FI 008 – Eletrodinâmica I

1º Semestre de 2020
05/05/2020
Aula 15

Radiação eletromagnética de distribuições localizadas de correntes

FONTES: $\rho(\bar{x},t)$, $\vec{J}(\vec{x},t)$ LOCALIZADAS NO ESPAÇO

$$\left.\begin{aligned} \rho(\bar{x},t) &= \rho(\bar{x})e^{-i\omega t} \\ \vec{J}(\vec{x},t) &= \vec{J}(\bar{x})\,e^{-i\omega t} \end{aligned}\right\}$$

DECOMPOSIÇÃO DE FOURIER NO TEMPO. $\rho(\bar{x})$ E $\vec{J}(\bar{x}) \in \mathbb{C}$

$$\bar{\nabla}\cdot\bar{J} + \frac{\partial\rho}{\partial t} = 0 \Rightarrow \boxed{\bar{\nabla}\cdot\bar{J} = i\omega\rho}$$

EQ. CONTINUIDADE DE CARGA

ESTAREMOS INTERESSADOS, PRINCIPALMENTE, NOS CAMPOS $\bar{E}(\bar{x})$ E $\bar{B}(\bar{x})$ FORA E LONGE DA DISTRIBUIÇÃO DE CARGAS E CORRENTES

$$t - \frac{R}{c} = t_{ret}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}',t-R/c)}{R} d^3x'$$

$$\boxed{R = |\mathbf{x}-\mathbf{x}'|}$$

$$\bar{J}(\bar{x}',t-R/c) = \bar{J}(\bar{x}')\, e^{-i\omega(t-R/c)} = \bar{J}(\bar{x}')\, e^{i\frac{\omega}{c}R}\, e^{-i\omega t}$$

$$k = \frac{\omega}{c}$$

$$\vec{A}(\bar{x},t) = \frac{\mu_0}{4\pi}\int \frac{\bar{J}(\bar{x}')\, e^{i\frac{\omega}{c}R}\, e^{-i\omega t}}{R} d^3x'$$

$$= e^{-i\omega t}\left[\frac{\mu_0}{4\pi}\int \frac{\bar{J}(\bar{x}')\, e^{ikR}}{R} d^3x'\right] = \bar{A}(\bar{x})\, e^{-i\omega t}$$

$$\boxed{\vec{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \bar{J}(\bar{x}')\frac{e^{ikR}}{R} d^3x'}$$

ELEMENTO CENTRAL DO CAPÍTULO

CAMPOS :

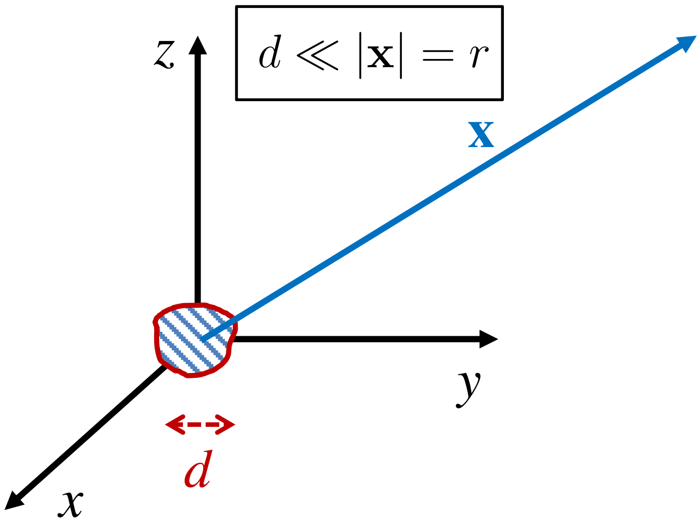
$$\vec{B}(\bar{x},t) = \bar{B}(\bar{x})\, e^{-i\omega t} \quad ; \quad \bar{E}(\bar{x},t) = \bar{E}(\bar{x})\, e^{-i\omega t}$$

$$\boxed{\bar{B}(\bar{x}) = \bar{\nabla} \times \bar{A}(\bar{x}) = \bar{\nabla} \times \left[\frac{\mu_0}{4\pi} \int \bar{J}(\bar{x}') \frac{e^{ikR}}{R} d^3x' \right]}$$

