EQUAÇÃO DE DIRAC

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1°

$$[\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, H]=0$$

Hamiltoniano de Dirac

$$H = \boldsymbol{\gamma}^0 \boldsymbol{\gamma}^i \cdot \boldsymbol{p} + \boldsymbol{\gamma}^0 m$$

$$H = \widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}} + \beta m$$

$$\widehat{\boldsymbol{\alpha}} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

Matriz Identidade

- Matrizes de Pauli ($\boldsymbol{\sigma}$);
- Matrizes 4x4 em formato 2x2.
- $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\Sigma}$

$$[\beta, \boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}]=0$$ → β é Matriz diagonal → $$\beta\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\beta = \begin{pmatrix} \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} & 0 \\ 0 & -\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} \end{pmatrix}$$

$$[\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}] = (\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}})(\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}) - (\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}})(\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}})$$

$$\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \cdot \widehat{\boldsymbol{p}}$$

$$\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \cdot \widehat{\boldsymbol{p}}$$

Projeta as componentes na direção do momento.

$$[\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}] = \begin{pmatrix} \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} \\ \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} & \mathbf{0} \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} \\ \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}} \end{pmatrix}$$

$$[\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}] = \begin{pmatrix} 0 & (\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}})^2 \\ (\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}})^2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} 0 & (\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}})^2 \\ (\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{p}})^2 & \mathbf{0} \end{pmatrix}$$

$$[\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, \widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{p}}] = [\boldsymbol{\Sigma} \cdot \widehat{\boldsymbol{p}}, H] = 0$$

2°

$[J, H] = 0$

Momento angular total

$$\boldsymbol{J} = \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma} = \mathbf{L} + \mathbf{S}$$

$$
\begin{aligned}
[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m, L_i] &= [\alpha_l p_l, \varepsilon_{ijk} x_j p_k] \\
&= \varepsilon_{ijk}\ \alpha_l\ [p_l, x_j]\ p_k \\
&= \varepsilon_{ijk}\ \alpha_l (-i\delta_{jl})\ p_k \\
[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m, L_i] &= -i\ \varepsilon_{ijk}\ \alpha_j\ p_k
\end{aligned}
$$

O hamiltoniano não comuta com o momento angular.

$$
\begin{aligned}
&[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m, \boldsymbol{\Sigma}_i] = [\alpha_k p_k, \boldsymbol{\Sigma}_i] = [\alpha_k, \boldsymbol{\Sigma}_i]\ p_k \\
&= 2\ i\ \varepsilon_{kij} \boldsymbol{\sigma}_j\ p_k \\
&= 2\ i\ \varepsilon_{ijk} \boldsymbol{\sigma}_j\ p_k
\end{aligned}
$$

O hamiltoniano não comuta com o momento angular de Spin.

$$[\boldsymbol{\sigma}_i, \boldsymbol{\Sigma}_j] = \begin{pmatrix} \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i \end{pmatrix} = [\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j] \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\boldsymbol{\sigma}_i, \boldsymbol{\Sigma}_j] = 2\ i\ \varepsilon_{ijk} \boldsymbol{\sigma}_k$$

$$[A, B + C] = [A, B] + [A, C]$$

$\hbar = 1$

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p}, L_i + \frac{\hbar}{2}\Sigma_i\right] = [\boldsymbol{\alpha}\cdot\boldsymbol{p}, L_i] + \frac{1}{2}[\boldsymbol{\alpha}\cdot\boldsymbol{p}, \Sigma_i]$$

$$= -i\,\varepsilon_{ijk}\,\alpha_j\,p_k + \frac{1}{2}\left(2\,i\,\varepsilon_{ijk}\alpha_j p_k\right)$$

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p}, L_i + \frac{\hbar}{2}\Sigma_i\right] = 0$$

$$[\boldsymbol{J}, H] = 0$$

3°

$$U_p = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \beta^\dagger$$

$$\mathcal{P} = \pi U_p$$ → Operador unitário e invariante sob uma transformação de paridade

