FI 008 – Eletrodinâmica I

1º Semestre de 2021

18/05/2021

Aula 17

Região de radiação

Fontes harmônicas:

$$\rho(\mathbf{x},t) = \rho(\mathbf{x})\, e^{-i\omega t}$$

$$\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})\, e^{-i\omega t}$$

$$\boldsymbol{\nabla}\cdot\ \mathbf{J} = i\omega\rho$$

Soluções harmônicas:

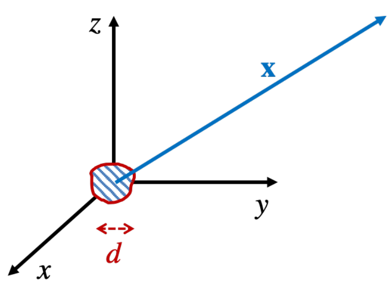
$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})\, e^{-i\omega t}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}(\mathbf{x})\, e^{-i\omega t}$$

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})\, e^{-i\omega t}$$

Hipóteses:

- Fontes pequenas
- Região de radiação



$$\begin{cases} d \ll r = |\mathbf{x}| \\ d \ll \lambda = \dfrac{2\pi}{k} = \dfrac{2\pi c}{\omega} \\ \lambda \ll r \end{cases}$$

Aulas passadas

Onda esférica:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{F}(k\hat{\mathbf{n}})$$

$$\mathbf{F}(k\hat{\mathbf{n}}) = \int \mathbf{J}(\mathbf{x}')\, e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'} d^3x'$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r}\hat{\mathbf{n}} \times \mathbf{F}(k\hat{\mathbf{n}})$$

$$\mathbf{E}(\mathbf{x}) = -\frac{\mu_0}{4\pi} i\omega \frac{e^{ikr}}{r}\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \mathbf{F}(k\hat{\mathbf{n}})\right]$$

$$c\mathbf{B}(\mathbf{x}) = \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{x})$$

$$\frac{dP}{d\Omega} = \frac{\mu_0\omega^2}{32\pi^2 c}\left|\hat{\mathbf{n}} \times \mathbf{F}(k\hat{\mathbf{n}})\right|^2$$

$$\mathbf{F}(k\hat{\mathbf{n}}) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int \mathbf{J}(\mathbf{x}')\left(\hat{\mathbf{n}}\cdot\mathbf{x}'\right)^n d^3x'$$

Aulas passadas

n=0: radiação de dipolo elétrico

$$\mathbf{p} = \int \mathbf{x}\rho(\mathbf{x})\, d^3x \in \mathbb{C}$$

$$\begin{aligned}
\mathbf{A}_{DE}(\mathbf{x}) &= -i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p} \\
\mathbf{B}_{DE}(\mathbf{x}) &= \frac{\mu_0\omega^2}{4\pi c}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\mathbf{p} \\
\mathbf{E}_{DE}(\mathbf{x}) &= -\frac{\mu_0\omega^2}{4\pi}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p}) \\
\frac{dP_{DE}}{d\Omega} &= \frac{\mu_0\omega^4}{32\pi^2 c}\left|\hat{\mathbf{n}}\times\mathbf{p}\right|^2 \\
P_{DE} &= \frac{\mu_0\omega^4}{12\pi c}\left|\mathbf{p}\right|^2
\end{aligned}$$

Aulas passadas

n=1: radiação de dip. magnético

$$\mathbf{m} = \frac{1}{2}\int \left[\mathbf{x}\times\mathbf{J}\left(\mathbf{x}\right)\right] d^3x \in \mathbb{C}$$

