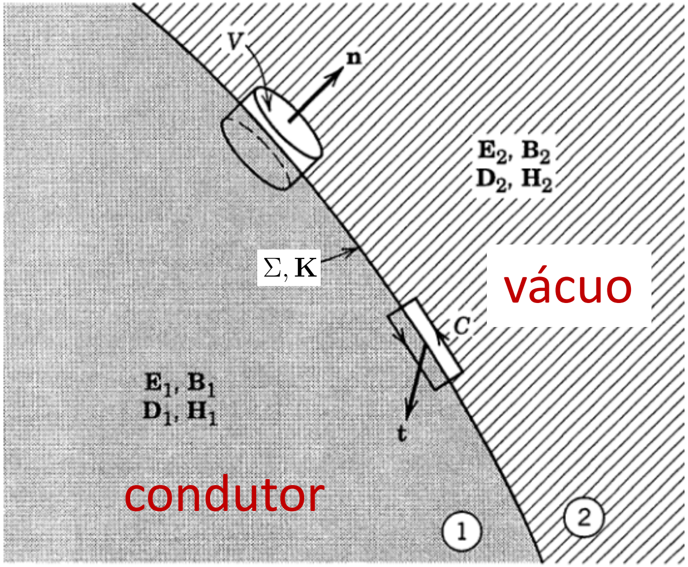
FI 008 – Eletrodinâmica I

1º Semestre de 2020

23/04/2020

Aula 12

Aula passada



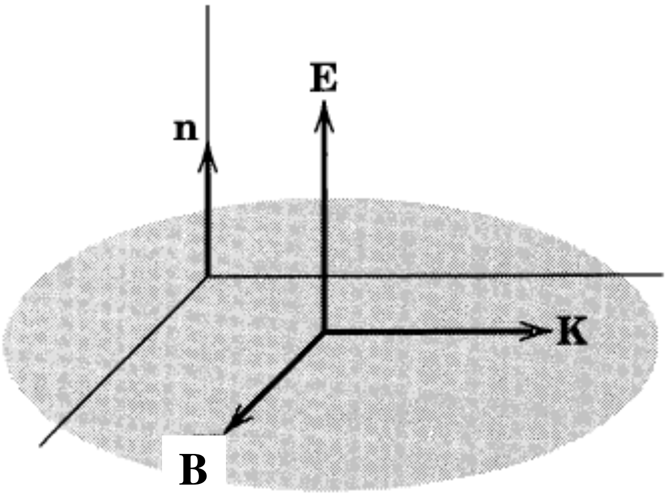
$$
\begin{aligned}
\hat{\mathbf{n}} \cdot \mathbf{B}_2 &= \hat{\mathbf{n}} \cdot \mathbf{B}_1 \\
\hat{\mathbf{n}} \times \mathbf{E}_2 &= \hat{\mathbf{n}} \times \mathbf{E}_1 \\
\hat{\mathbf{n}} \cdot \mathbf{E}_2 - \hat{\mathbf{n}} \cdot \mathbf{E}_1 &= \frac{\Sigma}{\epsilon_0} \\
\hat{\mathbf{n}} \times \mathbf{B}_2 - \hat{\mathbf{n}} \times \mathbf{B}_1 &= \mu_0 \mathbf{K}
\end{aligned}
$$

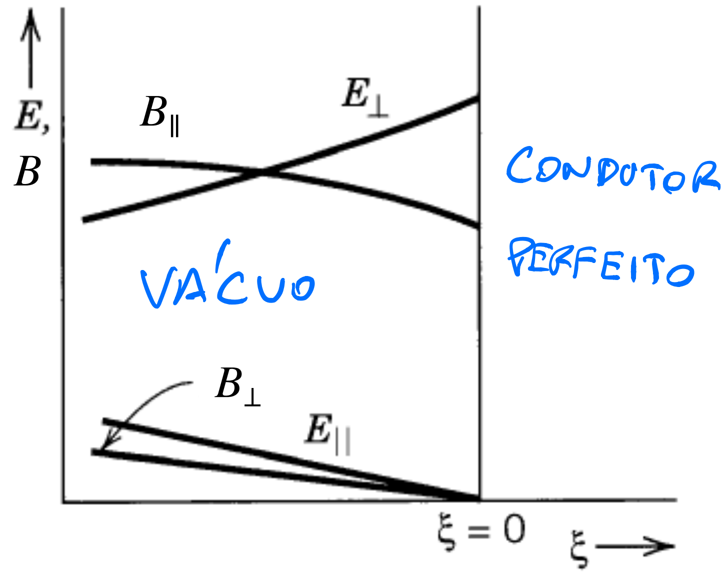
$$\mathbf{E}_1 = \mathbf{E}_c = 0; \; \mathbf{B}_1 = \mathbf{B}_c = 0$$

$$
\boxed{
\begin{aligned}
\hat{\mathbf{n}} \cdot \mathbf{B}_V &= 0 \\
\hat{\mathbf{n}} \times \mathbf{E}_V &= 0 \\
\hat{\mathbf{n}} \cdot \mathbf{E}_V &= \frac{\Sigma}{\epsilon_0} \\
\hat{\mathbf{n}} \times \mathbf{B}_V &= \mu_0 \mathbf{K}
\end{aligned}
}
$$

