FI 008 – Eletrodinâmica I

1º Semestre de 2021
30/03/2021
Aula 5

Magnetostática

Aula passada

$$\boldsymbol{\nabla} \cdot \mathrm{B} = 0 \iff \mathrm{B} = \boldsymbol{\nabla} \times \mathbf{A} \quad (1)$$

$$\boldsymbol{\nabla} \times \mathrm{B} = \mu_0 \mathbf{J} \quad (2)$$

Levando (1) na (2):

$$\boldsymbol{\nabla} \times \mathrm{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boxed{\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}}$$

Mas a Eq. (1) só nos dá o rotacional de $\mathbf{A}$.
Seu divergente continua indeterminado.

O teorema de Helmholtz diz que um campo vetorial só fica bem determinado se dermos seu rot e div.

Aula passada

Essa indeterminação de $\mathbf{A}$ fica clara, quando adicionamos a ele o gradiente de uma função escalar Λ qualquer:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda \Rightarrow \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A}' \quad (3)$$

$$\left.\begin{array}{rcl} \mathrm{B} & = & \boldsymbol{\nabla} \times \mathbf{A} \\ \mathrm{B}' & = & \boldsymbol{\nabla} \times \mathbf{A}' \end{array}\right\} \Rightarrow \mathrm{B} = \mathrm{B}'$$

A transformação (3) de $\mathbf{A}$ para $\mathbf{A'}$ é chamada de transformação de calibre (ou "gauge"). A invariância de $\mathbf{B}$ por essa transformação é chamada de invariância de calibre (ou "gauge"). Dizemos que a magnetostática é invariante por transformações de calibre (ou "gauge").

Aula passada

Podemos usar a liberdade de calibre para escolher o divergente de $\mathbf{A}$ igual a zero na equação:

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)-\nabla^2\mathbf{A}=\mu_0\mathbf{J}$$

donde: $\boldsymbol{\nabla}\cdot\mathbf{A}=0\Rightarrow\boxed{\nabla^2\mathbf{A}=-\mu_0\mathbf{J}}$

Cada componente dessa equação é uma eq. de Poisson:

$$\left.\begin{aligned}A_x\left(\mathbf{x}\right)&=\frac{\mu_0}{4\pi}\int_{T.E.}\frac{J_x\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\A_y\left(\mathbf{x}\right)&=\frac{\mu_0}{4\pi}\int_{T.E.}\frac{J_y\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\A_z\left(\mathbf{x}\right)&=\frac{\mu_0}{4\pi}\int_{T.E.}\frac{J_z\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'\end{aligned}\right\}\Rightarrow\boxed{\mathbf{A}\left(\mathbf{x}\right)=\frac{\mu_0}{4\pi}\int_{T.E.}\frac{\mathbf{J}\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x'}$$

Solução geral da magnetostática

Aula passada

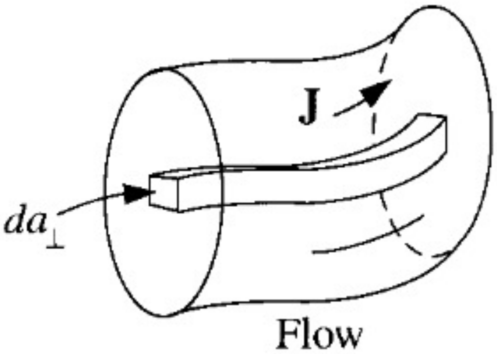
Uma vez obtido $\mathbf{A}$, podemos achar $\mathbf{B}$:

$$\mathbf{B}\left(\mathbf{x}\right) = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\int_{T.\,E.}\frac{\mathbf{J}\left(\mathbf{x}'\right)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3x'$$

$$\mathbf{B}\left(\mathbf{x}\right) = \frac{\mu_0}{4\pi}\int_{T.\,E.}\mathbf{J}\left(\mathbf{x}'\right)\times\frac{\left(\mathbf{x}-\mathbf{x}'\right)}{\left|\mathbf{x}-\mathbf{x}'\right|^3}d^3x'$$

(Lei de Biot-Savart)