FORA DA DISTRIBUIÇÃO DE CORRENTES:

$$\bar{\nabla} \times \bar{B} = \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} = \frac{1}{c^2} (-i\omega) \vec{E} = -i \frac{k}{c} \vec{E}(\bar{x})$$

$$\Rightarrow \boxed{\bar{E}(\bar{x}) = i \frac{c}{k} \bar{\nabla} \times \bar{B}}$$



TRÊS ESCALAS DE COMPRIMENTO RELEVANTES:

d: DIMENSÃO LINEAR DA FONTE

r: DISTÂNCIA DO PONTO DE OBSERVAÇÃO

$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$: COMPRIMENTO DE ONDA DA RADIAÇÃO

$\underline{d \ll r}$, $\underline{d \ll \lambda}$ (NEM SEMPRE)

$r \ll \lambda$: REGIÃO PRÓXIMA OU (QUASE-)ESTÁTICA

$r \sim \lambda$: REGIÃO INTERMEDIÁRIA OU DE INDUÇÃO

$r \gg \lambda$: REGIÃO DISTANTE OU DE RADIAÇÃO

Região estática

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int \bar{J}(\bar{x}') \frac{e^{ikR}}{R} d^3x'$$

$$kR = k|\bar{x}-\bar{x}'| \cong kr = 2\pi \frac{r}{\lambda} << 1 \quad \text{(ESTÁTICA)}$$

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\bar{x}')}{|\bar{x}-\bar{x}'|} d^3x'$$

EQUAÇÃO DA MAGNETOSTÁTICA

$$r << \lambda = \frac{2\pi c}{\omega} \Rightarrow \boxed{\omega r << 2\pi c}$$

Região de radiação

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \bar{J}(\bar{x}')\,\frac{e^{ikR}}{R}\,d^3x' \qquad \left(r >> \lambda = \frac{2\pi c}{\omega}\right)$$

$$R = |\bar{x}-\bar{x}'| = \left[r^2 + r'^2 - 2\,\bar{x}\cdot\bar{x}'\right]^{1/2}$$

$$= r\left[1 + \frac{r'^2}{r^2} - 2\,\frac{\bar{x}\cdot\bar{x}'}{r^2}\right]^{1/2} \cong r\left[1 - \frac{\bar{x}\cdot\bar{x}'}{r^2}\right]$$

$$= r - \frac{\bar{x}\cdot\bar{x}'}{r} = r - \hat{r}\cdot\bar{x}' = r - \hat{n}\cdot\bar{x}'$$

$$\boxed{\hat{n} \equiv \hat{r}}$$

$$\bar{A}(\bar{x}) \cong \frac{\mu_0}{4\pi}\int \bar{J}(\bar{x}')\,\frac{e^{ikr}\,e^{-ik\hat{n}\cdot\bar{x}'}}{r}\,d^3x'$$

$$\boxed{\bar{A}(x) = \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\underbrace{\int \bar{J}(\bar{x}')\,e^{-ik\hat{n}\cdot\bar{x}'}\,d^3x'}_{\bar{F}(k\hat{n})}}$$

ONDA ESFÉRICA DE VETOR DE ONDA DE MÓDULO $\underline{k}$

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{F}(k\hat{n})$$

ONDE $\vec{F}(k\hat{n}) = \int \bar{J}(\bar{x}')\, e^{-ik\hat{n}\cdot\bar{x}'} d^3x'$

$$\bar{B}(\bar{x}) = \bar{\nabla}\times\bar{A} = \frac{\mu_0}{4\pi} \bar{\nabla}\times\left[\frac{e^{ikr}}{r}\bar{F}(k\hat{n})\right]$$

$$= \frac{\mu_0}{4\pi}\left\{\bar{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\bar{F}(k\hat{n}) + \frac{e^{ikr}}{r}\bar{\nabla}\times\bar{F}\right\}$$