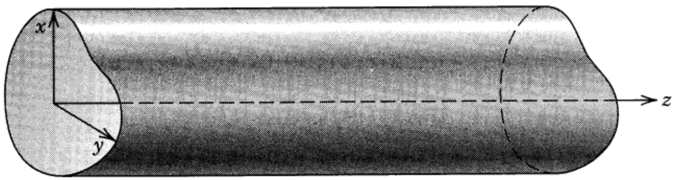
Aula passada

$$
\begin{aligned}
\hat{\mathbf{n}} \cdot \mathbf{B}_V &= 0 \\
\hat{\mathbf{n}} \times \mathbf{E}_V &= 0 \\
\hat{\mathbf{n}} \cdot \mathbf{E}_V &= \frac{\Sigma}{\epsilon_0} \\
\hat{\mathbf{n}} \times \mathbf{B}_V &= \mu_0 \mathbf{K}
\end{aligned}
$$





Aula passada



$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}(x,y)\,e^{i(\pm kz-\omega t)}$$
$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}(x,y)\,e^{i(\pm kz-\omega t)}$$

$$\boldsymbol{\nabla}_t = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$$

Das equações de Maxwell:

$$\boxed{\begin{aligned}\left(\nabla_t^2+\gamma^2\right)\mathbf{E}(x,y) &= 0\\ \left(\nabla_t^2+\gamma^2\right)\mathbf{B}(x,y) &= 0\end{aligned}}$$

$$\boxed{\gamma^2 = \frac{\omega^2}{c^2} - k^2}$$

$$\begin{cases}\mathbf{E} = \mathbf{E}_t + E_z\hat{\mathbf{z}}\\ \mathbf{B} = \mathbf{B}_t + B_z\hat{\mathbf{z}}\end{cases}$$

$$\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}_t\times\mathbf{E}_t) = i\omega B_z \quad (1)$$

$$\boldsymbol{\nabla}_t E_z \mp ik\mathbf{E}_t = i\omega\hat{\mathbf{z}}\times\mathbf{B}_t \quad (2)$$

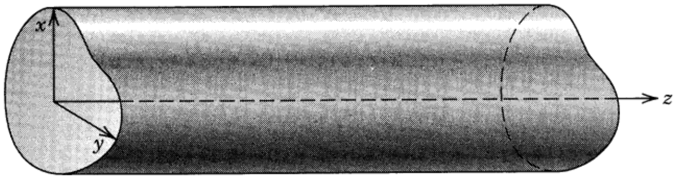
$$\hat{\mathbf{z}}\cdot(\boldsymbol{\nabla}_t\times\mathbf{B}_t) = -i\frac{\omega}{c^2}E_z \quad (3)$$

$$\boldsymbol{\nabla}_t B_z \mp ik\mathbf{B}_t = -i\frac{\omega}{c^2}\hat{\mathbf{z}}\times\mathbf{E}_t \quad (4)$$

$$\boldsymbol{\nabla}_t\cdot\mathbf{E}_t = \mp ikE_z \quad (5)$$

$$\boldsymbol{\nabla}_t\cdot\mathbf{B}_t = \mp ikB_z \quad (6)$$

Aula passada

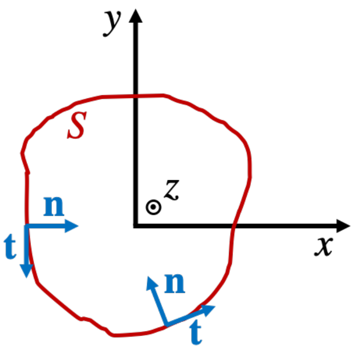


As componentes transversais são completamente determinadas pelas componentes z: basta achar E_z e B_z

$$\gamma^2 \mathbf{B}_t = i\left[\pm k \boldsymbol{\nabla}_t B_z + \frac{\omega}{c^2}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z\right] \quad (7)$$

$$\gamma^2 \mathbf{E}_t = i\left[\pm k \boldsymbol{\nabla}_t E_z - \omega \hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z\right] \quad (8)$$

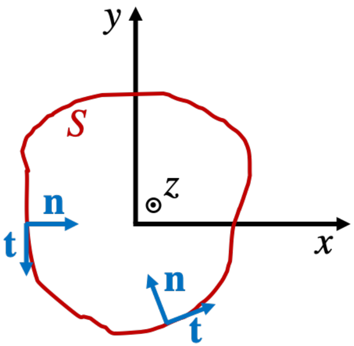
E_z e B_z satisfazem as seguintes condições de contorno no plano xy:



$$\boxed{\begin{aligned} E_z|_S &= 0 \\ \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}_t B_z|_S &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \left(\nabla_t^2 + \gamma^2\right) E_z &= 0 \\ \left(\nabla_t^2 + \gamma^2\right) B_z &= 0 \end{aligned}}$$

Aula passada



Os modos TM e TE são independentes e devem ser resolvidos separadamente: são problemas de auto-valores.