Elementos de corrente



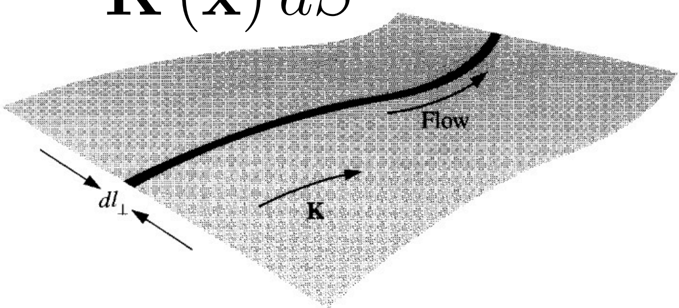
$$\mathbf{J}(\mathbf{x})\,d^3x$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.} \frac{\mathbf{J}(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.} \left[\mathbf{J}(\mathbf{x}')\,d^3x'\right] \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Correntes superficiais:

$$\mathbf{K}(\mathbf{x})\,dS$$

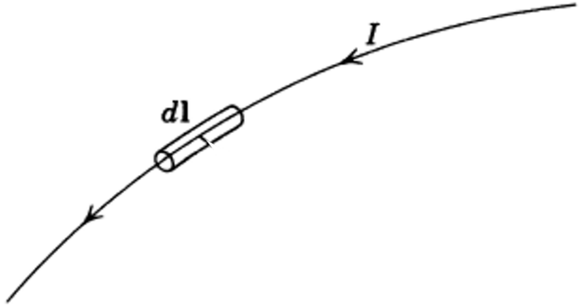


$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.} \frac{\mathbf{K}(\mathbf{x}')\,dS'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.} \left[\mathbf{K}(\mathbf{x}')\,dS'\right] \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Elementos de corrente

Correntes lineares (fios):



$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_{T.E.} \frac{d\mathbf{l}'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_{T.E.} d\mathbf{l}' \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\boxed{\mathbf{J}(\mathbf{x})\, d^3x \sim \mathbf{K}(\mathbf{x})\, dS \sim I d\mathbf{l}}$$

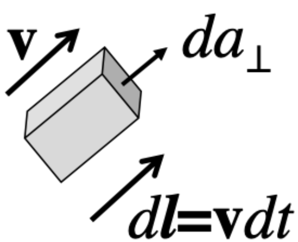
Força de Lorentz sobre correntes

Força magnética sobre elementos de carga:

$$\left.\begin{aligned} \mathbf{F} &= q\mathbf{v}\times\mathbf{B} \\ dq\mathbf{v} &= \rho(\mathbf{x})\,\mathbf{v}\,d^3x \end{aligned}\right\} \Rightarrow d\mathbf{F} = dq\mathbf{v}\times\mathbf{B} = \underbrace{\rho(\mathbf{x})\,\mathbf{v}}_{\vec{J}}\times\mathbf{B}d^3x$$

Suponha que um elemento de área ($da_\perp$) é normal à velocidade local da corrente. No intervalo dt, ele varre um volume dV:

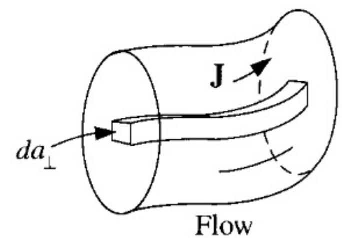
$$dV = dl\, da_\perp = v\, dt\, da_\perp$$



$$\rho\, dV = \rho\, v\, dt\, da_\perp = dq$$

$$J\, da_\perp = I = \frac{dq}{dt}$$

$$J = \frac{dq}{da_\perp dt} = \frac{\rho\, v\, \cancel{dt}\, \cancel{da_\perp}}{\cancel{da_\perp}\, \cancel{dt}} \Rightarrow J = \rho\, v \Rightarrow \boxed{\vec{J} = \rho\,\vec{v}}$$

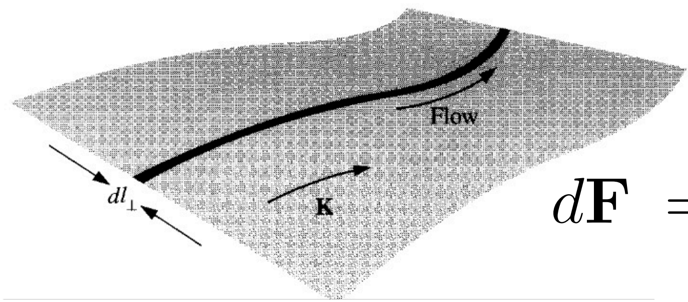


$$d\vec{F} = \vec{J}\times\vec{B}\; d^3x$$

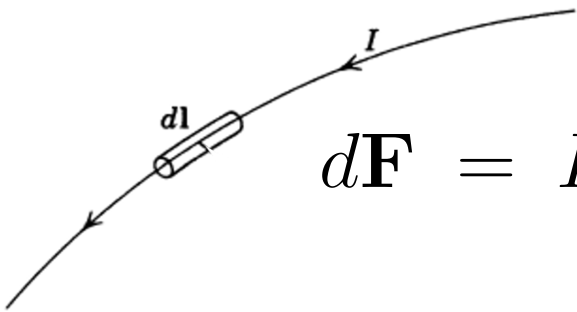
Força de Lorentz sobre correntes

$$\boxed{d\mathbf{F} = \mathbf{J} \times \mathbf{B} d^3x}$$

$$\boxed{\mathbf{J}(\mathbf{x})\, d^3x \sim \mathbf{K}(\mathbf{x})\, dS \sim I d\mathbf{l}}$$