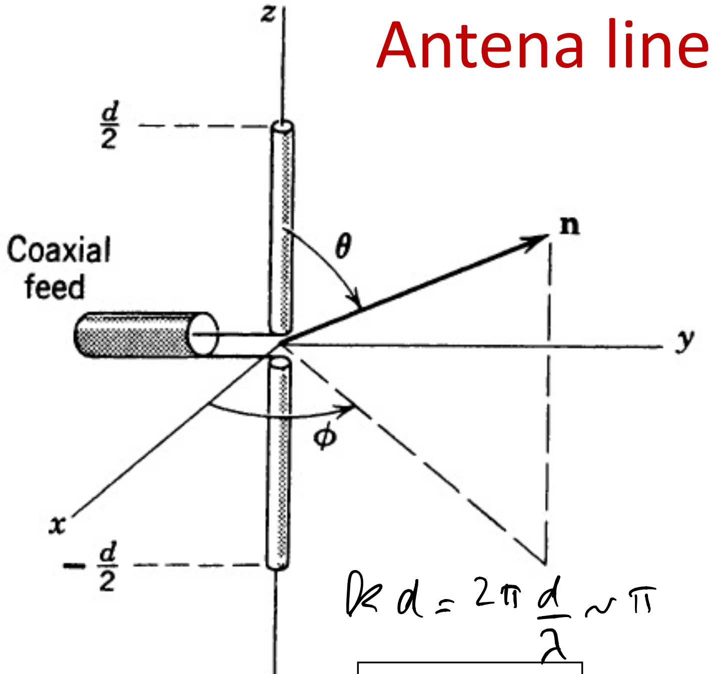
$$\begin{aligned}
\mathbf{A}_{DM} &= ik\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\hat{\mathbf{n}}\times\mathbf{m}\right)\\
\mathbf{B}_{DM} &= -k^2\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\mathbf{m}\right)\\
\mathbf{E}_{DM} &= -ck^2\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\hat{\mathbf{n}}\times\mathbf{m}\right)\\
\frac{dP_{DM}}{d\Omega} &= \frac{\mu_0 ck^4}{32\pi^2}\left|\hat{\mathbf{n}}\times\mathbf{m}\right|^2\\
P_{DM} &= \frac{\mu_0\omega^4}{12\pi c^3}\left|\mathbf{m}\right|^2
\end{aligned}$$

n=1: radiação de quad. elétrico

$$Q_{ij} = \int \left(3x_ix_j - \delta_{ij}r^2\right)\rho\left(\mathbf{x}\right) d^3x \in \mathbb{C}$$

$$\begin{aligned}
\mathbf{A}_{QE} &= -\frac{\omega k}{6}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right)\\
\mathbf{B}_{QE} &= -i\frac{\omega k^2}{6}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\left(\overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right)\\
\mathbf{E}_{QE} &= i\frac{\omega^2 k}{6}\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\left(\hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} - \overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right]\\
\frac{dP_{QE}}{d\Omega} &= \frac{\mu_0\omega^6}{1152\pi^2c^3}\left[\left|\overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right|^2 - \left|\hat{\mathbf{n}}\cdot\overline{\overline{\mathbf{Q}}}\cdot\hat{\mathbf{n}}\right|^2\right]\\
P_{QE} &= \frac{\mu_0\omega^6}{1440\pi c^3}\sum_{ij}\left|Q_{ij}\right|^2
\end{aligned}$$

Antena linear com corrente senoidal



$$\mathbf{F}(k\hat{\mathbf{n}}) = \int \mathbf{J}(\mathbf{x}')\, e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'} d^3x'$$

$$\mathbf{J}(\mathbf{x}) = I \sin\left(\frac{kd}{2} - k|z|\right) \delta(x)\,\delta(y)\,\hat{\mathbf{z}}$$

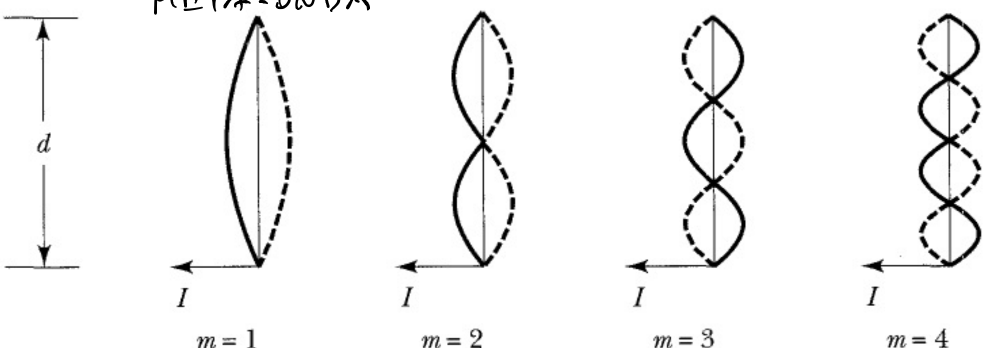
$$\mathbf{F}(k\hat{\mathbf{n}}) = 2I \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]\hat{\mathbf{z}}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c I^2}{8\pi^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2$$