Modo TM $(B_z = 0)$

$$\left(\nabla_t^2 + \gamma^2\right) E_z \;=\; 0$$

$$E_z\big|_S \;=\; 0$$

$$\gamma^2 \mathbf{E}_t \;=\; \pm ik \boldsymbol{\nabla}_t E_z$$

$$\mathbf{B}_t \;=\; \pm \frac{\omega}{c^2 k} \hat{\mathbf{z}} \times \mathbf{E}_t$$

Modo TE $(E_z = 0)$

$$\left(\nabla_t^2 + \gamma^2\right) B_z \;=\; 0$$

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}_t B_z\big|_S \;=\; 0$$

$$\gamma^2 \mathbf{B}_t \;=\; \pm ik \boldsymbol{\nabla}_t B_z$$

$$\mathbf{E}_t \;=\; \mp \frac{\omega}{k} \hat{\mathbf{z}} \times \mathbf{B}_t$$

Relações de dispersão e frequências de corte

PROBLEMAS DE VALOR DE CONTORNO PARA E_z E B_z LEVAM A QUE γ SÓ POSSA ASSUMIR VALORES DISCRETOS : $\gamma \rightarrow \gamma_\lambda$ $\quad \lambda$ = ROTULA OS AUTO-VALORES

$$\gamma_\lambda^2 = \frac{\omega^2}{c^2} - k_\lambda^2 \Rightarrow k_\lambda = \frac{1}{c}\sqrt{\omega^2 - c^2\gamma_\lambda^2}$$

DEFINO : $\omega_\lambda \equiv c\,\gamma_\lambda \Rightarrow k_\lambda = \frac{1}{c}\sqrt{\omega^2 - \omega_\lambda^2} \quad (*)$

$$\omega^2 = c^2 k_\lambda^2 + \omega_\lambda^2 \Rightarrow \omega = \sqrt{c^2 k_\lambda^2 + \omega_\lambda^2} \neq c\,k$$

VÁCUO

RELAÇÃO DE DISPERSÃO É DIFERENTE DO VÁCUO.

COMO $ck_\lambda = \sqrt{\omega^2 - \omega_\lambda^2} \Rightarrow k_\lambda \in \mathbb{R}$ APENAS SE $\boxed{\omega > \omega_\lambda}$

$k_\lambda \in \mathbb{C}$ SE $\omega < \omega_\lambda$

$k_\lambda = a + ib \Rightarrow e^{ik_\lambda z} = e^{iaz} e^{-bz}$

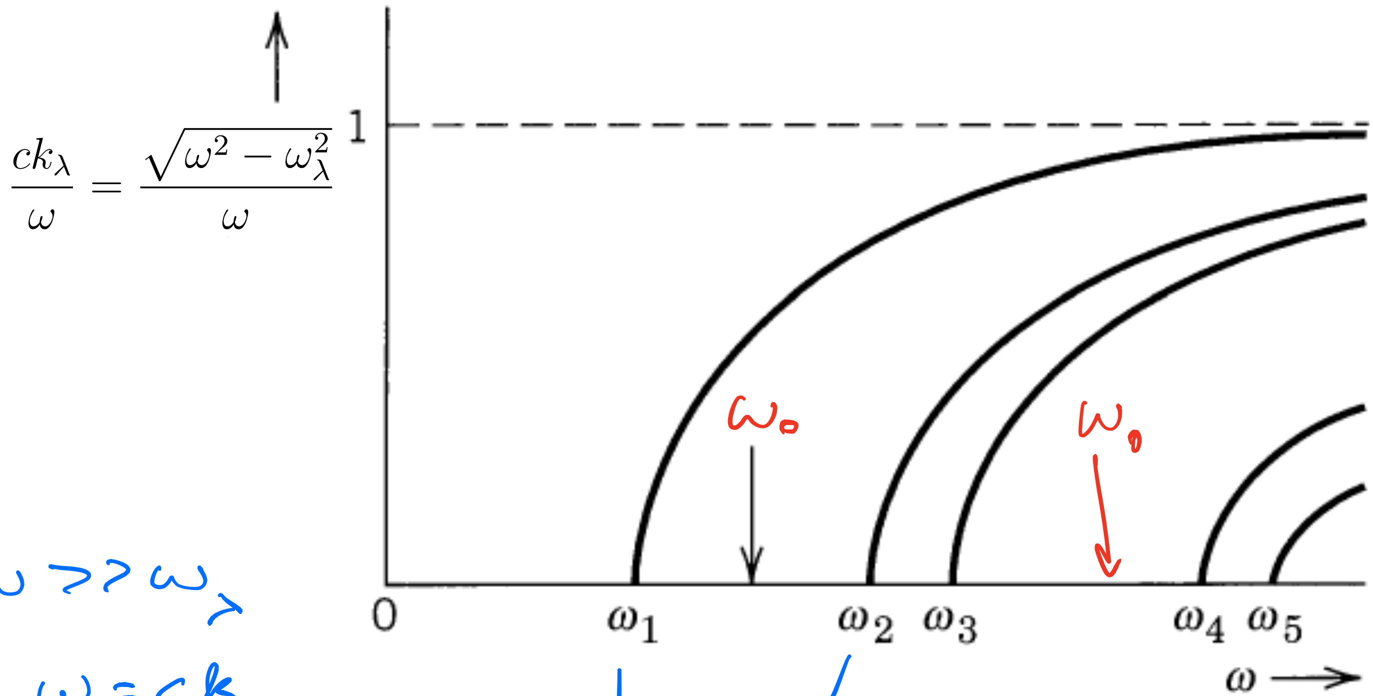
$k \in \mathbb{C} \Rightarrow$ ATENUAÇÃO DA ONDA

ω_λ = FREQUÊNCIA DE CORTE

$$v_p = \frac{\omega}{k_\lambda} = \frac{c\,\omega}{\sqrt{\omega^2 - \omega_\lambda^2}} > c$$

$$\sqrt{v_p v_g} = c$$

$$v_g = \frac{d\omega}{dk_\lambda} = \frac{\sqrt{\omega^2 - \omega_\lambda^2}}{\omega}\, c < c$$

