FI 008 – Eletrodinâmica I

1º Semestre de 2020
25/06/2020
Aula 27

Aulas passadas

Potenciais de Liénard-Wiechert:

$$\Phi(\mathbf{x},t) = \frac{e}{|\mathbf{x}-\mathbf{r}(t_r)|}\frac{1}{1-\frac{\dot{\mathbf{r}}(t_r)\cdot[\mathbf{x}-\mathbf{r}(t_r)]}{c|\mathbf{x}-\mathbf{r}(t_r)|}} = \left.\frac{e}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right|_{t=t_r}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{e/c}{|\mathbf{x}-\mathbf{r}(t_r)|}\frac{\dot{\mathbf{r}}(t_r)}{1-\frac{\dot{\mathbf{r}}(t_r)\cdot[\mathbf{x}-\mathbf{r}(t_r)]}{c|\mathbf{x}-\mathbf{r}(t_r)|}} = \left.\frac{e}{R}\frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right|_{t=t_r}$$

Campos de Liénard-Wiechert:

$$\mathbf{E} = \left.\frac{e}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\left\{\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2R^2}+\frac{1}{c}\frac{\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{R}\right\}\right|_{t=t_r}$$

$$\mathbf{B} = \left.(\hat{\mathbf{n}}\times\mathbf{E})\right|_{t=t_r}$$

Notação:

$$\mathbf{R} = \mathbf{x}-\mathbf{r}(t_r)$$

$$R = |\mathbf{x}-\mathbf{r}(t_r)|$$

$$\hat{\mathbf{n}} = \frac{\mathbf{R}}{R} = \frac{\mathbf{x}-\mathbf{r}(t_r)}{|\mathbf{x}-\mathbf{r}(t_r)|}$$

$$\boldsymbol{\beta} = \frac{\dot{\mathbf{r}}(t_r)}{c}$$

$$\dot{\boldsymbol{\beta}} = \frac{\ddot{\mathbf{r}}(t_r)}{c}$$

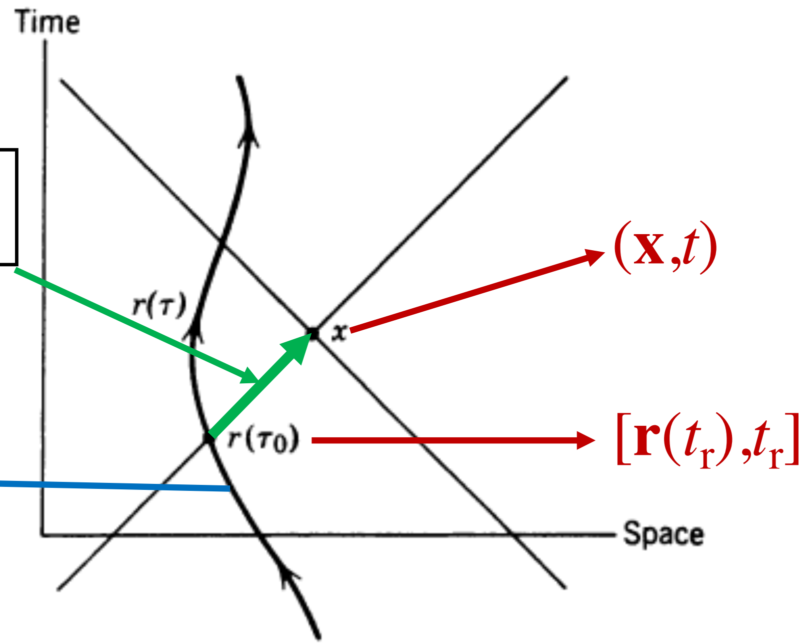
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Aulas passadas

O tempo retardado t_r:

$(\mathbf{x},t)$ está no cone de luz do futuro de $[\mathbf{r}(t_r),t_r]$

$$\boxed{c\left(t-t_r\right)=\left|\mathbf{x}-\mathbf{r}\left(t_r\right)\right|}$$



