FI 008 – Eletrodinâmica I

1º Semestre de 2020
09/06/2020
Aula 24

Aula passada

Formulação Lagrangiana para teorias de campo (passagem para o contínuo):

$$q_i\left(t\right) \;\rightarrow\; \phi_k\left(x^\mu\right)$$

$$\dot{q}_i\left(t\right) \;\rightarrow\; \partial^\alpha\phi_k\left(x^\mu\right)$$

$$L = \sum_i L_i\left(q_i, \dot{q}_i, t\right) \;\rightarrow\; \int \mathcal{L}\left[\phi_k, \partial^\alpha\phi_k, x^\nu\right] d^3x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \;\rightarrow \boxed{\partial^\alpha \frac{\partial \mathcal{L}}{\partial\left(\partial^\alpha\phi_k\right)} = \frac{\partial \mathcal{L}}{\partial \phi_k}}$$

Aula passada

Formulação Lagrangiana para a eletrodinâmica de Maxwell: $J_\mu(x^\alpha)$ dada

$$\begin{aligned}\mathcal{L}\left[A^\mu, \partial^\nu A^\mu, x^\alpha\right] &= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu\left(x^\alpha\right) A^\mu \\ &= -\frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \frac{1}{c} g^{\mu\beta} J_\mu\left(x^\alpha\right) A_\beta\end{aligned}$$

Eqs. de Euler-Lagrange: equações de Maxwell com fontes

$$\left.\begin{aligned}\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha A_\beta\right)} &= -\tfrac{1}{4\pi} F^{\alpha\beta} \\ \frac{\partial \mathcal{L}}{\partial A_\beta} &= -\tfrac{1}{c} J^\beta\end{aligned}\right\} \Rightarrow \partial_\alpha F^{\alpha\beta} = \partial_\alpha\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = \frac{4\pi}{c} J^\beta$$

Formulação em termos de $A^\alpha \Rightarrow$ as eqs. de Maxwell sem fontes são automaticamente satisfeitas:

$$\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \partial_\mu \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = 0$$

Leis de conservação e simetrias

Toda lei de conservação tem origem numa simetria

- Simetria de translação temporal: conservação de energia

$$L\left(q_i, \dot{q}_i, \cancel{t}\right) \Rightarrow \frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \Rightarrow H = \sum_i p_i \dot{q}_i - L = \text{const.}$$

- Simetria de translação espacial: conservação de momento linear

$$L\left(q_i^{\text{rel}}, \dot{q}_i^{\text{rel}}, \cancel{\mathbf{R}_{\text{CM}}}, \dot{\mathbf{R}}_{\text{CM}}, t\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}_{\text{CM}}}\right) = \frac{\partial L}{\partial \mathbf{R}_{\text{CM}}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\mathbf{R}}_{\text{CM}}} = \mathbf{P} = \text{const.}$$

- Simetria de rotação global: conservação de momento angular

Leis de conservação: discussão geral

SIMETRIA DE TRANSLAÇÃO ESPAÇO-TEMPORAL

PARA UMA TEORIA DE CAMPOS GENÉRICA:

$\mathcal{L}[\phi_i, \partial_\mu \phi_i]$ → NÃO HÁ DEPENDÊNCIA EXPLÍCITA COM x^μ

DERIVADA ESPAÇO-TEMPORAL DE $\mathcal{L}$:

$$\underbrace{\partial_\nu \mathcal{L}}_{\partial_\mu[\delta^\mu{}_\nu \mathcal{L}]} = \sum_i \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_i}}_{\partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]}} \partial_\nu \phi_i + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \partial_\nu (\partial_\mu \phi_i) \right]$$

$\partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]}$ (EQS. DE EULER-LAGRANGE)

$$= \sum_i \left\{ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \right] \partial_\nu \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu (\partial_\nu \phi_i) \right\}$$

$$= \sum_i \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \partial_\nu \phi_i \right\}$$

$$\Rightarrow \partial_\mu \left[\sum_i \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L} \right] = 0 \longrightarrow T^\mu{}_\nu$$

$$T^{\mu}{}_{\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \partial_\nu \phi_i - \delta^{\mu}{}_{\nu} \mathcal{L}$$