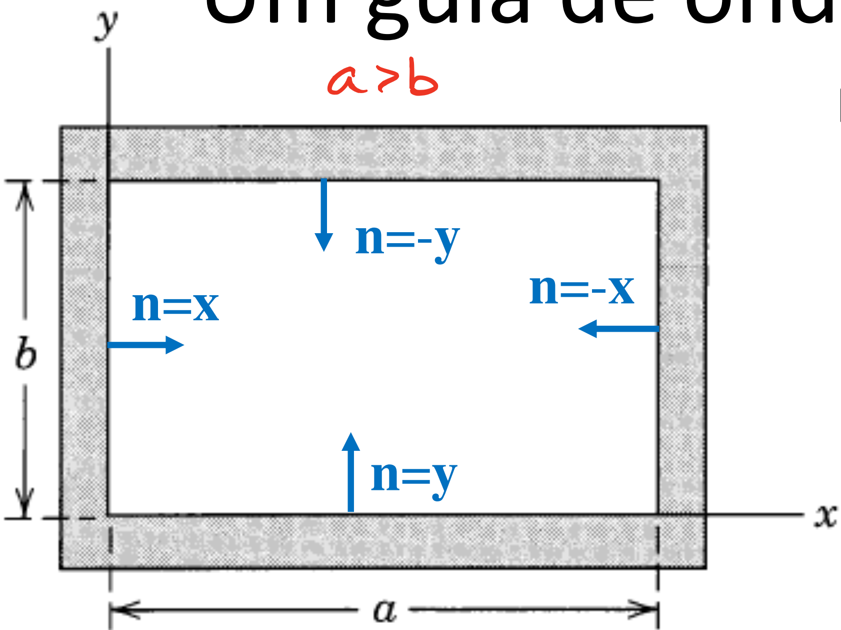


$\omega \gg \omega_\lambda$

$\omega = ck_\lambda$

FREQUÊNCIAS DE CORTE

Um guia de ondas retangular



Modos TM:

$$\left(\nabla_t^2+\gamma^2\right)E_z = 0$$

$$E_z\big|_S = 0$$

SEPARAÇÃO DE VARIÁVEIS

$$E_z(x,y) = X(x)\,Y(y)$$

$$\Rightarrow Y X'' + X Y'' + \gamma^2 X Y = 0$$

DIVIDO POR XY:

$$\frac{X''}{X}+\frac{Y''}{Y}+\gamma^2=0 \Rightarrow \frac{X''}{X} = -\gamma^2-\frac{Y''}{Y} = \text{CONSTANTE} = -\alpha^2$$

$$\boxed{\gamma^2=\alpha^2+\beta^2}$$

$$X''+\alpha^2 X=0 \quad (1) \qquad Y''+\underbrace{(\gamma^2-\alpha^2)}_{\beta^2}Y=0 \quad (2)$$

$$\left.\begin{aligned} X(x) &= A\cos(\alpha x) + B\sin(\alpha x) \\ Y(y) &= C\cos(\beta y) + D\sin(\beta y) \end{aligned}\right\} \quad E_z = X(x)\,Y(y)$$

CONDIÇÕES DE CONTORNO: $E_z(x=0,y) = E_z(x=a,y) = 0 \quad \forall y$

$E_z(x, y=0) = E_z(x, y=b) = 0 \quad \forall x$

(1) $E_z(x=0,y) = A\left[C\cos(\beta y) + D\sin\beta y\right] = 0 \quad \forall y \Rightarrow \boxed{A=0}$

(2) $E_z(x,y=0) = C\left[B\sin(\alpha x)\right] = 0 \quad \forall x \Rightarrow \boxed{C=0}$

$\Rightarrow E_z(x,y) = E_0 \sin(\alpha x)\sin(\beta y) \qquad E_0 = BD$

(3) $E_z(x=a,y) = E_0 \sin(\alpha a)\sin(\beta y) = 0 \quad \forall y$

$\alpha a = m\pi \quad (m = 1, 2, 3, \dots)$

(4) $\Rightarrow \beta b = n\pi \quad (n = 1, 2, 3, \dots) \Rightarrow \boxed{\gamma_{m,n} = \sqrt{\dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}}}$

$$\gamma_{m,n} = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad (m,n = 1,2,3,\ldots)$$

$$\omega_{m,n} = c\,\gamma_{m,n} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

FREQUÊNCIA DE CORTE TM

MENOR FREQUÊNCIA DE CORTE: $m = n = 1$

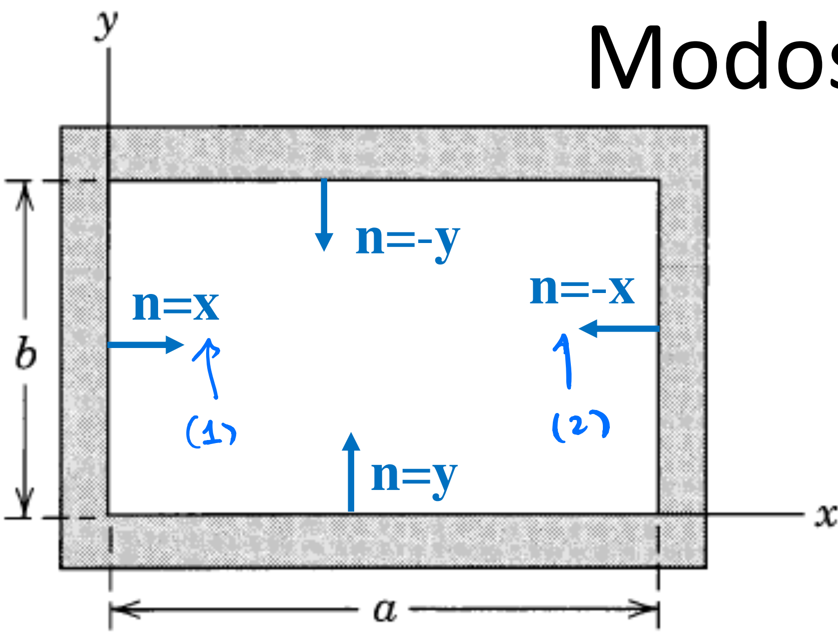
$$\omega_{1,1} = \pi c \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \qquad TM_{1,1}$$

