FI 008 – Eletrodinâmica I

1º Semestre de 2021
08/06/2021
Aula 21

Aulas passadas

Leis de transf. (Lorentz):

$$V'^{\alpha} = A^{\alpha}_{\lambda} V^{\lambda}$$

$$F'^{\alpha\beta} = A^{\alpha}_{\lambda} A^{\beta}_{\mu} F^{\lambda\mu} = A^{\alpha}_{\lambda} F^{\lambda\mu} \left(A^{T}\right)^{\mu}_{\beta} = \left(AFA^{T}\right)^{\alpha\beta}$$

Matricialmente:

$$\begin{pmatrix} V^{0\prime} \\ V^{1\prime} \\ V^{2\prime} \\ V^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{0} \\ V^{1} \\ V^{2} \\ V^{3} \end{pmatrix}$$

$$\begin{pmatrix} F'^{00} & F'^{01} & F'^{02} & F'^{03} \\ F'^{10} & F'^{11} & F'^{12} & F'^{13} \\ F'^{20} & F'^{21} & F'^{22} & F'^{23} \\ F'^{30} & F'^{31} & F'^{32} & F'^{33} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Aulas passadas

Quadri-nabla:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \to \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$$
$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} \to \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)$$

Quadri-divergente: $\partial_\alpha V^\alpha = \frac{1}{c}\frac{\partial V^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{V}$

Quadri-Laplaciano ou d’Alembertiano (“caixa”):

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

Covariância da Eletrodinâmica

Unidades gaussianas:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Força de Lorentz: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}\right)$

Conserv. da carga: $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

Potenciais:

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

Condição de Lorenz:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2\Phi &= 4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} &= \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

$\vec{F} = m\vec{a}$

A quadri-corrente

$$\boxed{J^\mu = (c\rho, \mathbf{J}) = \rho' (\gamma c, \gamma \mathbf{u}) = \rho' U^\mu}$$

ρ'= densidade de carga própria.

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad \Longrightarrow \quad \boxed{\partial_\mu J^\mu = 0}$$

O quadri-potencial

$$\boxed{A^\mu = (\Phi, \mathbf{A})}$$

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0 \quad \Rightarrow \quad \boxed{\partial_\mu A^\mu = 0}$$

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2\Phi &= 4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} &= \frac{4\pi}{c}\mathbf{J}\end{aligned} \quad \Rightarrow \quad \boxed{\left(\partial_\alpha\partial^\alpha\right) A^\mu = \Box A^\mu = \frac{4\pi}{c}J^\mu}$$

O quadri-tensor dos campos

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\Rightarrow \vec{E} \to -\vec{E}$

$\Downarrow$

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Transformações de Lorentz dos campos

$$F'^{\alpha\beta} = A^{\alpha}_{\lambda} A^{\beta}_{\mu} F^{\lambda\mu} = A^{\alpha}_{\lambda} F^{\lambda\mu} \left(A^T\right)^{\mu}_{\beta} = \left(AFA^T\right)^{\alpha\beta}$$

Para um "boost" na direção x:

$$\begin{pmatrix} 0 & -E'_1 & -E'_2 & -E'_3 \\ E'_1 & 0 & -B'_3 & B'_2 \\ E'_2 & B'_3 & 0 & -B'_1 \\ E'_3 & -B'_2 & B'_1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} E'_1 &= E_1 \\ E'_2 &= \gamma\left(E_2 - \beta B_3\right) \\ E'_3 &= \gamma\left(E_3 + \beta B_2\right) \end{aligned}$$

$$\begin{aligned} B'_1 &= B_1 \\ B'_2 &= \gamma\left(B_2 + \beta E_3\right) \\ B'_3 &= \gamma\left(B_3 - \beta E_2\right) \end{aligned}$$

Para um "boost" qualquer ($\boldsymbol{\beta}=\mathbf{v}/c$):

$$\mathbf{E}' = \gamma\left(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\underset{\beta\ll 1}{\longrightarrow}\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\underset{\beta\ll 1}{\longrightarrow}\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}$$

Eqs. de Maxwell com fontes em forma covariante

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

$$\Rightarrow \quad \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

O tensor de campos dual

$$e^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for } \alpha = 0,\ \beta = 1,\ \gamma = 2,\ \delta = 3, \text{ and} \\ & \text{any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases}$$

$$\boxed{\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}}$$

$$\mathcal{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\mathbf{E} \rightarrow \mathbf{B}$$

$$\mathbf{B} \rightarrow -\mathbf{E}$$

TRANSFORMAÇÃO
DE
DUALIDADE

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

Eqs. de Maxwell sem fontes em forma covariante

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad \boxed{\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = 0}$$

Equações de Maxwell

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\
\partial_\mu \widetilde{F}^{\mu\nu} &= 0
\end{aligned}$$

Força de Lorentz

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F} \cdot \mathbf{u} = q\mathbf{E} \cdot \mathbf{u} = \frac{dE}{dt}$$

$$\mathbf{p} \;=\; \gamma_u m \mathbf{u}$$

$$E \;=\; \gamma_u m c^2$$

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$$

$$\boxed{\frac{q}{c} F^{\mu\nu} U_\nu = \frac{dp^\mu}{d\tau}}$$

$$d\tau = \frac{dt}{\gamma}$$

$F^{\mu\nu} F_{\mu\nu}$ = ESCALAR DE LORENTZ

$F^{\mu\nu} \tilde{F}_{\mu\nu}$ = ESCALAR $\propto \vec{E} \cdot \vec{B}$

$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$ = ESCALAR

$$A^{\mu\nu\alpha} \quad B_{\lambda\rho\alpha} = F^{\mu\nu}{}_{\lambda\rho}$$