LEVANDO ν $\Rightarrow$ $T^{\mu\nu} = g^{\nu\alpha} T^{\mu}{}_{\alpha}$

$$\boxed{T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]} \partial^\nu \phi_i - g^{\mu\nu} \mathcal{L}}$$

$\Rightarrow$ $\boxed{\partial_\mu T^{\mu\nu} = 0}$ $\rightarrow$ LEI DE CONSERVAÇÃO

$$\partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0 \Rightarrow \frac{1}{c} \frac{\partial T^{0\nu}}{\partial t} + \frac{\partial T^{i\nu}}{\partial x^i} = 0$$

$$\Rightarrow \frac{\partial \rho^\nu}{\partial t} + \vec{\nabla} \cdot \vec{J}^\nu = 0 \quad (\nu = 0, 1, 2, 3)$$

ONDE: $\rho^\nu = \frac{1}{c} T^{0\nu}$

$(\vec{J}^\nu)_i = T^{i\nu}$

$$\frac{d}{dt}\Big[\underbrace{\int_V \rho^\nu d^3x}_{Q^\nu(V)}\Big] = -\oint_{S(V)} \vec{J}^\nu \cdot d\vec{S}$$

Leis de conservação na eletrodinâmica

PARA A ELETRODINÂMICA, A AUSÊNCIA DE DEPENDÊNCIA EXPLÍCITA COM x^μ IMPLICA EM AUSÊNCIA DE CORRENTES: $J^\mu(x^\alpha)$

ASSUMINDO $J^\mu(x^\alpha) = 0$, TEMOS:

$$\frac{\partial \mathcal{L}}{\partial[\partial_\mu A^\lambda]} = -\frac{1}{4\pi} F^\mu{}_\lambda \Rightarrow T^{\mu\nu} = -\frac{1}{4\pi} F^\mu{}_\lambda \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L}$$

$$\boxed{T^{\mu\nu} = -\frac{1}{4\pi}\left[F^\mu{}_\lambda \partial^\nu A^\lambda - g^{\mu\nu}\left(\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}\right)\right]}$$

É INTERESSANTE TRABALHAR COM UM TENSOR $T^{\mu\nu}$ QUE SEJA SIMÉTRICO. ISSO PODE SER FEITO SE HOUVER $V^{\mu\nu}$ TAL QUE: $\partial_\mu V^{\mu\nu} = 0$ E $T^{\mu\nu} + V^{\mu\nu}$ SEJA SIMÉTRICO

SEJA : $V^{\mu\nu} = \frac{1}{4\pi} F^{\mu}{}_{\lambda}\, \partial^{\lambda} A^{\nu}$

$$\partial_{\mu} V^{\mu\nu} = \frac{1}{4\pi}\, \partial_{\mu}\left(F^{\mu}{}_{\lambda}\, \partial^{\lambda} A^{\nu}\right) = \frac{1}{4\pi}\left[\underbrace{\left(\partial_{\mu} F^{\mu}{}_{\lambda}\right)}_{\frac{4\pi}{c} J_{\lambda}} \partial^{\lambda} A^{\nu} + F^{\mu}{}_{\lambda}\, \partial_{\mu} \partial^{\lambda} A^{\nu}\right]$$

2º TERMO: $\underbrace{F^{\mu\lambda}}_{\text{ANTI-SIMÉTRICO}}\, \underbrace{\partial_{\mu}\partial_{\lambda}}_{\text{SIMÉTRICO}} A^{\nu} = 0$

$$\Rightarrow \partial_{\mu} V^{\mu\nu} = \frac{1}{c} J_{\lambda}\left(\partial^{\lambda} A^{\nu}\right) \quad (1)$$