$$E_z^{m,n}(x,y) = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi y}{b}\right) e^{i(\pm \underset{k_{m,n}}{k} z - \omega t)}$$

Modos TM

$$
\begin{aligned}
E_z &= E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\pm i(k_{m,n}z-\omega t)}\\
E_x &= \pm i\frac{k_{m,n}}{\gamma_{m,n}}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\pm i(k_{m,n}z-\omega t)}\\
E_y &= \pm i\frac{k_{m,n}}{\gamma_{m,n}}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\pm i(k_{m,n}z-\omega t)}\\
\mathbf{B}_t &= \pm\frac{\omega}{c^2 k_{m,n}}\hat{\mathbf{z}}\times\mathbf{E}_t
\end{aligned}
$$

Modos TE



$$\left(\nabla_t^2 + \gamma^2\right) B_z = 0$$

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}_t B_z\big|_S = 0$$

$$B_z(x,y) = X(x)\,Y(y)$$

$$X(x) = A \cos \alpha x + B \sin(\alpha x)$$

$$Y(y) = C \cos \alpha y + D \sin(\alpha y)$$

$$\hat{n}\cdot\bar{\nabla}_t = \frac{\partial}{\partial x} \Rightarrow \left.\frac{\partial B_z}{\partial x}\right|_{x=0} = 0 \quad (1)$$

$$(1),\ (3) \Rightarrow B = D = 0$$

$$\hat{n}\cdot\bar{\nabla}_t = -\frac{\partial}{\partial x} \Rightarrow \left.\frac{\partial B_z}{\partial x}\right|_{x=a} = 0 \quad (2)$$

$$B_z(x,y) = B_0 \cos(\alpha x)\cos(\beta y)$$

$$(3) \quad \left.\frac{\partial B_z}{\partial y}\right|_{y=0} = 0 \qquad (4) \quad \left.\frac{\partial B_z}{\partial y}\right|_{y=b} = 0$$

$$(2) \quad \left.\frac{\partial B_z}{\partial x}\right|_{x=a} = -\alpha B \sin(\alpha a)\cos(\beta y) = 0$$

$$\Rightarrow \alpha a = m\pi \quad (m = 0, 1, 2, \dots)$$

$$(4) \quad \Rightarrow \beta b = n\pi \quad (n = 0, 1, 2, \dots)$$

TEMOS QUE EXCLUIR A SOLUÇÃO $m = n = 0$

PORQUE ELA DÁ $B_z = B_0$ $\vec{B}_t = \vec{E}_t = 0$

$m = 0, 1, 2, \dots$

$n = 0, 1, 2, \dots$ mas $m + n \neq 0$

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{i(\pm kz - \omega t)}$$

$$\omega_{m,n} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$: FREQUÊNCIAS DE CORTE

MENOR FREQ. DE CORTE :
(MENOR DE TODOS)

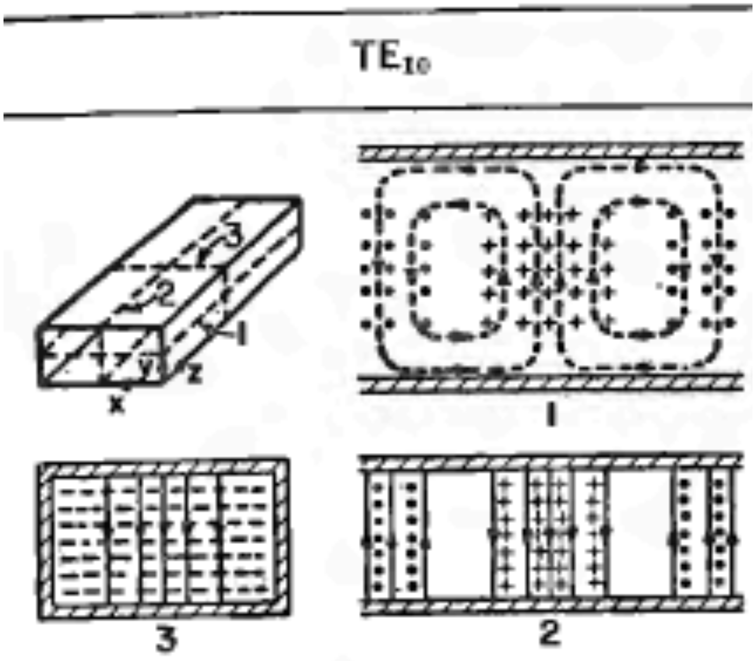
$$\boxed{\omega_{1,0} = \frac{\pi c}{a}} < \omega_{0,1} = \frac{\pi c}{b} \quad (a > b)$$