$$d\mathbf{F} = \mathbf{K} \times \mathbf{B} dS$$



$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

Força total: integrar

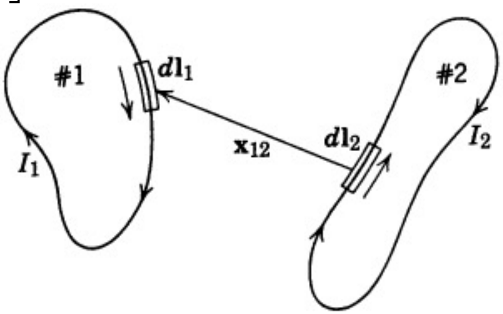
$$\vec{F} = \int d\vec{F} = \int_{T.E.} \vec{J} \times \vec{B}\, d^3x$$

$$\vec{F} = \int_{T.E} \vec{K} \times \vec{B}\, dS$$

$$\vec{F} = \int_{T.E} I\, d\vec{l} \times \vec{B}$$

Força entre 2 circuitos de corrente

$$\mathbf{F}_{2\to 1} = \int_{C_1} I_1 d\mathbf{l}_1 \times \mathbf{B}_2(\mathbf{x}_1) = \frac{\mu_0 I_1 I_2}{4\pi} \int_{C_1} d\mathbf{l}_1 \times \left[\int_{C_2} \frac{d\mathbf{l}_2 \times (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right]$$



$$d\vec{\ell}_1 \times [d\vec{\ell}_2 \times \underbrace{(\vec{x}_1 - \vec{x}_2)}_{\vec{x}_{12}}] =$$

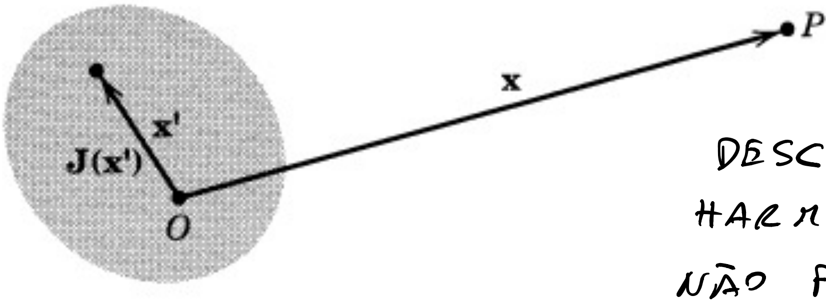
$$= (d\vec{\ell}_1 \cdot \vec{x}_{12})\, d\vec{\ell}_2 - (d\vec{\ell}_1 \cdot d\vec{\ell}_2)\, \vec{x}_{12}$$

NAS NOTAS EU PROVO QUE A CONTRIBUIÇÃO DO 1º TERMO SE ANULA.

$$\vec{F}_{2\to 1} = -\frac{\mu_0 I_1 I_2}{4\pi} \iint_{C_1 C_2} \frac{(d\vec{\ell}_1 \cdot d\vec{\ell}_2)(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} = -\vec{F}_{1\to 2}$$

3ª LEI DE NEWTON

Expansão multipolar



$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.\,E.}\frac{\mathbf{J}(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

DESCRIÇÃO COMPLETA REQUER HARMÔNICOS VETORIAIS, QUE NÃO FAREMOS.

$$\frac{1}{|\vec{x}-\vec{x}'|} \xrightarrow{|\vec{x}'|\ll|\vec{x}|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + O\left(\frac{r'^2}{r^4}\right)$$

$$\vec{A}(\vec{x}) \cong \frac{\mu_0}{4\pi}\left[\underbrace{\frac{1}{r}\int_{T.E.}\vec{J}(\vec{x}')\,d^3x'}_{1^{\underline{o}}\ \text{TERM}} + \underbrace{\frac{1}{r^3}\int_{T.E.}(\vec{x}\cdot\vec{x}')\,\vec{J}(\vec{x}')\,d^3x'}_{2^{\underline{o}}\ \text{TERMO}}\right]$$

SE $\vec{J}\,d^3x' = I\,d\vec{\ell}'$

$$\Rightarrow I\oint_{C_i} d\vec{\ell} = 0$$

Expansão multipolar

$$\nabla' \cdot (x_i' \mathbf{J}) = x_i' \nabla' \cdot \mathbf{J} + J_i \;\; (i = 1, 2, 3)$$

