FI 008 – Eletrodinâmica I

1º Semestre de 2021
08/04/2021
Aula 7

Aula passada

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}; \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}; \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0; \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

Usamos as equações sem fontes para definir os potenciais:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \iff \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A};$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \mathbf{A}}{\partial t} \iff \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \iff \mathbf{E} = -\boldsymbol{\nabla} \Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Aula passada

Levando nas equações com fontes obtemos: $\mu_0 \epsilon_0 = \frac{1}{c^2}$

$$\nabla^2 \Phi + \frac{\partial \left(\boldsymbol{\nabla} \cdot \mathbf{A} \right)}{\partial t} = -\frac{\rho}{\epsilon_0};$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla} \left[\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right] = -\mu_0 \mathbf{J}.$$

Mas existe enorme liberdade na definição dos potenciais:
invariância dos campos elétrico e magnético por transf. de calibre

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} \Lambda$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

Aula passada

Podemos impor a "condição de Lorenz":

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

Os potenciais então satisfazem a equação de onda não homogênea:

$$\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\epsilon_0}; \\ \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\mu_0 \boldsymbol{J}. \end{aligned}$$

Aula passada

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -f(\mathbf{x}, t)$$

Método das funções de Green:

$$\nabla^2 G(\mathbf{x}-\mathbf{x}', t-t') - \frac{1}{c^2}\frac{\partial^2 G(\mathbf{x}-\mathbf{x}', t-t')}{\partial t^2} = -\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$

$$G(\mathbf{x}-\mathbf{x}', t-t') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left[t' - (t - |\mathbf{x}-\mathbf{x}'|/c)\right]$$

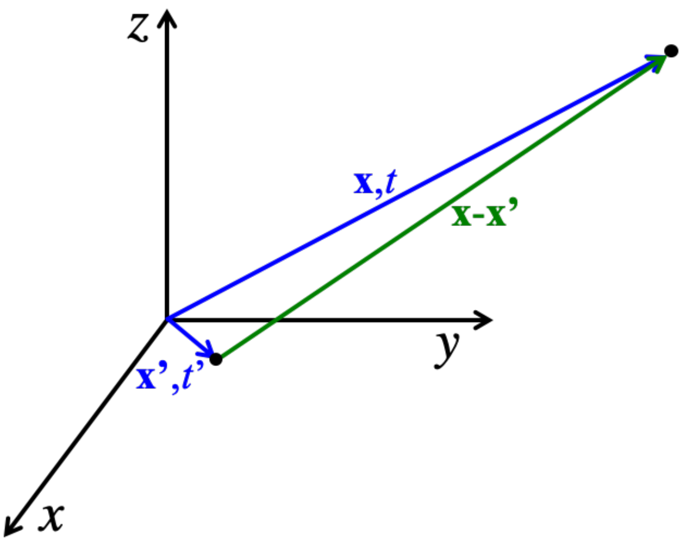
$$\begin{aligned}
\psi(\mathbf{x}, t) &= \int G(\mathbf{x}-\mathbf{x}', t-t')\, f(\mathbf{x}', t')\, d^3x' dt' \\
&= \int \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left[t' - (t - |\mathbf{x}-\mathbf{x}'|/c)\right] f(\mathbf{x}', t')\, d^3x' dt' \\
&= \int \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)\, d^3x'
\end{aligned}$$

Aula passada

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int_{T.E.}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho\left(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c\right)d^3x'$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int_{T.E.}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c\right)d^3x'$$

$$t_{ret} = t - |\mathbf{x}-\mathbf{x}'|/c$$



$$t - t' = |\mathbf{x}-\mathbf{x}'|/c$$

Leis de conservação

Taxa de realização de trabalho dos campos sobre a matéria:

$$\begin{aligned}\frac{dW_{C\to M}}{dt} &= \sum_i q_i\left[\mathbf{E}(\mathbf{x}_i)+\mathbf{v}_i\times\mathbf{B}(\mathbf{x}_i)\right]\cdot\mathbf{v}_i = \sum_i q_i\mathbf{E}(\mathbf{x}_i)\cdot\mathbf{v}_i \\ &= \int_V \mathbf{E}(\mathbf{x})\cdot\underbrace{\left[\rho(\mathbf{x})\,\mathbf{v}\right]}_{\mathbf{J}(\mathbf{x})} d^3x = \int_V \mathbf{E}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x})\,d^3x \equiv \int_V \frac{\partial u_{\mathrm{mec}}(\mathbf{x})}{\partial t}d^3x\end{aligned}$$