Modo $TE_{1,0}$

$$
\begin{aligned}
B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)} \\
B_x &= -i\left(\frac{k_{1,0}a}{\pi}\right) B_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)} \\
E_y &= icB_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)}
\end{aligned}
$$

Campo elétrico **E**: linhas contínuas

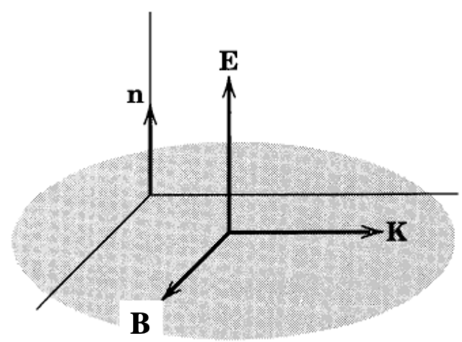
Campo magnético **B**: linhas tracejadas

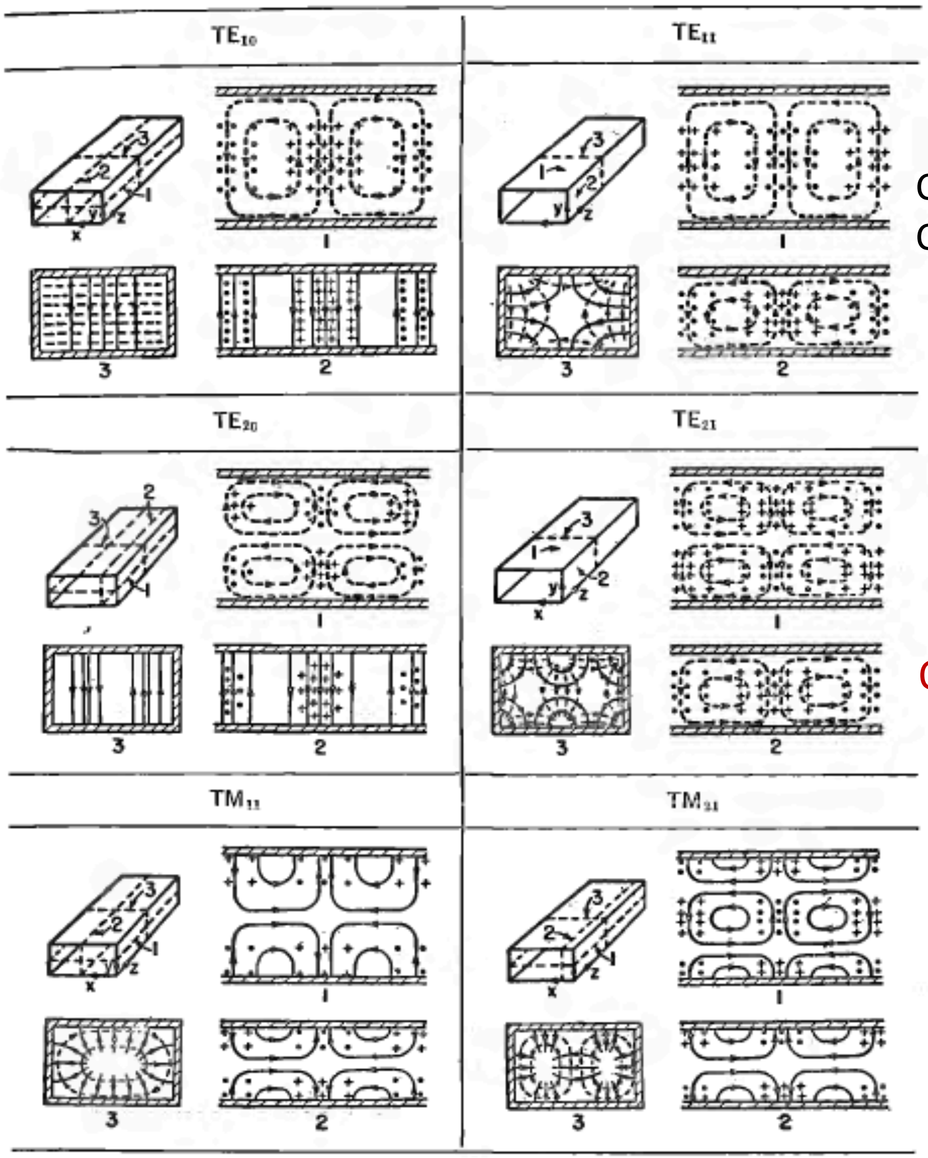


Modo $TE_{1,0}$

$$
\begin{aligned}
B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)}\\
B_x &= -i\left(\frac{k_{1,0}a}{\pi}\right) B_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)}\\
E_y &= icB_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_{1,0}z-\omega t)}
\end{aligned}
$$