$x_i' = x', y', z'$

$\nabla'(x_i') = \hat{x}_i'$ $\qquad \hat{x}_i' \cdot \vec{J} = J_i$

INTEGRO EM TODO O ESPAÇO:

$$\int_{T.E.} \vec{\nabla}' \cdot (x_i' \vec{J})\, d^3x' = \int_{S_\infty} (x_i' \vec{J}) \cdot \hat{n}_i'\, ds' = 0$$

CORRENTE LOCALIZADA

$$0 = \int_{T.E.} x_i' (\cancelto{0}{\vec{\nabla}' \cdot \vec{J}})\, d^3x' + \int_{T.E.} J_i\, d^3x' \Rightarrow \int_{T.E.} \vec{J}\, d^3x' = 0$$

CORRENTE ESTACIONÁRIA

TERMO DE MONOPOLO MAGNÉTICO É IDENTICAMENTE NULO.

Expansão multipolar

SOMA IMPLÍCITA SOBRE $\underline{i}$

$$x_i \int_{T.E.} x_i' \mathbf{J}(\mathbf{x}')\, d^3x' = -\frac{1}{2}\left[\mathbf{x} \times \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')\, d^3x'\right]$$ (ver dedução nas notas)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left(\frac{-1}{r^3}\right)\vec{x} \times \underbrace{\left[\frac{1}{2}\int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x'\right]}_{\text{MOMENTO DE DIPOLO MAGNÉTICO DA DISTRIBUIÇÃO } \vec{J}(\vec{x}) \Rightarrow \vec{m}}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{x}}{r^3}$$

COMPARE COM $\Phi(\vec{x}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}\cdot\vec{x}}{r^3}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

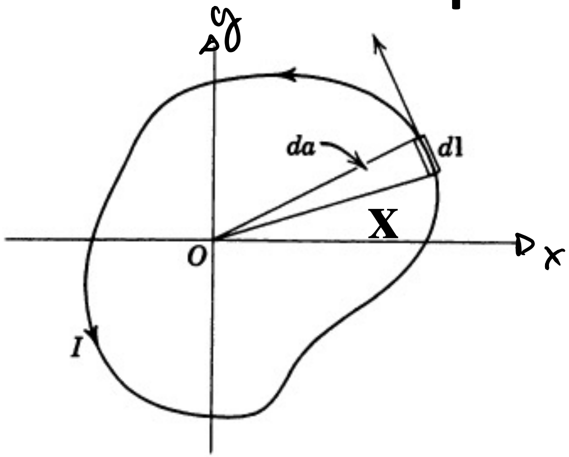
Expansão multipolar

Campos de dipolo magnético:

$$\mathbf{A}\left(\mathbf{x}\right) \approx \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{r^3},$$

$$\mathbf{B}\left(\mathbf{x}\right) \approx \frac{\mu_0}{4\pi}\left[\frac{3\left(\mathbf{m}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\mathbf{m}}{r^3}\right],\ \hat{\mathbf{n}}=\hat{\mathbf{r}}=\frac{\mathbf{x}}{r}.$$

Dipolo magnético de um circuito plano de corrente

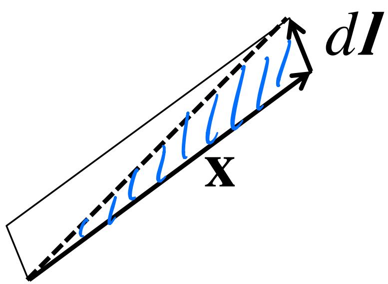


$$\mathbf{m} = \frac{I}{2}\oint \mathbf{x} \times d\mathbf{l}$$

$\vec{x} \times d\vec{\ell} \parallel \hat{z}$

$|\vec{x} \times d\vec{\ell}|$ = ÁREA DO PARALELOGRAMO DEFINIDO NA FIGURA AO LADO