NOTE QUE $\bar{\nabla}$ ATUA SOBRE

$\bar{F}(k\hat{n}) = \bar{F}(k\hat{r})$

$= \bar{F}\left(k\frac{\bar{x}}{r}\right)$

$$\bar{\nabla}\left(\frac{e^{ikr}}{r}\right) = \hat{r}\frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\right] = \hat{r}\left[ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right] \cong ik\frac{e^{ikr}}{r}\hat{r}$$

DESPREZANDO TERMOS QUE CAEM MAIS RAPIDAMENTE QUE $\left(\frac{1}{r}\right)$

ESTÁ MOSTRADO NAS NOTAS QUE DERIVADAS DE $\bar{F}(k\hat{n})$ CAEM COM $\left(\frac{1}{r}\right)$

$$\boxed{\bar{B}(\bar{x}) = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r}\hat{n}\times\vec{F}(k\hat{n})}$$

$$\bar{E}(\bar{x}) = \frac{ic}{k}\bar{\nabla}\times\bar{B} = -\frac{\mu_0}{4\pi}c\,\bar{\nabla}\times\left[\frac{e^{ikr}}{r}\hat{n}\times\bar{F}(k\hat{n})\right]$$

RETENDO APENAS TERMOS QUE CAEM COM $\frac{1}{r}$:

$$\bar{E}(\bar{x}) = -\frac{\mu_0 c}{4\pi}\underbrace{\left[\bar{\nabla}\times\left(\frac{e^{ikr}}{r}\right)\right]}_{ik\frac{e^{ikr}}{r}\hat{n}}\times(\hat{n}\times\bar{F})$$

$$\boxed{\bar{E}(\bar{x}) = -i\,ck\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\bar{F}) = -c\,\hat{n}\times\bar{B}}$$

TOMANDO $\hat{n}\times\bar{E}$:

$$\hat{n}\times\bar{E} = -c\,\hat{n}\times(\hat{n}\times\bar{B}) = -c\left[\underbrace{(\hat{n}\cdot\bar{B})}_{0}\hat{n} - \underbrace{(\hat{n}\cdot\hat{n})}_{1}\bar{B}\right]$$

$$= c\vec{B}$$

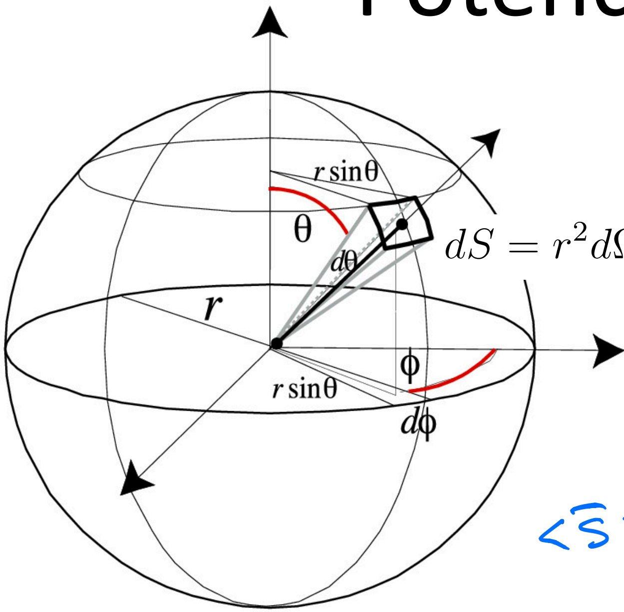
$$\boxed{\vec{B} = \frac{1}{c}\hat{n}\times\bar{E}}$$

$(\bar{E}, \bar{B}, \hat{n})$ FORMAM UMA TRÍADE

ORTOGONAL ENTRE SI :