Observe as condições de contorno

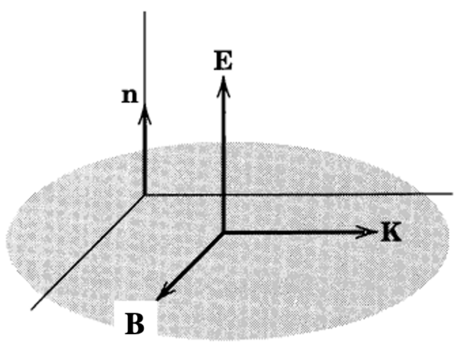




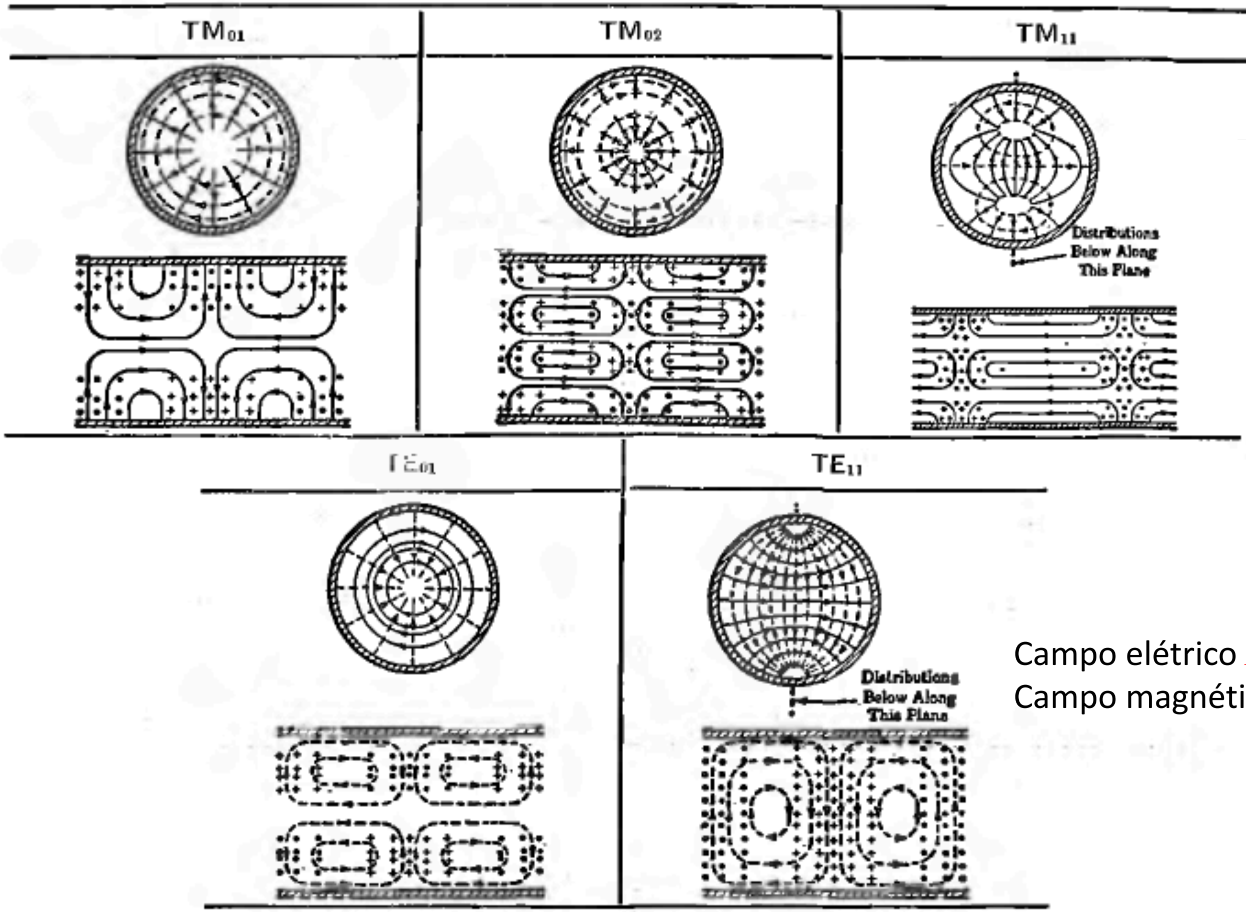
Campo elétrico **E**: linhas contínuas

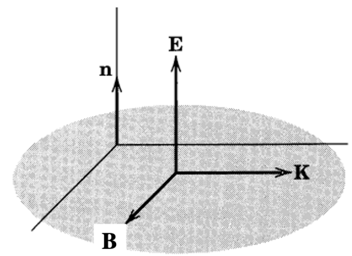
Campo magnético **B**: linhas tracejadas

Observe as condições de contorno



Guia com seção reta circular





Campo elétrico $\boldsymbol{E}$: linhas contínuas
Campo magnético $\boldsymbol{B}$: linhas tracejadas

Análise de potência em guias de ondas

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left[\vec{E} \times \vec{B}^*\right]$$

$$\vec{E} = \vec{E}_t + \vec{E}_z$$
$$\vec{B} = \vec{B}_t + \vec{B}_z$$

$$\Rightarrow \vec{E} \times \vec{B}^* = (\vec{E}_t + \vec{E}_z) \times (\vec{B}_t^* + \vec{B}_z^*)$$

$$= \underbrace{\vec{E}_t \times \vec{B}_t^*}_{\substack{\parallel \hat{z} \\ \langle \vec{S}_z \rangle}} + \underbrace{\vec{E}_t \times \vec{B}_z^* + \vec{E}_z \times \vec{B}_t^*}_{\substack{\perp \hat{z} \\ \langle \vec{S}_t \rangle}}$$

MODOS TM:

$$\vec{E}_t \times \vec{B}_t^* = \vec{E}_t \times \left[\pm \frac{\omega}{c^2 k} \hat{z} \times \vec{E}_t^*\right] = \pm \frac{\omega}{c^2 k} \vec{E}_t \times (\hat{z} \times \vec{E}_t^*) = \pm \frac{\omega}{c^2 k} |\vec{E}_t|^2 \hat{z}$$

$$\langle S_z \rangle = \pm \frac{\omega k}{2\mu_0 c^2 \gamma^4} |\vec{\nabla}_t E_z|^2 \hat{z}$$

