Equilíbrio e Estabilidade de um sistema de equações

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

$x_{n+1} = x_n = \bar{x}$ e $y_{n+1} = y_n = \bar{y}$

$$\bar{x} = f(\bar{x}, \bar{y})$$
$$\bar{y} = g(\bar{x}, \bar{y})$$

Modelo de Nicholson-Bailey

Hospedeiros - parasitóides

$\phi(H_n, P_n)$ = fração de hospedeiros ñ parasitados

$1-\phi(H_n, P_n)$ = " " " parasitados

$$\begin{cases} H_{n+1} = \lambda \phi(H_n, P_n) H_n \\ P_{n+1} = c\left(1 - \phi(H_n, P_n)\right) H_n \end{cases}$$

$$\phi(H_n, P_n) = e^{-aP_n}$$

$$\begin{cases} H_{n+1} = \lambda e^{-aP_n} H_n \\ P_{n+1} = c\left(1 - e^{-aP_n}\right) H_n \end{cases}$$

$$P_{n+1} = P_n = \bar{P}$$

$$H_{n+1} = H_n = \bar{H}$$

$$\begin{cases} \bar{H} = \lambda \bar{H} e^{-a\bar{P}} \\ \bar{P} = c\left(1 - e^{-a\bar{P}}\right) \bar{H} \end{cases}$$

$\bar{H} - \lambda \bar{H} e^{-a\bar{P}} = 0$

$(\bar{H}, \bar{P})$

$\bar{H} \left(1 - \lambda e^{-a\bar{P}}\right) = 0$

$\bar{H} = 0$ ou $1 - \lambda e^{-a\bar{P}} = 0$

$\lambda e^{-a\bar{P}} = 1$

$e^{-a\bar{P}} = \frac{1}{\lambda} = \lambda^{-1}$

$\ln e^{-a\bar{P}} = \ln \lambda^{-1}$

$+a\bar{P} = +\ln \lambda$

$\bar{P} = \frac{1}{a} \ln \lambda$

$\bar{P} = c\underline{\underline{\bar{H}}}\left(1 - e^{-a\bar{P}}\right)$

$\bar{P} = 0$

$(\bar{H}_0, \bar{P}_0) = (0, 0)$

$\bar{P} = c\left(1 - e^{-a\bar{P}}\right)\bar{H}$

$\frac{1}{a} \ln \lambda = c\left(1 - e^{-a\frac{1}{a}\ln\lambda}\right)\bar{H}$

$\frac{1}{a} \ln \lambda = c\left(1 - e^{-\ln\lambda}\right)\bar{H}$

$\frac{1}{a} \ln \lambda = c\left(1 - e^{\ln\lambda^{-1}}\right)\bar{H}$

$\frac{1}{a} \ln \lambda = c\left(1 - \lambda^{-1}\right)\bar{H}$

$\frac{1}{ac} \frac{\ln \lambda}{(1 - \lambda^{-1})} = \bar{H}$

$$\bar{H} = \frac{1}{ac} \ln\lambda \frac{1}{1-\lambda^{-1}} = \frac{1}{ac} \ln\lambda \left(\frac{\lambda}{\lambda-1}\right)$$

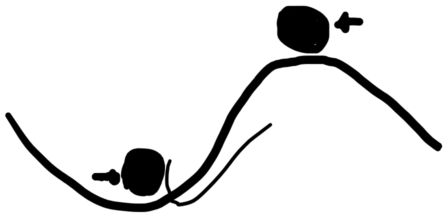
$$\downarrow$$

$$\frac{1}{1-\frac{1}{\lambda}}$$

$$\frac{1}{\frac{\lambda-1}{\lambda}} = \frac{\lambda}{\lambda-1}$$

$$(\bar{H}_1, \bar{P}_1) = \left(\frac{1}{ac}\frac{\lambda}{\lambda-1}\ln\lambda, \frac{1}{a}\ln\lambda\right)$$

Estabilidade ?



$$x_{n+1} = f(x_n)$$

$$\left|\frac{df}{dx}\Big|_{\bar{x}}\right| < 1 \text{ estável}$$

$$> 1 \text{ instável}$$

$$f(H_n, P_n)$$
$$g(H_n, P_n)$$

$$\begin{cases} x_0 = \underline{\bar{x} + \delta x_0} \\ y_0 = \underline{\bar{y} + \delta y_0} \end{cases}$$

