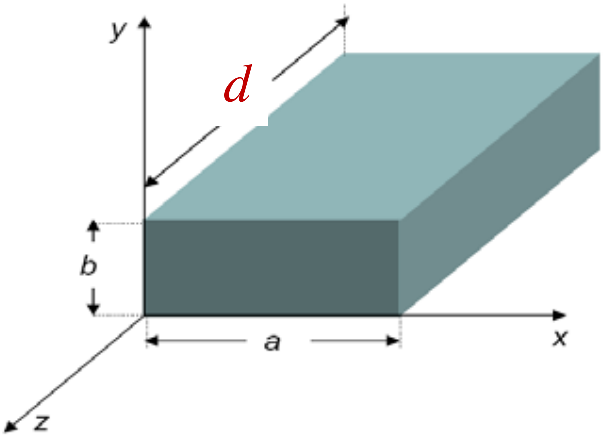
FI 008 – Eletrodinâmica I

1º Semestre de 2020
30/04/2020
Aula 14

Aula passada



Cavidades ressonantes como guias de ondas com "tampas" em z=0 e z=d: novas condições de contorno.

$$\boxed{\begin{aligned} \mathbf{E}_t\big|_{z=0,d} &= 0 \\ B_z\big|_{z=0,d} &= 0 \end{aligned}}$$

O confinamento na direção z transforma a onda propagante do guia numa onda estacionária na cavidade: $e^{\pm ikz} \rightarrow A\cos(kz) + B\sin(kz)$

As novas condições de contorno "quantizam" o vetor de onda k e impõem valores discretos de frequências:

$$k = \frac{p\pi}{d}, \text{ onde } p = \begin{cases} 0, 1, 2, \ldots \\ 1, 2, 3, \ldots \end{cases} \qquad \omega_{\lambda,p} = c\left[\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right]^{1/2}$$

Aula passada

Modos TM:

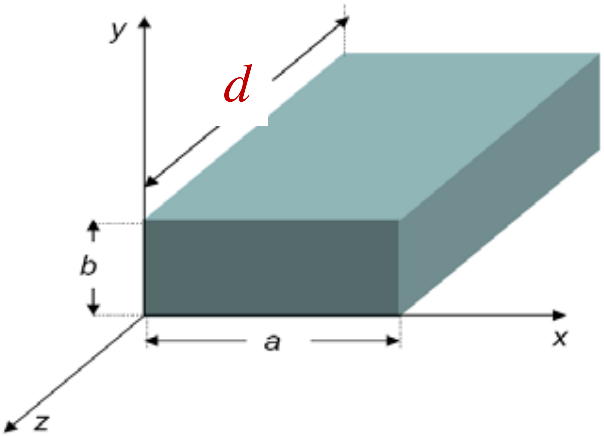
$$\begin{aligned}
E_z(\mathbf{x},t) &= E_z(x,y)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad (p=0,1,2,3,\ldots)\\
\mathbf{E}_t(\mathbf{x},t) &= -\frac{p\pi}{d\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\boldsymbol{\nabla}_t E_z(x,y)\,e^{-i\omega t}\\
\mathbf{B}_t(\mathbf{x},t) &= \frac{i\omega}{\gamma^2 c^2}\cos\left(\frac{p\pi z}{d}\right)\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t E_z(x,y)\,e^{-i\omega t}
\end{aligned}$$

Modos TE:

$$\begin{aligned}
B_z(\mathbf{x},t) &= B_z(x,y)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad (p=1,2,3,\ldots)\\
\mathbf{B}_t(\mathbf{x},t) &= \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\boldsymbol{\nabla}_t B_z(x,y)\,e^{-i\omega t}\\
\mathbf{E}_t(\mathbf{x},t) &= -\frac{i\omega}{\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t B_z(x,y)\,e^{-i\omega t}
\end{aligned}$$

$$\omega_{\lambda,p} = c\left[\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right]^{1/2}$$

Aula passada



Seção reta retangular:

$$\omega_{m,n,p} = \pi c \left[\underbrace{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}_{\gamma_{m,n}} + \left(\frac{p}{d}\right)^2 \right]^{1/2}$$