COMO $J_{\lambda} = 0 \Rightarrow \partial_{\mu} V^{\mu\nu} = 0$

ALÉM DISSO:

$$\Theta^{\mu\nu} = T^{\mu\nu} + V^{\mu\nu} = \frac{1}{4\pi}\left[\underbrace{-F^{\mu}{}_{\nu}\left(\partial^{\nu} A^{\lambda}\right) + F^{\mu}{}_{\lambda}\, \partial^{\lambda} A^{\nu}} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}\right]$$

$$F^{\mu}{}_{\lambda}\underbrace{\left(\partial^{\lambda} A^{\nu} - \partial^{\nu} A^{\lambda}\right)}_{F^{\lambda\nu}} = F^{\mu}{}_{\lambda} F^{\lambda\nu}$$ QUE É SIMÉTRICO!

$F^{\mu}{}_{\lambda} F^{\lambda\nu}$ É SIMÉTRICO:

$$F^{\nu}{}_{\lambda} F^{\lambda\mu} = F^{\lambda\mu} F^{\nu}{}_{\lambda} = -F^{\mu\lambda} F^{\nu}{}_{\lambda} = -F^{\mu}{}_{\lambda} F^{\nu\lambda} = F^{\mu}{}_{\lambda} F^{\lambda\nu}$$

$\Rightarrow F^{\mu}{}_{\lambda} F^{\lambda\nu}$ É SIMÉTRICO

$$\boxed{\partial_{\mu} \Theta^{\mu\nu} = 0}$$ ONDE

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu}{}_{\lambda} F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu}\left(F^{\alpha\beta} F_{\alpha\beta}\right)\right]$$

→ TENSOR DE TENSÕES CANÔNICO

Tensor de tensões canônico

$$\boxed{\Theta^{\mu\nu} = \frac{1}{4\pi}\left[g^{\mu\alpha}F_{\alpha\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] = \frac{1}{4\pi}\left[F^{\mu}{}_{\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]}$$

$$\Theta^{\mu\nu} \to \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{xx} & -T_{xy} & -T_{xz} \\ S_y/c & -T_{yx} & -T_{yy} & -T_{yz} \\ S_z/c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}$$

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right)$$ = DENSIDADE DE ENERGIA EM UNIDADES GAUSSIANAS

$$\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}$$ → VETOR DE POYNTING

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]$$

→ TENSOR DE TENSÕES DE MAXWELL

SEGUE QUE: $\theta^{0\nu}$

$$\Rightarrow \frac{1}{c}\frac{\partial \theta^{0\nu}}{\partial t} + \frac{\partial}{\partial x^i}\theta^{i\nu} = 0$$

$(\nu=0)$ $\Rightarrow \dfrac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$ (TEOREMA DE POYNTING)

$(\nu=i)$ $\Rightarrow \underbrace{\dfrac{1}{c^2}\dfrac{\partial S_i}{\partial t}}_{\frac{\partial g_i}{\partial t}} - \vec{\nabla}\cdot\vec{\vec{T}}^i = 0$

ONDE $\vec{g} = \dfrac{\vec{S}}{c^2}$ = DENSIDADE DE MOMENTO LINEAR DOS CAMPOS

$$\Rightarrow \frac{\partial \vec{g}}{\partial t} = \vec{\nabla}\cdot\vec{\vec{T}}$$

(LEI DE CONSERVAÇÃO DE MOMENTO LINEAR)

E na presença de cargas e correntes?

DE (1) E DO FATO DE $\partial_\mu T^{\mu\nu} = 0$ SEGUE QUE, NA PRESENÇA DE CORRENTES:

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu \left[T^{\mu\nu} + V^{\mu\nu} \right] = \partial_\mu V^{\mu\nu} = \frac{1}{c} J_\lambda F^{\lambda\nu}$$

EM TERMOS DOS CAMPOS:

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E}$$

$$\frac{\partial \vec{g}}{\partial t} - \vec{\nabla} \cdot \vec{\vec{T}} = - \left[\rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right]$$