$kd = 2\pi \frac{d}{\lambda} \sim \pi$

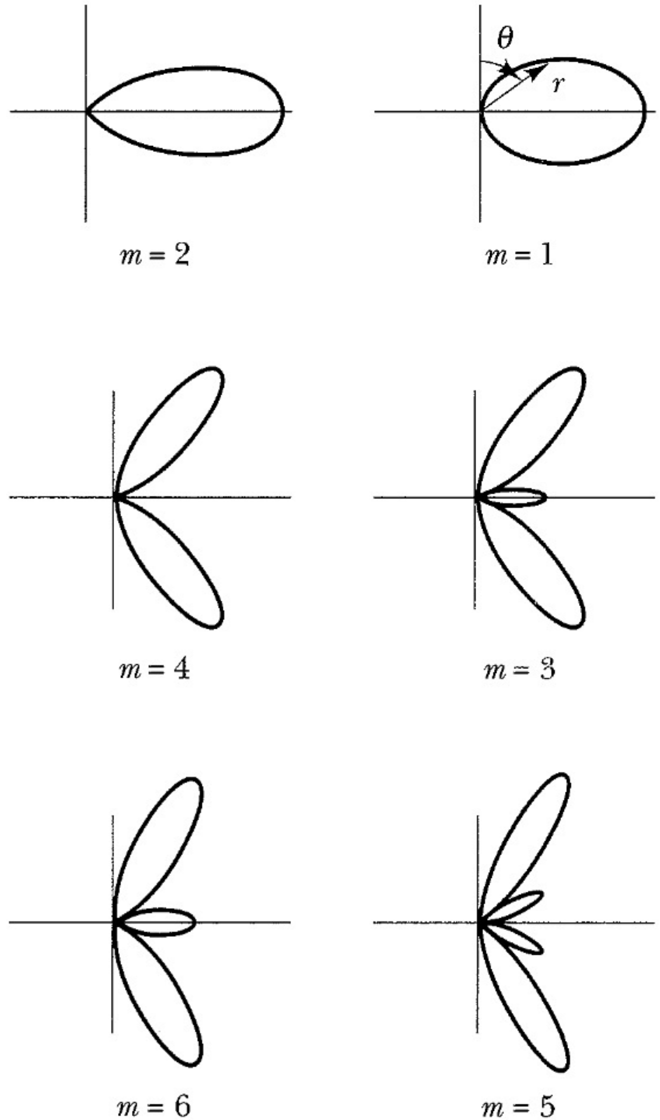
$$\boxed{kd = m\pi}$$

MEIA-ONDA



ONDA INTEIRA

$$kd = m\pi$$



9.1 A common textbook example of a radiating system (see Problem 9.2) is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not in the form of (9.1).

(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\mathbf{x}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (9.1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

(b) Consider a charge density $\rho(\mathbf{x}, t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\mathbf{x}, t) = \rho_0(\mathbf{x}) + \sum_{n=1}^{\infty} \mathrm{Re}[2\rho_n(\mathbf{x})e^{-in\omega_0 t}]$$

where

$$\rho_n(\mathbf{x}) = \frac{1}{T}\int_0^T \rho(\mathbf{x}, t)e^{in\omega_0 t}\, dt$$

This shows explicitly how to establish connection with (9.1).

(c) For a single charge q rotating about the origin in the x-y plane in a circle of radius R at constant angular speed ω_0, calculate the $l = 0$ and $l = 1$ multipole moments by the methods of parts a and b and compare. In method b express the charge density $\rho_n(\mathbf{x})$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

$$\begin{aligned} \rho(\mathbf{x}, t) &= \rho(\mathbf{x})e^{-i\omega t} \\ \mathbf{J}(\mathbf{x}, t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t} \end{aligned} \qquad (9.1)$$