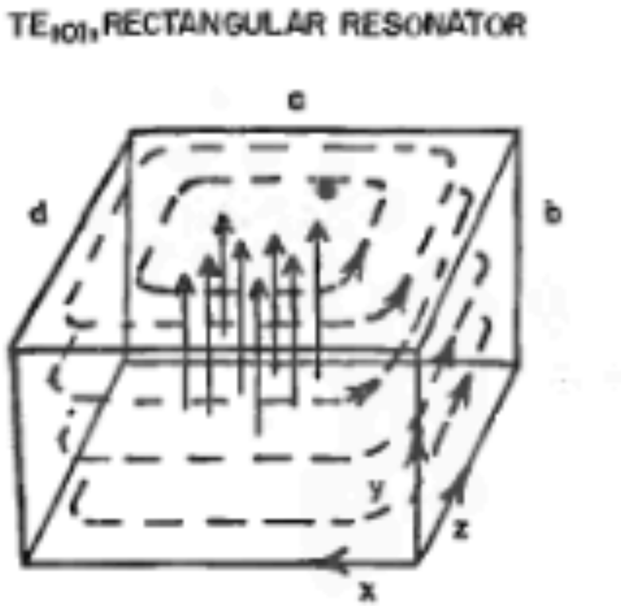
$$\text{TM} \; : \; \begin{cases} m = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \\ p = 0, 1, 2, \ldots \end{cases}$$

$$\text{TE} \; : \; \begin{cases} m = 0, 2, 3, \ldots \\ n = 0, 2, 3, \ldots \\ m + n \neq 0 \\ p = 1, 2, 3, \ldots \end{cases}$$

$$\omega^{\text{TM}}_{1,1,0} = \pi c \left[\frac{1}{a^2} + \frac{1}{b^2} \right]^{1/2}$$

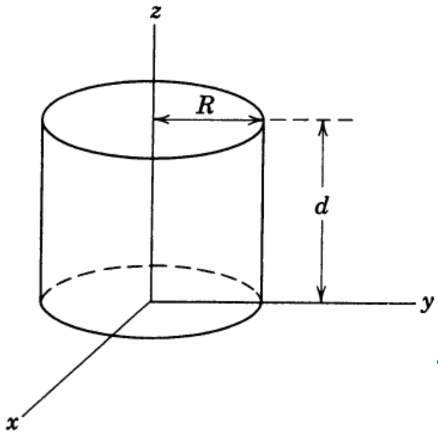
$$\omega^{\text{TE}}_{1,0,1} = \pi c \left[\frac{1}{a^2} + \frac{1}{d^2} \right]^{1/2} \quad (a > b)$$

Aula passada



Campo elétrico **E**: linhas contínuas
Campo magnético **B**: linhas tracejadas

Cavidade de seção reta circular



$$(\nabla_t^2 + \gamma^2)\psi = 0 \qquad \psi = \begin{cases} E_z & (TM) \\ B_z & (TE) \end{cases}$$

SIMETRIA DO PROBLEMA → COORD. CILÍNDRICAS (ρ, ϕ, z)

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \gamma^2 \psi = 0 \quad (1)$$

$$\psi(\rho, \phi) = R(\rho)\,\Phi(\phi)$$

$$\Rightarrow \frac{\Phi}{\rho}\frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] + \frac{R}{\rho^2}\frac{d^2\Phi}{d\phi^2} + \gamma^2 R\Phi = 0$$

$$\Rightarrow \frac{\rho}{R}\underbrace{\frac{d}{d\rho}\left[\rho R'\right]}_{*} + \underbrace{\frac{1}{\Phi}\Phi''}_{-m^2} + \gamma^2 \rho^2 = 0$$

$*: \rho R'' + R'$

$$\Rightarrow \Phi'' + m^2 \Phi = 0 \Rightarrow \Phi(\phi) = e^{\pm i m \phi}\ ;\ \Phi(0) = \Phi(2\pi) \Rightarrow m = 0, \pm 1, \pm 2, \ldots$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right)R = 0$$

TROCA DE VARIÁVEIS: $x = \gamma\rho$ (ADIMENSIONAL)

$$\frac{d}{d\rho} = \gamma\frac{d}{dx} \qquad \frac{d^2}{d\rho^2} = \gamma^2\frac{d^2}{dx^2} \qquad R(\rho) = \bar{R}(x = \gamma\rho)$$

$$\frac{d^2\bar{R}}{dx^2} + \frac{1}{x}\frac{d\bar{R}}{dx} + \left(1 - \frac{m^2}{x^2}\right)\bar{R} = 0$$