= 2 ÁREA DO TRIÂNGULO HACHURADO



$\vec{m} = I\hat{z}\int dA = IA\hat{z}$

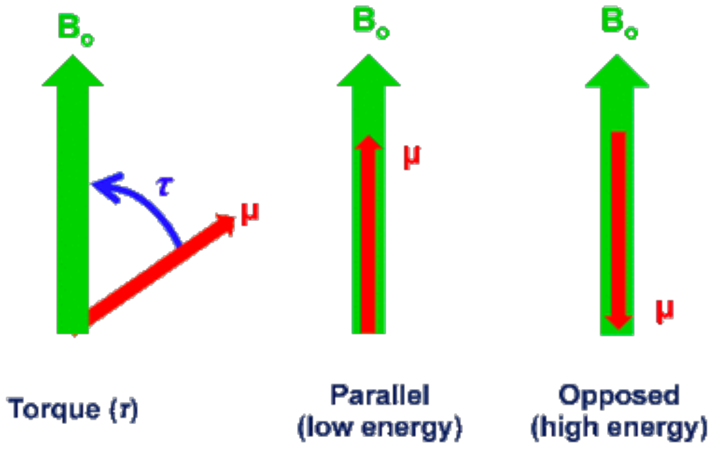
A = ÁREA CUJA BORDA É O CIRCUITO

$$\boxed{\vec{m} = IA\hat{z}}$$

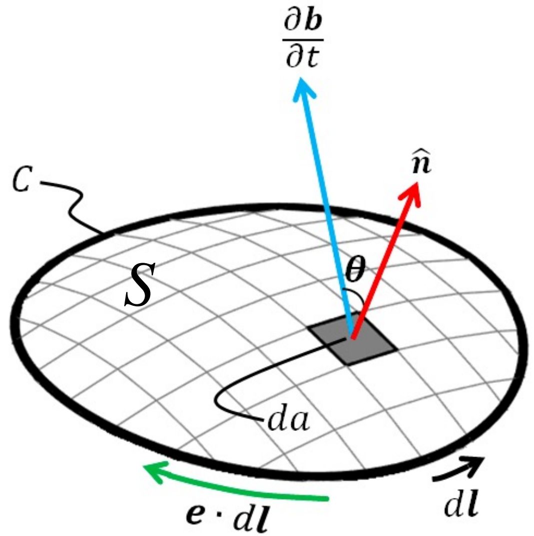
Energia de (e torque sobre) um dipolo magnético num campo externo

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_{\text{ext}}$$

$$W = -\mathbf{m} \cdot \mathbf{B}_{\text{ext}}$$



Lei de indução de Faraday



$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

FORMA INTEGRAL:

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n}\, dS = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot \hat{n}\, dS$$

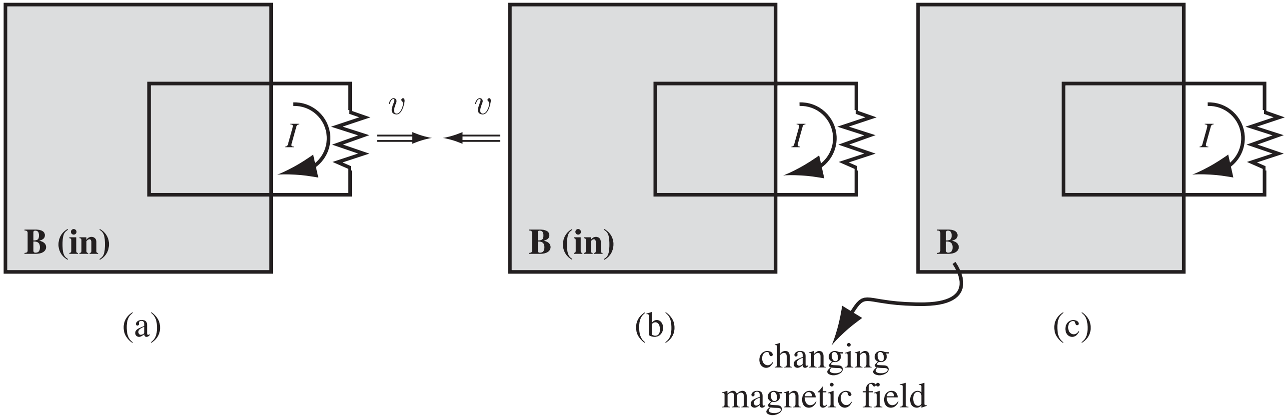
TEOREMA DE STOKES:

$$\mathcal{E} = \oint_{C(S)} \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \underbrace{\left[\int_S \vec{B} \cdot \hat{n}\, dS \right]}_{\Phi_B(S)} = -\frac{d\, \Phi_B(S)}{dt}$$

LEI DO FLUXO

$\mathcal{E}$ = FORÇA ELETROMOTRIZ

Lei do fluxo



(a) O circuito <u>não</u> está fixo no espaço. A força eletromotriz é consequência da força magnética (Lorentz) sobre as cargas.

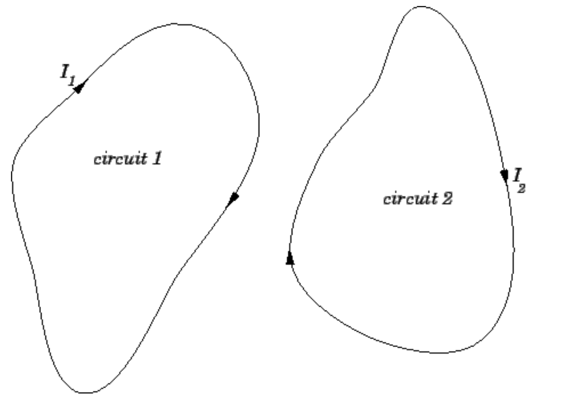
(b) e (c) O circuito está parado. A força eletromotriz vem da lei de indução de Faraday.