Trajetória: $[\mathbf{r}(t),t]$

Aula passada

Distribuição angular da potência total irradiada:

$$\frac{dP\left(t_r\right)}{d\Omega} = \frac{e^2}{4\pi c}\frac{1}{\left(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}\right)^5}\left.\left|\hat{\mathbf{n}}\times\left[\left(\hat{\mathbf{n}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\right|^2\right|_{\text{ret}}$$

$$\frac{dP_{\text{não}-\text{relat.}}}{d\Omega} = \frac{e^2}{4\pi c}\left|\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\dot{\boldsymbol{\beta}}\right]\right|^2 = \frac{e^2a^2}{4\pi c^3}\sin^2\theta$$

Potência total irradiada

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2-\left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right] \quad \text{(Fórmula de Liénard)}$$

$$P_{\text{não}-\text{relat.}} = \frac{2}{3}\frac{e^2}{c}\left|\dot{\boldsymbol{\beta}}\right|^2 = \frac{2}{3}\frac{e^2a^2}{c^3} \quad \text{(Fórmula de Larmor)}$$

Perdas em aceleradores lineares

$\vec{\beta} \parallel \dot{\vec{\beta}}$ $\qquad P = \frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2$

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = m(\dot{\gamma} v + \gamma \dot{v})$$

$$\dot{\gamma} = \frac{d}{dt}\left[\frac{1}{\sqrt{1-\beta^2}}\right] = \gamma^3 \beta \dot{\beta} = \frac{1}{c^2}\gamma^3 v \dot{v}$$

$$F = m\left[\frac{\gamma^3}{c^2}\dot{v} v^2 + \gamma \dot{v}\right]$$
$$= m\dot{v}\gamma\left[\gamma^2\frac{v^2}{c^2} + 1\right]$$
$$\simeq m\gamma\dot{v}\left[\gamma^2 + 1\right] = m\gamma^3\dot{v}$$
$$= mc\gamma^3\dot{\beta}$$

$$\Rightarrow \gamma^3\dot{\beta} = \frac{F}{mc} \Rightarrow P = \frac{2}{3}\frac{e^2}{c}\left(\frac{F}{mc}\right)^2$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}F^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$$

USEI: $\frac{dE}{dt} = Fv = F\frac{dx}{dt} \Rightarrow dE = F\,dx \Rightarrow F = \frac{dE}{dx} \Rightarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$

$$\frac{P_{LOSS}}{P_{GAIN}} = \frac{P_{LOSS}}{dE/dt} = \frac{P_{LOSS}}{\underbrace{\frac{dx}{dt}}_{v \cong c}\frac{dE}{dx}} = \frac{P_{LOSS}}{c\,\frac{dE}{dx}}$$

$$= \frac{1}{c\,\cancel{\frac{dE}{dx}}}\,\frac{2}{3}\,\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^{\cancel{2}} = \frac{2}{3}\,\frac{e^2}{\underbrace{m^2c^4}_{\left[2\times10^{14}\,\frac{\text{MeV}}{m}\right]^{-1}}}\left(\frac{dE}{dx}\right)$$

VALORES TÍPICOS EM LINACS DE $\frac{dE}{dx} = 50\,\frac{\text{MeV}}{m}$

$\Rightarrow \frac{P_{LOSS}}{P_{GAIN}} \sim 10^{-13} \ll 1 \Rightarrow$ PERDAS SÃO COMPLETAMENTE DESPREZÍVEIS EM LINACS

Perdas em aceleradores circulares

$\vec{\beta} \perp \dot{\vec{\beta}} \Rightarrow |\vec{\beta} \times \dot{\vec{\beta}}| = \beta\dot{\beta} \Rightarrow P = \frac{2}{3}\frac{e^2}{c}\gamma^6[\dot{\beta}^2 - \beta^2\dot{\beta}^2]$

$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2\underbrace{(1-\beta^2)}_{1/\gamma^2} = \frac{2}{3}\frac{e^2}{c}\gamma^4\dot{\beta}^2$

EM UM MCU: $a = \frac{v^2}{\rho} \cong \frac{c^2}{\rho} = c\dot{\beta} \Rightarrow \dot{\beta} = \frac{c}{\rho}$

$P = \frac{2}{3}\frac{e^2 c}{\rho^2}\gamma^4$

EM ACELERADORES CIRCULARES, O MOVIMENTO CIRCULAR É OBTIDO ATRAVÉS DE CAMPOS MAGNÉTICOS UNIFORMES (DIPOLOS). COMO VIMOS, NESSE CASO:

$\rho = \frac{pc}{eB} \cong \frac{E}{eB} = \frac{\gamma mc^2}{eB} \qquad E = \sqrt{p^2c^2 + m^2c^4} \xrightarrow[\beta\approx 1]{} pc$

$$P = \frac{2}{3}\frac{e^4 B^2}{m^2 c^3}\gamma^2$$

NUMA VOLTA COMPLETA, A ENERGIA TOTAL PERDIDA PARA RADIAÇÃO:

$$\delta E_{Loss} = P \times T \doteq P\,\frac{2\pi\rho}{c} = \frac{4\pi}{3}\frac{e^3 B}{m c^2}\gamma^3$$

PARA UM SÍNCROTRON TÍPICO:

$$\left.\begin{array}{l} E = 10\ \text{GeV} \\ \rho = 100\ \text{m} \\ B = 3.3\ \text{kG} \end{array}\right\} \Rightarrow \delta E_{Loss} \simeq 10\ \text{MeV}$$

ESSA PERDA É COMPARÁVEL COM A ACELERAÇÃO GERADA PELA RADIOFREQUÊNCIA USADA PARA COMPENSAR AS PERDAS.

Distribuição angular da radiação: velocidade e aceleração colineares

$\vec{\beta} \parallel \dot{\vec{\beta}}$:

$$\frac{dP(t_r)}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})|^2}{(1-\vec{\beta}\cdot\hat{n})^5}$$

$\vec{\beta} = \beta\hat{z}$; $\dot{\vec{\beta}} = \dot{\beta}\hat{z}$

$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$\frac{dP(t_r)}{d\Omega} = \frac{e^2\dot{\beta}^2}{4\pi c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

ESSE FATOR FAZ COM QUE A RADIAÇÃO FIQUE MUITO CONCENTRADA EM TORNO DE $\theta \simeq 0$, OU NA DIREÇÃO $\hat{z}$ DE $\vec{\beta}$ e $\dot{\vec{\beta}}$

POSSO EXPANDIR EM TORNO DE $\theta \simeq 0$:

$\sin\theta \simeq \theta$ $\quad \cos\theta \cong 1 - \frac{\theta^2}{2}$ $\quad \beta = \sqrt{1-\frac{1}{\gamma^2}} \cong 1 - \frac{1}{2\gamma^2}$

$$\frac{dP(t_r)}{d\Omega} \cong \frac{e^2 \dot{\beta}^2 8 \gamma^{10}}{\pi c} \frac{\theta^2}{(1+\gamma^2\theta^2)^5}$$

VARIAÇÃO ANGULAR SE DÁ NUMA ESCALA

$$\theta \sim \frac{1}{\gamma}$$

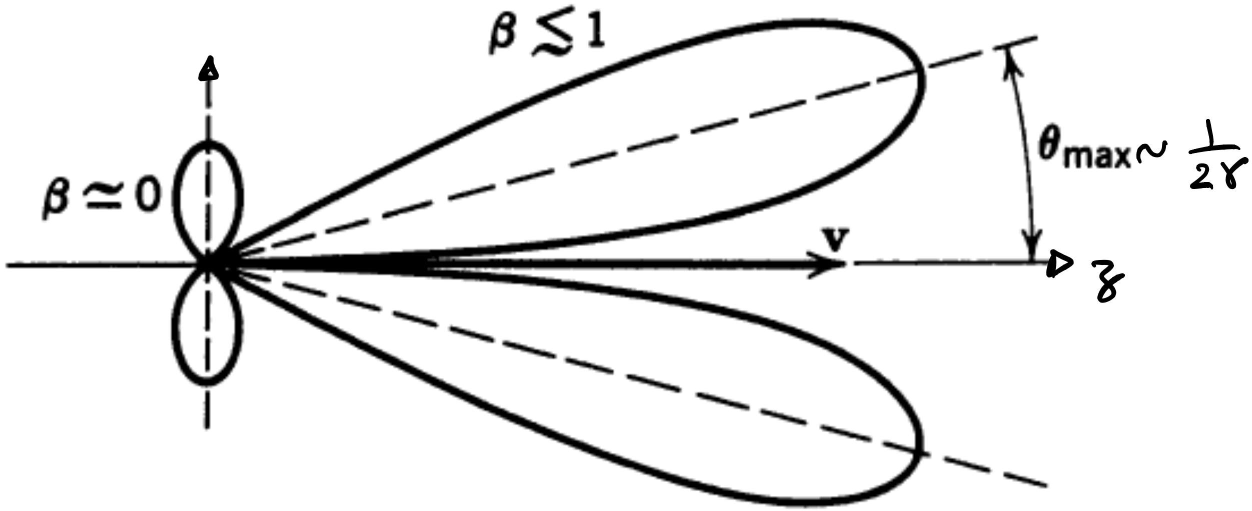
Distribuição angular da radiação para velocidade e aceleração paralelas