"EQUAÇÃO DE BESSEL"

$$\bar{R}(x) = A\,J_m(x) + B\,Y_m(x)$$

$J_m(x) \to$ FUNÇÃO DE BESSEL DE ORDEM m
DE 1º TIPO

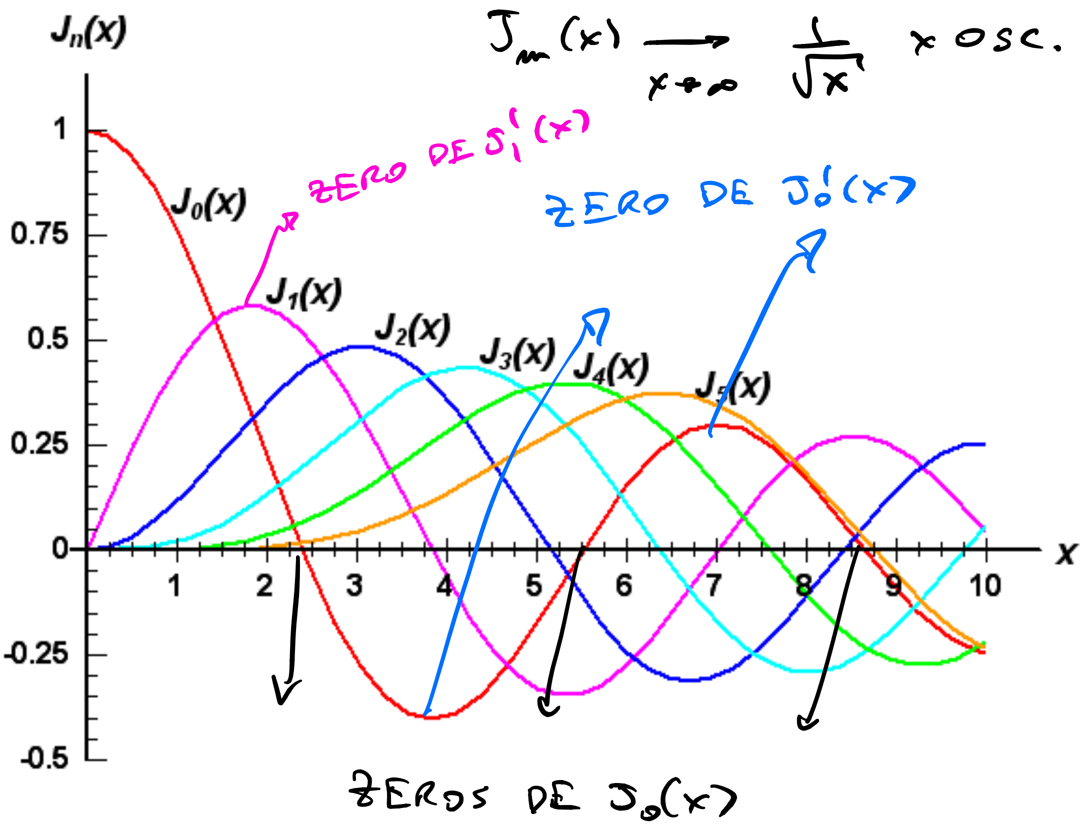
$Y_m(x) \to$ FUNÇÃO DE BESSEL DE ORDEM m
DE 2º TIPO

$\bar{R}(x)$ TEM QUE SER NÃO SINGULAR QUANDO $x \to 0$, $\rho \to 0$

$$\Rightarrow \bar{R}(x) = A\,J_m(x) \quad \Rightarrow \boxed{R(\rho) = A\,J_m(\gamma\rho)}$$