(a), (b) e (c) A lei do fluxo é sempre válida.

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$

Energia magnética

Trabalho realizado adiabaticamente contra as forças ~~externas~~ ELETROMAGNÉTICAS para estabelecer uma configuração de correntes.



$$\frac{d(\delta W)}{dt} = d\vec{F}_{ext}\cdot\vec{v} = -dq\,(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v} \qquad (\vec{v}\times\vec{B})\cdot\vec{v} = 0$$

$$= -dq\,\vec{E}\cdot\vec{v}$$

$$= -\rho\,\vec{E}\cdot\vec{v}\,d^3x = -\vec{J}\cdot\vec{E}\,d^3x$$

$$\Rightarrow \frac{dW}{dt} = -\int \underbrace{\vec{J}\cdot\vec{E}}_{T.E.}\,d^3x = -\frac{1}{\mu_0}\int(\vec{\nabla}\times\vec{B})\cdot\vec{E}\,d^3x$$

$$\frac{dW}{dt} = -\frac{1}{\mu_0}\int_{T.E.} (\nabla\times\vec{B})\cdot\vec{E}\, d^3x \qquad \nabla\cdot(\mathbf{E}\times\mathbf{B}) = (\nabla\times\mathbf{E})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{E}$$

$$= \frac{1}{\mu_0}\left[\int_{T.E.} \vec{\nabla}\cdot(\vec{E}\times\vec{B})\, d^3x - \int_{T.E.} \vec{B}\cdot(\nabla\times\vec{E})\, d^3x\right] = (*)$$

T. GAUSS

$$\oint_{S_\infty} (\vec{E}\times\vec{B})\cdot\hat{n}\, dS \to 0$$: PARA DISTRIBUIÇÕES LOCALIZADAS DE CARGAS E CORRENTES:

$$\vec{E} \sim \frac{1}{r^2} \qquad \vec{B} \sim \frac{1}{r^2}$$

$$\vec{E}\times\vec{B} \sim \frac{1}{r^4} \qquad \int dS \sim R^2$$

$$(*) = +\frac{1}{\mu_0}\int_{T.E.} \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\, d^3x = \frac{1}{\mu_0}\int_{T.E.} \frac{\partial}{\partial t}\left[\frac{\vec{B}\cdot\vec{B}}{2}\right] d^3x =$$

$$\frac{dW}{dt} = \frac{1}{2\mu_0}\frac{d}{dt}\left[\int_{T.E.} B^2\, d^3x\right] \Rightarrow \boxed{W = \int_{T.E.} \frac{B^2}{2\mu_0}\, d^3x}$$

COMPARE COM $W = \frac{\epsilon_0}{2}\int_{T.E.} E^2\, d^3x$

$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B}$ [redacted] $(\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{A}$

$$W = \frac{1}{2\mu_0}\int_{T.E.} \vec{B}\cdot\vec{B}\, d^3x$$

$$= \frac{1}{2\mu_0}\int_{T.E.}\left[(\vec{\nabla}\times\vec{A})\cdot\vec{B}\right] d^3x = \frac{1}{2\mu_0}\left[\underbrace{\int_{T.E.}\vec{\nabla}\cdot(\vec{A}\times\vec{B})\, d^3x}_{\oint_{S_\infty}(\vec{A}\times\vec{B})\cdot\hat{n}\, dS \to 0} + \int_{T.E.}\vec{A}\cdot(\vec{\nabla}\times\vec{B})\, d^3x\right]$$

$\vec{A}\sim\frac{1}{r^2}$ $\qquad \vec{A}\times\vec{B}\sim\frac{1}{r^5}$

$\vec{B}\sim\frac{1}{r^3}$

$$W = \frac{1}{2\mu_0}\int_{T.E.}\vec{A}\cdot\underbrace{(\vec{\nabla}\times\vec{B})}_{\mu_0\vec{J}}\, d^3x = \frac{1}{2}\int_{T.E.}\vec{A}\cdot\vec{J}\, d^3x$$

COMPARE COM: $W = \frac{1}{2}\int_{T.E.}\rho\,\Phi\, d^3x$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.} \frac{\mathbf{J}(\mathbf{x}')\, d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