Utilizando as eqs. de Maxwell:

$$-\int_V \mathbf{E}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x})\,d^3x = \int_V\left[\frac{\partial}{\partial t}\underbrace{\left(\frac{\epsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2\right)}_{u}+\nabla\cdot\underbrace{\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)}_{\mathbf{S}}\right]d^3x$$

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J} = Q_E = -\frac{\partial u_{mec}}{\partial t}$$

$$\frac{\partial}{\partial t}\left(u+u_{\mathrm{mec}}\right)+\nabla\cdot\mathbf{S}=0$$

$$\boxed{\begin{aligned} u &= \frac{\epsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2 \\ \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\end{aligned}}$$

Leis de conservação

Conservação de energia:

$$\frac{\partial u}{\partial t} + \frac{\partial u_{mec}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0$$

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Gaussiano:

$$u = \frac{1}{8\pi} \left(E^2 + B^2 \right)$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

Leis de conservação

Taxa de variação do momento linear da matéria devido à atuação dos campos:

$$\frac{d\mathbf{P}_M}{dt} = \sum_i q_i \left[\mathbf{E}\left(\mathbf{x}_i\right) + \mathbf{v}_i \times \mathbf{B}\left(\mathbf{x}_i\right)\right] = \int_V \left[\rho\left(\mathbf{x}\right)\mathbf{E}\left(\mathbf{x}\right) + \rho\left(\mathbf{x}\right)\mathbf{v} \times \mathbf{B}\left(\mathbf{x}\right)\right] d^3x$$

$$= \int_V \left[\rho\left(\mathbf{x}\right)\mathbf{E}\left(\mathbf{x}\right) + \mathbf{J}\left(\mathbf{x}\right) \times \mathbf{B}\left(\mathbf{x}\right)\right] d^3x \equiv \int_V \frac{\partial \mathbf{p}_{\text{mec}}\left(\mathbf{x}\right)}{\partial t} d^3x$$

Utilizando as eqs. de Maxwell:

$$-\int_V \left[\rho\left(\mathbf{x}\right)\mathbf{E}\left(\mathbf{x}\right) + \mathbf{J}\left(\mathbf{x}\right) \times \mathbf{B}\left(\mathbf{x}\right)\right] d^3x = \int_V \left[\frac{\partial \mathbf{p}}{\partial t} - \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}}\right] d^3x$$

$\overleftrightarrow{T} \sim \overline{\overline{T}}$

$$\frac{\partial \mathbf{p}}{\partial t} - \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}} = -\left(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}\right) = \mathbf{Q}$$

$-\frac{\partial \vec{p}_{mec}}{\partial t}$

$$\frac{\partial}{\partial t}\left(\mathbf{p} + \mathbf{p}_{\text{mec}}\right) - \boldsymbol{\nabla} \cdot \overleftrightarrow{\mathbf{T}} = 0$$

$$\mathbf{p} = \epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$T_{ij} = T_{ji}$

Leis de conservação

Conservação de momento linear: $$\frac{\partial}{\partial t}\left(\mathbf{p}+\mathbf{p}_{\mathrm{mec}}\right)-\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}} \;=\; 0$$

$$\mathbf{p} \;=\; \epsilon_0 \mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

$$T_{ij} \;=\; \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

Gaussiano:

$$\mathbf{p} \;=\; \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\mathbf{S}$$

$$T_{ij} \;=\; \frac{1}{4\pi}\left[\left(E_i E_j + B_i B_j\right) - \frac{1}{2}\delta_{ij}\left(E^2+B^2\right)\right]$$

O que significa esse divergente de um tensor?

$$\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\mathbf{T}}\right)_i = \sum_j \frac{\partial T_{ij}}{\partial x_j}$$

$$[T_{ij}] = \frac{P}{L^2 T} = \frac{F}{L^2} = \text{PRESSÃO}$$

$$P = \text{MOMENTO LINEAR} = M\frac{L}{T}$$

$$\frac{P}{T} = \text{FORÇA}$$

$$\vec{p} = \frac{\vec{S}}{c^2}$$

$$p = \frac{S}{c^2}$$

$$S = \frac{E}{T L^2}$$

$$J = \rho v$$

$$S = uv = cu$$

$$p = \frac{u}{c}$$

$\downarrow$

$$\frac{\text{MOMENTO}}{\cancel{V}} = \frac{\text{ENERGIA}}{\cancel{V}\, c}$$

$$\text{ENERGIA} = c\ \text{MOMENTO}$$

$$\cancel{h}\nu = c\left(\frac{\cancel{h}}{\lambda}\right) \Rightarrow \boxed{\lambda\nu = c}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_F$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

SE $\vec{J}_F = 0 \Rightarrow \vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla} \Phi_M$

$\vec{A} = ?$