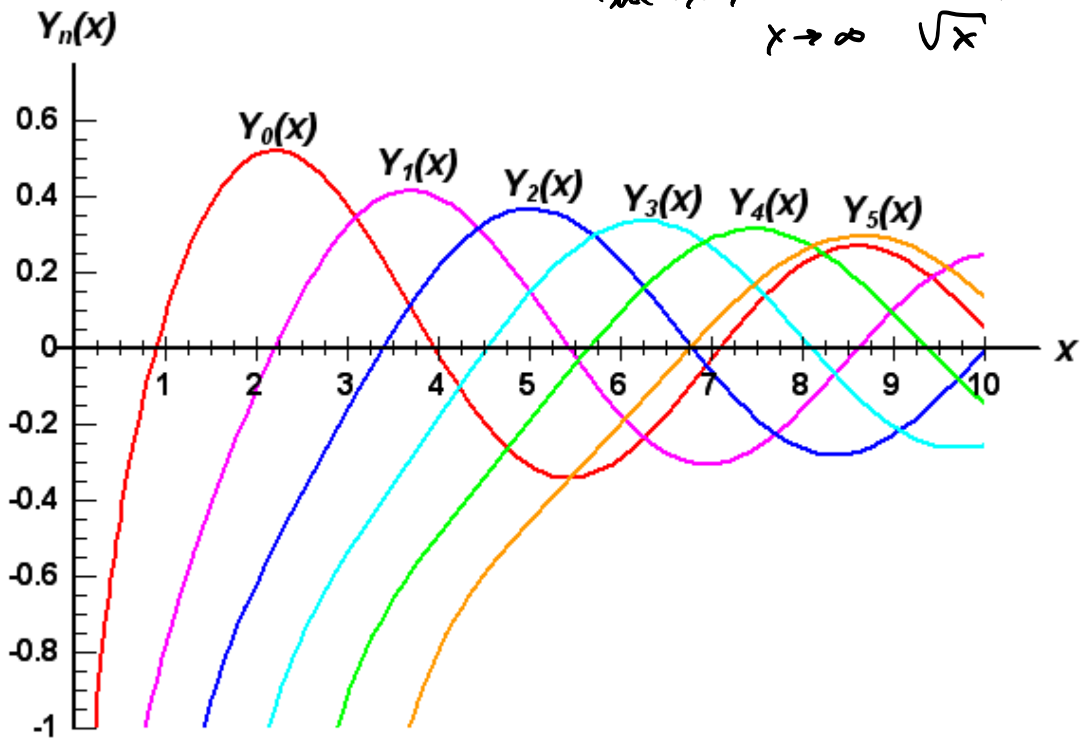
$$\psi(\rho,\phi) = A\, J_m(\gamma\rho)\, e^{\pm i m\phi} \qquad m = 0, 1, 2, \ldots$$

Funções de Bessel do 1o. tipo



Funções de Bessel do 2o. tipo

$Y_n(x) \xrightarrow[x\to\infty]{} \frac{1}{\sqrt{x}} \times$ osc.



↳ DIVERGEM QUANDO $x \to 0$

Modos TM

$$E_z(\rho,\phi) = A_\pm J_m(\gamma\rho)\, e^{\pm i m\phi} \qquad (m = 0, 1, 2, \dots)$$

COND. CONTORNO: $E_z\big|_S = 0 \Rightarrow E_z(\rho = R, \phi) = 0 \quad \forall\, \phi$

$$\Rightarrow A_\pm J_m(\gamma R)\, e^{\pm i m \phi} = 0 \quad \forall\, \phi$$

$$J_m(\gamma R) = 0 \Rightarrow \gamma R = x_{m,n} \quad (n = 1, 2, 3, \dots)$$

$$\gamma = \boxed{\gamma_{m,n} = \frac{x_{m,n}}{R}}$$

$$E_z(\rho,\phi) = A_\pm J_m\left(x_{m,n}\frac{\rho}{R}\right) e^{\pm i m\phi} \qquad \begin{array}{l} m = 0, 1, 2, \dots \\ n = 1, 2, 3, \dots \end{array}$$

PARA A CAVIDADE CILÍNDRICA:

$$\omega^{TM}_{m,n,p} = c\left[\frac{X^2_{m,n}}{R^2} + \left(\frac{p\pi}{d}\right)^2\right]^{1/2} \quad \begin{cases} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \\ p = 0, 1, 2, \ldots \end{cases}$$

FREQUÊNCIA MAIS BAIXA:

$$\omega^{TM}_{0,1,0} = c\left[\frac{X^2_{0,1}}{R^2}\right]^{1/2} = \frac{c\,X_{0,1}}{R}$$