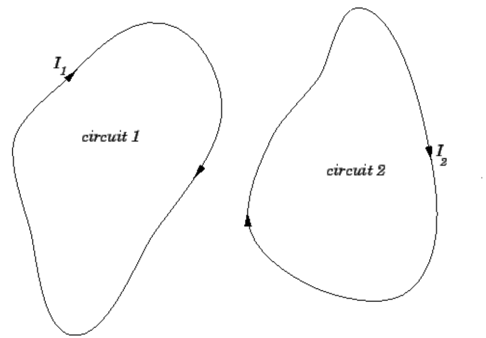
$$W = \frac{1}{2}\int \vec{A}(\bar{x})\cdot\vec{J}(\bar{x})\, d^3x$$

$$\boxed{W = \frac{\mu_0}{8\pi}\int \frac{\bar{J}(\bar{x})\cdot\vec{J}(\bar{x}')}{|\bar{x}-\bar{x}'|}\, d^3x\, d^3x'}$$

COMPARE CON:

$$W = \frac{1}{8\pi\epsilon_0}\int \frac{\rho(\bar{x})\,\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\, d^3x\, d^3x'$$

Indutâncias



$$\boxed{W=\frac{\mu_0}{8\pi}\int_{T.E.}\frac{\mathbf{J}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}}$$

N circuitos de correntes: $\mathbf{J}(\mathbf{x})=\sum_{i=1}^{N}\mathbf{J}_i(\mathbf{x})$

$$W=\left[\frac{\mu_0}{8\pi}\sum_{i=1}^{N}\int_{T.E.}\frac{\mathbf{J}_i(\mathbf{x})\cdot\mathbf{J}_i(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}+\frac{\mu_0}{8\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int_{T.E.}\frac{\mathbf{J}_i(\mathbf{x})\cdot\mathbf{J}_j(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}\right]$$

Como $J\sim I$:

$$\begin{aligned}M_{ij}&=\frac{\mu_0}{4\pi}\frac{1}{I_iI_j}\int_{T.E.}\frac{\mathbf{J}_i(\mathbf{x})\cdot\mathbf{J}_j(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}=M_{ji},\\ L_i&=\frac{\mu_0}{4\pi}\frac{1}{I_i^2}\int_{T.E.}\frac{\mathbf{J}_i(\mathbf{x})\cdot\mathbf{J}_i(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned}$$

Indutâncias mútuas

Auto-indutâncias

Puramente geométricas

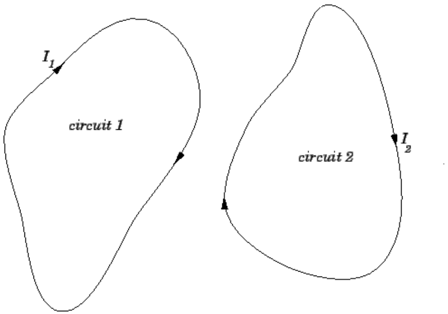
$$W=\left[\frac{1}{2}\sum_{i=1}^{N}L_iI_i^2+\sum_{i,j=1,i<j}^{N}M_{ij}I_iI_j\right]$$

$$M_{ij} = \frac{\mu_0}{4\pi}\frac{1}{I_i I_j}\int_{T.E.}\frac{\mathbf{J}_i(\mathbf{x})\cdot\mathbf{J}_j(\mathbf{x}')\,d^3x\,d^3x'}{|\mathbf{x}-\mathbf{x}'|},$$

Separando o potencial vetor devido ao circuito *j:*

$$\mathbf{A}_j(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{T.E.}\frac{\mathbf{J}_j(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$M_{ij} = \frac{1}{I_i I_j}\int_{T.E.}\mathbf{J}_i(\mathbf{x})\cdot\mathbf{A}_j(\mathbf{x})\,d^3x \qquad (\mathbf{J}_i d^3x \to I_i d\mathbf{l}_i)$$



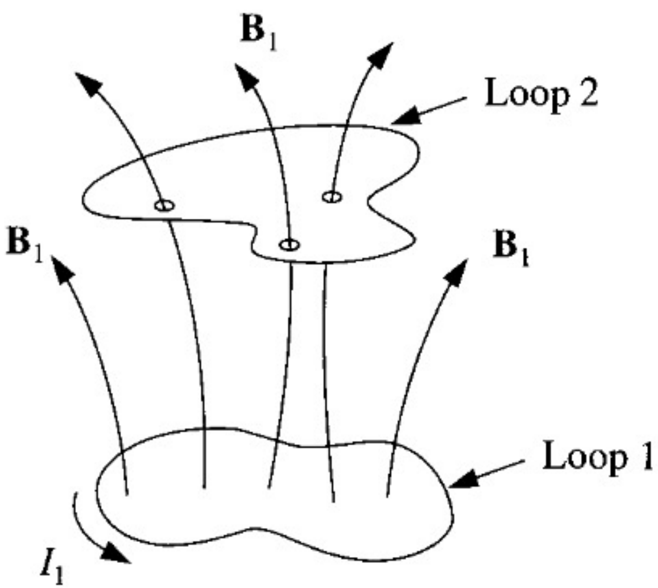
$$= \frac{1}{\cancel{I_i} I_j}\int \cancel{I_i}\, d\vec{\ell}_i \cdot \vec{A}_j(\vec{x})$$

