1+r)\cos\left(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta + \alpha/2\right)\hat{\epsilon}_1' - (1-r)\sin\left(\mathbf{k}\cdot\mathbf{x} - \omega t + \delta + \alpha/2\right)\hat{\epsilon}_2'\right]$$

$$\boxed{\frac{a}{b} = \left|\frac{1+r}{1-r}\right|}$$



# Os parâmetros de Stokes

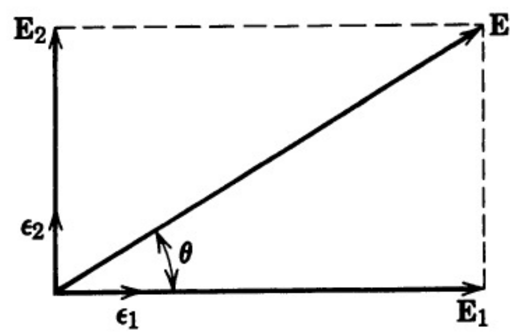
$$E_1 = a_1 e^{i\delta_1} \quad E_2 = a_2 e^{i\delta_2}$$

$$\begin{aligned} s_0 &= a_1^2 + a_2^2 \propto I \\ s_1 &= a_1^2 - a_2^2 \\ s_2 &= 2a_1 a_2 \cos(\delta_2 - \delta_1) \\ s_3 &= 2a_1 a_2 \sin(\delta_2 - \delta_1) \end{aligned}$$

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

a) polarização linear $\mathbf{E}_0 = |E_0| e^{i\delta} (\cos\theta \hat{\boldsymbol{\epsilon}}_1 + \sin\theta \hat{\boldsymbol{\epsilon}}_2)$

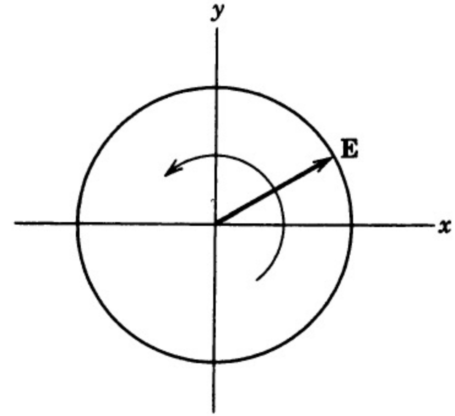
$$\begin{aligned} s_0 &= |E_0|^2 \propto I \\ s_1 &= |E_0|^2 \cos(2\theta) \\ s_2 &= |E_0|^2 \sin(2\theta) \\ s_3 &= 0 \end{aligned}$$



# Os parâmetros de Stokes

b) polarização circular $\mathbf{E}_0^{\pm} = \frac{|E_0|}{\sqrt{2}} e^{i\delta} \left(\hat{\boldsymbol{\epsilon}}_1 \pm i\hat{\boldsymbol{\epsilon}}_2\right)$

$$\begin{aligned} s_0 &= |E_0|^2 \propto I \\ s_1 &= 0 \\ s_2 &= 0 \\ s_3 &= \pm |E_0|^2 \end{aligned}$$




$\mathbf{E}(\mathbf{x}, t) = E_0(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$

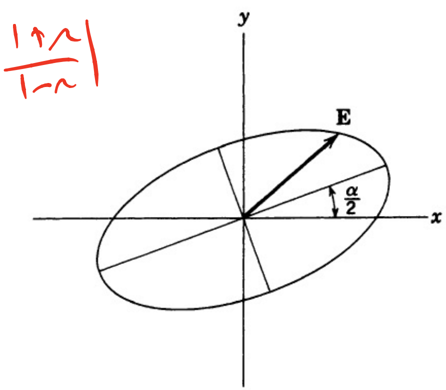
c) polarização elíptica $E_{\pm} = \frac{1}{\sqrt{2}}(E_1 \mp iE_2)$ $\frac{E_-}{E_+} = re^{i\alpha}$ $r, \alpha \in \mathbb{R}$

$$\begin{aligned} s_0 &= |E_1|^2 + |E_2|^2 \propto I \\ s_1 &= Is\cos\alpha \\ s_2 &= Is\sin\alpha \\ s_3 &= It \end{aligned}$$

$$\begin{aligned} s &= \frac{2r}{1+r^2} \\ t &= \frac{1-r^2}{1+r^2} \end{aligned}$$

$$s^2 + t^2 = 1$$

$\frac{a}{b} = \left|\frac{1+r}{1-r}\right|$



$$P\left[\vec{J}(\vec{x})\right] = -\vec{J}(-\vec{x})$$

$$\vec{J}_i(\vec{x}) \Rightarrow \quad a_{ij} J_j(R\vec{x}) = \vec{J}'_i(\vec{x}')$$

$$T\left[\vec{J}(\vec{x},t)\right] = -\vec{J}(\vec{x},-t)$$

$$\int \vec{J}(\vec{x})\, d^3x = 0 \qquad \text{SE } \vec{\nabla}\cdot\vec{J} = 0$$