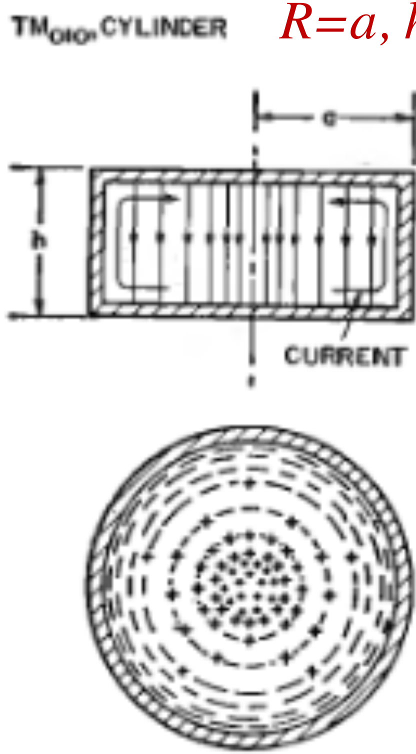
ESSA FREQUÊNCIA NÃO DEPENDE DE $\underline{d}$!

Raízes de $J_\nu(x) = 0$

$$
\begin{array}{llllll}
 & n = & 1 & 2 & 3 & \\
\nu = 0, & x_{0n} = & \underline{\underline{2.405}}, & 5.520, & 8.654, & \ldots \\
\nu = 1, & x_{1n} = & 3.832, & 7.016, & 10.173, & \ldots \\
\nu = 2, & x_{2n} = & 5.136, & 8.417, & 11.620, & \ldots
\end{array}
$$

$X_{m,n}$ É O n-ÉSIMO ZERO DE $J_m(x)$

$R=a,\ h=d$



TM$_{010}$

$$E_z = E_0 J_0\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}$$

$$B_\phi = -\frac{i}{c}E_0 J_1\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}$$

Campo elétrico **E**: linhas contínuas
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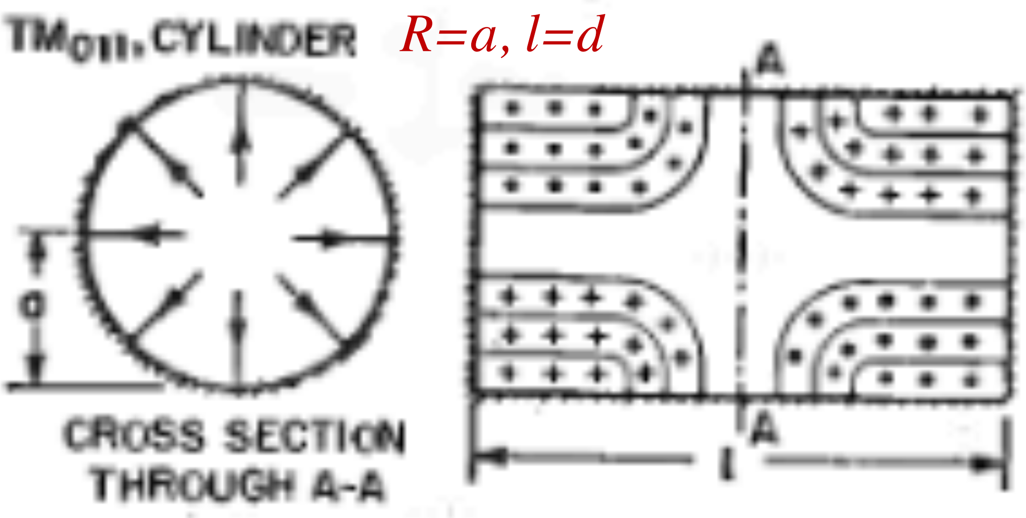

Campo elétrico **E**: linhas contínuas
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Modos TE

$$B_z(\rho,\phi) = B_0^{\pm}\, J_m(\gamma\rho)\, e^{\pm i m\phi} \qquad m = 0, 1, 2, \ldots$$

COND. CONTORNO : $\hat{n}\cdot\vec{\nabla}_t \vec{B}_z\Big|_S = 0 \Rightarrow \left.\frac{\partial B_z}{\partial\rho}\right|_{\rho=R} = 0 \quad \forall\,\phi$

