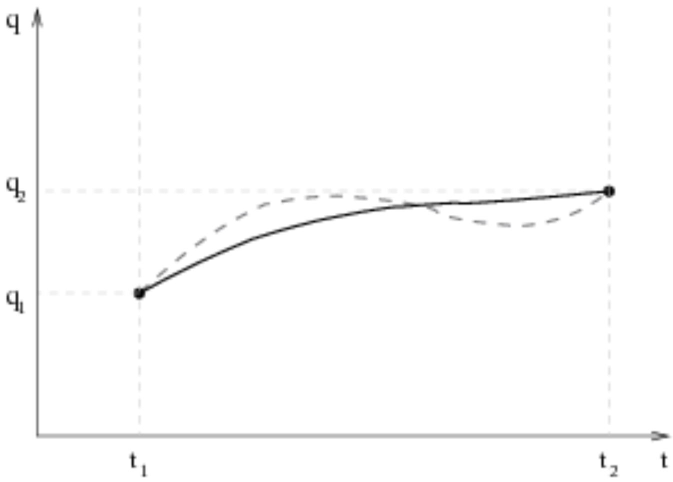
FI 008 – Eletrodinâmica I

1º Semestre de 2020
04/06/2020
Aula 23

Aula passada

Formulação Lagrangiana: Princípio de Hamilton



$$L\left[q_i\left(t\right),\dot{q}_i\left(t\right),t\right]\equiv L\left(q,\dot{q},t\right)$$

$$S=\int_{t_1}^{t_2}L\left(q,\dot{q},t\right)dt$$

Consideramos variações em torno de uma trajetória dada $q_0(t)$, que se anulam nos extremos:

$$\begin{aligned} q_i\left(t\right) &= q_i^0\left(t\right)+\delta q_i\left(t\right) \\ \dot{q}_i\left(t\right) &= \dot{q}_i^0\left(t\right)+\delta \dot{q}_i\left(t\right) \end{aligned}$$

$$\delta q_i\left(t_1\right)=\delta q_i\left(t_2\right)=0$$

Aula passada

Se impusermos que a variação da ação se anula em primeira ordem para qualquer variação, a trajetória $q_0(t)$, é a trajetória clássica.

$$\delta S = S\left[q_i^0(t) + \delta q_i(t), \dot{q}_i(t) + \delta \dot{q}_i(t), t\right] - S\left[q_i^0(t), \dot{q}_i^0(t), t\right] = 0, \text{ para todo } \delta q_i(t)$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] = 0. \quad \text{Equações de movimento (Euler-Lagrange)}}$$

Aula passada

Lagrangiana de uma partícula de massa *m* e carga *e* em campos eletromagnéticos dados $(\mathbf{\Phi},\mathbf{A})$

$$L\left[\mathbf{x},\dot{\mathbf{x}}\right]=-mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}-e\Phi\left(\mathbf{x},t\right)+\frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}\left(\mathbf{x},t\right)$$

$$\begin{aligned} U_\mu &= \gamma_u\left(c,-\mathbf{u}\right) \\ A^\mu &= \left(\Phi,\mathbf{A}\right) \end{aligned}$$

$$\boxed{\gamma_u L\left[x^\mu,U^\mu\right]=-mc^2-\frac{e}{c}U_\mu A^\mu\left(x^\mu\right)}$$

Formulação Lagrangiana para teorias de campo: caso uni-dimensional

$$\boxed{q_i(t) \quad (i = 1, 2, ..., N) \Longrightarrow \phi(x, t)}$$

O ÍNDICE DISCRETO $i = 1, 2, \ldots, N$ PASSARÁ A ASSUMIR VALORES NUM CONTÍNUO $x \in (-\infty, +\infty)$

$$L = \sum_{i=1}^{N} L_i \longrightarrow L = \int_{-\infty}^{+\infty} \mathcal{L}\, dx$$

$\mathcal{L}$ = DENSIDADE LAGRANGIANA

$$L[q_i(t), \dot{q}_i(t), t] \longrightarrow \mathcal{L}\left[\phi(x,t), \frac{\partial\phi}{\partial t}(x,t), \frac{\partial\phi}{\partial x}(x,t), x, t\right]$$

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2}\int_{-\infty}^{+\infty} \mathcal{L}\, dx\, dt \xrightarrow[t_2 \to +\infty]{t_1 \to -\infty} S = \int \mathcal{L}\, dx\, dt = \int \mathcal{L}\, d^2x$$

