FI 008 – Eletrodinâmica I

1º Semestre de 2021
17/06/2021
Aula 24

Aula passada

Formulação Lagrangiana para teorias de campo (passagem para o contínuo):

$$q_i\left(t\right) \;\rightarrow\; \phi_k\left(x^\mu\right)$$

$$\dot{q}_i\left(t\right) \;\rightarrow\; \partial^\alpha \phi_k\left(x^\mu\right)$$

$$L = \sum_i L_i\left(q_i, \dot{q}_i, t\right) \;\rightarrow\; \int \mathcal{L}\left[\phi_k, \partial^\alpha \phi_k, x^\nu\right] d^3x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \;\rightarrow\; \boxed{\partial^\alpha \frac{\partial \mathcal{L}}{\partial\left(\partial^\alpha \phi_k\right)} = \frac{\partial \mathcal{L}}{\partial \phi_k}}$$

Aula passada

Formulação Lagrangiana para a eletrodinâmica de Maxwell: $J_\mu(x^\alpha)$ dada

$$\begin{aligned}\mathcal{L}\left[A^\mu, \partial^\nu A^\mu, x^\alpha\right] &= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu\left(x^\alpha\right) A^\mu \\ &= -\frac{1}{16\pi} g^{\mu\alpha} g^{\nu\beta}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \frac{1}{c} g^{\mu\beta} J_\mu\left(x^\alpha\right) A_\beta\end{aligned}$$

Eqs. de Euler-Lagrange: equações de Maxwell com fontes

$$\left.\begin{aligned}\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha A_\beta\right)} &= -\tfrac{1}{4\pi} F^{\alpha\beta} \\ \frac{\partial \mathcal{L}}{\partial A_\beta} &= -\tfrac{1}{c} J^\beta\end{aligned}\right\} \Rightarrow \partial_\alpha F^{\alpha\beta} = \partial_\alpha\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = \frac{4\pi}{c} J^\beta$$

Formulação em termos de $A^\alpha \Rightarrow$ as eqs. de Maxwell sem fontes são automaticamente satisfeitas:

$$\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = 0$$

Leis de conservação e simetrias

Toda lei de conservação tem origem numa simetria

- Simetria de translação temporal: conservação de energia

$$L\left(q_i, \dot{q}_i, \cancel{t}\right) \Rightarrow \frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \Rightarrow H = \sum_i p_i \dot{q}_i - L = \text{const.}$$

- Simetria de translação espacial: conservação de momento linear

$$L\left(q_i^{\text{rel}}, \dot{q}_i^{\text{rel}}, \cancel{\mathbf{R}_{\text{CM}}}, \dot{\mathbf{R}}_{\text{CM}}, t\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}_{\text{CM}}}\right) = \frac{\partial L}{\partial \mathbf{R}_{\text{CM}}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\mathbf{R}}_{\text{CM}}} = \mathbf{P} = \text{const.}$$

- Simetria de rotação global: conservação de momento angular

Leis de conservação: discussão geral

Dens. de Lagrangiana sem dependência explícita com as coordenadas do espaço-tempo:

$$\mathcal{L}\left[\phi_i\left(x^\nu\right), \partial_\mu \phi_i\left(x^\nu\right), \cancel{x^\nu}\right]$$

Então:

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi_i\right]} \partial^\nu \phi_i - g^{\mu\nu}\mathcal{L} \qquad \Rightarrow \boxed{\partial_\mu T^{\mu\nu} = 0}$$

Lei de conservação (como a da carga): $\partial_\mu J^\mu = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

$$\frac{\partial\left(T^{0\nu}/c\right)}{\partial t} + \frac{\partial T^{i\nu}}{\partial x^i} = \frac{\partial\left(T^{0\nu}/c\right)}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T}^\nu = 0 \qquad (\mathbf{T}^\nu \to T^{i\nu})$$

$$\rho^\nu \equiv \frac{T^{0\nu}}{c}$$

$$\oint_{S(V)} \mathbf{T}^\nu \cdot d\mathbf{S} = -\frac{d}{dt}\left[\int_V \left(\frac{T^{0\nu}}{c}\right) d^3x\right] = -\frac{dQ^\nu(V)}{dt}$$

$$\left(\mathbf{J}^\nu\right)_i \equiv T^{i\nu}$$

Leis de conservação na eletrodinâmica

Na ausência de fontes:

$J_\mu = 0$

$$\mathcal{L}\left[A^{\mu}, \partial^{\nu} A^{\mu}, x^{\alpha}\right] = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_{\mu}\left(x^{\alpha}\right) A^{\mu}$$

$$T^{\mu\nu} = -\frac{1}{4\pi}\left[F^{\mu}_{\lambda} \partial^{\nu} A^{\lambda} - \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}\right] \Rightarrow \partial_{\mu} T^{\mu\nu} = 0$$

Mas, também existe:

$$V^{\mu\nu} = \frac{1}{4\pi} F^{\mu}_{\lambda} \partial^{\lambda} A^{\nu} \Rightarrow \partial_{\mu} V^{\mu\nu} = 0$$

Logo: $\Rightarrow \partial_{\mu}\left(T^{\mu\nu} + V^{\mu\nu}\right) \equiv \partial_{\mu} \Theta^{\mu\nu} = 0$

Tensor de tensões canônico:

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu}_{\lambda} F^{\lambda\nu} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}\right]$$
$$\Theta^{\mu\nu} = \Theta^{\nu\mu}$$

Tensor de tensões canônico

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left[g^{\mu\alpha}F_{\alpha\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] = \frac{1}{4\pi}\left[F^{\mu}{}_{\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

$$\Theta^{\mu\nu} \rightarrow \begin{pmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & -T_{xx} & -T_{xy} & -T_{xz} \\ S_y/c & -T_{yx} & -T_{yy} & -T_{yz} \\ S_z/c & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}$$

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right)$$ densidade de energia

$$\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}$$ vetor de Poynting

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]$$ tensor de tensões de Maxwell

Leis de conservação de energia de momento linear dos campos

$$
\begin{aligned}
(\nu = 0) \quad \frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} &= 0 \\
(\nu = i) \quad \frac{\partial g^i}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{T}^i &= 0 \\
\mathbf{g} &= \frac{\mathbf{S}}{c^2}
\end{aligned}
$$

E na presença de cargas e correntes?

$$\mathcal{L}\left[A^{\mu},\partial^{\nu}A^{\mu},x^{\alpha}\right] \;=\; -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}-\frac{1}{c}J_{\mu}\left(x^{\alpha}\right)A^{\mu}$$

$$\boxed{\partial_{\mu}\Theta^{\mu\nu}=\frac{1}{c}J_{\lambda}F^{\lambda\nu}}$$

$$\begin{aligned}
(\nu=0)\quad & \frac{\partial u}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{S} \;=\; -\mathbf{J}\cdot\mathbf{E}\\
(\nu=i)\quad & \frac{\partial g^{i}}{\partial t}-\boldsymbol{\nabla}\cdot\mathbf{T}^{i} \;=\; -\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}\right)
\end{aligned}$$

Movimento de partículas em campos constantes e uniformes

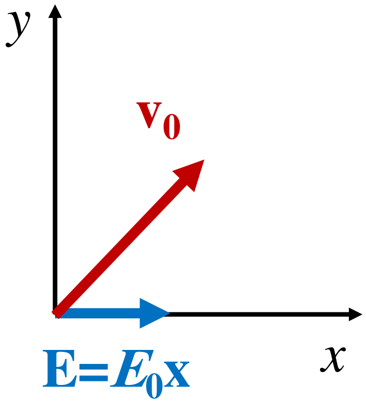
$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

$$\frac{dE}{dt} = e\mathbf{v} \cdot \mathbf{E}$$

$$\mathbf{p} = \gamma_v m \mathbf{v}$$

$$E = \gamma_v m c^2$$

Caso 1: **B=0, E≠0**



Movimento hiperbólico:

$$\frac{v_x(t)}{c} = \frac{eE_0t + p_{0x}}{\sqrt{m^2c^2 + p_{0y}^2 + (eE_0t + p_{0x})^2}} \xrightarrow[t\to\infty]{} 1$$

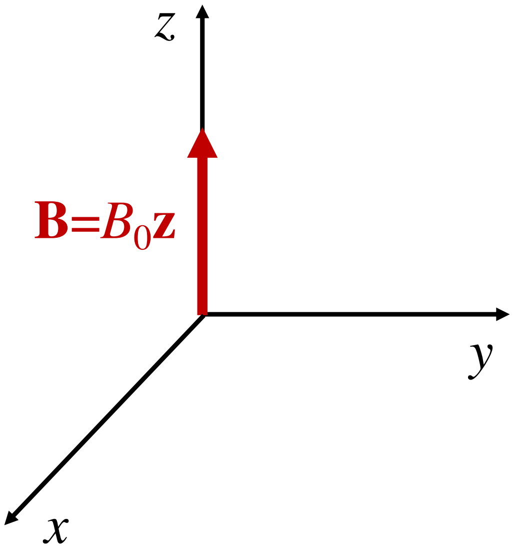
$$\frac{v_y(t)}{c} = \frac{p_{0y}}{\sqrt{m^2c^2 + p_{0y}^2 + (eE_0t + p_{0x})^2}} \xrightarrow[t\to\infty]{} 0$$

