# FI 008 – Eletrodinâmica I

1º Semestre de 2021

15/04/2021

Aula 9

# Monopolos magnéticos

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_m$$
$$\nabla \cdot \mathbf{B} = \rho_m$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot \vec{J}_m = 0$$

Mudança de unidades: forma mais simétrica

$$c\mathbf{B} \to \mathbf{B}$$
$$\frac{\rho}{\sqrt{\epsilon_0}} \to \rho$$
$$\frac{\rho_m}{\sqrt{\mu_0}} \to \rho_m$$
$$\frac{\mathbf{J}}{\sqrt{\epsilon_0}} \to \mathbf{J}$$
$$\frac{\mathbf{J}_m}{\sqrt{\mu_0}} \to \mathbf{J}_m$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\sqrt{\epsilon_0}} \rho$$
$$\nabla \cdot \mathbf{B} = \frac{1}{\sqrt{\epsilon_0}} \rho_m$$
$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} - \sqrt{\mu_0} \mathbf{J}_m$$
$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \sqrt{\mu_0} \mathbf{J}$$

# Transformações de dualidade

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

$$\begin{pmatrix} \rho' \\ \rho'_m \end{pmatrix} = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \begin{pmatrix} \rho \\ \rho_m \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{J}' \\ \mathbf{J}'_m \end{pmatrix} = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} \begin{pmatrix} \mathbf{J} \\ \mathbf{J}_m \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{1}{\sqrt{\epsilon_0}}\rho \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= \frac{1}{\sqrt{\epsilon_0}}\rho_m \\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} - \sqrt{\mu_0}\mathbf{J}_m \\ \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \sqrt{\mu_0}\mathbf{J} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E}' &= \frac{1}{\sqrt{\epsilon_0}}\rho' \\ \boldsymbol{\nabla}\cdot\mathbf{B}' &= \frac{1}{\sqrt{\epsilon_0}}\rho'_m \\ \boldsymbol{\nabla}\times\mathbf{E}' &= -\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t} - \sqrt{\mu_0}\mathbf{J}'_m \\ \boldsymbol{\nabla}\times\mathbf{B}' &= \frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t} + \sqrt{\mu_0}\mathbf{J}' \end{aligned}$$

Covariância por transformação de dualidade

# As quantidades quadráticas relevantes são invariantes por transf. dualidade

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) = \frac{\epsilon_0}{2}\left(E^2 + c^2 B^2\right) \to \frac{\epsilon_0}{2}\left(E^2 + B^2\right)$$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}\times(c\mathbf{B}) \to \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}\times\mathbf{B}$$

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}\left(E^2 + c^2 B^2\right)\right] \to \epsilon_0\left[E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}\left(E^2 + B^2\right)\right]$$

# Razão carga elétrica/magnética constante

Se <u>todas</u> as partículas tiverem a <u>mesma</u> razão carga elétrica/magnética

$$q_m = rq \Rightarrow \rho_m = r\rho,\ \mathbf{J}_m = r\mathbf{J}$$

então é possível fazer uma transformação de dualidade tal que:

$$\Rightarrow \rho'_m = \mathbf{J}'_m = 0$$

| Nesse caso, monopolos magnéticos não são observáveis. |
|---|

Deve-notar também que as eqs. de Maxwell no vácuo (ondas) são covariantes por transformações de dualidade.