$$\frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}$$

$$\gamma \gg 1 \qquad \frac{dP(t')}{d\Omega} \simeq \frac{8}{\pi}\frac{e^2\dot{v}^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{(1 + \gamma^2\theta^2)^5}$$

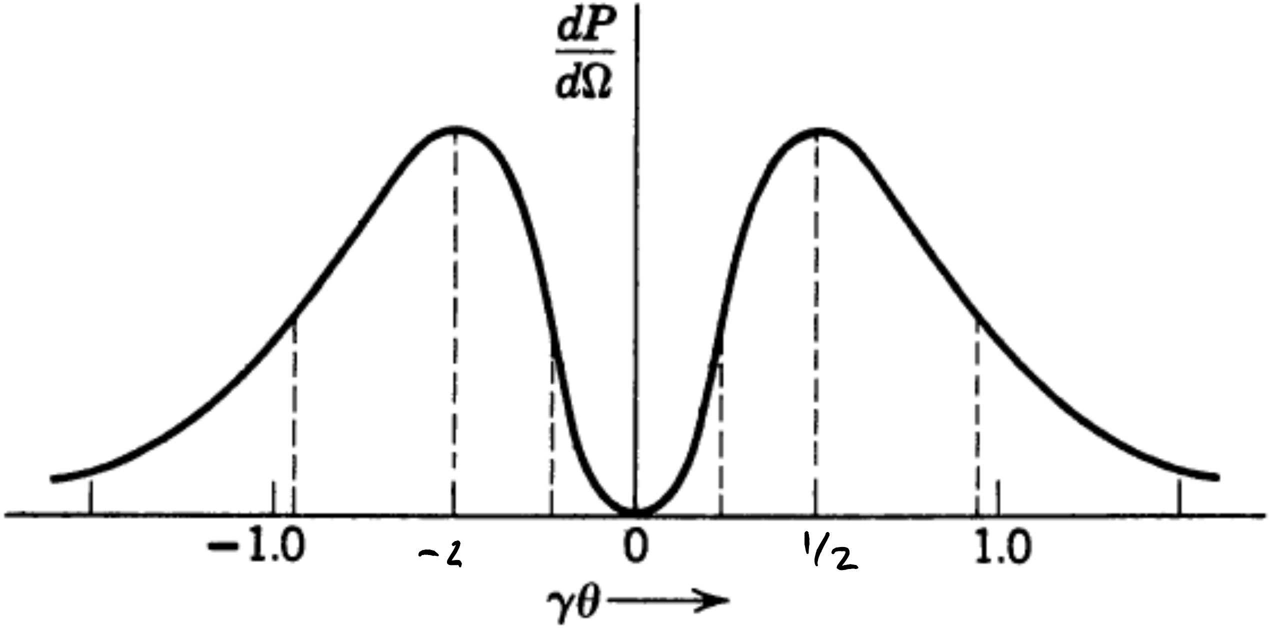
$$\theta_{\mathrm{max}} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1 + 15\beta^2} - 1\right)\right] \to \frac{1}{2\gamma}$$

Distribuição angular da radiação para velocidade e aceleração paralelas



$$\theta_{\max} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1 + 15\beta^2} - 1\right)\right] \rightarrow \frac{1}{2\gamma}$$