Operador Paridade Total

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}$$

$$U_p \boldsymbol{\alpha} U_p^\dagger = -\boldsymbol{\alpha}$$

$$U_p \beta U_p^\dagger = \beta$$

$$U_p^2 = 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4°

$\tilde{C}\tilde{C}^{-1} = 1$ $\tilde{C}(\boldsymbol{\gamma}^{\mu})^{*}\tilde{C}^{-1} = -\boldsymbol{\gamma}^{\mu}$ $\tilde{C} = \boldsymbol{i\gamma}^{2}$

Klein-Gordon

$\psi_{partícula}(x,t) \equiv \psi_{E>0}(x,t)$ → $\left(i\,\boldsymbol{\gamma}^{\mu}\partial_{\mu} - e\boldsymbol{\gamma}^{\mu}A_{\mu} - m\right)\psi(x,t) = 0$

$\psi_{antipartícula}(x,t) \equiv \psi^{*}_{E<0}\,(x,t)$ → $\left(-i\,(\boldsymbol{\gamma}^{\mu})^{*}\partial_{\mu} - e(\boldsymbol{\gamma}^{\mu})^{*}A_{\mu} - m\right)\psi(x,t)^{*} = 0$

Equação de Dirac com campo eletromagnético

↓

$\tilde{C}\tilde{C}^{-1} = 1$ $\tilde{C}(\boldsymbol{\gamma}^{\mu})^{*}\tilde{C}^{-1} = -\boldsymbol{\gamma}^{\mu}$ ←

$$\left(i\boldsymbol{\gamma}^{\mu}\partial_{\mu} + e\boldsymbol{\gamma}^{\mu}A_{\mu} - m\right)\tilde{C}\,\psi(x,t)^{*} = 0$$

Equação do Pósitron

$$\boldsymbol{i\gamma^2}=\begin{pmatrix} 0 & \boldsymbol{i\sigma_2} \\ -i\boldsymbol{\sigma_2} & \mathbf{0} \end{pmatrix}= \tilde{C}$$

$$\boldsymbol{\sigma_2}=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boldsymbol{\gamma^\mu} =\begin{pmatrix} 0 & \boldsymbol{\sigma}_\mu \\ -\boldsymbol{\sigma}_\mu & 0 \end{pmatrix}$$

$$\tilde{C}\tilde{C}^{-1} = i\begin{pmatrix} 0 & \boldsymbol{\sigma_2} \\ -\boldsymbol{\sigma_2} & \mathbf{0} \end{pmatrix} i\begin{pmatrix} 0 & \boldsymbol{\sigma_2} \\ -\boldsymbol{\sigma_2} & \mathbf{0} \end{pmatrix}$$

$$= i^2\begin{pmatrix} -\boldsymbol{\sigma_2}^2 & 0 \\ 0 & -\boldsymbol{\sigma_2}^2 \end{pmatrix}$$

$$=(-1)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{C}\tilde{C}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{C}(\boldsymbol{\gamma^\mu})^*\tilde{C}^{-1} = i\begin{pmatrix} 0 & \boldsymbol{\sigma_2} \\ -\boldsymbol{\sigma_2} & \mathbf{0} \end{pmatrix}\begin{pmatrix} 0 & -\boldsymbol{\sigma}_\mu \\ \boldsymbol{\sigma}_\mu & \mathbf{0} \end{pmatrix} i\begin{pmatrix} 0 & \boldsymbol{\sigma_2} \\ -\boldsymbol{\sigma_2} & \mathbf{0} \end{pmatrix}$$

$$= (i)^2\begin{pmatrix} 0 & \boldsymbol{\sigma_2} \\ -\boldsymbol{\sigma_2} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}_\mu\boldsymbol{\sigma_2} & 0 \\ 0 & \boldsymbol{\sigma}_\mu\boldsymbol{\sigma_2} \end{pmatrix}$$

$$= (-1)\begin{pmatrix} 0 & \boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_2^2 \\ -\boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_2^2 & 0 \end{pmatrix}$$

$$\tilde{C}(\boldsymbol{\gamma^\mu})^*\tilde{C}^{-1} = (-1)\begin{pmatrix} 0 & \boldsymbol{\sigma}_\mu \\ -\boldsymbol{\sigma}_\mu & 0 \end{pmatrix}$$

Referências

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