PARA DIPOLO ELÉTRICO:

a) $\mathbf{p}(t) = \int \mathbf{x}\rho(\mathbf{x},t)\, d^3x \quad \rho(\mathbf{x},t) \in \mathbb{R}$

$$\mathbf{p}(t) = \mathbf{p}_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\mathbf{p}_n e^{-in\omega t}\right] \Rightarrow \begin{cases} \mathbf{p}_{\text{cap.9}} \in \mathbb{C} \to 2\mathbf{p}_n \text{ (frequentemente, por inspeção)} \\ \omega_{\text{cap.9}} \to n\omega \end{cases}$$

$$\mathbf{p}_n = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \mathbf{p}(t)\, e^{in\omega t} dt$$

b) $$\rho(\mathbf{x},t) = \rho_0(\mathbf{x}) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(\mathbf{x})\, e^{-in\omega t}\right] \Rightarrow \begin{cases} \mathbf{p}_{\text{cap.9}} \in \mathbb{C} \to \int \mathbf{x}\left[2\rho_n(\mathbf{x})\right] d^3x \\ \omega_{\text{cap.9}} \to n\omega \end{cases}$$

$$\rho_n(\mathbf{x}) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \rho(\mathbf{x},t)\, e^{in\omega t} dt$$

Atenção com o fator de 2!

Analogamente para $\mathbf{m}(t)$, $Q_{ij}(t)$,...

Usando $q_{l,-m}(t) = (-1)^m q^*_{lm}(t)$

$$
\begin{aligned}
p_x(t) &= \sqrt{\frac{2\pi}{3}}\left[q_{1-1}(t) - q_{11}(t)\right] = -2\sqrt{\frac{2\pi}{3}}\mathrm{Re}\left[q_{11}(t)\right] \\
p_y(t) &= -i\sqrt{\frac{2\pi}{3}}\left[q_{1-1}(t) + q_{11}(t)\right] = 2\sqrt{\frac{2\pi}{3}}\mathrm{Im}\left[q_{11}(t)\right] \\
p_z(t) &= \sqrt{\frac{4\pi}{3}}q_{10}(t)
\end{aligned}
$$

$$
\begin{aligned}
Q_{11}(t) &= \sqrt{\frac{6\pi}{5}}\left[q_{2-2}(t) + q_{22}(t)\right] - \sqrt{\frac{4\pi}{5}}q_{20}(t) = 2\sqrt{\frac{6\pi}{5}}\mathrm{Re}\left[q_{22}(t)\right] - \sqrt{\frac{4\pi}{5}}q_{20}(t) \\
Q_{22}(t) &= -\sqrt{\frac{6\pi}{5}}\left[q_{2-2}(t) + q_{22}(t)\right] - \sqrt{\frac{4\pi}{5}}q_{20}(t) = -2\sqrt{\frac{6\pi}{5}}\mathrm{Re}\left[q_{22}(t)\right] - \sqrt{\frac{4\pi}{5}}q_{20}(t) \\
Q_{33}(t) &= 2\sqrt{\frac{4\pi}{5}}q_{20}(t) \\
Q_{12}(t) &= i\sqrt{\frac{6\pi}{5}}\left[q_{22}(t) - q_{2-2}(t)\right] = -2\sqrt{\frac{6\pi}{5}}\mathrm{Im}\left[q_{22}(t)\right] \\
Q_{13}(t) &= \sqrt{\frac{6\pi}{5}}\left[q_{2-1}(t) - q_{21}(t)\right] = -2\sqrt{\frac{6\pi}{5}}\mathrm{Re}\left[q_{21}(t)\right] \\
Q_{23}(t) &= -i\sqrt{\frac{6\pi}{5}}\left[q_{2-1}(t) + q_{21}(t)\right] = 2\sqrt{\frac{6\pi}{5}}\mathrm{Im}\left[q_{21}(t)\right]
\end{aligned}
$$

$$p_x(t) = p_0(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \in \mathbb{R}$$

$$\cos\omega t \to \mathrm{Re}[e^{-i\omega t}] \qquad \sin\omega t \to \mathrm{Re}[+i\, e^{-i\omega t}]$$

$$p_x(t) = p_0\left(\hat{x}\,\mathrm{Re}[e^{-i\omega t}] + \hat{y}\,\mathrm{Re}[i\, e^{-i\omega t}]\right)$$

$$= \mathrm{Re}\left[\underbrace{p_0(\hat{x} + i\hat{y})}_{\vec{P}_{\text{CAP. 9}}} e^{-i\omega t}\right]$$

COM FREQÜÊNCIA ω