Distribuição angular da radiação para velocidade e aceleração paralelas



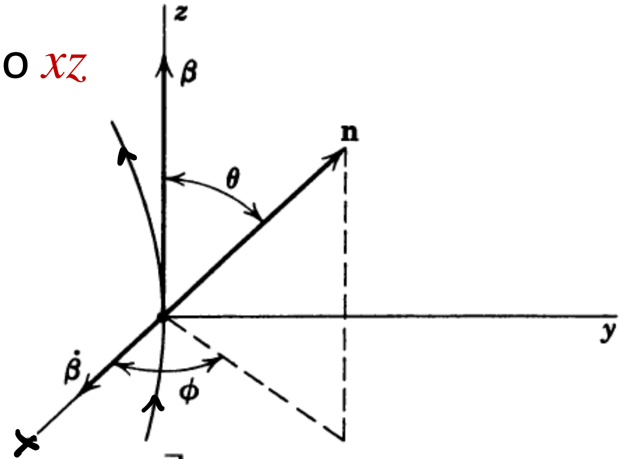
$$\theta_{\max} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] \rightarrow \frac{1}{2\gamma} \qquad \langle\theta^2\rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$

Distribuição angular da radiação: velocidade e aceleração perpendiculares

Tomando o movimento circular no plano *xz*

$$\boldsymbol{\beta} = \beta \hat{\boldsymbol{z}}$$

$$\dot{\boldsymbol{\beta}} = \dot{\beta} \hat{\boldsymbol{x}}$$



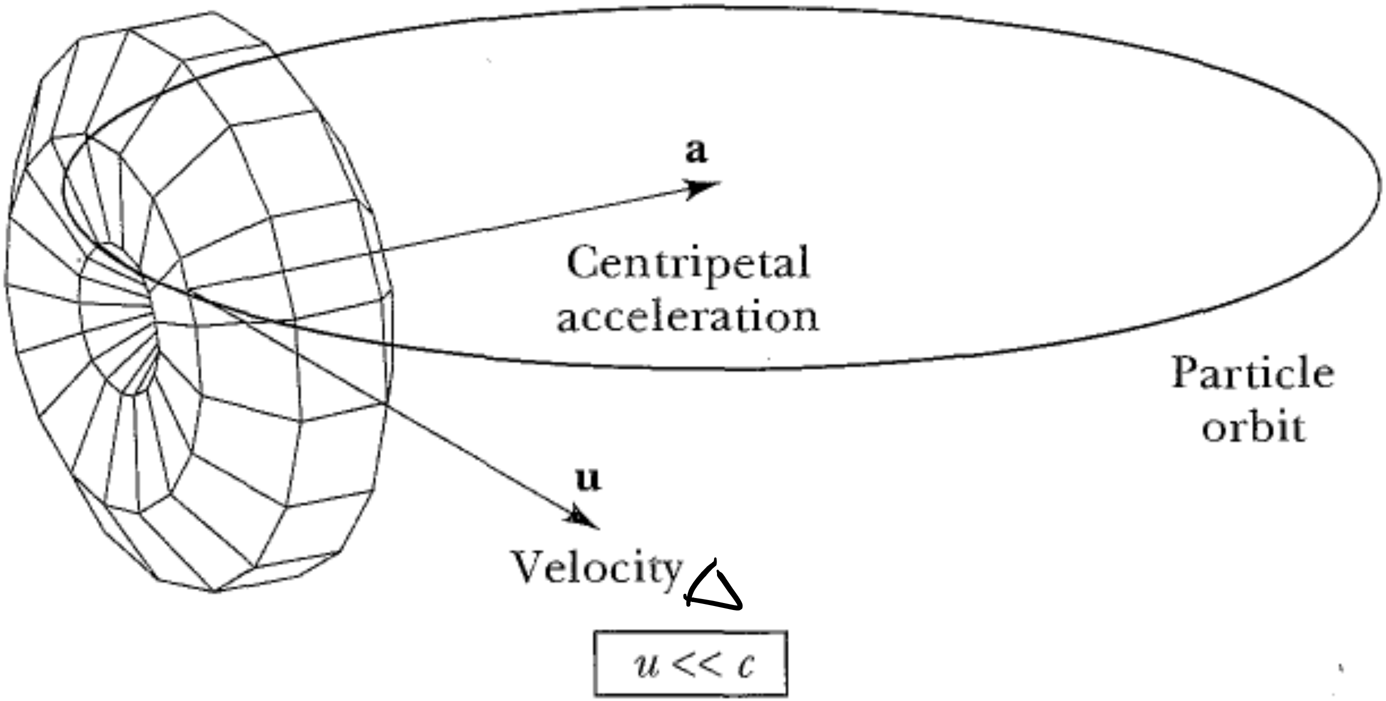
$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1 - \beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

$$\gamma \gg 1 \qquad \frac{dP(t')}{d\Omega} \simeq \frac{2}{\pi}\frac{e^2}{c^3}\gamma^6 \frac{|\dot{\mathbf{v}}|^2}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]$$