Ver tratamento Hamiltoniano da eletrodinâmica nas notas

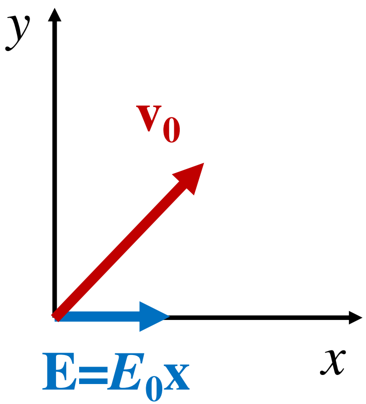
Movimento de partículas em campos constantes e uniformes

CONSIDERAÇÕES GERAIS: EQUAÇÕES

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \qquad \vec{p} = \gamma m \vec{v} \; ; \; \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\frac{dE}{dt} = e\,\vec{v}\cdot\vec{E} \qquad \vec{E} = \gamma m c^2$$

Caso 1: B=0 , E≠0



$$\frac{d\vec{p}}{dt} = e\vec{E} = eE_0\,\hat{x}$$

$$\Rightarrow \frac{dp_z}{dt} = 0 \qquad p_z = 0 \Rightarrow p_z = 0 = \text{CONST.}$$

O MOVIMENTO É CONFINADO AO PLANO xy

$$\frac{d(\gamma m v_x)}{dt} = eE_0 \qquad \frac{d(\gamma m v_y)}{\partial t} = 0$$

$$\Rightarrow \gamma m v_x = eE_0 t + B \quad (2)$$

$$\gamma m v_y = A \quad (3)$$

ONDE A E B SÃO CONSTANTES DETERMINADAS PELAS CONDIÇÕES INICIAIS ($A = p_y(0)$; $B = p_x(0)$)

$$\gamma^{-1} = \sqrt{1 - \left(\frac{v_x^2 + v_y^2}{c^2}\right)}$$

QUERO EXTRAIR γ.

DE (2) E (3):

$$\frac{v_x^2}{c^2} = \left(\frac{eE_0t + B}{mc}\right)^2 (1-\beta^2) \quad (4) \qquad \beta^2 = \frac{v_x^2 + v_y^2}{c^2}$$

$$\frac{v_y^2}{c^2} = \left(\frac{A}{mc}\right)^2 (1-\beta^2) \quad (5)$$

SOMANDO (4) E (5):

$$\beta^2 = (1-\beta^2)\underbrace{\left[\left(\frac{A}{mc}\right)^2 + \left(\frac{eE_0t + B}{mc}\right)^2\right]}_{x}$$

$$\Rightarrow \beta^2 = (1-\beta^2)x \Rightarrow \beta^2 = \frac{x}{1+x} \Rightarrow 1-\beta^2 = \frac{1}{\gamma^2} = \frac{1}{1+x}$$

$$\Rightarrow \gamma = \sqrt{1+x}$$

VOLTANDO A (2) E (3):

$$\left.\begin{aligned} \frac{v_x}{c} &= \frac{eE_0t + B}{\left[m^2c^2 + A^2 + (eE_0t + B)^2\right]^{1/2}} \\ \frac{v_y}{c} &= \frac{A}{\left[\quad\right]^{1/2}} \end{aligned}\right\}$$

QUANDO $t \to \infty$

$v_y \to 0$

$v_x \to c$

INTEGRANDO DE NOVO NO TEMPO:

$$\frac{x(t)}{c} = \frac{1}{eE_0}\left[m^2c^2 + A^2 + (eE_0t + B)^2\right]^{1/2}$$

$$\frac{y(t)}{c} = \frac{A}{eE_0}\sinh^{-1}\left[\frac{eE_0t + B}{\sqrt{A^2 + m^2c^2}}\right]$$