Modos TM

$$\gamma^2 \mathbf{E}_t = \pm ik \boldsymbol{\nabla}_t E_z$$

$$\mathbf{B}_t = \pm \frac{\omega}{c^2 k} \hat{\mathbf{z}} \times \mathbf{E}_t$$

$$\langle \vec{S}_t \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left[\vec{E}_z \times \vec{B}_t^{*} \right]$$

$$= \frac{1}{2\mu_0}\left(\pm \frac{\omega}{c^2 k}\right) \mathrm{Re}\left[E_z \hat{z} \times (\hat{z} \times \vec{E}_t^{*}) \right]$$

$$= \mp \frac{\omega}{2\mu_0 c^2 k} \mathrm{Re}\left[E_z \vec{E}_t^{*} \right] \propto \mathrm{Re}\left[\underbrace{i E_z \vec{\nabla}_t E_z}_{\text{PURAMENTE IMAGINÁRIA}} \right]$$

$$\Longrightarrow \boxed{\langle \vec{S}_t \rangle = 0}$$

$$S_z = \frac{E}{t A_\perp} = \frac{P}{A_\perp} \Rightarrow P = \int_{S.R.} \langle S_z \rangle \underbrace{dx\,dy}_{da} = \pm \frac{\omega k}{2\mu_0 c^2 \gamma^4} \int_{S.R.} |\vec{\nabla}_t E_z|^2 \, da$$

S.R. ↓ SEÇÃO RETA

NAS NOTAS, PROVO QUE

$$\int_{S.R.} |\nabla_t E_z|^2 \, da = \gamma^2 \int_{S.R.} |E_z|^2 \, da$$

$$P = \pm \frac{\omega k}{2\mu_0 c^2 \gamma^2} \int_{S.R.} |E_z|^2 \, da$$

$$P = \pm \frac{\omega}{2\mu_0 c} \frac{\sqrt{\omega^2 - \omega_\lambda^2}}{\omega_\lambda^2} \int_{S.R.} |E_z|^2 \, da$$

Modos TE

$$P = \pm \frac{c\,\omega\sqrt{\omega^2-\omega_\lambda^2}}{2\mu_0\,\omega_\lambda^2}\int_{S.R.} |B_z|^2\,da$$

Potência em guias de ondas

$$P_{TM} = \pm\frac{\omega\sqrt{\omega^2-\omega_\lambda^2}}{2\mu_0 c\omega_\lambda^2}\int_{S.R.}|E_z|^2\,da,$$

$$P_{TE} = \pm\frac{c\omega\sqrt{\omega^2-\omega_\lambda^2}}{2\mu_0\omega_\lambda^2}\int_{S.R.}|B_z|^2\,da.$$

Energia por unidade de comprimento do guia

$$\langle u \rangle = \frac{\epsilon_0}{4} \mathrm{Re}[\vec{E} \cdot \vec{E}^*] + \frac{1}{4\mu_0} \mathrm{Re}[\vec{B} \cdot \vec{B}^*]$$

$$\langle u \rangle = \mathrm{Re}\left\{ \frac{\epsilon_0}{4}\left(\vec{E}_t \cdot \vec{E}_t^* + |E_z|^2\right) + \frac{1}{4\mu_0}\left(\vec{B}_t \cdot \vec{B}_t^* + |B_z|^2\right) \right\}$$

$u = \dfrac{U}{V}$ $\quad V = A_\perp L \quad$ L = COMPRIMENTO AO LONGO DO GUIA

$$\frac{U}{L} = \frac{dU}{dz} = u A_\perp \longrightarrow \boxed{\frac{dU}{dz} = \int_{S.R.} \langle u \rangle \, da}$$

$\downarrow$

ENERGIA ARMAZENADA POR UNIDADE DE COMPRIMENTO DO GUIA

$$\frac{dU_{TM}}{dz} = \frac{\epsilon_0}{2} \frac{\omega^2}{\omega_\lambda^2} \int_{S.R.} |E_z|^2 da \qquad \text{(MODOS TM)}$$

$$\frac{dU_{TE}}{dz} = \frac{1}{2\mu_0} \frac{\omega^2}{\omega_\lambda^2} \int_{S.R.} |B_z|^2 da \qquad \text{(MODOS TE)}$$

Potência e energia em guias de ondas

$$P_{TM} = \pm\frac{\omega\sqrt{\omega^2-\omega_\lambda^2}}{2\mu_0 c\omega_\lambda^2}\int_{S.R.}|E_z|^2\,da,$$

$$P_{TE} = \pm\frac{c\omega\sqrt{\omega^2-\omega_\lambda^2}}{2\mu_0\omega_\lambda^2}\int_{S.R.}|B_z|^2\,da.$$

$$\frac{dU_{TM}}{dz} = \frac{\varepsilon_0}{2}\frac{\omega^2}{\omega_\lambda^2}\int_{S.R.}|E_z|^2\,da,$$

$$\frac{dU_{TE}}{dz} = \frac{1}{2\mu_0}\frac{\omega^2}{\omega_\lambda^2}\int_{S.R.}|B_z|^2\,da.$$

$$\boxed{\frac{P}{dU/dz} = \pm\frac{\sqrt{\omega^2-\omega_\lambda^2}}{\omega}c = v_g}$$