$$x(t) = \frac{c}{eE_0}\sqrt{m^2c^2 + p_{0y}^2 + (eE_0t + p_{0x})^2} - X$$

$$y(t) = \frac{cp_{0y}}{eE_0}\sinh^{-1}\left(\frac{eE_0t + p_{0x}}{\sqrt{m^2c^2 + p_{0y}^2}}\right) - Y$$

$$x + X = \frac{\sqrt{m^2c^4 + c^2p_{0y}^2}}{eE_0}\cosh\left[\frac{eE_0(y+Y)}{cp_{0y}}\right]$$

Caso 2: **E=0 , B≠0**



Movimento helicoidal:

$$E = \text{const.} \Rightarrow |\mathbf{v}| = \text{const.}$$

$$\mathbf{v}_{\perp}(t) = v_{0\perp}\left[\cos\left(\omega_B t + \delta\right)\hat{\mathbf{x}} - \sin\left(\omega_B t + \delta\right)\hat{\mathbf{y}}\right]$$

$$\mathbf{v}_{\parallel}(t) = v_{0\parallel}\hat{\mathbf{z}}$$

$$\boxed{\omega_B = \frac{eB}{\gamma_v mc} = \frac{ecB}{E}}$$

$$\mathbf{x}_{\perp}(t) = \frac{v_{0\perp}}{\omega_B}\left[\sin\left(\omega_B t + \delta\right)\hat{\mathbf{x}} + \cos\left(\omega_B t + \delta\right)\hat{\mathbf{y}}\right]$$

$$\mathbf{x}_{\parallel}(t) = \left(v_{0\parallel} t\right)\hat{\mathbf{z}}$$

$$\boxed{a = \frac{v_{0\perp}}{\omega_B} = \frac{\gamma m c v_{0\perp}}{eB} = \frac{c p_{0\perp}}{eB}}$$

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} (\partial_\mu \phi_i)(\partial^\mu \phi_i)$$

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\mathcal{L}_1 = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$$

E-L: $\Rightarrow \Box A^\mu = \frac{4\pi}{c} J^\mu \Rightarrow \Box(\partial_\mu A^\mu) = \frac{4\pi}{c}\partial_\mu J^\mu$

$= 0$

$\Rightarrow \boxed{\partial_\mu A^\mu = 0}$

SE $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ (1)

ENTÃO:

$$\left\{ \begin{array}{l} \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \\ \text{E} \\ \epsilon^{\alpha\beta\mu\nu}\partial_\alpha[\partial_\mu A_\nu - \partial_\nu A_\mu] = 0 \end{array} \right\}$$ EQ.S DE MAXWELL

(2) $\partial_\mu J^\mu = 0$

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = 0 = \mu_0\,\bar{\nabla}\cdot\bar{J} + \mu_0\epsilon_0\frac{\partial(\bar{\nabla}\cdot\bar{E})}{\partial t}$$

$$\frac{\partial}{\partial t}\left[\bar{\nabla}\cdot\bar{E} - \frac{\rho}{\epsilon_0}\right] = 0$$

$$\frac{\partial}{\partial t}\left[\epsilon_0\bar{\nabla}\cdot\bar{E}\right] + \bar{\nabla}\cdot\bar{J} = 0$$

$$\bar{\nabla}\cdot\bar{J} = -\frac{\partial\rho}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\epsilon_0\bar{\nabla}\cdot\bar{E} - \rho\right] = 0 \quad \checkmark$$

INVARIANTES: (1) $F^{\mu\nu}F_{\mu\nu} \propto (E^2 - B^2)$

(2) $F^{\mu\nu}\tilde{F}_{\mu\nu} \propto (\vec{E}\cdot\vec{B})$

$\vec{B} = 0$ NUM REFERENCIAL

(1) $E^2 > 0$
(2) $= 0$

NOUTRO REFERENCIAL
POSSO TER $\vec{B}' \neq 0$, MAS

(1) $E'^2 - B'^2 > 0 \Rightarrow E' > B'$

(2) $\vec{E}'\cdot\vec{B}' = 0 \Rightarrow \vec{E}' \perp \vec{B}'$

ONDAS PLANAS: (1) $= 0$
(2) $= 0$