$\hat{n} \downarrow -\hat{\rho}$

$$\Rightarrow B_0^{\pm}\, \left.\frac{d}{d\rho} J_m(\gamma\rho)\right|_{\rho=R} e^{\pm i m\phi} = 0 \qquad \forall\,\phi$$

$$\Rightarrow \gamma R = X'_{m,n} = \text{ZERO/RAIZ DA DERIVADA } J'_m(x)$$

$$n = 1, 2, 3, \ldots$$

$$\gamma_{m,n} = \frac{X'_{m,n}}{R}$$

$$B_z(\rho,\phi) = B_0^{\pm}\, J_m\!\left(\frac{X'_{m,n}}{R}\rho\right) e^{\pm i m\phi} \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{array}$$

$$\omega^{TE}_{m,n,p} = c\left[\left(\frac{X'_{m,n}}{R}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right]^{1/2}$$

$m = 0, 1, 2, \ldots$

$n = 1, 2, 3, \ldots$

$p = 1, 2, 3, \ldots$

MENOR FREQUÊNCIA:

$$\omega^{TE}_{111} = c\left[\left(\frac{X'_{11}}{R}\right)^2 + \left(\frac{\pi}{d}\right)^2\right]^{1/2}$$

QUE DEPENDE DE <u>d</u>

Roots of $J'_m(x) = 0$

$$
\begin{aligned}
m &= 0: & x'_{0n} &= 3.832,\ 7.016,\ 10.173,\ \ldots \\
m &= 1: & x'_{1n} &= \underline{\underline{1.841}},\ 5.331,\ 8.536,\ \ldots \\
m &= 2: & x'_{2n} &= 3.054,\ 6.706,\ 9.970,\ \ldots \\
m &= 3: & x'_{3n} &= 4.201,\ 8.015,\ 11.336,\ \ldots
\end{aligned}
$$

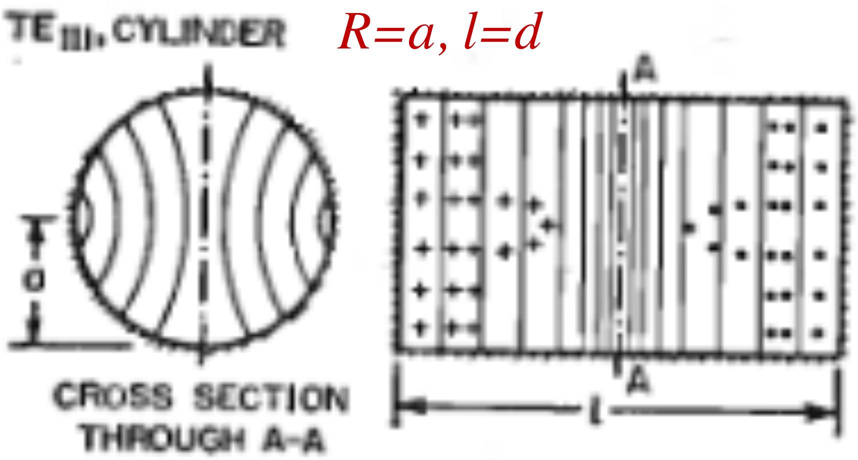
$$B_z = B_0 J_1\left(\frac{1.841\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)e^{-i\omega t}$$

TE_{111}

$$\mathbf{E}_t = -\frac{i\omega}{\gamma_{11}^2}\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t B_z$$

$$\mathbf{B}_t = \frac{1}{\gamma_{11}^2}\boldsymbol{\nabla}_t\frac{\partial B_z}{\partial z}$$

R=a, l=d



Campo elétrico **E**: linhas contínuas
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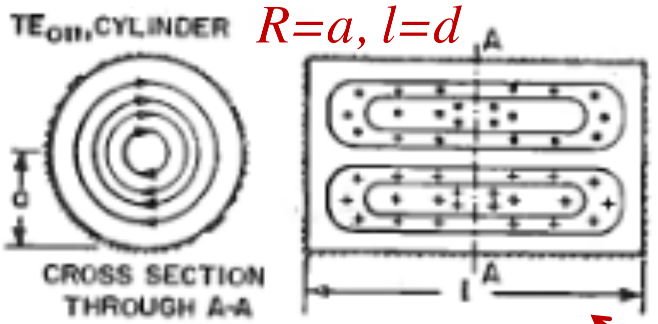


Campo elétrico $\mathbf{E}$: linhas contínuas
Campo magnético $\mathbf{B}$: linhas tracejadas

Linhas de $\mathbf{E}$ e $\mathbf{B}$ trocadas?

Atenuação em cavidades ressonantes

EM CAVIDADES IDEAIS, CADA MODO DE OSCILAÇÃO PODE OSCILAR ETERNAMENTE.