ONDA ESFÉRICA TRANSVERSAL

Potência irradiada [redacted]



$$dS = r^2 d\Omega = r^2 \sin\theta\, d\theta\, d\phi$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left[\vec{E} \times \vec{B}^*\right]$$

$$dP = \langle \vec{S} \rangle \cdot \hat{r}\, dS$$

$$= \langle \vec{S} \rangle \cdot \hat{n}\, dS$$

$$dP = \langle \vec{S} \rangle \cdot \hat{n}\, r^2 d\Omega$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \mathrm{Re}\left[-c(\hat{n} \times \vec{B}) \times \vec{B}^*\right]$$

$$= \frac{c}{2\mu_0} \mathrm{Re}\left[\vec{B}^* \times (\hat{n} \times \vec{B})\right]$$

$$= \frac{c}{2\mu_0} \mathrm{Re}\left[|\vec{B}|^2 \hat{n} - \cancelto{0}{(\vec{B}^* \cdot \hat{n})} \vec{B}\right] = \frac{c}{2\mu_0} |\vec{B}|^2 \hat{n}$$

$$= \frac{\mu_0 c k^2}{32\pi^2 r^2} |\hat{n} \times \vec{F}|^2 \hat{n}$$

$$\frac{dP}{d\Omega} = \langle \bar{S} \rangle \cdot \hat{n}\, r^2 = \frac{\mu_0 c k^2}{32\pi^2} |\hat{n} \times \bar{F}|^2 = \frac{\mu_0 \omega^2}{32\pi^2 c} |\hat{n} \times \bar{F}|^2$$

POTÊNCIA TOTAL IRRADIADA:

$$P = \int \frac{dP}{d\Omega} d\Omega = \int \frac{dP}{d\Omega} \sin\theta \, d\theta \, d\phi$$

$$\bar{F}(k\hat{n}) = \int \bar{J}(\bar{x}')\, e^{-ik\hat{n}\cdot\bar{x}'}\, d^3x'$$

$$k\hat{n}\cdot\bar{x}' \sim kd = 2\pi\frac{d}{\lambda} << 1$$

$$\bar{F}(k\hat{n}) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \bar{J}(\bar{x}')\, (\hat{n}\cdot\bar{x}')^n\, d^3x'$$

USO A EXPANSÃO, PARANDO NO PRIMEIRO TERMO NÃO NULO.

Resumo

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{F}(k\hat{\mathbf{n}})$$

$$\mathbf{F}(k\hat{\mathbf{n}}) = \int \mathbf{J}(\mathbf{x})\, e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'} d^3x'$$

$$\mathbf{B}(\mathbf{x}) = ik\hat{\mathbf{n}} \times \mathbf{A}(\mathbf{x})$$

$$\mathbf{E}(\mathbf{x}) = -ick\hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times \mathbf{A}(\mathbf{x})] = -c\hat{\mathbf{n}} \times \mathbf{B}(\mathbf{x})$$

$$c\mathbf{B}(\mathbf{x}) = \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{x})$$

$$\frac{dP}{d\Omega} = \frac{\mu_0\omega^2}{32\pi^2 c}\left|\hat{\mathbf{n}} \times \mathbf{F}(k\hat{\mathbf{n}})\right|^2$$

$$\boxed{\mathbf{F}(k\hat{\mathbf{n}}) = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int \mathbf{J}(\mathbf{x}')\,(\hat{\mathbf{n}}\cdot\mathbf{x}')^n\, d^3x'}$$

Primeiro termo: dipolo elétrico

$m=0$: $\vec{F}(k\hat{n}) = \int \vec{J}(\vec{x}')\, d^3x'$

USANDO: $\vec{\nabla}\cdot(x_i\vec{J}) = [\underbrace{\vec{\nabla}(x_i)}_{\hat{x}_i}]\cdot\vec{J} + x_i\,\vec{\nabla}\cdot\vec{J} = J_i + x_i(\vec{\nabla}\cdot\vec{J})$

$$F_i = \int J_i(\vec{x}')\, d^3x' = \int \left[\underbrace{\vec{\nabla}'\cdot(x_i'\vec{J})}\; - x_i'(\vec{\nabla}'\cdot\vec{J})\right] d^3x'$$