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

$$\bar{x} = f(\bar{x}, \bar{y})$$
$$\bar{y} = g(\bar{x}, \bar{y})$$

$$x_1 = f(x_0, y_0) = f(\bar{x} + \delta x_0, \bar{y} + \delta y_0) = \underline{\bar{x} + \delta x_1}$$

$$y_1 = g(x_0, y_0) = g(\bar{x} + \delta x_0, \bar{y} + \delta y_0) = \underline{\bar{y} + \delta y_1}$$

$$\boxed{f(\bar{x} + \delta x_0) \approx f(\bar{x}) + \frac{df}{dx}(\bar{x})\,\delta x_0}$$ Aproximação linear

$$f(\bar{x} + \delta x_0, \bar{y} + \delta y_0) \approx f(\bar{x}, \bar{y}) + \overbrace{\frac{df}{dx}(\bar{x}, \bar{y})}^{a_{11}}\delta x_0 + \overbrace{\frac{df}{dy}(\bar{x}, \bar{y})}^{a_{12}}\delta y_0$$

$$g(\bar{x} + \delta x_0, \bar{y} + \delta y_0) \approx g(\bar{x}, \bar{y}) + \underbrace{\frac{dg}{dx}(\bar{x}, \bar{y})}_{a_{21}}\delta x_0 + \underbrace{\frac{dg}{dy}(\bar{x}, \bar{y})}_{a_{22}}\delta y_0$$

$$x_1 = \bar{x} + \delta x_1 = f(\bar{x}+\delta x_0, \bar{y}+\delta y_0) = f(\bar{x},\bar{y}) + a_{11}\delta x_0 + a_{12}\delta y_0$$

$$y_1 = \bar{y} + \delta y_1 = g(\bar{x}+\delta x_0, \bar{y}+\delta y_0) = g(\bar{x},\bar{y}) + a_{21}\delta x_0 + a_{22}\delta y_0$$

$$\cancel{f(\bar{x},\bar{y})} + \delta x_1 = \cancel{f(\bar{x},\bar{y})} + a_{11}\delta x_0 + a_{12}\delta y_0$$

$$\cancel{g(\bar{x},\bar{y})} + \delta y_1 = \cancel{g(\bar{x},\bar{y})} + a_{21}\delta x_0 + a_{22}\delta y_0$$

$$\begin{cases} \delta x_1 = a_{11}\delta x_0 + a_{12}\delta y_0 \\ \delta y_1 = a_{21}\delta x_0 + a_{22}\delta y_0 \end{cases}$$

$$\vec{\delta}_1 = A\vec{\delta}_0$$

$$\begin{pmatrix} \delta x_1 \\ \delta y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\boxed{A}} \begin{pmatrix} \delta x_0 \\ \delta y_0 \end{pmatrix}$$

$$\vec{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

autovalores e autovetores

$$\vec{v}_{n+1} = \underline{\underline{A}}\, \vec{v}_n \quad \rightsquigarrow \quad \vec{v}_n = C_+ R_+^n + C_- R_-^n$$

$$= C'_+ v_+ \lambda_+^n + C'_- v_- \lambda_-^n$$

perturbação cresce $\Rightarrow$ sist instável $\rightarrow$ autovalor dominante em módulo > 1

perturbação decresce $\Rightarrow$ sist estável $\rightarrow$ autovalor dominante em módulo é < 1

A nossa matriz A das perturbações é dada por $\left.\begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}\right|_{(\bar{x},\bar{y})}$

matriz jacobiana

$$\begin{cases} H_{n+1} = \lambda H_n e^{-aP_n} = f(H_n, P_n) \\ P_{n+1} = c\left(1 - e^{-aP_n}\right) H_n = g(H_n, P_n) \end{cases}$$

Matriz jacobiana

$$J = \begin{pmatrix} \frac{df}{dH} & \frac{df}{dP} \\ \frac{dg}{dH} & \frac{dg}{dP} \end{pmatrix} = \begin{pmatrix} \lambda e^{-aP} & -a\underbrace{\lambda H e^{-aP}}_{H} \\ C(1 - e^{-aP}) & cH(ae^{-aP}) \end{pmatrix} = \begin{pmatrix} \lambda e^{-aP} & -aH \\ c(1 - e^{-aP}) & \frac{ac}{\lambda} H \end{pmatrix}$$

$$\Rightarrow (H_0, P_0) = (0,0) \qquad J\Big|_{(H_0,P_0)} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow J_{(0,0)} = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}}$$