$$= \frac{1}{I_j}\int_{C_i}\vec{A}_j \cdot d\vec{\ell}_i = \frac{1}{I_j}\int_{S(C_i)}\overbrace{(\vec{\nabla}\times\vec{A}_j)}^{\vec{B}_j}\cdot d\vec{S}_i$$

$$M_{ij} = \frac{1}{I_j}\Phi_B(j\to i) = \frac{1}{I_i}\Phi_B(i\to j)$$

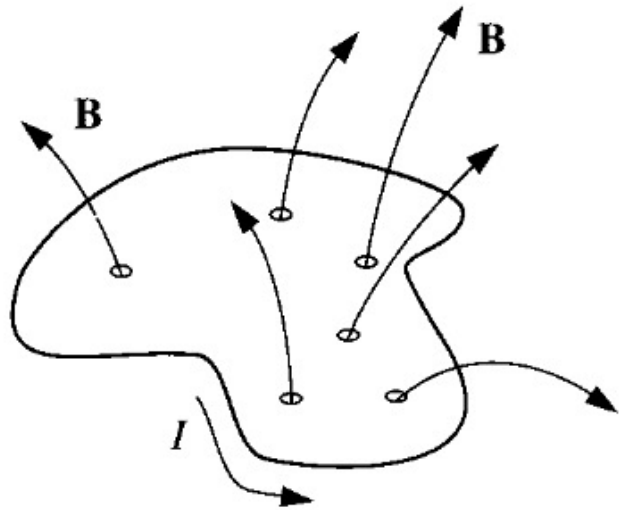
$$L_i = \frac{1}{I_i}\Phi_B(i\to i)$$

Indutância mútua



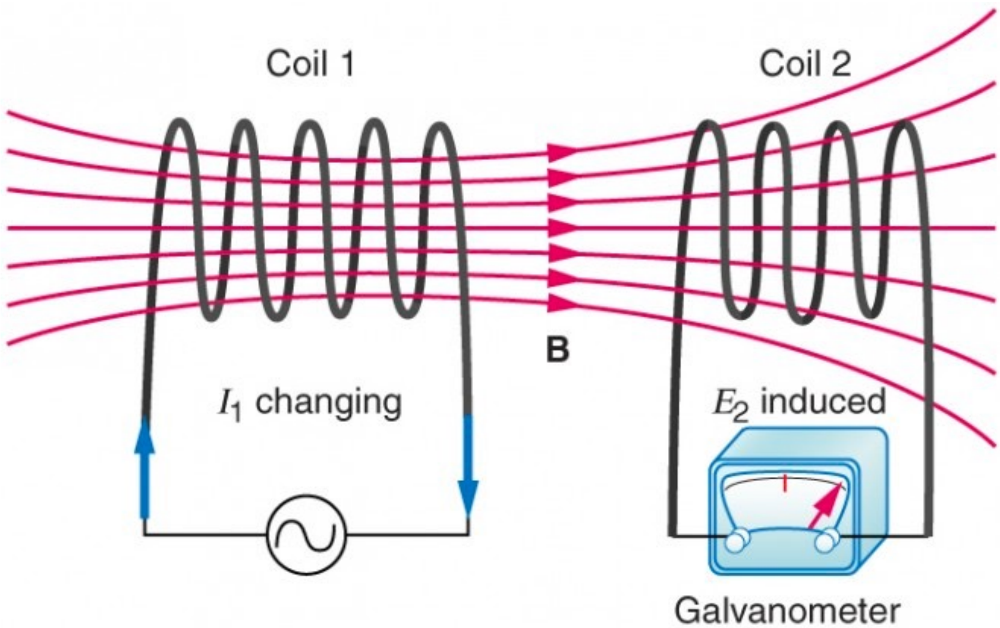
$$\Phi_B\,(j \to i) \;=\; M_{ij} I_j$$

Auto-indutância



$$\Phi_B\,(i \to i) \;=\; L_i I_i$$

Indutância



$$\Phi_B(1 \to 2) = M_{12} I_1$$

$$\Rightarrow \underbrace{\frac{d\Phi_B}{dt}(1 \to 2)}_{-\mathcal{E}_2} = M_{12} \frac{dI_1}{dt}$$

$$\Rightarrow \boxed{\mathcal{E}_2 = -M_{12} \frac{dI_1}{dt}}$$

$$\boxed{\mathcal{E}_1 = -L_1 \frac{dI_1}{dt}}$$