$\int_{S(\infty)} x_i'\,\vec{J}\, d^3x' \longrightarrow 0$ PARA DIST. LOCALIZADAS

$$F_i = -\int x_i'\,(\underbrace{\vec{\nabla}'\cdot\vec{J}}_{i\omega\rho(\vec{x}')})\, d^3x' = -i\omega \underbrace{\int x_i'\,\rho(\vec{x}')\, d^3x'}_{p_i} = -i\omega\, p_i$$

$\boxed{\vec{F} = -i\omega\vec{p}}$ ONDE $\boxed{\vec{p} = \int \vec{x}\,\rho(\vec{x})\, d^3x} \in \mathbb{C}$!

$$\vec{A}_D(\vec{x}) = -\frac{i\mu_0\omega}{4\pi r} e^{ikr}\vec{p}$$

$$\vec{B}_D(\vec{x}) = \frac{\omega^2\mu_0}{4\pi c}\frac{e^{ikr}}{r}(\hat{n}\times\vec{p})$$

$$\vec{E}_D(\vec{x}) = -\frac{\mu_0\omega^2}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p})$$

$$\frac{dP_D}{d\Omega} = \frac{\mu_0\omega^4}{32\pi^2 c}|\hat{n}\times\vec{p}|^2$$

$$|\hat{n}\times\vec{p}|^2 = (\hat{n}\times\vec{p}^*)\cdot(\hat{n}\times\vec{p}) = \hat{n}\cdot[\vec{p}\times(\hat{n}\times\vec{p}^*)]$$

$$= \hat{n}\cdot[|\vec{p}|^2\hat{n} - (\vec{p}\cdot\hat{n})\vec{p}^*] = |\vec{p}|^2 - (\vec{p}\cdot\hat{n})(\vec{p}^*\cdot\hat{n})$$

$$= |\vec{p}|^2 - |\vec{p}\cdot\hat{n}|^2$$

$$\frac{dP_D}{d\Omega} \propto |\vec{p}|^2 - |\vec{p}\cdot\hat{n}|^2$$

SUPONHAMOS QUE OS COMPONENTES DE $\vec{p}$ TÊM TODOS A MESMA FASE:

$$\vec{p} = e^{i\alpha}\left(\bar{p}_x \hat{x} + \bar{p}_y \hat{y} + \bar{p}_z \hat{z}\right) \quad \text{ONDE } \bar{p}_x, \bar{p}_y, \bar{p}_z \in \mathbb{R}$$

$$|\vec{p}|^2 = (\bar{p}_x)^2 + (\bar{p}_y)^2 + (\bar{p}_z)^2 \equiv \bar{p}^2$$

$$|\vec{p}\cdot\hat{n}|^2 = |\vec{\bar{p}}\cdot\hat{n}|^2 = \bar{p}^2\cos^2\theta \quad \text{ONDE}$$

$\boxed{\theta = \text{ÂNGULO ENTRE } \vec{\bar{p}} \text{ E } \hat{n}}$

$$\frac{dP_D}{d\Omega} = \frac{\mu_0\omega^4}{32\pi^2 c}\left[\bar{p}^2(1-\cos^2\theta)\right] = \frac{\mu_0\bar{p}^2\omega^4}{32\pi^2 c}\sin^2\theta$$

$$\frac{dP_D}{d\Omega} = \begin{cases} 0 & \theta = 0, \pi \\ \text{MÁXIMA} & \theta = \frac{\pi}{2} \end{cases}$$