autovalores dessa matriz

matriz diagonal tem autovalores dados pela diagonal

autovalores de $J(0,0) = \{\lambda, 0\}$

$|\lambda| < 1 \Rightarrow (0,0)$ é estável

$|\lambda| > 1 \Rightarrow (0,0)$ é instável

$$\Rightarrow (H_1, P_1) = \left(\frac{1}{ac}\frac{\lambda}{\lambda-1}\ln\lambda \; , \; \frac{\ln\lambda}{a} \right)$$

$$J\Big|_{(H_1,P_1)} = \begin{pmatrix} \frac{df}{dH} = \lambda e^{-aP} & \frac{df}{dP} = -aH \\ \frac{dg}{dH} = c(1-e^{-aP}) & \frac{dg}{dH} = \frac{ac}{\lambda}H \end{pmatrix}_{(H_1,P_1)}$$

$$\begin{pmatrix} \lambda e^{-a\frac{1}{a}\ln\lambda} = \lambda e^{+\ln\lambda^{-1}} & -\cancel{a}\left(\frac{1}{ac}\frac{\lambda}{\lambda-1}\ln\lambda\right) \\ c\left(1-e^{+a\frac{1}{a}\ln\lambda^{-1}}\right) & \frac{\cancel{ac}}{\cancel{\lambda}}\left(\frac{1}{ac}\frac{\lambda}{(\lambda-1)}\ln\lambda\right) \end{pmatrix} = \begin{pmatrix} \lambda\lambda^{-1} & -\frac{1}{c}\frac{\lambda}{\lambda-1}\ln\lambda \\ c(1-\lambda^{-1}) & \frac{\ln\lambda}{(\lambda-1)} \end{pmatrix}$$

$$J_{(H_1,P_1)} = \begin{pmatrix} \underset{\sim}{1} & \dfrac{-\lambda}{c(\lambda-1)} \ln \lambda \\ c\left(1-\dfrac{1}{\lambda}\right) & \underset{\sim}{\dfrac{\ln \lambda}{\lambda-1}} \end{pmatrix}$$

Traço de uma matriz qualquer

é $\sum_i d_{ii} = \sum_i \Gamma_i$

Det de uma matriz qualquer

é $\underset{\sim}{\det(A)} = \prod \Gamma_i$

$x^2 - \beta x + \gamma = 0$ → Polinômio característico

β → traço da matriz

γ → determinante da matriz

$$J\Big|_{(H_1,P_1)} \Rightarrow \det(J_{H,P}) = \frac{\ln \lambda}{\lambda-1} - \cancel{c}\left(\frac{\cancel{\lambda-1}}{\cancel{\lambda}}\right)\left(\frac{-\cancel{\lambda}}{\cancel{c}(\cancel{\lambda-1})}\ln \lambda\right)$$

$$\frac{\ln \lambda}{\lambda-1} + \ln \lambda$$

$$\Gamma_1 \Gamma_2 = \frac{\ln \lambda}{\lambda - 1} - \ln \lambda$$

$|\Gamma_1|$ e $|\Gamma_2| < 1 \rightarrow$ estável

$$|\Gamma_1 \Gamma_2| < 1$$

$$\ln \lambda \left(\frac{1}{\lambda - 1} - 1 \right)$$

$$\ln(\underbrace{1 + x}_{\lambda}) \sim x \qquad x = \lambda - 1$$

$$\ln \lambda \left(\frac{1 - \lambda + 1}{\lambda - 1} \right)$$

$$\Gamma_1 \Gamma_2 = \underbrace{\ln \lambda \left(\frac{\lambda}{\lambda - 1} \right)} = \ln(1 + x) \left(\frac{1 + x}{x} \right)$$

$$= \cancel{x} \, \frac{1 + x}{\cancel{x}} = 1 + x \Big| > 1$$

x $\lambda > 1 \Rightarrow \Gamma_1$ e $\Gamma_2 > 1$ é $\therefore$ (x_1, p_1) é instável

- $(H_0, P_0) = (0,0)$ é estável quando $|\lambda| < 1$

  é instável quando $|\lambda| > 1$

fertilidade dos hosp.

- $(H_1, P_1) = \left(\frac{1}{ac}\frac{\lambda}{\lambda-1}\ln\lambda,\ \frac{1}{a}\ln\lambda\right)$ é instável se $\lambda > 1$