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= 0 \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{aligned}$$

$$\begin{pmatrix}\mathbf{E}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}$$

# Condição de quantização de Dirac

## Covariância de calibre na mecânica quântica

Equação de Schrödinger na presença de campos eletromagnéticos:

$$\frac{1}{2m}\left[\frac{\hbar}{i}\boldsymbol{\nabla} - q\mathbf{A}(\mathbf{x},t)\right]^2 \psi(\mathbf{x},t) + \left[V(\mathbf{x}) + q\Phi(\mathbf{x},t)\right]\psi(\mathbf{x},t) = i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}$$

Covariância de calibre:

$$\mathbf{A}(\mathbf{x},t) \;\rightarrow\; \mathbf{A}'(\mathbf{x},t) = \mathbf{A}(\mathbf{x},t) + \boldsymbol{\nabla}\Lambda(\mathbf{x},t)$$

$$\Phi(\mathbf{x},t) \;\rightarrow\; \Phi'(\mathbf{x},t) = \Phi(\mathbf{x},t) - \frac{\partial\Lambda(\mathbf{x},t)}{\partial t}$$

$$\psi(\mathbf{x},t) \;\rightarrow\; \psi'(\mathbf{x},t) = e^{iq\Lambda(\mathbf{x},t)/\hbar}\psi(\mathbf{x},t)$$

$(\vec{p} - q\vec{A})$ INVARIANTE POR CALIBRE

$\langle\vec{p}\rangle$?

# Dinâmica quântica de uma partícula carregada na presença de um monopolo magnético

Um monopolo magnético na origem:

$$\nabla \cdot \mathbf{B} = g\delta^{(3)}(\mathbf{x})$$
$$\mathbf{B}(\mathbf{x}) = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}$$

Não é possível definir um potencial vetor em todo o espaço, mas:

$$\mathbf{A}(\mathbf{x}) = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \Rightarrow \nabla \times \mathbf{A}(\mathbf{x}) = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} \qquad \boxed{\mathbf{A}(\mathbf{x}) \to \infty \text{ se } \theta \to \pi}$$

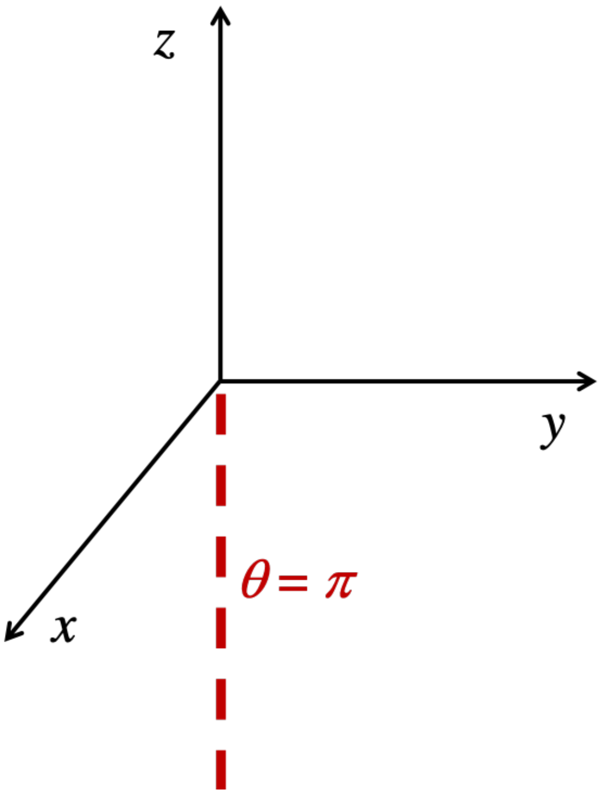
$$\mathbf{A}'(\mathbf{x}) = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \Rightarrow \nabla \times \mathbf{A}'(\mathbf{x}) = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} \qquad \boxed{\mathbf{A}'(\mathbf{x}) \to \infty \text{ se } \theta \to 0}$$