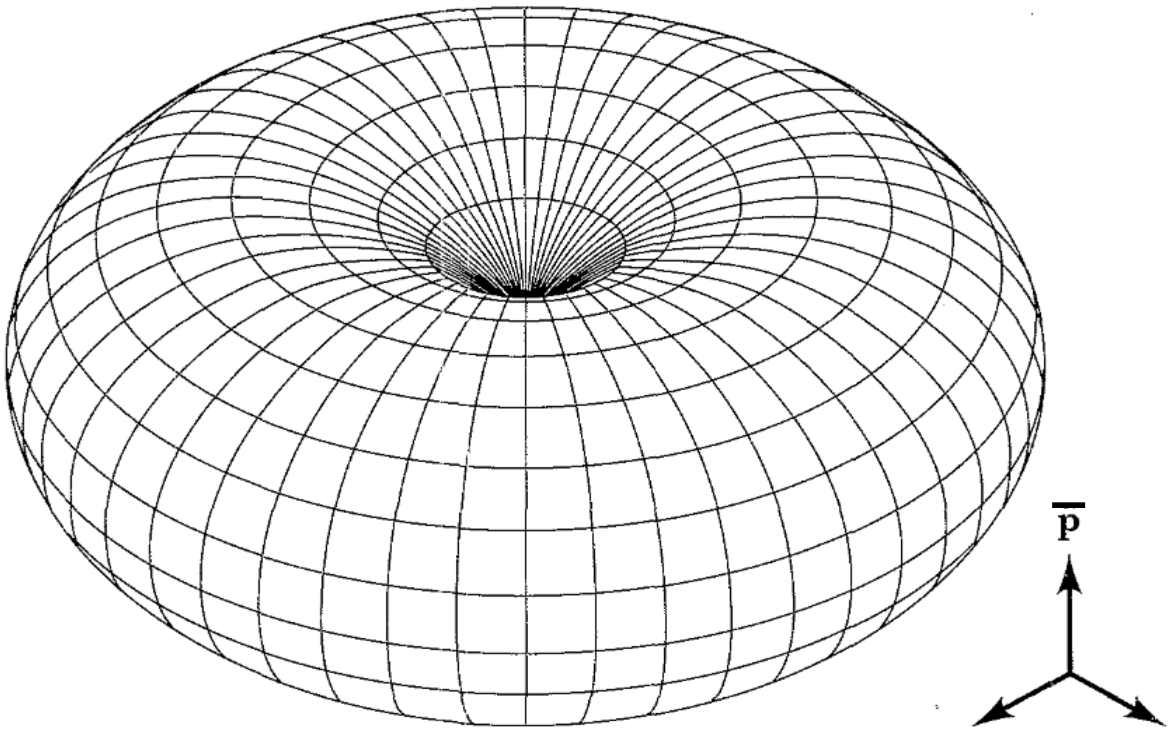
A POTÊNCIA TOTAL IRRADIADA:

$$P_D = \int \frac{\mu_0 \omega^4}{32\pi^2 c} |\hat{n}\times\vec{p}|^2 \sin\theta \, d\theta \, d\phi$$

$$P_D = \frac{\mu_0 \omega^4}{12\pi c} |\vec{p}|^2$$

(VÁLIDO PARA UM $\vec{p}$ QUALQUER
VER NOTAS PARA A DEDUÇÃO)

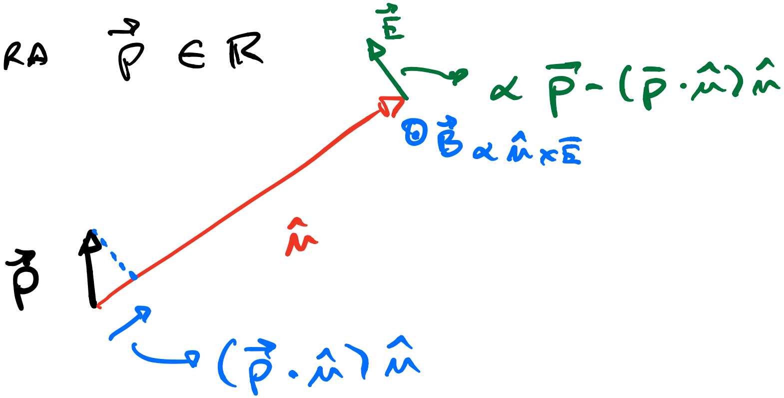
Distribuição angular de radiação de dipolo elétrico