Princípio de Hamilton para campos

$\delta S = 0 \quad \forall \quad \delta\phi(x,t)$ EM 1ª ORDEM

$$\delta\phi(x,t) \xrightarrow[\substack{x\to\infty \\ t\to\infty}]{} 0$$

$$S[\phi, \partial_x\phi, \partial_t\phi] = \int dx\,dt\, \mathcal{L}[\phi, \partial_x\phi, \partial_t\phi, x, t]$$

$$\delta S = \int dx\,dt \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial[\partial_x\phi]} \underbrace{\delta(\partial_x\phi)}_{\partial_x(\delta\phi)} + \frac{\partial \mathcal{L}}{\partial[\partial_t\phi]} \underbrace{\delta(\partial_t\phi)}_{\partial_t(\delta\phi)} \right]$$

$$\int dx\,dt\, \frac{\partial \mathcal{L}}{\partial[\partial_x\phi]}\partial_x(\delta\phi) = \int dt \left[\cancel{\frac{\partial \mathcal{L}}{\partial[\partial_x\phi]}\delta\phi \Bigg|_{x=-\infty}^{x=+\infty}} - \int \partial_x\left(\frac{\partial \mathcal{L}}{\partial[\partial_x\phi]}\right)\delta\phi\, dx \right]$$

0

$$\Rightarrow \delta S = \int dx\,dt \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_x\left[\frac{\partial \mathcal{L}}{\partial[\partial_x\phi]}\right] - \partial_t\left[\frac{\partial \mathcal{L}}{\partial[\partial_t\phi]}\right] \right\}\delta\phi = 0 \quad \forall\, \delta\phi$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_x\left[\frac{\partial \mathcal{L}}{\partial[\partial_x\phi]}\right] - \partial_t\left[\frac{\partial \mathcal{L}}{\partial[\partial_t\phi]}\right] = 0$$

EQS. DE EULER-LAGRANGE

Generalizando para $D=3+1$ e N campos

$\phi_i(\vec{x},t) = \phi_i(x^\mu) \qquad i = 1, 2, \ldots N$

$\mathcal{L}[\phi_i, \partial_\mu \phi_i, x^\mu] \longrightarrow S = \int \mathcal{L}\, d^4x = \int \mathcal{L}\, d^3x\, dt$

$\delta\phi_i(x^\mu) \xrightarrow[x^\mu \to \infty]{} 0$

PRINCÍPIO DE HAMILTON: $\delta S = 0 \quad \forall\ \delta\phi_i$ EM 1ª ORDEM

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right] = 0 \qquad (i = 1, 2, \ldots N)$$

Formulação Lagrangiana para teorias de campo

$$\mathcal{L} = \mathcal{L}\left[\phi_i\left(x^\nu\right), \partial_\mu \phi_i\left(x^\nu\right), x^\nu\right]$$

$$S = \int d^4x \mathcal{L}\left[\phi_i\left(x^\nu\right), \partial_\mu \phi_i\left(x^\nu\right), x^\nu\right]$$

$$\begin{aligned} \phi_i\left(x^\nu\right) &= \phi_i^0\left(x^\nu\right) + \delta\phi_i\left(x^\nu\right) \\ \partial_\mu \phi_i\left(x^\nu\right) &= \partial_\mu \phi_i^0\left(x^\nu\right) + \partial_\mu \delta\phi_i\left(x^\nu\right) \end{aligned}$$

$$\delta\phi_i\left(x^\nu\right) = 0 \text{ nas bordas do espaço} - \text{tempo}$$

$$\delta S = 0 \Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} = 0 \ \ (i = 1, 2, \ldots, N)}$$

Exemplo: o campo escalar

$$\mathcal{L} = \frac{1}{2}\overset{(\partial^\mu\phi)(\partial_\mu\phi)}{\cancel{\partial_\mu\partial^\mu\phi}} - \frac{m^2}{2}\phi^2 = \frac{1}{2}g^{\mu\nu}\overset{(\partial_\mu\phi)(\partial_\nu\phi)}{\cancel{\partial_\mu\partial_\nu\phi}} - \frac{m^2}{2}\phi^2 \qquad \frac{\partial}{\partial x^\mu} = \partial_\mu$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial[\partial_\alpha\phi]} = \frac{1}{2}g^{\mu\nu}\left[\delta^\alpha_\mu(\partial_\nu\phi) + \delta^\alpha_\nu(\partial_\mu\phi)\right]$$