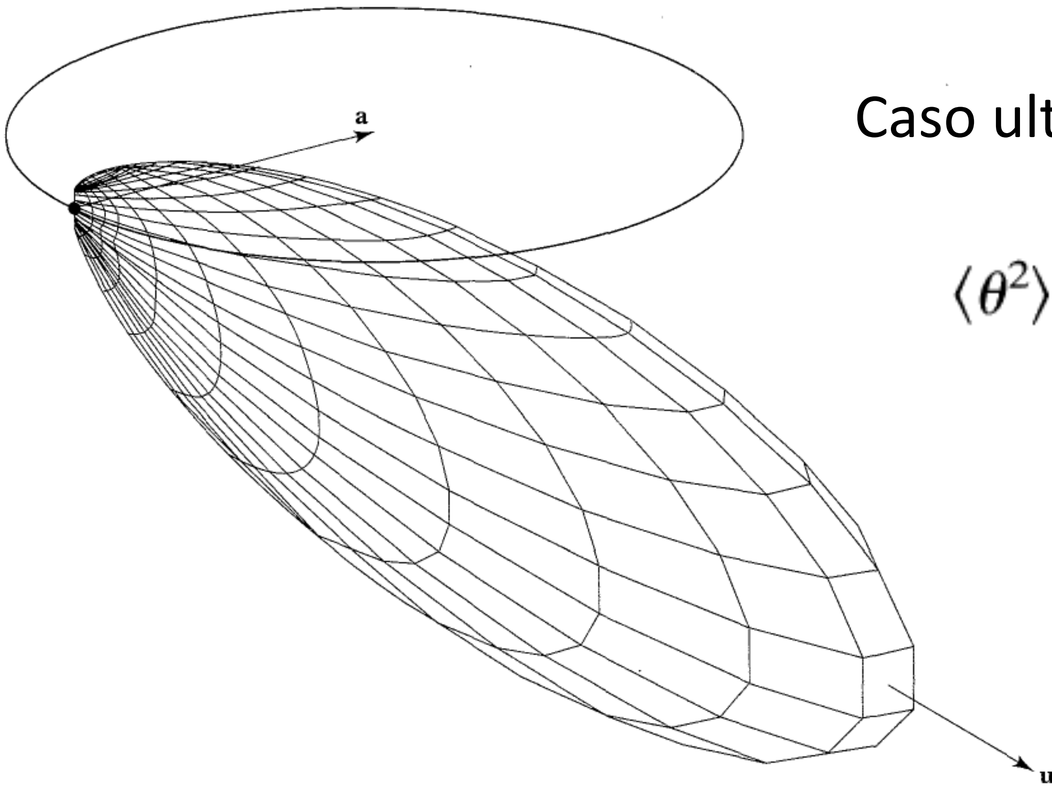
$$\langle\theta^2\rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$

Distribuição angular da radiação: velocidade e aceleração perpendiculares

Caso não-relativístico:



Distribuição angular da radiação: velocidade e aceleração perpendiculares



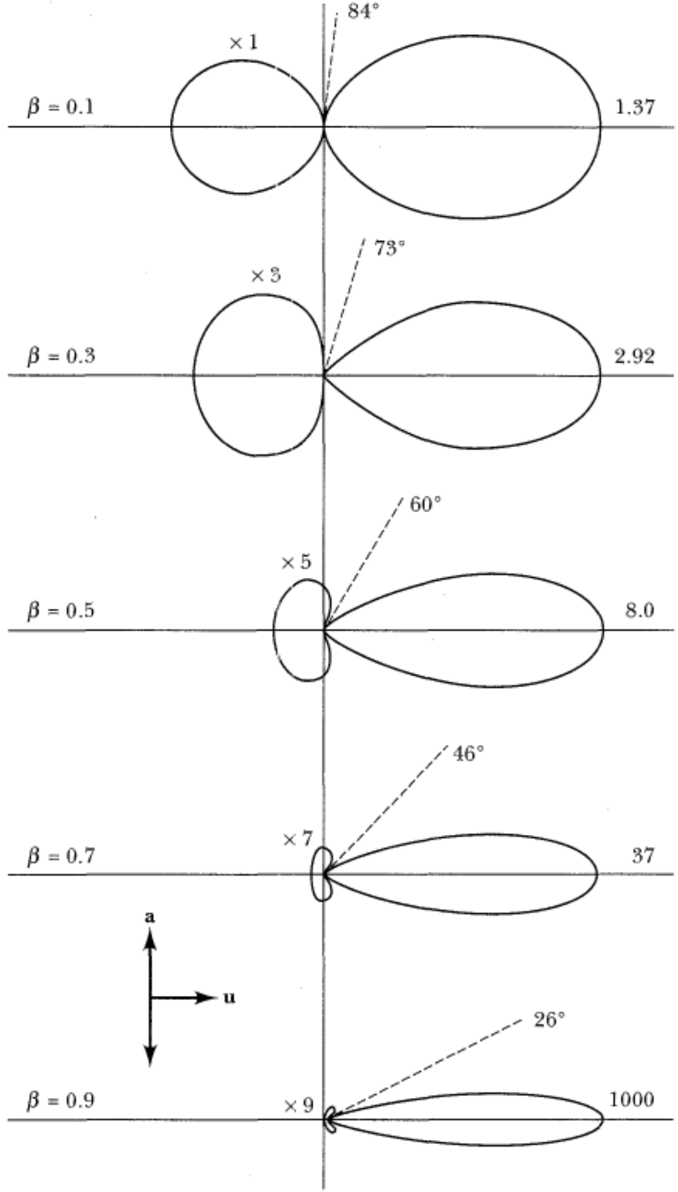
$u \to c$

Caso ultra-relativístico:

$$\langle \theta^2 \rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$

Dependência com a velocidade:

$$\langle \theta^2 \rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$



Distribuição espectral da radiação

$$\frac{dP(t)}{d\Omega} = R^2 \hat{\mathbf{n}} \cdot \mathbf{S} \qquad \mathbf{S} = \frac{c}{4\pi} |\mathbf{E}|^2 \hat{\mathbf{n}}$$

$$\Rightarrow \frac{dP(t)}{d\Omega} = \frac{c}{4\pi} \left| \, |R\vec{E}|^2 \right|_{t_r}$$

ENERGIA TOTAL IRRADIADA É:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} \frac{dP(t)}{d\Omega} dt = \frac{c}{4\pi} \int_{-\infty}^{+\infty} |\underbrace{R\vec{E}}_{\vec{G}(t)}|^2 dt \equiv \frac{c}{4\pi} \int_{-\infty}^{+\infty} |\vec{G}(t)|^2 dt$$

A INTEGRAL SÓ TEM CONTRIBUIÇÕES NUM INTERVALO CURTO DE TEMPO DURANTE O QUAL A PARTÍCULA É ACELERADA.

VAMOS OBSERVAR A RADIAÇÃO DE UM PONTO MUITO DISTANTE DA PARTÍCULA QUE ESTÁ SENDO ACELERAÇÃO

USANDO TRANSFORMADAS DE FOURIER:

$$\vec{G}(\omega) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t}\, \vec{G}(t)$$

$$\vec{G}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{-i\omega t}\, \vec{G}(\omega)$$

TEOREMA DE PARSEVAL: $\displaystyle\int_{-\infty}^{+\infty} dt\, |\vec{G}(t)|^2 = \int_{-\infty}^{+\infty} dt\, \vec{G}^*(t)\cdot\vec{G}(t)$

$$= \int_{-\infty}^{+\infty} dt \left[\int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{+i\omega t}\, \vec{G}^*(\omega)\right]\cdot\left[\int_{-\infty}^{+\infty} \frac{d\omega'}{\sqrt{2\pi}}\, e^{-i\omega' t}\, \vec{G}(\omega')\right] = (*)$$

$$\int_{-\infty}^{+\infty} dt\, e^{it(\omega-\omega')} = 2\pi\,\delta(\omega-\omega')$$

$$(*) = \int_{-\infty}^{+\infty} d\omega\, \vec{G}^*(\omega)\cdot\vec{G}(\omega) = \int_{-\infty}^{+\infty} d\omega\, |\vec{G}(\omega)|^2$$

$$\frac{dW}{d\Omega} = \frac{c}{4\pi}\int_{-\infty}^{+\infty} d\omega\, |\vec{G}(\omega)|^2 = \frac{c}{4\pi}\int_{0}^{\infty} d\omega \left[|\vec{G}(\omega)|^2 + |\vec{G}(-\omega)|^2\right]$$