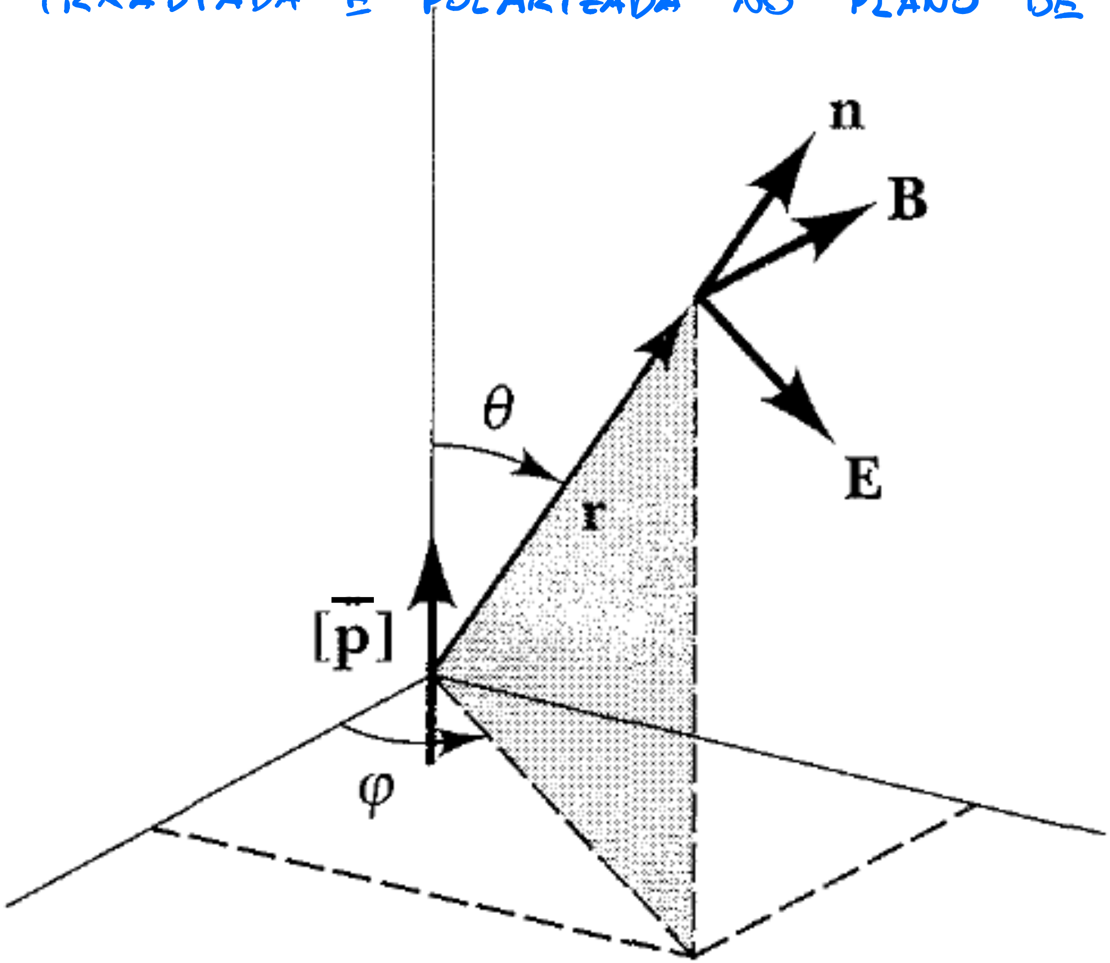
Polarização da radiação de dipolo elétrico