$$= \frac{1}{2}\left[g^{\alpha\nu}\partial_\nu\phi + g^{\mu\alpha}\partial_\mu\phi\right]$$

$$= \frac{1}{2}\left[\partial^\alpha\phi + \partial^\alpha\phi\right] = \partial^\alpha\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial[\partial_\alpha\phi]}\right] = 0 \Rightarrow -m^2\phi - \underbrace{\partial_\alpha\partial^\alpha}_{\Box}\phi = 0$$

$$\Rightarrow \boxed{\Box\phi + m^2\phi = 0}$$

EQUAÇÃO DE KLEIN-GORDON

$$E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

$$\phi(x^\mu) \propto e^{ip^\mu x_\mu} = e^{-i\omega t}e^{+i\vec{p}\cdot\vec{x}} \Rightarrow (-p^\alpha p_\alpha + m^2) = 0 \Rightarrow -E^2 + \vec{p}^{\,2} + m^2 = 0$$

Formulação Lagrangiana da eletrodiâmica

1. Fontes são dadas: $J^\mu = (c\rho, \mathbf{J})$
2. Queremos ao final recuperar as equações de Maxwell:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{4\pi}{c} J^\nu$$

EQUAÇÕES COM FONTES

$$\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = 0$$

EQUAÇÕES SEM FONTES

3. $\mathcal{L}$ deve ser um escalar de Lorentz
4. $A^\mu = (\Phi, \mathbf{A})$ são as variáveis dinâmicas: $\mathcal{L}[A^\mu, \partial^\nu A^\mu, J^\mu]$
5. Invariância sob transformações de calibre:

$$\left.\begin{array}{lcl} \Phi & \rightarrow & \Phi + \frac{1}{c}\frac{\partial\lambda}{\partial t} = \Phi + \frac{\partial\lambda}{\partial x^0} \\ \mathbf{A} & \rightarrow & \mathbf{A} - \nabla\lambda \end{array}\right\} \Rightarrow A^\mu \rightarrow A^\mu + \partial^\mu\lambda$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$$

$$S = \int \mathcal{L}\, d^4x$$

$S \rightarrow$ ESCALAR DE LORENTZ

$d^4x = d^3x\, dt$ $\qquad K \rightarrow K'$

$$\left.\begin{aligned} dt &= \gamma\, dt' \\ dx &= \frac{dx'}{\gamma} \\ dy &= dy' \\ dz &= dz' \end{aligned}\right\} d^4x = d^4x'$$

$\Rightarrow \mathcal{L} \rightarrow$ ESCALAR DE LORENTZ

PRECISAMOS CONSTRUIR ESCALARES COM, A^μ, $\partial^\mu A^\nu$, J^μ QUE SATISFAÇAM A CONDIÇÃO 5:

- $A_\mu A^\mu$: NÃO É INVARIANTE DE CALIBRE

$$A^\mu A_\mu \longrightarrow (A^\mu + \partial^\mu \lambda)(A_\mu + \partial_\mu \lambda) = A^\mu A_\mu + 2A^\mu \partial_\mu \lambda + (\partial^\mu \lambda)(\partial_\mu \lambda)$$

QUE NÃO É INVARIANTE DE CALIBRE

- $J^\mu A_\mu$: PARECE NÃO SER INVARIANTE SOB TRANSF. DE CALIBRE. MAS, NA VERDADE, ELE APARECE INTEGRADO NA AÇÃO

$$\int J^\mu A_\mu d^4x \longrightarrow \int J_\mu A^\mu d^4x + \underbrace{\int J_\mu \partial^\mu \lambda \, d^4x}_{(*)}$$

$$(*) = \underbrace{\int \partial^\mu (J_\mu \lambda) d^4x}_{} - \int \lambda \underbrace{(\partial^\mu J_\mu)}_{(**)} d^4x$$

$$\oint_{S(\infty)} (\lambda J_\mu) d^\mu S = 0$$ SE AS CORRENTES SÃO LOCALIZADAS

$$(**) \quad \partial^\mu J_\mu = 0 \Rightarrow \bar{\nabla} \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0$$ CONSERVAÇÃO DA CARGA

$\boxed{\int J^\mu A_\mu d^4x}$ É INVARIANTE DE CALIBRE ✓

- $\partial^\mu A^\nu$: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ JÁ É INVARIANTE DE CALIBRE