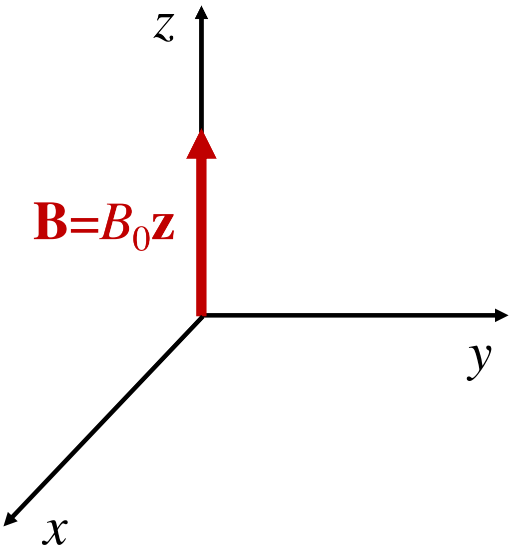
EQ. DA TRAJETÓRIA:

$$\left.\begin{array}{l} Dx = \cosh\left[\left(\frac{eE_0}{cA}\right)y\right] \\ \text{ONDE} \quad D = \frac{eE_0}{c\sqrt{A^2 + m^2c^2}} \end{array}\right\} \text{EQ. DA CATENÁRIA}$$

SE x, y SÃO PEQUENOS

$$Dx \cong 1 + \left(\frac{eE_0}{cA}\right)^2 \frac{y^2}{2} \quad \sim \text{PARÁBOLA}$$

Caso 2: E=0, B≠0



$$\frac{dE}{dt}=0 \Rightarrow E = \text{CONST.}$$
$$\gamma = \text{CONST.}$$
$$|\vec{v}| = \text{CONST.}$$

$$\frac{d(\gamma m \vec{v})}{dt} = \frac{e}{c}\vec{v}\times\vec{B}$$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{e}{\gamma m c}\vec{v}\times\vec{B} \equiv \vec{v}\times\vec{\omega}_B$$

ONDE: $\vec{\omega}_B = \frac{e}{\gamma m c}\vec{B} = \left(\frac{ec}{E}\right)\vec{B} \Rightarrow \boxed{\frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B}$

$$\vec{v} = \vec{v}_\perp + v_z\hat{z} \Rightarrow \frac{d\vec{v}_\perp}{dt} + \left(\frac{dv_z}{dt}\right)\hat{z} = \vec{v}_\perp \times \vec{\omega}_B$$

NÃO TEM COMPONENTE NA DIREÇÃO z

$$\Rightarrow \frac{dv_z}{dt} = 0 \Rightarrow v_z = v_z(0) = \text{CONST.}$$

$$z = z_0 + v_z(0)\, t$$

$$\frac{d\vec{v}_\perp}{dt} = \vec{v}_\perp \times \vec{\omega}_B \quad (6)$$

TOMANDO $(\vec{v}_\perp \cdot)$ COM OS DOIS LADOS:

$$\Rightarrow \vec{v}_\perp \cdot \frac{d\vec{v}_\perp}{dt} = \frac{1}{2}\frac{d(\vec{v}_\perp \cdot \vec{v}_\perp)}{dt} = 0 \Rightarrow |\vec{v}_\perp| = \text{CONST.}$$

A SOLUÇÃO DE (6) É:

$$\vec{v}_\perp(t) = v_\perp^0 \left[\cos(\omega_B t + \delta)\,\hat{x} - \sin(\omega_B t + \delta)\,\hat{y}\right]$$

INTEGRO NOVAMENTE NO TEMPO:

$$\vec{x}_\perp(t) = \vec{x}_\perp^0 + \frac{v_\perp^0}{\omega_B}\left[\sin(\omega_B t + \delta)\,\hat{x} + \cos(\omega_B t + \delta)\,\hat{y}\right]$$

$\Rightarrow$ MOVIMENTO HELICOIDAL

O RAIO DA HÉLICE É:

$$a = \frac{v^0_\perp}{\omega_B} = \frac{\gamma m c\, v^0_\perp}{eB} = \boxed{\frac{c\, p^0_\perp}{eB} = a}$$

A MEDIDA DO RAIO <u>a</u> POSSIBILITA DETERMINAR O MOMENTO TRANSVERSO $p^0_\perp$.