$\bar{E}_D \propto -\hat{n}\times(\hat{n}\times\bar{p}) = \bar{p} - (\hat{n}\cdot\bar{p})\hat{n}$ DIREÇÃO DA POLARIZAÇÃO

PARA $\vec{p} \in \mathbb{R}$



A LUZ IRRADIADA É POLARIZADA NO PLANO DE $\vec{p}$ E $\hat{n}$



A sequência temporal da radiação de dipolo

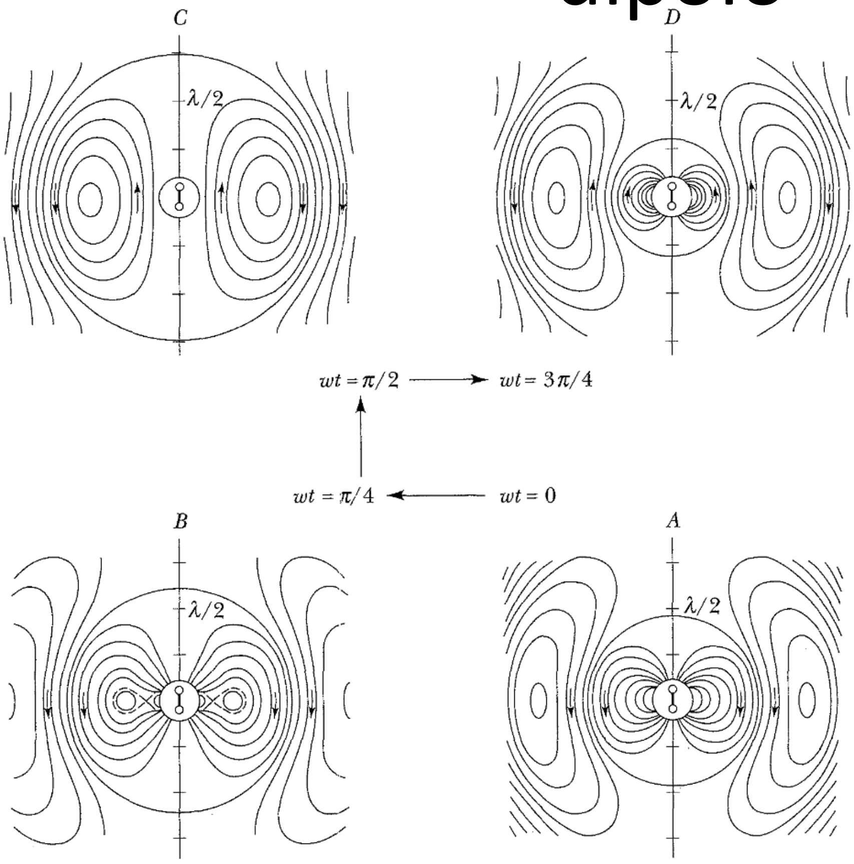
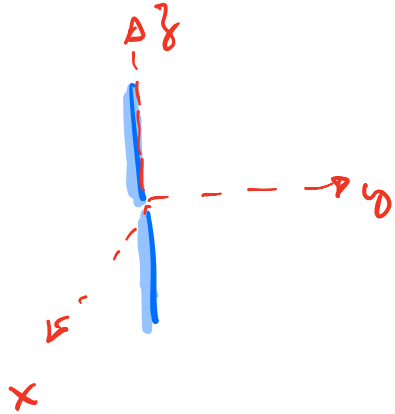


FIGURE 9-6. Snapshots of oscillating dipole. [From Hertz, *Wiedemann's Ann.* **36,** 1 (1889); reprinted in (He62).]

Ler aplicação para antena linear curta no final da Seção 9.2 do Jackson. Em particular, ver o conceito de resistência de radiação.



$$P_D = R_D I^2 \Rightarrow R_D = \frac{P_D}{I^2}$$