$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ TAMBÉM É INV. CALIBRE

$\boxed{F^{\mu\nu}F_{\mu\nu} \rightarrow \text{ESCALAR DE LORENTZ} \checkmark}$

$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \rightarrow$ ESCALAR DE LORENTZ, MAS É $\propto F^{\mu\nu}F_{\mu\nu}$

$F^{\mu\nu}\tilde{F}_{\mu\nu} \rightarrow$ ESCALAR DE LORENTZ

MAS, PODE-SE PROVAR (VER NOTAS) QUE

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial^\mu H_\mu$$

$$\Rightarrow \int \partial^\mu H_\mu \, d^4x = \int_{S(\infty)} H_\mu \, d^\mu S$$

QUE SÓ DEPENDE DOS CAMPOS NA BORDA DO ESPAÇO-TEMPO

NO PRINCÍPIO DE HAMILTON, NÃO CONTRIBUI $\Rightarrow$ NÃO AFETA AS EQS. DE EULER-LAGRANGE

ISSO NOS LEVA A:

$$\mathcal{L} = a F^{\mu\nu} F_{\mu} + b J^{\mu} A_{\mu}$$

PARA FICAR COMPATÍVEL COM AS EQS. DE MAXWELL:

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu}$$

↳ VER A LAGRANGIANA DE PARTÍCULA

$$F^{\mu\nu} F_{\mu\nu} = -2 \left(E^2 - B^2 \right)$$

Equações de Euler-Lagrange

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu}$$

$$\mathcal{L}[A^{\mu}, \partial^{\mu} A^{\nu}, J^{\mu}] = -\frac{1}{16\pi} \underbrace{(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})}_{g_{\mu\alpha} g_{\nu\beta} (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})(\partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha})} - \frac{1}{c} J_{\mu} A^{\mu}$$

$$\frac{\partial \mathcal{L}}{\partial A^{\mu}} = -\frac{1}{c} J_{\mu}$$

$$\frac{\partial \mathcal{L}}{\partial [\partial^{\rho} A^{\sigma}]} = g_{\mu\alpha} g_{\nu\beta} \Big\{ (\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \delta^{\mu}_{\sigma}) \underbrace{(\partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha})}_{F^{\alpha\beta}}$$

$$+ \underbrace{(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})}_{F^{\mu\nu}} (\delta^{\alpha}_{\rho} \delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho} \delta^{\alpha}_{\sigma}) \Big\}$$

$$= \Big\{ (g_{\rho\alpha} g_{\sigma\beta} - g_{\sigma\alpha} g_{\rho\beta}) F^{\alpha\beta} + (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) F^{\mu\nu} \Big\}$$

$$= F_{\rho\sigma} - F_{\sigma\rho} + F_{\rho\sigma} - F_{\sigma\rho} = 4 F_{\rho\sigma} = 4 (\partial_{\rho} A_{\sigma} - \partial_{\sigma} A_{\rho})$$

FINALMENTE:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial A^\mu}}_{-\frac{1}{c}J_\mu} - \partial^\alpha \left[\underbrace{\frac{\partial \mathcal{L}}{\partial [\partial^\alpha A^\mu]}}_{} \right] = 0$$

$$-\frac{1}{c} J_\mu - \partial^\alpha \left[-\frac{1}{4\pi} F_{\alpha\mu} \right] \Rightarrow \partial^\alpha F_{\alpha\mu} = \frac{4\pi}{c} J_\mu$$

$$\boxed{\partial_\alpha F^{\alpha\mu} = \frac{4\pi}{c} J^\mu}$$

QUE SÃO AS EQS. DE MAXWELL COM FONTES

E AS EQS. DE MAXWELL SEM FONTES?

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 0$$

TRIVIALMENTE QUANDO SE TRABALHA DIRETAMENTE COM O 4-POTENCIAL A^μ

COMO A FORMULAÇÃO LAGRANGIANA É CONSTRUÍDA COM A^μ, ESSAS EQS. SÃO TRIVIAIS

Lagrangiana da eletrodinâmica

$$\begin{aligned}\mathcal{L}\left[A^\mu, \partial^\nu A^\mu\right] &= -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J_\mu A^\mu \\ &= \frac{1}{8\pi}\left[\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \left(\nabla \times \mathbf{A}\right)^2\right] - \rho\Phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A}\end{aligned}$$

Equações de Euler-Lagrange são as equações de Maxwell com fontes:

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{4\pi}{c}J^\nu$$

Equações de Maxwell sem fontes são automaticamente satisfeitas

$$\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = 0$$