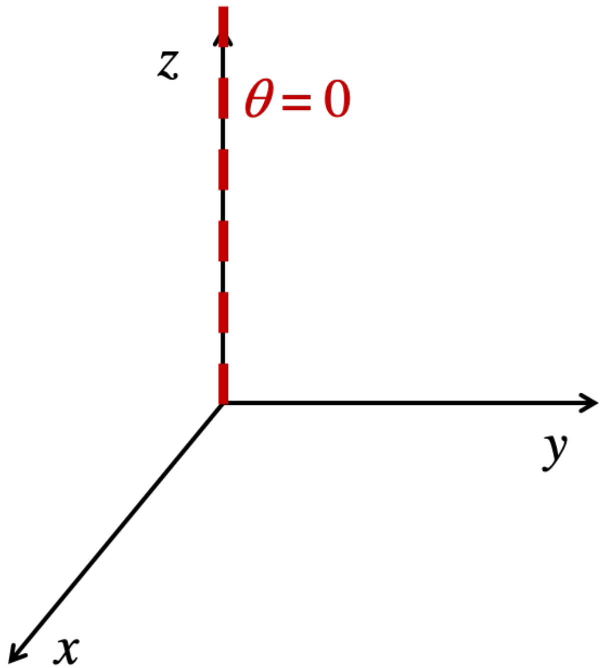
$$\mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \frac{g}{2\pi r}\frac{1}{\sin\theta}\hat{\boldsymbol{\phi}} = \mathbf{A}(\mathbf{x}) + \nabla\Lambda(\mathbf{x}) \qquad \boxed{\Lambda(\mathbf{x}) = -\frac{g\phi}{2\pi}}$$

# Duas possíveis cordas de Dirac

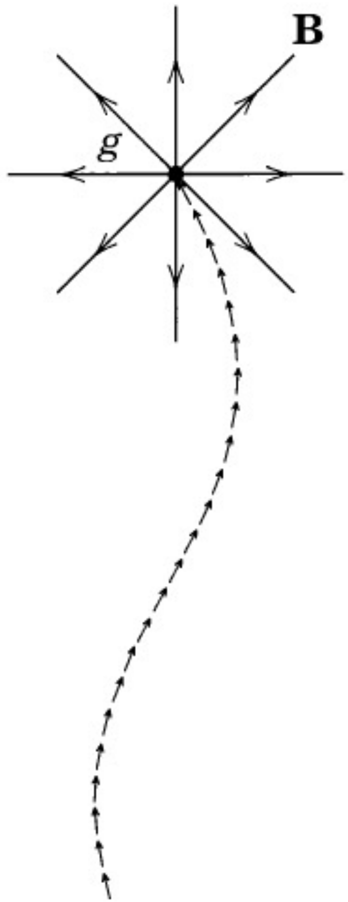
$$\mathbf{A}(\mathbf{x}) = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \qquad \mathbf{A}(\mathbf{x}) \to \infty \text{ se } \theta \to \pi$$