COMO $\vec{G}(t) \in \mathbb{R}$:

$$\vec{G}(-\omega) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}}\, e^{-i\omega t}\, \vec{G}(t) = \Big[\underbrace{\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t}\, \vec{G}(t)}_{\vec{G}(\omega)} \Big]^*$$

$$\Rightarrow \boxed{\vec{G}(-\omega) = \vec{G}^*(\omega)}$$

$$\Rightarrow \frac{dW}{d\Omega} = \frac{c}{2\pi} \int_0^{\infty} |\vec{G}(\omega)|^2\, d\omega$$

COM ISSO, A ENERGIA IRRADIADA POR ÂNGULO SÓLIDO E INTERVALO DE FREQUÊNCIA ENTRE ω E $\omega + d\omega$:

$$\frac{d^2 I(\omega, \Omega)}{d\omega\, d\Omega} = \frac{c}{2\pi} |\vec{G}(\omega)|^2$$

$$= \frac{c}{2\pi} \left| \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t} \left[R \vec{E}(t) \right] \right|^2$$

$$= \frac{c}{4\pi^2} \left| \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \left[R \vec{E}(t) \right] \right|^2$$

MAS: $R\vec{E}(t) = \frac{e}{c} \left. \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3} \right|_{t_r}$

$$\vec{C} = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \frac{e}{c} \left. \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3} \right|_{t_r}$$

TRANSFORMANDO VARIÁVEIS DE t PARA t_r:

$$t = t_r + \frac{1}{c} R(t_r)$$

$$\frac{dt}{dt_r} = 1 - \vec{\beta}\cdot\hat{n}\Big|_{t_r}$$

$$\vec{C} = \int_{-\infty}^{+\infty} dt_r\, e^{i\omega t_r} e^{i\frac{\omega}{c}R(t_r)} \frac{e}{c} \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^2}$$

APROXIMAÇÕES PARA A REGIÃO DE RADIAÇÃO

$|\vec{x}| >> |\vec{r}(t_r)| \Rightarrow R = |\vec{x}-\vec{r}(t_r)| \cong x - \hat{n}\cdot\vec{r}(t_r)$

$\hat{n} = \frac{\vec{x}-\vec{r}(t_r)}{|\vec{x}-\vec{r}(t_r)|} \cong \frac{\vec{x}}{x}$ QUE NÃO DEPENDE DE t_r

$$\vec{C} = \frac{e}{c}\, e^{i\frac{\omega}{c}x} \int_{-\infty}^{+\infty} dt_r\, e^{i\omega t_r}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)}\, \underbrace{\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\beta}]}{(1-\vec{\beta}\cdot\hat{n})^2}}_{\frac{d}{dt_r}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right]}$$

ONDE $\hat{n}$ NÃO MAIS DEPENDE DE t_r

(VER NOTAS)

$$\vec{C} = \frac{e}{c}\, e^{i\frac{\omega}{c}x} \int_{-\infty}^{+\infty} dt_r\, e^{i\omega t_r}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)}\, \frac{d}{dt_r}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{(1-\vec{\beta}\cdot\hat{n})}\right]$$

INTEGRANDO POR PARTES:

$$\vec{C} = -\frac{e}{c}\, e^{i\frac{\omega}{c}x} \int_{-\infty}^{+\infty} dt_r\, \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\, \underbrace{\frac{d}{dt_r}\left[e^{i\omega t_r}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)}\right]}_{e^{i\omega t_r}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)} \times (+i\omega)(1-\hat{n}\cdot\vec{\beta}(t_r))}$$

$$\vec{C} = -i\omega\frac{e}{c}\, e^{i\frac{\omega}{c}x} \int_{-\infty}^{+\infty} dt_r\, e^{i\omega t_r}\, e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{r}(t_r)}\, \hat{n}\times(\hat{n}\times\vec{\beta}(t_r))$$

Distribuição espectral e angular

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta})\, e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c]} dt\right|^2$$