EM CAVIDADES REAIS, OS MODOS DECAEM COM O TEMPO DEVIDO A PERDAS ÔHMICAS NAS PAREDES.

ANÁLOGO AO DECAIMENTO DE ESTADOS EXCITADOS DE ÁTOMOS.

$$\omega_0 \to \omega_0 + \Delta\omega - i\frac{\Gamma}{2}$$

$$\Rightarrow B(t), E(t) \sim e^{-i(\omega_0+\Delta\omega)t}\, e^{-\frac{\Gamma}{2}t} \xrightarrow[t\to\infty]{} 0$$

ENERGIA ARMAZENADA:

$$U(t) \propto |E(t)|^2 \propto e^{-\Gamma t} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\Gamma = \frac{-dU/dt}{U_0}}$$

$\tau = \Gamma^{-1}$ = MEIA-VIDA DO MODO

DEFINE-SE UMA MEDIDA ADIMENSIONAL DA ATENUAÇÃO:

$$Q = \frac{\omega_0}{\Gamma}$$ ⇒ "Q" DA CAVIDADE OU FATOR DE QUALIDADE

Q = "NÚMERO" DE OSCILAÇÕES DO MODO ANTES DE DECAIR

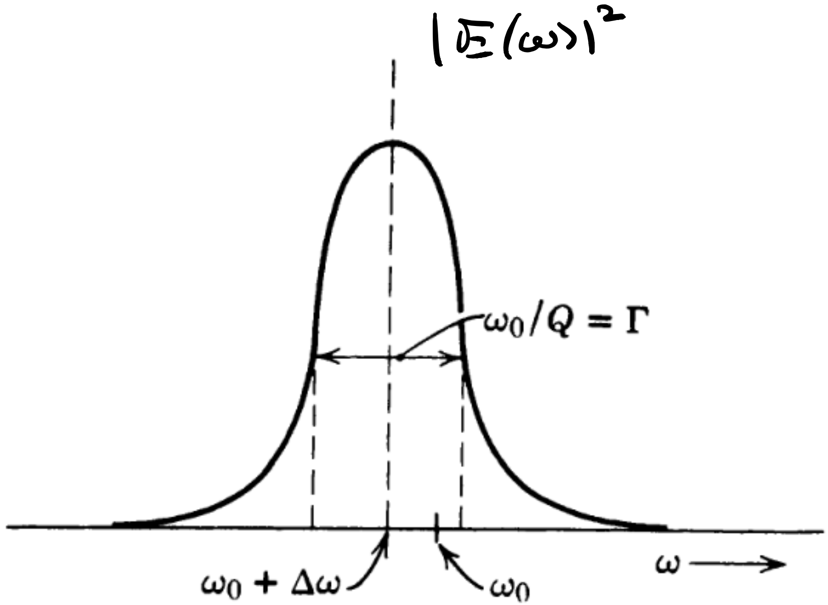
$$E(t) \propto e^{-i(\omega_0+\Delta\omega)t}\, e^{-\frac{\Gamma}{2}t}$$

$$E(\omega) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} E(t)\, e^{i\omega t} \propto \frac{1}{(\omega-\omega_0-\Delta\omega)+i\frac{\Gamma}{2}}$$

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0-\Delta\omega)^2+\Gamma^2/4} = \frac{1}{(\omega-\omega_0-\Delta\omega)^2+\left(\frac{\omega_0}{2Q}\right)^2}$$

LORENTZIANA CENTRADA EM $\omega_0+\Delta\omega$ DE LARGURA $\frac{\omega_0}{2Q}$ OU $\frac{\Gamma}{2}$

ESPECTRO DE POTÊNCIA



Energia eletromagnética em guias de ondas:

$$u = \frac{\epsilon_0}{2}\left(\mathbf{E}_t^2 + c^2\mathbf{B}_t^2 + E_z^2 + c^2 B_z^2\right)$$

QUERO INTEGRAR u NO VOLUME DA CAVIDADE: ÁREA DA S.R. × d

GUIA

$$\frac{dU_{TM}}{dz} = \frac{\varepsilon_0}{2}\frac{\omega^2}{\omega_\lambda^2}\int_{S.R.} |E_z|^2 \, da,$$

$$\frac{dU_{TE}}{dz} = \frac{1}{2\mu_0}\frac{\omega^2}{\omega_\lambda^2}\int_{S.R.} |B_z|^2 \, da.$$