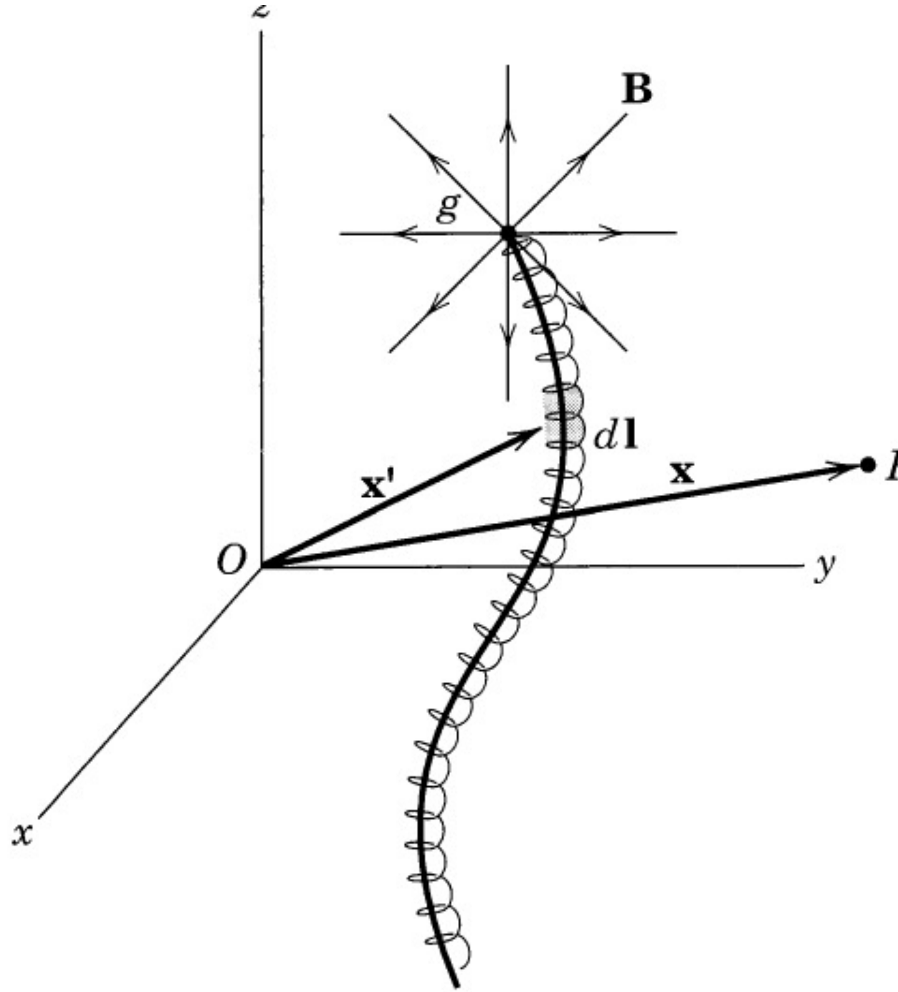
$$\mathbf{A}'(\mathbf{x}) = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}} \qquad \mathbf{A}'(\mathbf{x}) \to \infty \text{ se } \theta \to 0$$



# De maneira geral….



# Mudança de calibre entre cordas de Dirac

$$\mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \frac{g}{2\pi r}\frac{1}{\sin\theta}\hat{\phi} = \mathbf{A}(\mathbf{x}) + \nabla\Lambda(\mathbf{x}) \qquad \Lambda(\mathbf{x}) = -\frac{g\phi}{2\pi}$$

Partícula de carga *e* na presença do monopolo:

$$\boxed{\psi'(\mathbf{x},t) = e^{-ieg\phi/h}\psi(\mathbf{x},t)}$$

Mas a função de onda tem de satisfazer, no plano *xy* ($\theta = \pi/2$):

$$\begin{aligned}
\psi\left(r,\theta=\frac{\pi}{2},\phi,t\right) &= \psi\left(r,\theta=\frac{\pi}{2},\phi+2\pi,t\right)\\
\psi'\left(r,\theta=\frac{\pi}{2},\phi,t\right) &= \psi'\left(r,\theta=\frac{\pi}{2},\phi+2\pi,t\right)\\
e^{-ieg\phi/h}\psi\left(r,\theta=\frac{\pi}{2},\phi,t\right) &= e^{-ieg\phi/h}e^{-i2\pi eg/h}\psi\left(r,\theta=\frac{\pi}{2},\phi+2\pi,t\right)
\end{aligned}$$

$$\boxed{e^{-i2\pi eg/h} = 1 \Rightarrow 2\pi eg/h = 2\pi n \Rightarrow e = \frac{hn}{g},\quad n = 1,2,3,\ldots}$$

0, −1, −2, …

A existência de um único monopolo magnético explicaria a quantização da carga elétrica!

$$\vec{A} = \vec{B} \times \vec{C} \Rightarrow A_i = \epsilon^{ijk} B_j C_k$$

$$\epsilon^{ijk} \epsilon^{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\epsilon^{ijk} T_{jk} = 0$$

$$\int_V \nabla_j \left( A_{jklm} \right) d^3x = \int_{S(V)} A_{jklm} n_j \, dS$$