→ INTEGRAR NA DIREÇÃO z

FORAM CALCULADAS PARA $E_z, B_z \propto e^{\pm ikz}$

NA CAVIDADE $e^{\pm ikz} \rightarrow \cos(kz)$ OU $\sin(kz)$

$\Rightarrow \cos^2(kz)$ OU $\sin^2(kz)$

QUANDO INTEGRADAS EM z:

$$\int_0^d dz \cos^2\left(\frac{p\pi z}{d}\right) = \frac{d}{2} \qquad p \neq 0$$

$$\int_0^d dz \sin^2\left(\frac{p\pi z}{d}\right) = \frac{d}{2}$$

$$\int_0^d dz = d \qquad (p=0,\ TM)$$

ALÉM DISSO:

$$\frac{\omega^2}{\omega_\lambda^2} = \frac{c^2\gamma_\lambda^2 + c^2k^2}{c^2\gamma_\lambda^2} = 1 + \frac{k^2}{\gamma_\lambda^2} = 1 + \left(\frac{p\pi}{\gamma_\lambda d}\right)^2$$

Energia armazenada em cavidades

$$U_{TM} = \frac{d\varepsilon_0}{4}\left[1+\left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\int_{S.R.} |E_z|^2\, da \ \ (p\neq 0)\,,$$

$$U_{TM} = \frac{d\varepsilon_0}{2}\int_{S.R.} |E_z|^2\, da \ \ (p = 0)\,,$$

$$U_{TE} = \frac{d}{4\mu_0}\left[1+\left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\int_{S.R.} |B_z|^2\, da.$$

NUMERADOR DE Γ

Perdas ôhmicas nas paredes: $-\frac{dU}{dt} = P_{Loss}$

$$\frac{dP_{loss}}{da} = \frac{1}{2\mu_0^2\sigma\delta}\left|B_{\parallel V}\right|^2 \qquad \delta = \sqrt{\frac{2}{\mu_c\sigma\omega}}$$

$$-\frac{dU}{dt} = \int_{\substack{\text{PAREDES}\\ \text{DA CAVIDADE}}} \frac{dP_{Loss}}{da}\, dS = \frac{1}{2\mu_0^2\sigma\delta}\Bigg[\underbrace{\oint_C dl \int_0^d dz\, |B_{V\parallel}|^2}_{\text{PAREDES LATERAIS}} +$$

$$\underbrace{+\,2\int_{\substack{\text{TAMPAS}\\ \text{EM } z=0 \text{ OU } d}} |B_{V\parallel}|^2\, da}_{\text{TAMPAS}}\Bigg]$$

Modos TM:

$$-\frac{dU}{dt} = \frac{1}{2\mu_0^2\sigma\delta_\lambda}\frac{1}{c^2\gamma_\lambda^2}\left[1+\left(\frac{p\pi}{\gamma_\lambda d}\right)^2\right]\left\{\frac{d}{2}\left(1+\delta_{p,0}\right)\oint\left|\frac{\partial E_z}{\partial n}\right|^2 dl + 2\gamma_\lambda^2\int_{S.R.}\left|E_z\right|^2 da\right\}$$

Modos TE:

$$-\frac{dU}{dt} = \frac{1}{2\mu_0^2\sigma\delta_\lambda}\left\{\frac{d}{2}\oint\left[\left|B_z\right|^2+\left(\frac{p\pi}{\gamma_\lambda^2 d}\right)^2\left|\hat{\boldsymbol{n}}\times\nabla_t B_z\right|^2\right]dl + 2\left(\frac{p\pi}{\gamma_\lambda^2 d}\right)^2\int_{S.R.}\left|\nabla_t B_z\right|^2 da\right\}$$

$$\delta_\lambda = \sqrt{\frac{2}{\mu_c\sigma\omega_\lambda}}$$

Intuição física sobre Q

$$Q \propto \frac{V}{S\delta}$$

V = VOLUME DA CAVIDADE
(VEM DO U_e)

S = ÁREA DA SUPERFÍCIE TOTAL DAS
PAREDES DA CAVIDADE

δ = SKIN DEPTH

$S\delta$ = VOLUME DA PELÍCULA QUE
"ENVOLVE" A PAREDE DO GUIA
E QUE TEM ESPESSURA δ